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Financial equilibrium with differential information in a production economy: a basic model of 'generic' existence

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Abstract

We study the existence of equilibria in two-period production economies, where asymmetrically informed agents exchange securities, on incomplete financial markets, and commodities, on spot markets, with a perfect foresight of future prices. Extending our pure-exchange existence theorems, we show that equilibria exist for an open dense set of economies, parametrized by the assets' payoffs, and for all economies, whose assets are nominal or numeraire. The model covers all types of private ownership - sole proprietorship, partnership or corporations - and all sectors consistent with competition, i.e., with non-increasing returns to scale. It is a step towards proving existence in stochastic production economies, and the full existence of sequential equilibria with production, when perfect price foresight fails to prevail.

Key words: sequential equilibrium, production economies, perfect foresight, existence, rational expectations, financial markets, asymmetric information, arbitrage.

JEL Classification: D52

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1 Introduction

This paper extends De Boisdeffre's (2007 and 2017) existence theorems of two-period pure-exchange financial economies with differential information to similar economies with production. It shows that equilibrium exists for an open dense set of economies parametrized by assets' payoffs, and for all economies, whose financial structure is nominal or numeraire. We call this existence property "*weakly generic*".

The model has two periods, with an a priori uncertainty upon which state of nature will prevail tomorrow, out of a finite space, S. There are finite sets, I, of consumers, and J, of producers. Asymmetric information amongst them is represented, ex ante, by idiosyncratic private signals, $S_k \subset S$, which correctly inform every agent, $k \in I \cup J$, that tomorrow's true state will lie in S_k. Non restrictively, from De Boisdeffre (2016), the signals, (S_k), preclude all arbitrage opportunity on the financial market, where agents may trade, unrestrictively, nominal or real assets.

Agents exchange finitely many goods and services on spot markets, serving as inputs or outputs in production, or as final consumption goods, and whose prices are commonly observed or perfectly anticipated. The means and fruits of production reward sole proprietors, or partners, in joint ventures, or the shareholders of corporations. Consistently with competition, the model covers all sectors with non-increasing returns to scale. Consumers' preferences need not be ordered. The current existence proof, building on De Boisdeffre's (2017), displays specific complexities due to production. It is a step towards proving existence in stochastic production economies, and the full existence when anticipations fail to be perfect. The following Section 2 presents the model, Section 3 states and proves our Theorem, Section 4 deals with numeraire assets and an Appendix proves a Lemma.

2 The model

We consider a production economy with two periods, $t \in \{0, 1\}$, and an ex ante uncertainty about which state of nature will prevail ex post. Agents exchange goods and services, serving as inputs or final consumption goods. They trade assets of all kinds on typically incomplete financial markets. The sets, I, J, S, H and J_0, respectively, of consumers, producers, states of nature, goods and services, and assets, are all finite. The non random state at the first period ($t = 0$) is denoted by $s = 0$ and we let $\Sigma' := \{0\} \cup \Sigma$, for every subset, Σ, of S. Similarly, $l = 0$ denotes the unit of account and we let $H' := \{0\} \cup H$.

2.1 Markets and information

Producers and consumers, $k \in K := I \cup J$, exchange goods and services, $h \in H$, on both periods' spot and labour markets, for the purpose of the final consumption of consumers, or the use of inputs by producers, which include raw materials, intermediary goods and labour. To simplify exposition, we assume that H is the union of H_1, the set of final consumption goods (including services & leisure), and H_2, that of inputs. We restrict, at no cost, spot prices to the set, $\Delta := \{p \in \mathbb{R}^H_+ : \|p\| \leqslant 1\}$. We refer to a pair of state and price, $\omega := (s, p_s) \in S \times \Delta$, as a forecast, and let $\Omega := S \times \Delta$ be their set.

Producers, $j \in J := J_1 \cup J_2$, are of two types: corporations (when $j \in J_1$), whose shares (called equities) can be exchanged on the stock market, and all other producers, $j \in J_2$, consisting of sole proprietors and joint ventures. Consumers may exchange, at $t = 0$, finitely many assets, or securities, $j \in J_0$ (with $\#J_0 \leqslant \#\underline{\mathbf{S}}$), whose yields, at $t = 1$, are exogenous and conditional on the realizations of forecasts, $\omega \in \Omega$.

They may also exchange equities on a stock market, or participations in corporations, $j \in J_1$, whose conditional yields across forecasts are endogenous. The equities' payoffs, and bounded portfolio set, $[0,1]^{J_1}$, are presented below. The generic agent's portfolio, $z := (z_0, z_1) := ((z_0^j), (z_1^j)) \in Z^I := \mathbb{R}^{J_0} \times [0,1]^{J_1}$, summarizes the positions that she takes on each asset or equity, positive, if bought, negative, if sold short. Producer's portfolio set, $Z^J \subsetneq Z^I$, will be restricted. At market price $q \in \mathbb{R}^{J_0} \times \mathbb{R}^{J_1}$, the purchase of a portfolio, $z \in Z^I$, costs $q \cdot z$ units of account at $t = 0$, against delivery of conditional payoffs at $t = 1$.

Assets' payoffs at $t = 1$ may be nominal (i.e., pay in cash) or real (pay in goods) or a mix of both. They define a matrix, V, which is identified to the continuous map, $V : \Omega \to \mathbb{R}^{J_0}$, relating forecasts, $\omega := (s, p) \in \Omega$, to the rows, $V(\omega) \in \mathbb{R}^{J_0}$, of all assets' cash payoffs, delivered if state s and price p obtain.

At $t = 0$, each agent, $k \in K$, receives a private information signal, $S_k \subset S$, which correctly informs her that tomorrow's true state will be in S_k, and we let $\underline{\mathbf{S}} := \cap_{k \in K} S_k$ be their pooled information. We assume costlessly, from De Boisdeffre (2016), that, at the time of trading and given price expectations, agents have inferred all information required to preclude unlimited arbitrage opportunities on financial markets.

For every price, $p := (p_s) \in \Delta^S$, we let $V(p)$ be the $\underline{\mathbf{S}} \times J_0$ payoff matrix, whose generic row is $V(s, p_s)$ (for $s \in \underline{\mathbf{S}}$) and $< V(p) >$ be its span in $\mathbb{R}^{\underline{\mathbf{S}}}$. Before presenting agents' behaviours, we recall well-behaved properties of the financial structure, in the following Claim 1. We let $\mathcal{V}$ be the set of $(S \times H') \times J_0$ exogenous payoff matrixes defined as the matrix, V, above. That set is equiped with the same notations as above (for $V \in \mathcal{V}$). For more details, we refer the reader to De Boisdeffre (2017).

Claim 1 *Let* $\Lambda := \{\widetilde{V} \in \mathcal{V} : rank\ \widetilde{V}(p) = \#J_0,\ \forall p := (p_s) \in \Delta^S\}$ *and* $M \in \Lambda$ *be given. The following Assertions hold:*

(i) Λ *is open and everywhere dense, in the set* $\mathcal{V}$*;*

(ii) $\nexists((z_k), p) \in (\mathbb{R}^{J_0})^K \backslash \{0\} \times \Delta^S : \sum_{k \in K} z_k = 0$ *and* $M(s, p_{s_k}) \cdot z_k \geqslant 0,\ \forall (k, s_k) \in K \times S_k$.

Proof The proof is given under Claim 1 in De Boisdeffre (2017). □

Each agent, $k \in K$, forms idiosyncratic anticipations, $p_{ks} \in \mathbb{R}^{H_1}_{++}$, of commodity prices in each (possible) state, $s \in S_k \backslash \underline{\mathbf{S}}$. To alleviate notations, we assume that $p_{ks} = p_{k's} := (\overline{p}^h_s)_{h \in H_1}$, for every triple $(k, k', s) \in K \times K \times S_k \cap S_{k'} \backslash \underline{\mathbf{S}}$. Thus, we restrict tomorrow's prices to $P := \{p := (p^h_s) \in \Delta^S : p^h_s = \overline{p}^h_s,\ \forall (s, h) \in S \backslash \underline{\mathbf{S}} \times H_1\}$. Agents' symmetric forecasts across states, $s \in S \backslash \underline{\mathbf{S}}$, simplifies exposition w.l.o.g. We restrict first period prices to $P_0 := \{(p_0, q) \in \Delta \times \mathbb{R}^{J_0} \times \mathbb{R}^{J_1} : \|q\| \leqslant 1\}$, whose bounds are normalized to one for convenience and could be replaced by any positive value.

Given (S_k), the generic consumer, $i \in I$, has for consumption set $X^o_i := (\mathbb{R}^{H_1}_+ \times \{0\}^{H_2})^{S'_i}$. Similarly, each producer, $j \in J$, elects a production plan within a production set, $Y^o_j \subset (\mathbb{R}^H)^{S'_j}$, representing her technology constraints.

2.2 The producer's behaviour

Throughout a generic producer, $j \in J$, is given, and always referred to.

Agent j has a production set, $Y^o_j \subset (\mathbb{R}^{H_1}_+ \times (-\mathbb{R}^{H_2}_+))^{S'_j}$, characterizing the feasible input-output bundle pairs, $(y_0, y_s) \in \mathbb{R}^H \times \mathbb{R}^H$, across states, $s \in S_j$, that her technology permits. The components of a production plan, $y := (y^h_s) \in Y^o_j$, are positive, if h is produced, and negative, if used as an input. Many goods and services are not used or produced, so appear with zero components in production plans, $y \in Y^o_j$. If production demands time, inputs will typically be used at $t = 0$ and outputs released at $t = 1$. We make standard assumptions on technology as follows:

Assumption A1, *Y_j^o is closed and convex;*

Assumption A2, *$Y_j^o \cap (\mathbb{R}_+^H)^{S'_j} = \{0\}$ and $Y_j^o - (\mathbb{R}_+^H)^{S'_j} \subset Y_j^o$;*

Assumption A3, *$\forall (j, y) \in J \times (\mathbb{R}_+^H)^{S'_j}$, $(y + Y_j^o) \cap (\mathbb{R}_+^H)^{S'_j}$ is bounded.*

The above conditions have a clear economic meaning and imply, as standard, the technology's non-increasing returns to scale, consistently with competition. Moreover, from Assumption *A3* and the limited quantity of total inputs and endowments in the economy, production will be bounded.

The producer values each state, $s \in S$, of a state price, π_s^j, such that, $\pi_s^j > 0$, for every $s \in S_j$, $\pi_s^j = 0$, for every $s \in S \backslash S_j$, and $\sum_{s \in S_j} \pi_s^j \leqslant 1$. At prices $(p_s) \in \Delta \times P$, her discounted profit of a production plan, $y := (y_s) \in Y_j^o$, is thus: $p_0 \cdot y_0 + \sum_{s \in S_j} \pi_s^j \; p_s \cdot y_s$.

Once all agents have inferred their arbitrage-free information signals, (S_k), as assumed above, any restriction of access to the stock market, defined as the joint financial and equity markets, for a firm, does not change the equilibrium outcome, provided the firm had no deficit constraint. This is because the producer keeps an indirect, yet full, access to the stock market via the eventual owners, namely, consumers, whose participations to the stock market are unrestricted (see, e.g., Magill & Quinzii, 1996, chap. 6). Hereafter, we assume that producers have a portfolio set, Z^J, which is a sub-vector space of $\mathbb{R}^{J_0} \times \{0\}^{J_1}$. Hence, producers' participations to the asset market may be restricted or not. We do not allow producers to take crossed participations in corporations not only because they eventually belong to consumers, but also (and mainly) for expositional purposes. Our existence Theorem would be unchanged if we did allow crossed participations, but definitions and notations would be overwhelming, as will be apparent later. Typically, producers would have access to a loan and credit market, as in the real world, to start a business.

Restricted participation is not, hereafter, innocuous because our model is one of '*limited liability*'. That is, once a company is created (with the possible contributions of owners or shareholders at $t = 0$), it is not allowed to bankruptcy, at $t = 1$. If, moreover, the producer is endowed with physical wealth, she may offer it as collateral for a (possible) loan and make profit in any state. Consistently with the fact that a company runs its business with available stocks, capital, equipment, etc., we make the following technical assumption, that the producer is endowed with some physical assets, $e_j := (e_{js}) \in (\mathbb{R}_+^H)^{S'_j}$, which grant the bundle of goods, $e_{js} \in \mathbb{R}_+^H$, in each state $s \in S'_j$, if it prevails. This endowment is such that:

Assumption A4, *$\forall j \in J,\ e_j \in (\mathbb{R}_{++}^H)^{S'_j}$, where $\mathbb{R}_{++} := \{x \in \mathbb{R} : x > 0\}$.*

Remark 1 Assumption *A4* (and *A6* below) are tantamount to assuming that every agent is endowed with some wealth (or cash) in any state. This assumption is natural for companies, which always detain physical assets. Indeed, the spot price is observed or perfectly anticipated and, with some wealth available, agents can always exchange the total endowment of the economy on spot markets, without changing their strategies, to meet the above Assumptions.

Thus, for all price system, $\varpi := ((p_0, q), p := (p_s)) \in P_0 \times P$, the firm's budget set is:

$$B_j(\varpi) = \{(y, z = (z_0, 0)) \in Y_j^o \times Z^J : p_0 \cdot (y_0 + e_{i0}) - q \cdot z \geqslant 0,\ \ p_s \cdot (y_s + e_{is}) + V(s, p_s) \cdot z_0 \geqslant 0, \forall s \in S_j\}$$

From Assumptions *A4*, the interior of $B_j(\varpi)$ may not be empty for non-zero spot prices. The generic producer, $j \in J$, has an objective function, her profit, or present return of her strategy, namely, for every $\varpi := ((p_0, q, p := (p_s)) \in P_0 \times P$ and every $(y, z := (z_0, 0)) \in B_j(\varpi)$:

$$\Pi_j(\varpi, (y, z)) = p_0 \cdot (y_0 + e_{j0}) - q \cdot z + \textstyle\sum_{s \in S_j} \pi_s^j\ (p_s \cdot (y_s + e_{is}) + V(s, p_s) \cdot z_0).$$

Her behaviour is to maximise her profit in the budget set, that is, to select one element in $\eta_j(\varpi) := \arg\max \Pi_j(\varpi, (y, z))$ for $(y, z) \in B_j(\varpi)$, if nonempty. We will show this set is, indeed, non empty at clearing market prices. In any case, the producer makes a decision, that is, chooses one strategy $(y_j, z_j := (z_{j0}, 0)) \in B_j(\varpi)$, henceforth, set as given for all other agents. This strategy results in the endogeneous yields, $r_{0j}(\varpi, (y_j, z_j)) := (p_0 \cdot (y_{j0} + e_{j0}) - q \cdot z_j)$, at $t = 0$, $r_{sj}(\varpi, (y_j, z_j)) := (p_s \cdot (y_{js} + e_{js}) + V(s, p_s) \cdot z_{j0})$, in each state $s \in S_j$, and $r_{sj}(\varpi, (y_j, z_j)) := 0$, in each state $s \in S \backslash S_j$.

2.2 The consumer's behaviour and concept of equilibrium

Throughout, a generic consumer, $i \in I$, is given.

Agent i receives an endowment, $e_i := (e_{is}^h)$, granting the conditional bundles of goods and services, $e_{i0} \in \mathbb{R}_+^H$ at $t = 0$, and $e_{is} \in \mathbb{R}_+^H$, in each state, $s \in S_i$, if it prevails. The endowment in services consists in an amount of labour with certain skills, called workforce, that she may offer to producers. The agent consumes leisure if she does not offer her full workforce. Her consumption set, $X_i^o := (\mathbb{R}_+^{H_1} \times \{0\}^{H_2})^{S_i'}$, lets every consumption in inputs - only used by firms - be zero.

In addition to their endowments, the consumer may receive dividends. Indeed, each firm, $j \in J$, belongs to consumers, either exclusively, or as partners or shareholders. Each agent, $i \in I$, detains initial shares (which may be zero), $\overline{z}_i^j \in [0, 1]$, of each company, $j \in J$, which satisfy $\sum_{i \in I} \overline{z}_i^j = 1$. Most of these shares should be zero. For each consumer, $i \in I$, we denote by $\overline{z}_i := (0, ..., 0, (\overline{z}_i^j)_{j \in J_1}) \in \{0\}^{J_0} \times [0, 1]^{J_1} \subset Z^I$ her portfolio endowment. We recall ownership breaks down into three categories:

*** sole proprietorship**

It occurs if a production unit, $j \in J_2$, is owned by one person, $i \in I$ (i.e., $\overline{z}_{2i}^j = 1$). Then, there is no managerial conflict, for the sole proprietor decides in her own will of the production strategy, $(y_j, z_j) \in B_j(\varpi)$, given prices, $\varpi \in P_0 \times P$. However, selling the company, or shares of it, may turn out to be difficult and the owner is, hence, assumed to keep her property until the second period.

*** partnership**

It occurs when a limited number of partners, $i \in I_j \subset I$, have agreed to create a joint venture, $j \in J_2$, and on the shares, $\overline{z}^j_{2i} > 0$, of each member. Partners may also have difficulties in retrading their shares, which they keep until the next period. To the difference of sole owners, partners may have different assessments of future income streams, resulting in potential management disagreements. Conflicts can often be resolved by side payments, whose study is beyond our scope. Joint ventures only create if partners have reached a managerial agreement.

Partners would be expected to share their information so that $S_i = S_j$, for every $i \in I_j$. If, eventually, partners do not share the same beliefs, the shareholder, $i \in I_j$, of the firm, $j \in J_2$, expects to receive her share of profits in any state $s \in S_i \cap S_j$.

*** corporations**

Corporations' shares may be exchanged on the stock market by the generic consumer, deciding to keep or change her initial shares, $(\overline{z}^j_i) \in \mathbb{R}^{J_1}$, for new ones, along her perceived interests, at a market price, $q_1 \in \mathbb{R}^{J_1}$. Corporations are run by an appointed manager and owned by private shareholders, (possibly) meeting in boards and always free to exchange their participations on the stock market.

To the difference of assets $(j \in J_0)$, corporations $(j \in J_1)$ have endogenous yields, defined above, and their purchase and sale are bounded in practice. Indeed, corporations are physical units, which cannot be bought or sold short an unlimited number of times. Transactions are thus bounded. W.l.o.g. on the bounds, we assume that a corporation cannot be sold short and cannot be bought more than one time by any agent, that is, consumers' porfolio set is $Z^I := \mathbb{R}^{J_0} \times [0,1]^{J_1}$. □

Given prices, $\varpi := ((p_0, q), p := (p_s)) \in P_0 \times P$, and the producers' decisions, $Y := [(y_j, z_j)] \in \times_{j \in J} B_j(\varpi)$, the generic i^{th} agent's budget set is:

$$B_i(\varpi, Y) := \{(x := (x_s), z := (z_0, z_1)) \in X_i^o \times Z^I :$$
$$p_0 \cdot (x_0 - e_{i0}) \leqslant -q \cdot (z - \overline{z}_i) + \textstyle\sum_{j \in J_1} z_1^j \; r_{0j}(\varpi, (y_j, z_j)) + \sum_{j \in J_2} \overline{z}_i^j \; r_{0j}(\varpi, (y_j, z_j)) \quad and$$
$$p_s \cdot (x_s - e_{is}) \leqslant V(s, p_s) \cdot z_0 + \textstyle\sum_{j \in J_1} z_1^j \; r_{sj}(\varpi, (y_j, z_j)) + \sum_{j \in J_2} \overline{z}_i^j \; r_{sj}(\varpi, (y_j, z_j)), \; \forall s \in S_i\}.$$

Each consumer, $i \in I$, has preferences, $\prec_i$, represented, for all $x \in X_i^0$, by the set, $P_i(x) := \{y \in X_i^o : x \prec_i y\}$, of strictly preferred consumptions to x. In the above economy, denoted by $\mathcal{E} = \{(I, J_0, J_1, J_2, H), V, (S_k)_{k \in K}, (Y_j^o)_{j \in J_1 \cup J_2}, (X_i^o, e_i, \prec_i)_{i \in I}\}$, agents optimise their objective in their budget sets. So the concept of equilibrium:

Definition 1 *A collection of prices,* $\varpi := ((p_0, q := (q_0, q_1)), p := (p_s)) \in P_0 \times P$, *and strategies,* $Y := [(y_j, z_j)] \in \times_{j \in J} B_j(\varpi)$ *and* $X := [(x_i, z_i)] \in \times_{i \in I} B_i(\varpi, Y)$, *defines an equilibrium of the economy,* $\mathcal{E}$, *if the following conditions hold:*

(a) $\forall j \in J, \; (y_j, z_j) \in \arg\max \Pi_j(\varpi, (y, z))$ *for* $(y, z) \in B_j(\varpi)$*;*

(b) $\forall i \in I, \; (x_i, z_i) \in B_i(\varpi, Y)$ *and* $P_i(x_i) \times Z^I \cap B_i(\varpi, Y) = \varnothing$*;*

(c) $\sum_{i \in I} (x_{is} - e_{is}) = \sum_{j \in J} (y_{js} + e_{js}), \; \forall s \in \underline{\mathbf{S}}'$*;*

(d) $\sum_{k \in K} z_k = \sum_{i \in I} \overline{z}_i$*.*

The economy, $\mathcal{E}$, is called standard under conditions ***A1*** to ***A4*** and the following:

Assumption A5*, (monotonicity):* $\forall (i, x, y) \in I \times X_i^o \times X_i^o, \; (x \leqslant y, \; x \neq y) \Rightarrow (x \prec_i y)$;

Assumption A6*,* $\forall i \in I, \; e_i \in (\mathbb{R}_{++}^H)^{S_i'}$;

Assumption A7*,* $\forall i \in I$, $\prec_i$ *is lower semicontinuous convex-open-valued and such that* $x \prec_i x + \lambda(y - x)$, *whenever* $(x, y, \lambda) \in X_i^o \times P_i(x) \times]0, 1]$.

3 The existence Theorem and proof

Theorem 1 *If its payoff map, $p \in P \mapsto V(p)$, is constant, or payoff matrix, V, is in the open dense set, Λ, of Claim 1, a standard economy, $\mathcal{E}$, admits an equilibrium.*

Under the condition of Claim 1-(ii), which is equivalent to the no-arbitrage condition and always assumed above, non restrictively along De Boisdeffre (2016), the proof of Theorem 1 is the same whether assets be nominal, or the payoff matrix, V, be in Λ. So, w.l.o.g., we henceforth assume that the economy, $\mathcal{E}$, is standard and that $V \in \Lambda$, which is set as given throughout. Some parts of the proof are akin to De Boisdeffre's (2017), to which we will refer, so as to avoid unnecessary lengths. Other parts are specific to production economies, and will be detailed hereafter.

3.1 Bounding the economy

For every $(i, j, \varpi := ((p_0, q := (q_0, q_1)), p, Y := [(y_j, z_j)]) \in I{\times}J{\times}P_0{\times}P{\times}(\times_{j\in J}Y_j{\times}Z_0)$, let:

$$\overline{B}_j(\varpi) := \{(y, z := (z_0, 0)) \in Y_j^o{\times}Z^J : \ p_0{\cdot}y_0 - q{\cdot}z + 1 \geqslant 0 \ \ and \ \ p_s{\cdot}y_s + V(s, p_s){\cdot}z_0 + 1 \geqslant 0, \forall s \in S_j\};$$

$$\overline{B}_i(\varpi, Y) := \{ \ (x := (x_s), z := (z_0, z_1) \in X_i^o \times Z^I :$$
$$p_0 \cdot (x_0 - e_{i0}) \leqslant 1 - q \cdot (z - \overline{z}_i) + \textstyle\sum_{j\in J_1} z_1^j \ |r_{0j}(\varpi, (y_j, z_j))| + \sum_{j\in J_2} \overline{z}_i^j \ |r_{0j}(\varpi, (y_j, z_j))|,$$
$$p_s{\cdot}(x_s - e_{is}) \leqslant 1{+}V(p, s){\cdot}z_0{+}\textstyle\sum_{j\in J_1} z_1^j \ |r_{sj}(\varpi, (y_j, z_j))|{+}\sum_{j\in J_2} \overline{z}_i^j|r_{sj}(\varpi, (y_j, z_j))|, \forall s \in S_i\},$$

where $|r_{sj}(\varpi, (y_j, z_j))| := \sqrt{r_{sj}(\varpi, (y_j, z_j))^2}$ (for all $(j, s) \in J \times S_i'$) and let

$$\overline{\mathcal{A}}(\varpi, Y) := \{ \ [(x_i, z_i)] \in \times_{i\in I} \ \overline{B}_i(\varpi, Y) : \textstyle\sum_{i\in I}(x_{is} - e_{is}) \leqslant \sum_{j\in J} \ (y_{js}{+}e_{js}), \forall s \in \underline{\mathbf{S}}' \ \ and \ \ \sum_{k\in K} z_k = \sum_{i\in I} \overline{z}_i \ \}.$$

Lemma 1 $\exists r > 0 : \forall \varpi \in P_0{\times}P, \ \forall Y \in \times_{j\in J}\overline{B}_j(\varpi), \ \forall X \in \overline{\mathcal{A}}(\varpi, Y), \ \|X\| + \|Y\| < r$

Proof : see the Appendix. □

Lemma 1 permits to bound the economy. Thus, we define (along Lemma 1), for every $(i, j, \varpi, Y) \in I{\times}J{\times}P_0{\times}P{\times}(\times_{j\in J}\overline{B}_j(\varpi))$, the following convex compact sets:

$X_i := \{x \in X_i^o : \|x\| \leqslant r\}$ and $Y_j := \{y \in Y_j^o : \|y\| \leqslant r\}$,

$Z_0 := \{z \in Z^J : \|z\| \leqslant r\}$ and $Z_1 := \{z \in Z^I : \|z\| \leqslant r\}$;

$\mathcal{A}(\varpi, Y) := \overline{\mathcal{A}}(\varpi, Y) \cap (\times_{i\in I} X_i \times Z_1)$.

3.2 The existence proof

For every $(i, j, \varpi := ((p_0, q := (q_0, q_1)), p, Y := [(y_j, z_j)]) \in I \times J \times P_0 \times P \times (\times_{j\in J} Y_j \times Z_0)$, we start with the following definitions of modified budget sets for every agent:

$$B'_j(\varpi) := \{(y, z) \in Y_j^o \times Z^J : \ p_0 \cdot (y_0 + e_{j0}) - q \cdot z + \gamma_{(p_0,q)} \geqslant 0 \ \ and \ \ p_s \cdot (y_s + e_{js}) + V(s, p_s) \cdot z_0 + \gamma_{(s,p_s)} \geqslant 0, \forall s \in S_j\};$$

$$B''_j(\varpi) := \{(y, z) \in Y_j^o \times Z^J : \ p_0 \cdot (y_0 + e_{j0}) - q \cdot z + \gamma_{(p_0,q)} > 0 \ \ and \ \ p_s \cdot (y_s + e_{js}) + V(s, p_s) \cdot z_0 + \gamma_{(s,p_s)} > 0, \forall s \in S_j\};$$

$$B'_i(\varpi, Y) := \{ \ (x := (x_s), z := (z_0, z_1)) \in X_i^o \times Z_1 :$$
$$p_0 \cdot (x_0 - e_{i0}) \leqslant \gamma_{(p_0,q)} - q \cdot (z - \overline{z}_i) + \sum_{j\in J_1} z_1^j |\, r_{0j}(\varpi, (y_j, z_j)) + \gamma_{(p_0,q)}| + \sum_{j\in J_2} \overline{z}_i^j |r_{0j}(\varpi, (y_j, z_j)) + \gamma_{(p_0,q)}|,$$
$$p_s \cdot (x_s - e_{is}) \leqslant \gamma_{(s,p_s)} + V(p, s) \cdot z_0 + \sum_{j\in J_1} z_1^j \, |r_{sj}(\varpi, (y_j, z_j)) + \gamma_{(s,p_s)}| + \sum_{j\in J_2} \overline{z}_i^j |r_{sj}(\varpi, (y_j, z_j)) + \gamma_{(s,p_s)}|, \forall s \in S_i\};$$

$$B''_i(\varpi, Y) := \{ \ (x := (x_s), z := (z_0, z_1)) \in X_i^o \times Z_1 :$$
$$p_0 \cdot (x_0 - e_{i0}) < \gamma_{(p_0,q)} - q_0 \cdot z_0 - q_1 \cdot (z_1 - \overline{z}_i) + \sum_{j\in J_1} z_1^j \, |r_{0j}(\varpi, (x_j, z_j)) + \gamma_{(p_0,q)}| + \sum_{j\in J_2} \overline{z}_i^j |r_{0j}(\varpi, (x_j, z_j)) + \gamma_{(p_0,q)}|,$$
$$p_s \cdot (x_s - e_{is}) < \gamma_{(s,p_s)} + V(p, s) \cdot z_0 + \sum_{j\in J_1} z_1^j |r_{sj}(\varpi, (x_j, z_j)) + \gamma_{(s,p_s)}| + \sum_{j\in J_2} \overline{z}_i^j |r_{sj}(\varpi, (x_j, z_j)) + \gamma_{(s,p_s)}|, \forall s \in S_i\},$$

where $\gamma_{(p_0,q)} := 1 - \min(1, \|(p_0, q)\|)$, $\gamma_{(s,p_s)} := 1 - \|p_s\|$, for each $s \in \underline{\mathbf{S}}$ and $\gamma_{(s,p_s)} := 0$, for each $s \in S \backslash \underline{\mathbf{S}}$. The above correspondences satisfy the following properties:

Claim 2 *Let* $(i, j) \in I \times J$, $\varpi := ((p_0, q), p) \in P_0 \times P$ *and* $Y \in \times_{j\in J} Y_j \times Z_0$ *be given. The following Assertions hold:*

(i) $B''_i(\varpi, Y) \neq \varnothing$ *and* B''_i *is lower semicontinuous at* (ϖ, Y);

(i) $B''_j(\varpi) \neq \varnothing$ *and* B''_j *is lower semicontinuous at* ϖ;

(iii) B'_j *and* B'_i *are upper semicontinuous at* ϖ *and* (ϖ, Y), *respectively.*

Proof : The proof is given, mutatis mutandis, under De Boisdeffre's (2017) Claims 2 to 4, to which we refer the reader. □

Then, we introduce the following reaction correspondences on the convex compact set, $\Theta := P_0 \times P \times (\times_{j\in J} Y_j \times Z_0) \times (\times_{i\in I} X_i \times Z_1)$, namely, for every $(i,j) \in I \times J$ and every $\theta := (\varpi := ((p_0, q), p := (p_s)), Y := [(y_j, z_j)], [(x_i, z_i)]) \in \Theta$, we let:

$$\Psi_0(\theta) := \{((p_0', q'), p') \in P_0 \times P : \sum_{s\in \underline{\mathbf{S}}'} (p_s' - p_s) \cdot (\sum_{i\in I} (x_{is} - e_{is}) - \sum_{J\in J} (y_{js} + e_{js}))$$
$$+ (q' - q)) \cdot (\sum_{k\in K} z_k - \sum_{i\in I} \overline{z}_{1i}) > 0\};$$

$$\Psi_j(\theta) := \left\{ \begin{array}{lll} B_j'(\varpi) & if & (y_j, z_j) \notin B_j'(\varpi) \\ \{(y,z) \in B_j''(\varpi) : \Pi_j(\varpi, (y,z)) > \Pi_j(\varpi, (y_j, z_j))\} & if & (y_j, z_j) \in B_j'(\varpi) \end{array} \right\};$$

$$\Psi_i(\theta) := \left\{ \begin{array}{lll} B_i'(\varpi, Y) & if & (x_i, z_i) \notin B_i'(\varpi, Y) \\ B_i''(\varpi, Y) \cap P_i(x_i) \times Z_1 & if & (x_i, z_i) \in B_i'(\varpi, Y) \end{array} \right\}.$$

The above correspondences meet the following Claims 3 and 4.

Claim 3 *The following Assertions hold:*

(i) for each $k \in K \cup \{0\}$, Ψ_k is lower semicontinuous;

(ii) $\exists\ \theta^* := (\varpi^* := ((p_0^*, q^*), p^*), Y^* := [(y_j^*, z_j^*)],\ X^* := [(x_i^*, z_i^*)]) \in \Theta$, such that:

$*\ \forall ((p_0, q), p := (p_s)) \in P_0 \times P,\ \sum_{s\in \underline{\mathbf{S}}'} (p_s^* - p_s) \cdot (\sum_{i\in I} (x_{is}^* - e_{is}) - \sum_{J\in J} (y_{js}^* + e_{js}))$

$+ (q^* - q) \cdot (\sum_{k\in K} z_k^* - \sum_{i\in I} \overline{z}_i) \geqslant 0;$

$*\ \forall i \in I,\ (x_i^*, z_i^*) \in B_i'(\varpi^*, Y^*)$ *and* $B_i''(\varpi^*, Y^*) \cap P_i(x_i^*) \times Z_1 = \varnothing;$

$*\ \forall j \in J,\ (y_j^*, z_j^*) \in B_j'(\varpi^*)$ *and* $\{(y,z) \in B_j''(\varpi^*) : \Pi_j(\varpi^*, (y,z)) > \Pi_j(\varpi^*, (y_j^*, z_j^*))\} = \varnothing.$

Proof The proof is technical and given, mutatis mutandis, under De Boisdeffre's (2017) Claims 5 and 6, to which we refer the reader. □

Claim 4 *The following Assertions hold:*

(i) $\sum_{k\in K} z_k^* = \sum_{i\in I} \overline{z}_i;$

(ii) $X^* \in \overline{\mathcal{A}}(\varpi^*, Y^*)$, hence, $\|X^*\| + \|Y^*\| < r$;

(iii) $\forall s \in \underline{\mathbf{S}}',\ \sum_{i\in I} (x_{is}^* - e_{is}) = \sum_{J\in J} (y_{js}^* + e_{js})$ may be assumed.

Proof Assertion (i) From Claim 3, the following relations hold:

$p_s^*{\cdot}(\sum_{i\in I}(x_{is}^*\text{-}e_{is}){-}\sum_{J\in J}(y_{js}^*{+}e_{js}))\geqslant 0$, for every $s\in\underline{\mathbf{S}}'$, and $q^*{\cdot}(\sum_{k\in K}z_k^*{-}\sum_{i\in I}\overline{z}_i)\geqslant 0$.

From Claim 3, summing up budget constraints at $t=0$, for each $i\in I$, yields:

$\sum_{i\in I}p_0^*{\cdot}(x_{i0}^*-e_{i0}^*)\leqslant\sum_{i\in I}(\gamma_{(p_0^*,q^*)}-q^*{\cdot}(z_i^*-\overline{z}_i))+\sum_{j\in J_1}[r_{0j}(\varpi^*,(y_j^*,z_j^*)){+}\gamma_{(p_0^*,q^*)}]$
$+\sum_{j\in J_2}[r_{0j}(\varpi^*,(y_j^*,z_j^*)){+}\gamma_{(p_0^*,q^*)}])$, that is,

$\sum_{i\in I}p_0^*{\cdot}(x_{i0}^*-e_{i0}^*)\leqslant\#K\gamma_{(p_0^*,q^*)}-q^*\cdot\sum_{k\in K}\ z_k^*+q^*\cdot\sum_{i\in I}\ \overline{z}_i+\sum_{j\in J}\ p_0^*{\cdot}(y_{j0}^*+e_{j0}^*)$, and, from above, $0\leqslant\sum_{i\in I}p_0^*{\cdot}(x_{i0}^*\text{-}e_{i0}^*){-}\sum_{j\in J}\ p_0^*{\cdot}(y_{j0}^*{+}e_{j0}^*)\leqslant\#K\gamma_{(p_0^*,q^*)}{-}q^*{\cdot}(\sum_{k\in K}z_k^*{-}\sum_{i\in I}\overline{z}_i)$.

Assume, by contraposition, that $\sum_{k\in K}z_k^*\neq\sum_{i\in I}\overline{z}_i$. Then, from Claim 3, the relations $\gamma_{(p_0^*,q^*)}=0$ and $q^*\cdot(\sum_{k\in K}z_k^*-\sum_{i\in I}\overline{z}_i)>0$ hold, in contradiction with the above relations. This contradiction proves that $\sum_{k\in K}z_k^*=\sum_{i\in I}\overline{z}_i$.

Assertion (ii) Let $s\in\underline{\mathbf{S}}$ be given. Assume, by contraposition, that there exists $h\in H$, such that $\sum_{i\in I}(x_{is}^{*h}-e_{is}^h)-\sum_{J\in J}(y_{js}^{*h}+e_{js}^h)>0$. Then, from Claim 3-$(ii)$, the relations $\gamma_{(s,p_s^*)}=0$ and $p_s^*{\cdot}(\sum_{i\in I}\ (x_{is}^*-e_{is})-\sum_{J\in J}\ (y_{js}^*+e_{js}))>0$ hold. Summing up budget constraints in state s, for every $i\in I$, yields, from Assertion (i) and above: $\sum_{i\in I}p_s^*{\cdot}(x_{is}^*-e_{is}^*)\leqslant\sum_{j\in J}\ p_s^*{\cdot}(y_s^*+e_{js}^*)$, in contradiction with the above relation. By the same token, we show $\sum_{i\in I}p_0^*{\cdot}(x_{i0}^*-e_{i0}^*)\leqslant\sum_{j\in J}\ p_0^*{\cdot}(y_0^*+e_{j0}^*)$. Assertion (ii) follows.

Assertion (iii) Let $\in\underline{\mathbf{S}}$ be given. Assume, by contraposition, that there exists $h\in H_1$, such that $\sum_{i\in I}\ (x_{is}^{*h}-e_{is}^h)-\sum_{J\in J}\ (y_{js}^{*h}+e_{js}^h)<0$. Then, from Claim 3-$(ii)$, the relation $p_s^{*h}=0$ holds and, from Assumption *A5*, and Assertion (ii), given $i\in I$, there exists $(x,z_i^*)\ \in P_i(x_i^*)\times Z_1\cap B_i'(\varpi^*,Y^*)$. Let $(x',z')\in B_i''(\varpi^*,Y^*)$ be given, a non-empty set, from Claim 2, above. Then, from Assumption *A7*, for $\lambda>0$, small enough, the relation $[1-\lambda](x,z_i^*)+\lambda(x',z')\in P_i(x_i^*){\times}Z_1\cap B_i''(\varpi^*,Y^*)$ holds and contradicts Claim 3.

We have, thus, shown that $\sum_{i\in I}(x_{is}^{*h}-e_{is}^{h})-\sum_{J\in J}(y_{js}^{*h}+e_{js}^{h})=0$, for every $h\in H_1$. Assume, now, that $\sum_{i\in I}(x_{is}^{*h}-e_{is}^{h})-\sum_{J\in J}(y_{js}^{*h}+e_{js}^{h})<0$, for some $h\in H_2$. Again, $p_s^{*h}=0$ and, from Assumption *A2*, the excess supply can, then, be redistributed amongst producers, until all markets clear, and without affecting any property of Claim 3-(ii). We let the reader check, as immediate from the fact that the total endowment is given and finite, that this redistribution is always possible within the sets $B_j'(\varpi^*)$, for every $j\in J$, by taking a sufficiently large bound, r, along Lemma 1 at the outset. So, the allocation, $[(x_{is}^*),(y_{js}^*)]$ may, indeed, be assumed to clear on all spot markets in state $s\in\underline{\mathbf{S}}$. By the same arguments, spot markets at $t=0$ may also be assumed to clear, that is, $\sum_{i\in I}p_s^*\cdot(x_{is}^*-e_{is}^*)=\sum_{j\in J}p_s^*\cdot(y_s^*+e_{js}^*)$, for every $s\in\underline{\mathbf{S}}'$. □

The following Claim completes the proof of Theorem 1.

Claim 5 *The above collection, θ^*, of prices and actions, is an equilibrium of the economy, $\mathcal{E}$, such that $1\leqslant\|(p_0^*,q^*)\|\leqslant 2$, $p_s^*\in(\mathbb{R}_{++}^{H_1})\times(\mathbb{R}_{+}^{H_2})$ and $\|p_s^*\|=1$, for all $s\in\underline{\mathbf{S}}$.*

Proof Given Claim 4 above and its proof, from which we infer that $p^*\in((\mathbb{R}_{++}^{H_1})\times(\mathbb{R}_{+}^{H_2}))^{S'}$, the proof of Claim 5 is given, mutatis mutandis, under De Boisdeffre's (2017) Claims 10 to 12, to which we refer the reader. □

4 The existence Theorem with numeraire assets

We consider a standard production economy, $\mathcal{E}$, of the above type, where assets only pay off in a numeraire, $e\in\mathbb{R}_+^H$ (with $\|e\|=1$). Agents' preferences are now represented by continuous, strictly concave, strictly increasing, separable functions, $u_i:X_i^o\to\mathbb{R}$, for each $i\in I$, and we let $u_i(x)=\sum_{s\in S_i}u_i^s(x_0,x_s)$, for every $x\in X_i^o$.

From the above Theorem 1, for every $n \in \mathbb{N}$, there exists an equilibrium, $\mathcal{C}^n := (\varpi^n := ((p_0^n, q^n), p^n := (p_s^n)), V^n, Y^n := [(y_j^n, z_j^n)], X^n := [(x_i^n, z_i^n)])$, for some payoff matrix $V^n \in \Lambda$ along Claim 1, such that $\|V^n - V\| \leqslant 1/n$, where V is the numeraire asset payoff matrix of the economy. The price sequence, $\{((p_0^n, q^n), p^n := (p_s^n))\}$, may be assumed to converge to some system, $((p_0^*, q^*), p^* := (p_s^*)) \in P_0 \times P$, such that $1 \leqslant \|(p_0^*, q^*)\| \leqslant 2$ and $\|p_s^*\| = 1$, for each $s \in \underline{\mathbf{S}}$. We assume costlessly that the payoff and information structure, $[V, (S_i)]$, is arbitrage-free along De Boisdeffre (2016).

The above sequence of equilibria, $\{\mathcal{C}^n\}$, meets the following properties.

Claim 6 *The following Assertions hold:*

(i) $\exists M > 0,\ \forall (n, i, s) \in \mathbb{N} \times I \times \underline{\mathbf{S}}',\ x_{is}^n \in [0, M]^H$;

(ii) it may be assumed to exist $X^* := [(x_i^*, z_i^*)] = \lim_{n\to\infty} X^n := [(x_i^n, z_i^n)]$;

(iii) it may be assumed to exist $Y^* := [(y_i^*, z_j^*)] = \lim_{n\to\infty} Y^n := [(y_j^n, z_j^n)]$;

(iv) for each $s \in \underline{\mathbf{S}}'$, $\sum_{i\in I} (x_{is}^* - e_{is}) = \sum_{j\in J} (y_{js}^* + e_{js})$ *and* $\sum_{k\in K} z_k^* = \sum_{i\in I} \overline{z}_i$;

(v) $\exists \varepsilon > 0 : \forall s \in \underline{\mathbf{S}},\ p_s^* \in [\varepsilon, 1]^{H_1} \times [0, 1]^{H_2}$;

(vi) $\mathcal{C}^* := ((p_0^*, q^*), p^*, V, Y^*, X^*)$ *is an equilibrium of the economy,* $\mathcal{E}$.

Claim 6-(vi) states the full existence property of the numeraire asset equilibrium.

Proof Assertion (i) is standard, from market clearance conditions of equilibrium and from the fact that the total goods and services available for input are bounded, and so is the total output, from Assumption *A3*.

Assertion (ii)-(iii): the fact that the sequences $\{X^n\}$ and $\{X^n\}$ are bounded, thus assumed to converge, results from Lemma 1 (see the Appendix).

Assertion (iv) states market clearance conditions on $\{\mathcal{C}^n\}$, passed to the limit.

Assertion (v): we let the reader check the proof is a corollary of Lemmata 1 in De Boisdeffre (2017), replacing the numeraire by any $h \in H_1$, and passing to limit.

Assertion (vi): the fact that X^* meets condition (b) of Definition 1 of equilibrium is proved, mutatis mutandis, under Theorem 2 in De Boisdeffre (2017). From Assertion (iv), $\mathcal{C}^*$, also meets conditions (c)-(d) of Definition 1.

Thus, we need only prove that $\mathcal{C}^*$ meets condition (a) of Definition 1, proceeding as follows. Let $\varpi^* := ((p_0^*, q^*), p^*)$, along Claim 6, $\mathcal{V}^o := \{M \in \mathcal{V} : \|M\text{-}V\| \leqslant 1\}$ and $j \in J$ be given. Consider the correspondence $\eta_j : (\varpi, M) \in P_0 \times P \times \mathcal{V}^o \mapsto \eta_j(\varpi, M) := \arg\max \Pi_j(\varpi, M, (y, z))$ for $(y, z = (z_0, 0)) \in B_j(\varpi, M)$, where $\varpi := ((p_0, q), p := (p_s)) \in P_0 \times P$, $\Pi_j(\varpi, M, (y, z)) := (p_0 \cdot (y_0 + e_{j0}) - q \cdot z) + \sum_{s \in S_j} \pi_s^j (p_s \cdot (y_s + e_{js}) + M(s, p_s) \cdot z_0)$ and $B_j(\varpi, M) := \{(y, z) \in Y_j^o \times Z^J : p_0 \cdot (y_0 + e_{j0}) - q \cdot z \geqslant 0 \; and \; p_s \cdot (y_s + e_{js}) + M(s, p_s) \cdot z_0 \geqslant 0, \forall s \in S_j\}$.

From Lemma 1 (see the Appendix), the set $Y_j^o \times Z^J$ may be assumed (restricted) to be compact. The scalar product and mapping, Π_j, are continuous and the correspondence, B_j, which has a closed graph, is upper semicontinuous. Then, the equilibrium relations $(y_j^n, z_j^n) \in B_j(V^n, \varpi^n)$, for all $n \in \mathbb{N}$, yield in the limit: $(y_j^*, z_j^*) \in B_j(\varpi^*, V)$.

We now show that the correspondence, $(\varpi, M) \in P_0 \times P \times \mathcal{V}^o \mapsto B_j(\varpi, M)$, is lower semicontinuous at (ϖ^*, V). Let $(y^*, z^*) \in B_j(\varpi^*, V)$ be given and, for each $k \in \mathbb{N}$, let $(\varpi^k, M^k) \in P_0 \times P \times \mathcal{V}^o$ be such that $\|(\varpi^k, M^k) - (\varpi^*, V)\| \leqslant 1/k$. From Assumption *A4* and Claim 6-(v), the interior, $B_j''(\varpi^*, V)$, of $B_j(\varpi^*, V)$ is non-empty. So, we set as given $(y, z) \in B_j''(\varpi^*, V)$, and we let $(y_j^m, z_j^m) := ([1\text{-}\frac{1}{m}]y^* + \frac{1}{m}y, \; [1\text{-}\frac{1}{m}]z^* + \frac{1}{m}z) \in B_j''(\varpi^*, V)$, for every $m \in \mathbb{N}$, converge to $(y^*, z^*) \in B_j(\varpi^*, V)$. From the continuity of the scalar product, for every $m \in \mathbb{N}$, there exists $k_m \in \mathbb{N}$, such that $(y_j^m, z_j^m) \in B_j''(\varpi^k, M^k)$, for every $k \geqslant k_m$. The latter relations imply, from the definition, that B_j is lower semicontinuous at (ϖ^*, V).

We have shown that the mapping, Π_j, and correspondence, B_j, are continuous at (ϖ^*, V) with non-empty compact values. Then, from Berge's Theorem (see, e.g., Debreu, 1959, p. 19), η_j is upper semi-continuous at (ϖ^*, V), which yields, in the limit, $(y_j^*, z_j^*) \in \eta_j(\varpi^*, V) := \arg\max \Pi_j(\varpi^*, V, (y,z))$ for $(y,z) \in B_j(\varpi^*, V)$, from the above relations, $(y_j^n, z_j^n) \in \eta_j(V^n, \varpi^n)$, for $n \in \mathbb{N}$. That is, condition (a) of Definition 1 holds. $\square$

Appendix

Lemma 1 $\exists r > 0 : \forall \varpi \in P_0 \times P,\ \forall Y \in \times_{j\in J} \overline{B}_j(\varpi),\ \forall X \in \overline{\mathcal{A}}(\varpi, Y),\ \|X\| + \|Y\| < r$

Proof

Lemma 1 in the general setting of of Section 3.

- Since total available inputs are uniformly bounded, so are outputs, from Assumption *A3*, for every $\varpi \in P_0 \times P$ and $Y \in \times_{j\in J} \overline{B}_j(\varpi)$, such that $\overline{\mathcal{A}}(\varpi, Y) \neq \varnothing$. So, we may assume that production sets are bounded and let $Q := (\sum_{j\in J} \sup_{y_j \in Y_j^o} \|y_j\|)$. Then, from the definition, consumptions of the set $\overline{\mathcal{A}}(\varpi, Y)$ are uniformly bounded, in $\varpi \in P_0 \times P$, $Y \in \times_{j\in J} \overline{B}_j(\varpi)$ and $s \in \underline{\mathbf{S}}'$.

- From above and the definition of P, Lemma 1 will be proved if we show portfolios are bounded independently of ϖ. Portfolios in equities $(z_1 \in [0,1]^{J_1})$ are bounded, from the definition. We show the same for assets.

- Let $\delta = (2 + \|(\overline{p}_s)\|)(1 + Q + \|(e_k)_{k\in K}\|)$. Assume, by contraposition, that, for every $n \in \mathbb{N}$, there exists $[(x_i^n, z_i^n := (z_{i0}^n, z_{i1}^n)] \in \overline{\mathcal{A}}(\varpi^n, Y^n)$, for some $\varpi^n := ((p_0^n, q^n), p^n) \in P_0 \times P$, and some $Y^n := [(y_j^n, z_j^n := (z_{j0}^n, 0)] \in \times_{j\in J} B_j(\varpi^n)$, such that $\sum_{k\in K} \|z_{k0}^n\| > n$. Such relations yield: $\sum_{k\in K}\ z_{k0}^n = 0$, and $V(s_k, p_{s_k}^n) \cdot z_{k0}^n \geqslant -\delta,\ \forall (k, s_k, n) \in K \times S_k \times \mathbb{N}$.

- The rest of the proof is identical to De Boisdeffre's (2017) for its Lemma 1. $\square$

Lemma 1 for the numeraire asset economy.

- As above, we need only show portfolios are bounded, but across all economies, $\mathcal{E}^n$. We assume, by contraposition, that there exist double indexed sequences of prices, $\varpi^{(n,m)} := ((p_0^{(n,m)}, q^{(n,m)}), p^{(n,m)}) \in P_0 \times P$, and strategies, $Y^{(n,m)} := [(y_j^{(n,m)}, z_j^{(n,m)})] \in \times_{j \in J} B_j(\varpi^{(n,m)})$ and $[(x_i^{(n,m)}, z_i^{(n,m)})] \in \overline{\mathcal{A}}(\varpi^{(n,m)}, Y^{(n,m)}, V^n)$, with obvious notations, such that $\|(z_{k0}^{(n,m)})\| > m$. Then, the following relations hold from the definitions: $\sum_{k \in K} z_{k0}^{(n,m)} = 0$, $V^n(s_k, p_{s_k}^{(n,m)}) \cdot z_{k0}^{(n,m)} \geqslant -\delta, \forall (k, s_k, n, m) \in K \times S_k \times \mathbb{N}^2$.
- The rest of the proof is identical to De Boisdeffre's (2017) for its Lemma 1. □

References

[1] Debreu, G., Theory of Value, Cowles foundation monograph 17, Yale University Press, New Haven, 1959.

[2] De Boisdeffre, L., No-arbitrage equilibria with differential information: an existence proof, Economic Theory 31, 255-269, 2007.

[3] De Boisdeffre, L., Learning from arbitrage, Econ Theory Bull 4, 111-119, 2016.

[4] De Boisdeffre, L., Financial equilibrium with differential information: an existence theorem, mimeo, 2017.

[5] Magill, M., Quinzii, M., Theory of incomplete markets, the MIT Press, Cambridge Mas., London, 1996.