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# On Time-Consistent Collective Choice with Heterogeneous Quasi-Hyperbolic Discounting*

Jean-Pierre Drugeon† & Bertrand Wigniolle‡

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## Abstract

A general setup is considered where agents are characterised by quasi-hyperbolic discounting and by heterogeneous bias for the present and heterogenous discounting parameters. Consumptions are moreover subject to a standard feasibility constraint. A *collective utility function* is defined as a function of the intertemporal utilities of the selves of the different agents, the elementary unit being thus the self of a given period for a given agent. The analysis is further specialized to *time-independent* collective utility functions. Such a framework generating a tension between Pareto-optimality and time-consistency for the optimal allocations, two approaches are suggested in order to tackle this issue. The first one imposes restrictions on the collective utility function that ensure the time-consistency of the optimal decisions. The second one builds from an *a* priori time-inconsistent collective utility function. The benevolent planner is then to be considered as a sequence of successive incarnations, any of these incarnations being endowed with its own objective. The associated optimal policy is the equilibrium of a game between the successive incarnations of the planner when the players follow Markovian strategies. The results obtained for both solution concepts are compared through an example that also shows how they can be recovered through a competitive equilibrium.



Keywords: Heterogeneities, hyperbolic discounting, collective choice.

JEL Classification: E32, C62.


## Contents




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# 1. Introduction

## 1.1 Focus

Consider a community that is to undertake some collective decisions such as consumptions over time, optimal policies for retirement or defining an optimal investment strategy. That community is made of heterogenous agents that differ in their evaluation of future outcomes, *i.e.*, in their discount factors. These agents are also *present-biased*, that results in a temporal inconsistency for their preferences. Is it however possible to aggregate these individual preferences through a collective utility function? Is it further conceivable to define an optimal inter-temporal policy? It is to the examination of such questions that this article is devoted.

Dating from the important contributions of Phelps & Pollack [32], Laibson [27] and Frederick, Loewenstein & O'Donoghue [15], the benchmark model for analyzing temporal inconsistencies got anchored on the *quasi-hyperbolic discounting hypothesis* and gained popularity through the so-called $\beta - \delta$ *model*. Numerous experiments[1] have consistently illustrated how such a model could provide a more accurate description of actual choices and arbitrages over time than the canonical model as settled by Samuelson [35].

A large number of studies (see Frederick, Loewenstein & O'Donoghue [15]) have also ruled field or laboratory experiments in order to evaluate the $\beta$ and $\delta$ parameters. They all document a strong heterogeneity between the individuals regarding these parameters. The contributions of Augenblieck, Niederle and Sprenger [2], Balakrishnan, Hausofer & Jakiela [3], Bozio & Laroque [8] and Grignon [16] are noticeable in that perspective.

A quasi-hyperbolic discounting formulation results in a time-inconsistency issue for the preferences of the agents. In order to provide the associated decision rules, it is a common hypothesis to assume that the agent is split between an infinity

[1] See Thaler [37], Benzion, Rapoport & Yagil [7], Chapman & Elstein [10] or Frederick, Loewenstein O'Donoghue [15] for a general presentation.

of successive selves that undertake their decisions in a non cooperative manner. Following this *sophisticated* behaviour, the decisions result from a Nash equilibrium between the successive selves. Any of the selves being the basis unit for the analysis, the eventual decision is sub-optimal, *i.e.*, a non Pareto-optimal one reached in a non-cooperative way.

Within this environment, there is hence a large role for active economic policies. It is however worth emphasizing that the definition of such policies presupposes to be first able to define a collective utility function on the different selves. Starting from Phelps & Pollack [32] or Laibson [27], the literature has usually selected a simple benchmark, namely the sole objective of the initial self and subsequently analyzed the policies that would allow for a decentralization of the commitment solution where the initial self would undertake the whole sequence of decisions. As an illustration, one may refer to the large group of literature on savings choices or retirement systems with quasi-hyperbolic discounting — Laibson, Repetto & Tobacman [28], Salanie & Treich [34], Diamond & Kaszegi [11]—.

In opposition to this literature, this article is aimed at building a *collective utility function* through the aggregation of the different selves without allowing for any *dictatorial role* for the initial self. Another important component of the current argument results from the heterogeneity of the preferences parameters between the agents. Even without any present biais for the agents, and as this has been well documented in the literature–cf. Jackson & Yariv [25], Heal & Millner [23], Alcala [1], Drugeon & Wigniolle [14]–, the sole heterogeneity between the discount factors of the agents is problematic in the aim of defining a collective utility function. For heterogenous discount factors and for a time-independent collective utility function, some tension emerges between Pareto-optimality and time-consistency as soon as there is no dictatorship—cf. Jackson & Yariv [25], Zuber [38]—by some agent.

The current approach introduces a general setup where agents are endowed with

quasi-hyperbolic discounting and with heterogeneous bias for the present and discounting parameters. At any date, consumptions are submitted to a *feasibility* constraint that builds from a neoclassical production technology which uses labour and capital. Along the literature, the analysis is specialized to *time-independent* collective utility functions: under this time-invariance assumption and as this was recently clarified by Halevy [21], time-consistency does correspond to the stationary property of Koopmans [26] for the collective utility function. Finally, collective utility is defined as a function of the inter-temporal utilities of the selves of the different agents, the elementary unit being thus the self of period $t \in \mathbb{N}$ for a given agent.

In a consistent way with the above definition of the collective utility function, the approach is based upon a *selves-Pareto* axiom: a feasible allocation will be referred to as being selves-Pareto optimal if there does not exist another feasible allocation such that none of the selves of any of the agents would suffer from a welfare loss and at least one self of one agent would benefit from a welfare improvement. Finally, the analysis will be specialized to additively separable collective utility functions.

It is firstly established that the satisfaction of the selves-Pareto optimality axiom requires a collective utility function that writes down as a weighted sum of inter-temporal utilities of the selves of the different agents. To this general collective utility concept is then added another *t-selves concerned* collective utility one. This notion corresponds to a situation in which the planner evaluates the welfare of all of the selves of all the different agents based upon the sole selves of period $t$.

Two different ways are then suggested in order to face with the tension between Pareto-optimality and time-consistency. The first one directly pertains to the definition of the collective utility function that states as a sum of the weighted inter-temporal utilities of the different selves of the different agents. Restrictions on the weights are first analyzed that could allow for recovering a time-consistency property. These weights can be understood as a way of imposing a common discounting parameter $\gamma$ to the whole set of agents, where $\gamma \geqslant \sup_i(\delta_i)$, for $\delta_i$ the discounting

parameter of agent $i$. The planner will then implement an optimal policy—labelled as *Fully Time-Consistent Pareto-optima* (FTCPO)—that corresponds to a degree of patience that is greater or equal to the one of all of the agents.

A second approach to remedy to the tension between Pareto-optimality and time-consistency is entirely distinct as it builds upon a collective utility function that is not *a priori* temporally consistent. The planner is then to be considered as being made of a sequence of successive incarnations, any of these incarnations being endowed with its own objective. Following such a hypothesis, it is assumed that the objective of the $t$-incarnation of the social planner is a $t$-selves concerned collective utility. The associated solution is the equilibrium of a game between the successive incarnations of the planner when the players follow Markovian strategies that solely depend upon the state variable. That solution corresponds to the one that maximizes the weighted sum of the inter-temporal utilities of the selves of a given period under a time-consistency constraint on their choice, hence a label of *Time-Consistent Constrained Pareto-optima* (TCCPO).

It is worth noticing that the first FTCPO approach recovered time-consistency through constraints on the collective utility function independently of the nature of the feasibility constraints. In opposition to this, the second TCCPO approach introduces a solution where the collective utility function is not *per se* constrained to time consistency. That property is rather recovered by restricting the planners incarnations strategies to be markovian, the feasibility constraints being embedded in their definition. The results associated to this second approach strongly differ from the earlier ones and can be understood as the choice that would result from a planner that would apply a common discount rate to the instantaneous utilities of the different agents. That discount factor would in turn emerge as the weighted average of the $\beta_i$ and $\delta_i$ coefficients of the different agents. More precisely, it is possible to determine lower and upper bounds anchored on weighted averages of $\beta_i\delta_i$ and $\delta_i$ for that discount rate.

In order to illustrate such results, an example with a Cobb-Douglas production technology and a logarithmic utility function is analyzed. It allows for an explicit characterization of the solution concepts and for an extensive comparison of their properties. It is also possible to establish that the two FTCPO and TCCPO planning solutions can be decentralized as the equilibrium of an environment with a distortionary system of taxes on the capital stock, with public debt and with initial transfers to the agents. For the FTCPO solution, the optimal policy leads to subsidy capital accumulation at every date, that scheme being financially supported by a negative transfer of wealth at the initial date, that in turn allows the government to start with a negative debt. That asset hence eventually allows for supporting the subsidy to the capital stock. For the TCCPO solution concept, the distortionary tax on the capital stock can be either positive or negative: the most patient agents are taxed while the most impatient ones are subsidized by the central authority.

## 1.2 Related Literature

Numerous related contributions have been interested in the problem of collective choice when individual preferences are heterogenous from the discounting standpoint. Within such an environment, it is a well-documented fact that a conflict emerges between Pareto-optimality, time-consistency and the time-dependency properties of the collective utility function. Gollier & Zeckhauser [17] establish that the aggregation of individual preferences leads to a variable social discount rate when agents are characterised by heterogenous discount rates. Jackson & Yariv [25] and Zuber [38] shed light upon the conflict between Pareto-optimality and time-consistency when the collective utiity function is time-invariant. While Jackson & Yariv [25] analyze this issue within a choice setup with a consumption sequence that is common to the whole population of agents, Zuber [38] considers an environment where any of the agents is endowed with a specific sequence of consumptions.

There does further exist a formal analogy between the model with quasi-hyperbolic discounting $\beta - \delta$ and a model with generations of altruistic agents and the same

range of concerns about the definition of a collective utility function hence potentially emerges into that setup. Galperti & Strulovici [18] provide an axiomatic approach that can be used to provide some fundamentals to these two classes of models. As for the definition of the collective utility function, they consider the specific case of an economy with homogeneous agents. The current FTCPO solution concept can hence be understood as a generalisation to theirs. Feng & Ke [20] have recently equally provided a definition of the social discount rate within a finite horizon generations model where the altruism function encompasses the $\beta - \delta$ model as a special case. They infer a general class of restrictions to be satisfied by the social discount function that are closely related to the current ones for the FTCPO solution concept of the $\beta - \delta$ model.

From a more general perspective, Herings & Rohde [24] introduce time-inconsistent preferences in a general equilibrium framework in finite horizon and suggest four possible efficiency concepts. In a related environment, Dziewulski [13] is interested in the welfare properties of equilibria with time-dependent preferences and presents a version of the First Welfare Theorem for these economies. Both authors are however not directly concerned with the definition of a collective utility function but rather with the characterization of Pareto optima.

An alternative approach builds upon the abandonment of the time-invariance of the collective utility function. Heal & Millner [23] accordingly illustrate how the heterogeneity between the individual discount factors may result in a conflict between the time-consistency of the decisions and the time-invariance of the collective utility function. Within an environment related to the currrent one, Alcala [1] introduces a specific solution to the time-inconsistency issue that emerges for heterogenous discount rates. He indeed considers a planner's objective that is formulated as weighted sum of the individual utilities where the weights vary at every date so that the choice of the planner at the initial date ends up corresponding to the choices of all the future planners.

# 2. The Model

## 2.1 Basic Assumptions

Time is discrete. The $n$ infinitely-lived agents are indexed by $i \in \{1, 2, \ldots, n\}$. Let $\mathscr{C}^i := (c^i_\tau)_{\tau\in\mathbb{N}}$ denote the consumption sequence of individual $i$. The consumption sequence $\mathscr{C}^i$ belongs to $\ell^\infty_+$, the space of non-negative bounded sequences. Consider further its $t$-truncated expression as ${}_t\mathscr{C}^i := (c^i_\tau)_{\tau\geqslant t}$. The preferences of the $n$ agents are associated with quasi-hyperbolic discounting. For a given agent $i \in \{1, 2, \ldots, n\}$, his *self* at time $t \in \mathbb{N}$ ranks consumption sequences according to:

$$\mathrm{U}^i({}_t\mathscr{C}^i) = u^i(c^i_t) + \beta_i\Big[\sum_{\tau=1}^{+\infty}(\delta_i)^\tau u^i(c^i_{t+\tau})\Big]. \tag{2.1}$$

Agents are heterogenous according to their felicity functions $u^i(\cdot)$ and their parameters $(\beta_i, \delta_i)$ where $\delta_i \in ]0, 1[$ pictures the rate of discount parameter whilst $\beta_i \in ]0, 1]$ features the *bias for the present*.

Equation (2.1) defines a sequence of utility functions $(\mathrm{U}^i_t)_{t\in\mathbb{N}}$, $i = 1, \ldots, n$ according to:

$$\mathrm{U}^i_t(\mathscr{C}^i) = \mathrm{U}^i({}_t\mathscr{C}^i) \tag{2.2}$$

Let further $\mathbb{U}$ denote the set of sequences of functions $(\mathrm{U}^i_t)_{t\in\mathbb{N}}$, $i = 1, \ldots, n$ satisfying Equations (2.1) and (2.2).

From now on, a sequence $(\mathrm{U}^i_t)_{t\in\mathbb{N}}$ will be denoted by $\underset{\sim}{\mathrm{U}}^i$. It is fully characterised by the three elements $(u^i, \beta_i, \delta_i)$.

Assumption 1. *The instantaneous utility function $u^i : \mathbb{R}_+ \to \mathbb{R}$, $i = 1, \ldots, n$ is continuous, strictly increasing and concave. At the origin, either $u^i(0) = 0$ or $u^i(0) = -\infty$. $u^i$ is of class $\mathrm{C}^2$ on $\mathbb{R}^*_+$. If $u^i(0) = 0$, then $\lim_{c^i\to 0^+} \mathrm{D}u^i(c^i) = +\infty$.*[2]

Agents have time inconsistent preferences for $\beta_i \in ]0, 1[$: every agent is to be perceived as a *sequence of selves*, any self being characterised by his own preferences.

[2] Note that, if $u^i(0) = -\infty$, then $\lim_{c^i\to 0^+} \mathrm{D}u^i(c^i) = +\infty$.

This economy also integrates at any period $t \in \mathbb{N}$ a competitive representative firm with a production function $F(K_t, L_t)$, for $K_t$ the capital stock and $L_t$ the quantity of labour, that satisfies:

Assumption 2. $F(K, L)$ is homogeneous of degree one and increases with K and L.

For simplicity, the total labour supply of the $n$ individuals $L_t$ is normalized to 1 at every period of time. Also let $\mu \in [0, 1]$ denote the rate of depreciation of the capital stock.

Assumption 3. $F(K, 1)$ is of class $C^2$ on $\mathbb{R}_+^*$, strictly increasing and concave.

Assumption 4. $F(0, 1) = 0$, $\lim_{K \to 0^+}[D_K F(K, 1) + (1 - \mu)] > 1/\inf_{i \in \{1, \ldots, n\}}(\delta_i)$, $\lim_{K \to +\infty} D_K F(K, 1) = 0$.

The resource constraint at date $t \in \mathbb{N}$ states as:

$$K_{t+1} + \sum_{i=1}^{n} c_t^i = F(K_t, 1) + (1 - \mu)K_t, \quad \mu \in ]0, 1]. \tag{2.3}$$

A date 0, $K_0 \geqslant 0$ is given. Under assumptions 2, 3, 4, there does exist a maximal level of sustainable capital $\bar{K}$, defined as the limit of the sequence $(K_t)_{t \in \mathbb{N}}$ such that $K_{t+1} = F(K_t, 1) + (1 - \mu)K_t$. Without loss of generality, the set of attainable capital stocks for $K_t$ is restricted to the interval $[0, \bar{K}]$. This implies that the set of feasible consumptions and utilities is bounded from above. For a given $K_0 \geqslant 0$ , a capital sequence $(K_t)_{t \in \mathbb{N}}$ is feasible if, $\forall\, t \in \mathbb{N}$, $K_{t+1} \in \Gamma(K_t)$, where

$$\Gamma(K_t) := \left\{K_{t+1} \in [0, \bar{K}] \text{ such that } 0 \leqslant K_{t+1} \leqslant F(K_t, 1) + (1 - \mu)K_t\right\}.$$

Let $\Pi(K_0)$ denote the set of feasible capital sequences from $K_0$:

$$\Pi(K_0) := \left\{(K_t)_{t \in \mathbb{N}} \in [0, \bar{K}]^{\mathbb{N}} \text{ such that } \forall t \in \mathbb{N},\ K_{t+1} \in \Gamma(K_t),\ K_0 \text{ given}\right\}.$$

The set of feasible consumption sequences is similarly introduced as:

$$\Omega(K_0) := \Big\{(\mathscr{C}^1, \ldots, \mathscr{C}^n) \in (\ell_+^\infty)^n \text{ such that } \forall\, t \in \mathbb{N}, \sum_{i=1}^{n} c_t^i \leqslant F(K_t, 1) + (1 - \mu)K_t - K_{t+1}, \text{ with } (K_t)_{t \in \mathbb{N}} \in \Pi(K_0)\Big\}.$$

## 2.2 Collective Utility Function

This article will specialize to time-independent collective utility functions. Time consistency hence boils down to the Koopmans [1960] stationarity axiom:

Definition 1. *Let $\succeq$ denote the preferences order corresponding to some collective utility function. $\succeq$ is stationary [time-consistent] if:*

$$\begin{aligned}
&\forall K_1 \in \Gamma(K_0), \quad \forall (\mathscr{C}^1, \dots, \mathscr{C}^n), (\mathscr{C}^{1\prime}, \dots, \mathscr{C}^{n\prime}) \in \Omega(K_1), \\
&\qquad (\mathscr{C}^1, \dots, \mathscr{C}^n) \succeq (\mathscr{C}^{1\prime}, \dots, \mathscr{C}^{n\prime}) \\
&\iff \forall (c_0^1, \dots, c_0^n) \quad \text{such that } \sum_{i=1}^{n} c_0^i \leqslant F(K_0, 1) + (1-\mu)K_0 - K_1, \\
&\qquad ((c_0^1, \mathscr{C}^1), \dots, (c_0^n, \mathscr{C}^n)) \succeq ((c_0^{1\prime}, \mathscr{C}^{1\prime}), \dots, (c_0^{n\prime}, \mathscr{C}^{n\prime})).
\end{aligned}$$

The *collective utility function* of the benevolent planner is defined by an aggregation of the objective functions of the selves of the different agents. It states as a function

$$\begin{aligned}
&V : \mathbb{U}^n \times \Omega(K_0) \longrightarrow \mathbb{R} \\
&\qquad \left((\underset{\sim}{U}^1, , \dots, \underset{\sim}{U}^n), (\mathscr{C}^1, \dots, \mathscr{C}^n)\right) \longmapsto V[(\underset{\sim}{U}^1, \dots, \underset{\sim}{U}^n)](\mathscr{C}^1, \dots, \mathscr{C}^n)
\end{aligned}$$

According to he following axiom, the elementary unit corresponds to the self of a given agent $i \in \{1, 2, \dots, n\}$ at a given date $t \in \mathbb{N}$:

Axiom 1 (Selves -Pareto). The collective utility function V is said to be selves-Pareto if, for every pair of feasible consumption sequences $(\mathscr{C}^1, \dots, \mathscr{C}^n)$, $(\check{\mathscr{C}}^1, \dots, \check{\mathscr{C}}^n) \in \Omega(K_0)$ such that, $\forall i = 1, \dots, n$, $\forall t \in \mathbb{N}$, $U_t^i(\mathscr{C}^i) \geqslant U_t^i(\check{\mathscr{C}}^i)$,

$$V[(\underset{\sim}{U}^1, \dots, \underset{\sim}{U}^n)](\mathscr{C}^1, \dots, \mathscr{C}^n)) \geqslant V[(\underset{\sim}{U}^1, \dots, \underset{\sim}{U}^n)](\check{\mathscr{C}}^1, \dots, \check{\mathscr{C}}^n).$$

Furthermore, if there exists $i \in \{1, \dots, n\}$ and $t_0 \in \mathbb{N}$ such that $U_{t_0}^i(\mathscr{C}^i) > U_{t_0}^i(\check{\mathscr{C}}^i)$, then

$$V[(\underset{\sim}{U}^1, \dots, \underset{\sim}{U}^n)](\mathscr{C}^1, \dots, \mathscr{C}^n) > V[(\underset{\sim}{U}^1, \dots, \underset{\sim}{U}^n)](\check{\mathscr{C}}^1, \dots \check{\mathscr{C}}^n).$$

This article will focus on a special form of collective utility function called *additively separable*.

Definition 2. A collective utility function V is said to be *additively separable* if there exists $n$ sequences $(\Delta_t^i)_{t\in\mathbb{N}} \in \ell^1$ with $\Delta_t^i > 0$, $\forall i \in \{1,\dots,n\}$, $\forall\, t \in \mathbb{N}$, such that:

$$V[(\underset{\sim}{U}^1,\dots,\underset{\sim}{U}^n)](\mathscr{C}^1,\dots\mathscr{C}^n) = \sum_{t=0}^{+\infty}\sum_{i=1}^{n}\Delta_t^i u^i(c_t^i). \tag{2.4}$$

A collective utility function must be a weighted average of the objectives of the different selves in order to be Selves-Pareto in the separably additive case:

Proposition 2.1. *Consider an additively separable collective function (2.4):*

(i) *It is selves-Pareto if and only if there exist n sequences* $(\eta_t^i)_{t\in\mathbb{N}} \in \ell^1$ *with* $\eta_t^i > 0$, $\forall i \in \{1,\dots,n\}$, $\forall\, t \in \mathbb{N}$, *such that:*

- $\eta_0^i = \Delta_0^i$,
- $\forall t \geq 1$ *and* $i = 1,..,n$,

$$\Delta_t^i - \delta_i\Delta_{t-1}^i = \eta_t^i - \delta_i(1-\beta_i)\eta_{t-1}^i \tag{2.5}$$

*or, equivalently,*

$$\eta_t^i = \Delta_t^i - \sum_{\tau=0}^{t-1}\Delta_\tau^i\left[\delta_i(1-\beta_i)\right]^{t-1-\tau}\beta_i\delta_i. \tag{2.6}$$

(ii) *Moreover, the objective (2.4) can be written as:*

$$\sum_{t=0}^{+\infty}\sum_{i=1}^{n}\eta_t^i\left\{u^i\left(c_t^i\right) + \beta_i\sum_{\tau=1}^{+\infty}(\delta_i)^\tau\, u^i\left(c_{t+\tau}^i\right)\right\} = \sum_{t=0}^{+\infty}\sum_{i=1}^{n}\eta_t^i U_t^i(\mathscr{C}^i). \tag{2.7}$$

In Equation (2.7), $\eta_t^i$ denotes the weight of the objective function of self $t$ for agent $i$.

## 2.3 $t$–Selves Concerned Collective Utility Functions

For a collective utility function V, a *$t$-selves concerned* collective utility function $V^t$ is now going to be introduced. It features a case where a benevolent planner would only be concerned with the selves of period $t$.

Let $\underset{\sim}{\mathrm{U}}^i \in \mathbb{U}$ and consider the function $S_t : \mathbb{U} \to \mathbb{U}$ such that to every $\underset{\sim}{\mathrm{U}}^i = (\mathrm{U}^i_\tau)_{\tau\in\mathbb{N}} \in \mathbb{U}$, it associates

$$S_t(\underset{\sim}{\mathrm{U}}^i) = (\mathrm{U}^i_t, \mathrm{U}^i_t, \ldots, \mathrm{U}^i_t, \ldots).$$

The *t-selves concerned* collective utility function $\mathrm{V}^t$ is then defined as:

$$\begin{aligned}\mathrm{V}^t : \mathbb{U}^n \times \Omega(\mathrm{K}_0) &\longrightarrow \mathbb{R},\\ \Big((\underset{\sim}{\mathrm{U}}^1, \ldots, \underset{\sim}{\mathrm{U}}^n), (\mathscr{C}^1, \ldots, \mathscr{C}^n)\Big) &\longmapsto \mathrm{V}^t\big[(\underset{\sim}{\mathrm{U}}^1, \ldots, \underset{\sim}{\mathrm{U}}^n)\big](\mathscr{C}^1, \ldots \mathscr{C}^n)\\ &= \mathrm{V}\big[S_t(\underset{\sim}{\mathrm{U}}^1), \ldots S_t(\underset{\sim}{\mathrm{U}}^n)\big](\mathscr{C}^1, \ldots, \mathscr{C}^n).\end{aligned}$$

It is worth mentioning that, for a given selves-Pareto collective utility function V, the ensued *t*-selves concerned collective utility function cannot be selves-Pareto. It is however conceivable to introduce a better suited concept of *t*-selves-Pareto collective utility function that is only concerned with the utilities of the selves of period *t*.

Axiom 2 (*t*-selves-Pareto). For a given $t \in \mathbb{N}$, the *t*-selves concerned collective utility function $\mathrm{V}^t$ is said to be *t*-selves-Pareto if, for every pair of feasible consumption sequences $(\mathscr{C}^1, \ldots, \mathscr{C}^n)$, $(\check{\mathscr{C}}^1, \ldots, \check{\mathscr{C}}^n) \in \Omega(\mathrm{K}_0)$ such that, $\forall i = 1, \ldots, n$, $\mathrm{U}^i_t(\mathscr{C}^i) \geqslant \mathrm{U}^i_t(\check{\mathscr{C}}^i)$,

$$\mathrm{V}^t\big[(\underset{\sim}{\mathrm{U}}^1, \ldots, \underset{\sim}{\mathrm{U}}^n)\big](\mathscr{C}^1, \ldots, \mathscr{C}^n)\big) \geqslant \mathrm{V}^t\big[(\underset{\sim}{\mathrm{U}}^1, \ldots, \underset{\sim}{\mathrm{U}}^n)\big](\check{\mathscr{C}}^1, \ldots, \check{\mathscr{C}}^n).$$

Furthermore, if there exists $i \in \{1, \ldots, n\}$ such that $\mathrm{U}^i_t(\mathscr{C}^i) > \mathrm{U}^i_t(\check{\mathscr{C}}^i)$, then

$$\mathrm{V}^t\big[(\underset{\sim}{\mathrm{U}}^1, \ldots, \underset{\sim}{\mathrm{U}}^n)\big](\mathscr{C}^1, \ldots, \mathscr{C}^n) > \mathrm{V}^t\big[(\underset{\sim}{\mathrm{U}}^1, \ldots, \underset{\sim}{\mathrm{U}}^n)\big](\check{\mathscr{C}}^1, \ldots \check{\mathscr{C}}^n).$$

A straightforward result is first available: if the collective utility function V is selves-Pareto, then the associated *t*-selves concerned collective utility function $\mathrm{V}^t$ is *t*-selves-Pareto. An immediate corollary of Proposition 2.1 is further available:

Corollary 1. *If* V *is additively separable and Selves-Pareto, the associated t-selves concerned collective utility function* $\mathrm{V}^t$ *formulates as:*

$$\mathrm{V}^t\big[(\underset{\sim}{\mathrm{U}}^1, \ldots, \underset{\sim}{\mathrm{U}}^n)\big](\mathscr{C}^1, \ldots \mathscr{C}^n) = \sum_{i=1}^{n} \eta^i \mathrm{U}^i_t(\mathscr{C}^i), \tag{2.8}$$

*where* $\eta^i = \sum_{\tau=0}^{+\infty} \eta^i_\tau$.

The $t$-selves concerned collective utility function $V^t$ is a weighted sum of the intertemporal utilities of the $t$-period selves.

## 3. Fully Time-Consistent Pareto Optima [FTCPO]

Let a benevolent planner be endowed with a collective utility function that is Selves-Pareto and additively separable. This section is aimed at proving that it is possible to restrict the weights $(\eta_t^i)_{t\in\mathbb{N}}$ in order for the objective (2.7) to be temporally consistent along Definition 1. The case of a single agent will first be considered before moving to the general heterogeneous agents case. The associated Pareto-Optima, labelled as Fully Time-Consistent Pareto Optima, or FTCPO, are finally analyzed.

### 3.1 The One-Agent Case

Consider an agent described by a 3-uple $(u, \beta, \delta)$. Assume that the collective utility function is separably additive and Selves-Pareto. From Proposition 2.1, it corresponds to a weighted sum of the objective functions of the different selves:

$$\sum_{t=0}^{+\infty} \eta_t \left[ u(c_t) + \beta \sum_{\tau=1}^{+\infty} u(c_{t+\tau})\delta^\tau \right] \tag{3.1}$$

Proposition 3.1. *The Collective Utility Function (3.1) is time-consistent iff there exists $\gamma$ with $\gamma \in [\delta, 1[$ such that the sequence $(\eta_t)_{t\in\mathbb{N}}$ satisfies:*

$$\eta_t = \eta_0 \left\{ \frac{(\gamma-\delta)}{\gamma-\delta(1-\beta)}\gamma^t + \frac{\beta\delta}{\gamma-\delta(1-\beta)}[\delta(1-\beta)]^t \right\} \tag{3.2}$$

*Moreover, the objective (3.1) becomes:*

$$\eta_0 \sum_{t=0}^{+\infty} \gamma^t u(c_t). \tag{3.3}$$

Proposition 3.1 generalises the result of Galperti & Strulovici [18] by allowing the discount factor $\gamma$ to eventually differ from $\delta$. Whilst the configuration $\gamma = \delta$ was the one analysed by these authors, the whole range of the values of $\gamma$ for which $\delta < \gamma < 1$ is equally admissible.

## 3.2 The Multiple Agents Case

Consider now the case of multiple agents characterised by a 3-uple $(u^i, \beta_i, \delta_i)$, $i = 1, \ldots, n$. The collective utility function being Selves-Pareto and additively separable, from Proposition 2.1, it formulates as:

$$\sum_{i=1}^{n} \sum_{t=0}^{+\infty} \eta_t^i \Big[ u^i(c_t^i) + \beta_i \sum_{\tau=1}^{+\infty} u^i(c_{t+\tau}^i)(\delta^i)^\tau \Big]. \tag{3.4}$$

Proposition 3.2. — *Consider the Collective Utility Function* (3.4)*:*

(i) *It is time-consistent if and only if there exists* $\gamma \in [\max_{i \in \{1,\ldots,n\}}(\delta_i), 1[$ *such that, for every* $i \in \{1, \ldots, n\}$*,*

$$\eta_t^i = \eta_0^i \Big\{ \frac{(\gamma - \delta_i)}{\gamma - \delta_i(1 - \beta_i)} \gamma^t + \frac{\beta_i \delta_i [\delta_i(1 - \beta_i)]^t}{\gamma - \delta_i(1 - \beta_i)} \Big\}. \tag{3.5}$$

(ii) *It then formulates as:*

$$\sum_{t=0}^{+\infty} \gamma^t \Big\{ \sum_{i=1}^{n} \eta_0^i u^i(c_t^i) \Big\}. \tag{3.6}$$

Objective (3.6) can be understood as a weighted sum of the inter-temporal utilities of $n$ agents without present biais—$\beta_i = 1$ for every $i \in \{1, \ldots, n\}$—and with discount factors equal to $\gamma$, for $\gamma \geqslant \sup_{i \in \{1,\ldots,n\}}(\delta_i)$. According to Proposition 3.2, this objective is obtained from Equation (3.4) and through appropriate weights $\eta_t^i$ for the utilities of the selves of the different agents— —Equation (3.5). Everything happens as if the original agents had been transformed into more patient ones without present biais.

## 3.3 Pareto Optima

The maximization program of a planner using a collective utility function provided by (3.6) states as:

$$\max \sum_{t=0}^{+\infty} \gamma^t \Big\{ \sum_{i=1}^{n} \eta_0^i u^i(c_t^i) \Big\} \text{ s.t. } (\mathscr{C}^1, \ldots, \mathscr{C}^n) \in \Omega(K_0).$$

This is a standard concave problem of optimal growth that leads to a unique solution under Assumptions 1,2, 3 and 4. The optimal capital sequence $(K_t^*)_{t\geqslant 0}$ converges in the long run towards a modified golden rule $K^*$ defined from:

$$D_K F(K^*, 1) + (1 - \mu) = \frac{1}{\gamma}.$$

The optimal solution is both Pareto optimal for the successive incarnations of the agents and temporally consistent. The rate of time preference of the planner is however unrelated to the ones of the agents for $\gamma > \sup_{i\in\{1,\ldots,n\}}(\delta_i)$. When $\gamma = \sup_{i\in\{1,\ldots,n\}}(\delta_i)$, the rate of time preference of the planner is the one of the most patient agent.

## 4. Time-Consistent Constrained Pareto Optima [TCCPO]

This second main solution concept is based upon a $t$-selves concerned collective utility function. Assume that a benevolent planner is to undertake decisions building upon a collective utility function that is not time-consistent along definition 1. Within such a configuration, the planner is to be understood as being made of successive incarnations. The date-$t$ incarnation of the planner being in charge of the decisions for period $t$, it is only concerned by the well-being of the $t$-selves of the different agents. The decision of the $t$-incarnation of the planner is based upon a $t$-selves concerned collective utility function (2.8), or

$$\sum_{i=1}^{n} \eta^i u^i(c_t^i) + \sum_{i=1}^{n} \eta^i \beta^i \sum_{\tau=1}^{+\infty} (\delta_i)^\tau u^i(c_{t+\tau}^i). \tag{4.1}$$

This section introduces a new solution concept labelled *Time-Consistent Constrained Pareto Optima [TCCPO]*. The decision result from an equilibrium in the game played by the successive incarnations of the social planner. Time-consistency is constrained as the strategies of the successive incarnations are assumed to be Markovian and only depend on the state variable $K_t$. A first subsection will introduce the TCCPO solution concept. A second subsection will recall some features of the benchmark time-inconsistent solution.

## 4.1 The Temporally Consistent Solution

Definition 3. The *Time-Consistent Constrained Pareto Optimal solution* (TCCPO) is the Nash equilibrium of a game defined as follows:

(i) An infinity of players indexed by $t \in \mathbb{N}$, where player $t$ is the $t$-incarnation of the social planner endowed with the $t$-selves concerned utility function (4.1).

(ii) Player $t$ uses $C^1$ Markovian strategies $c_t^i = \vartheta_C^i(K_t)$ and $K_{t+1} = \vartheta_K(K_t)$ subject to a feasibility constraint:

$$\vartheta_K(K_t) + \sum_{i=1}^{n} \vartheta_C^i(K_t) = F(K_t, 1) + (1-\mu)K_t \tag{4.2}$$

The following three lemmas—their proofs are obvious and omitted—will enable to characterize the optimal strategies as the solutions of some optimization program.

Lemma **4.1.** *Consider a TCCPO solution with a value of* $K_t$ *for date* $t \in \mathbb{N}$*:*

(i) *The decision rules* $\vartheta_C^i$ *and* $\vartheta_K$ *allow to determine the sequences* $(c_x^i)_{x \geq t}$ *and* $(K_x)_{x \geq t+1}$ *with respect to* $K_t$*:*

$$c_x^i = \vartheta_C^i \circ (\vartheta_K)^{x-t}(K_t)$$
$$K_x = (\vartheta_K)^{x-t}(K_t)$$

(ii) *The inter-temporal payoff of the* $t$*-self of agent* $i \in \{1, \ldots, n\}$ *is obtained as:*

$$J^i(K_t) = u^i[\vartheta_C^i(K_t)] + \beta_i \sum_{\tau=1}^{+\infty} (\delta_i)^\tau u^i[\vartheta_C^i \circ (\vartheta_K)^\tau (K_t)] \tag{4.3}$$

(iii) *The inter-temporal payoff of the* $t$*-incarnation of the social planner is obtained as:*

$$W(K_t) = \sum_{i=1}^{n} \eta^i J^i(K_t), \tag{4.4}$$

Lemma **4.2.** *Consider a TCCPO solution with a value of* $K_t$ *for date* $t \in \mathbb{N}$ *and let*

$$\mathscr{J}^i(K_t) = u^i[\vartheta_C^i(K_t)] + \sum_{\tau=1}^{+\infty} (\delta_i)^\tau u^i[\vartheta_C^i \circ (\vartheta_K)^\tau (K_t)] \tag{4.5}$$

(i) $\mathscr{J}^i[\vartheta_{\mathrm{K}}(\mathrm{K}_t)]$ *corresponds to the inter-temporal payoff from period* $t+1$ *onward as measured by the* $t$*-self of agent* $i$*:*

$$\mathrm{J}^i(\mathrm{K}_t) = u^i[\vartheta^i_{\mathrm{C}}(\mathrm{K}_t)] + \beta_i\delta_i\,\mathscr{J}^i(\vartheta_{\mathrm{K}}(\mathrm{K}_t)) \tag{4.6}$$

(ii) *Moreover:*

$$\mathrm{J}^i(\mathrm{K}_t) = (1-\beta_i)u^i[\vartheta^i_{\mathrm{C}}(\mathrm{K}_t)] + \beta_i\,\mathscr{J}^i(\mathrm{K}_t).$$

Lemma **4.3.** *Consider a TCCPO solution with a value of* $K_t$ *for date* $t \in \mathbb{N}$ *and let*

$$\mathscr{W}(\mathrm{K}_t) = \sum_{i=1}^{n} \eta^i\beta_i\delta_i\,\mathscr{J}^i(\mathrm{K}_t), \tag{4.7}$$

*the inter-temporal payoff of the* $t$*-incarnation of the social planner satisfies:*

$$\mathrm{W}(\mathrm{K}_t) = \sum_{i=1}^{n} \eta^i u^i[\vartheta^i_{\mathrm{C}}(\mathrm{K}_t)] + \mathscr{W}[\vartheta_{\mathrm{K}}(\mathrm{K}_t)]$$

The following proposition then directly results from the above sequence of lemmas.

Proposition 4.1. *Consider a TCCPO solution with a value of* $K_t$ *for date* $t \in \mathbb{N}$*. Decision rules* $\vartheta^i_{\mathrm{C}}(\mathrm{K}_t)$ *and* $\vartheta_{\mathrm{K}}(\mathrm{K}_t)$ *are solutions of the program:*

$$\mathrm{W}(\mathrm{K}_t) = \max_{(c^i_t,\mathrm{K}_{t+1})} \sum_{i=1}^{n} \eta^i u^i(c^i_t) + \mathscr{W}(\mathrm{K}_{t+1}) \tag{P}$$
$$\text{s.t. } \mathrm{K}_{t+1} = \mathrm{F}(\mathrm{K}_t, 1) + (1-\mu)\mathrm{K}_t - \sum_{i=1}^{n} c^i_t$$

It is worth noticing that, in Program (P), functions W and $\mathscr{W}$ do differ and do not satisfy the Bellman equation, the latter being only recovered for the specific case where $\beta_i = \beta_j = 1$, $\delta_i = \delta_j$ for every $i, j \in \{0, \ldots, n\}$.

No existence result was completed for the functions $\mathrm{J}^i(\cdot)$, $\mathrm{W}(\cdot)$ and $\mathscr{W}(\cdot)$ in the general case. However, and as this shall be illustrated in Section 5, an explicit characterization is available for logarithmic utility functions and Cobb-Douglas production technologies. In the general case, some further assumptions on $\mathscr{W}(\cdot)$ will allow for the determination of some properties of the optimal strategies $\vartheta^i_{\mathrm{C}}(\cdot)$.

Assumption 5. *$\mathscr{W}(\cdot)$ is a well-defined function from $\mathbb{R}_+^*$ into $\mathbb{R}_+$ that is of class $C^2$, strictly increasing and concave. Moreover,* $\lim_{K\to 0} D\mathscr{W}(K) = +\infty$.

Lemma **4.4.** *A solution to the Program* (P) *satisfies the following first-order and associated envelope conditions:*

$$\eta^i Du^i(c_t^i) = D\mathscr{W}((K_{t+1}), \tag{4.8a}$$

$$DW(K_t) = D\mathscr{W}(K_{t+1})\big[D_K F(K_t, 1) + (1-\mu)\big]. \tag{4.8b}$$

Proposition 4.2. *Under Assumption 5:*

(i) *The decision rules $\vartheta_C^i(K_t)$ and $\vartheta_K^i(K_t)$ are uniquely defined on $]0, \bar{K}]$.*

(ii) *$\vartheta_C^i(\cdot)$ and $\vartheta_K^i(\cdot)$ are increasing and $C^1$ functions such that $\lim_{K_t\to 0}\vartheta_C^i(K_t) = 0$ and $\lim_{K_t\to 0}\vartheta_K(K_t) = 0$.*

(iii) *Moreover:*

$$DJ^i(K_t) = Du[\vartheta^i(K_t)]D\vartheta_C^i(K_t) + \beta_i\delta_i D\mathscr{J}^i(K_{t+1})D\vartheta_K(K_t), \tag{4.9a}$$

$$D\vartheta_K(K_t) = D_K F(K_t, 1) + (1-\mu) - \sum_{i=1}^{n} D\vartheta_C^i(K_t), \tag{4.9b}$$

$$DJ^i(K_t) = (1-\beta_i)Du[\vartheta^i(K_t)]D\vartheta_C^i(K_t) + \beta_i D\mathscr{J}^i(K_t). \tag{4.9c}$$

Henceforward focusing on the long-run steady state and taking advantage of Proposition 4.2, it does correspond to a modified golden rule.

Proposition 4.3. — *Consider the previous environment:*

(i) *A steady state solution $(K^*, c^{i*})_{i\in\{1,\dots,n\}}$ corresponds to a modified golden rule such that $\zeta = D_K F(K^*, 1) + (1-\mu)$ is solution of:*

$$\sum_{i=1}^{n}(1-\beta_i)D\vartheta_C^i(K^*) = \sum_{i=1}^{n}\frac{D\vartheta_C^i(K^*)\beta_i\,(\delta_i\zeta - 1)}{1-\delta_i\big[\zeta - \sum_{j=1}^{n} D\vartheta_C^j(K^*)\big]} \tag{4.10}$$

(ii) *When $\beta_i = 1$ for every $i \in \{1,\dots,n\}$, Equation* (4.10) *assumes a unique solution $\zeta_1$ with*

$$\zeta_1 \in \left]\frac{1}{\sup_{i\in\{1,\dots,n\}}(\delta_i)}, \frac{1}{\inf_{i\in\{1,\dots,n\}}(\delta_i)}\right[.$$

(iii) *For every* $i \in \{1, \ldots, n\}$*, let* $\beta_i < 1$ *and* $\beta_i$ *be close from* 1*, then Equation* (4.10) *assumes a unique solution with* $\zeta > \zeta_1$.

It is first to be recalled that the FTCPO of Section 3 results in a stationary state such that $\zeta = 1/\gamma$ with $\gamma \geqslant \sup_{i \in \{1,\ldots,n}(\delta_i)$. In opposition to this, Proposition 4.3 shows how the steady state of the TCCPO solution results in a value of $\zeta$ that belongs to $\left]1/\sup_{i \in \{1,\ldots,n\}}(\delta_i), 1/\inf_{i \in \{1,\ldots,n\}}(\delta_i)\right[$ for $\beta_i = 1$. More generally, Equation 4.10 makes clear that the value of $\zeta$ will depend upon the whole distribution of pairs $(\delta_i, \beta_i)$.

Remark 1. *These results can be compared with the temporally inconsistent solution. For the latter, the date-*$t$ *incarnation of the centralized planner solves the following maximization program:*

$$\max \sum_{i=1}^{n} \eta^i u^i(c_t^i) + \sum_{i=1}^{n} \eta^i \beta^i \sum_{\tau=1}^{+\infty} (\delta_i)^\tau u^i(c_{t+\tau}^i) \quad s.t.\ (\mathscr{C}^1, \ldots, \mathscr{C}^n) \in \Omega(K_0).$$

*The trajectory then converges towards a modified golden rule* $\tilde{K}$ *such that:*

$$D_K F(\tilde{K}, 1) + 1 - \mu = \frac{1}{\bar{\delta}}, \text{ with } \bar{\delta} = \sup_{i \in \{1,\ldots,n\}} (\delta_i).$$

*The most patient agent preferences will hence determine the characteristics of the long run. This solution is* $t$*-selves Pareto optimal but turns out as being not temporally consistent.*

The following subsection is going to make clear that the TCCPO solution concept mimics the solution that would result from a benevolent planner that would incorporate a weighed average of the various $(\delta_i, \beta_i)$.

## 4.2 Recovering the Time-Consistent Constrained Optimum through a Standard Discounted Optimisation Problem

This section shows that the Time-Consistent Constrained Optimum (TCCPO) can be recovered through a standard discounted optimisation problem, with a time-dependent discount factor that depends on all discount factors $\delta_i$ and bias $\beta_i$. Con-

sider the following centralized planification program:

$$\max_{\{c_t^i\}} \sum_{t=0}^{+\infty} \Delta_t \sum_{i=1}^{n} \eta^i u^i(c_t^i) \tag{PP}$$
$$\text{s.t. } K_{t+1} = F(K_t, 1) + (1-\mu)K_t - \sum_{i=1}^{n} c_t^i,$$
$$K_0 \text{ given},$$

with $(\Delta_t)_{t\in\mathbb{N}}$ a sequence defined by the holding of $\Delta_0 = 1$ and $\Delta_{t+1} = \delta_t \Delta_t$ and $(\delta_t)_{t\in\mathbb{N}}$ a sequence such that $\lim_{t\to+\infty} \delta_t = \delta$ with $\delta < 1$. The problem (PP) features a standard concave optimal planner's problem with a rate of discount $\delta_t$ that is explicitly time-dependent. The following lemma then characterizes the solution to this program:

Lemma 4.5. *The optimal solution of the program* (PP) *is characterised by a sequence* $(c_t^i, K_t)_{t\in\mathbb{N}}$, $K_0$ *given, such that, for every* $i = 1, \dots, n$, $j = 1, \dots, n$ *and for every* $t \in \mathbb{N}$,

$$\eta^i Du^i(c_t^i) = \eta^j Du^j(c_t^j), \tag{4.11a}$$
$$Du^i(c_{t-1}^i) = \delta_{t-1}[D_K F(K_t, 1) + 1 - \mu]Du^i(c_t^i), \tag{4.11b}$$
$$\lim_{t\to+\infty} \Delta_t Du^i(c_t^i)[D_K F(K_t, 1) + 1 - \mu] = 0, \tag{4.11c}$$
$$K_{t+1} = F(K_t, 1) + (1-\mu)K_t - \sum_{i=1}^{n} c_t^i, \tag{4.11d}$$

The following proposition then establishes that the TCCPO $(K_t^*, c_t^{i*})_{t\geqslant 0, i=1,\dots,n}$ can be recovered as a solution to the planner's program (PP) for a well-chosen sequence of discount factors $(\delta_t)_{t\in\mathbb{N}}$.

Proposition 4.4. *Assume that the Time Consistent Constrained Optimum is described by a sequence* $(K_t^*, c_t^{i*})_{t\in\mathbb{N}, i=1,\dots,n}$ *that converges to a non-zero stationary state* $(K^*, c^{i*})_{i=1,\dots,n}$.

(i) $(K_t^*, c_t^{i*})_{t\in\mathbb{N}, i=1,\dots,n}$ *is a solution to the problem (PP) for a sequence of discount factors* $(\delta_t)_{t\in\mathbb{N}}$ *equal to:*

$$\delta_t = \frac{D\mathscr{W}(K_{t+1}^*)}{DW(K_{t+1}^*)}. \tag{4.12}$$

(ii) $(\delta_t)_{t\in\mathbb{N}}$ *converges to* $\delta$ *such that:*

$$\sum_{i=1}^{n}(1-\beta_i)D\vartheta_C^i(K^*) = \sum_{i=1}^{n} D\vartheta_C^i(K^*)\beta_i\left(\frac{\delta_i}{\delta}-1\right) \Big/ \left\{1-\delta^i\left[\frac{1}{\delta}-\sum_{j=1}^{n} D\vartheta_C^j(K^*)\right]\right\} \quad (4.13)$$

(iii) *Moreover,*

$$\sum_{i=1}^{n}\beta_i\delta_i\frac{\eta^i D\mathscr{J}^i(K^*_{t+1})}{\sum_{j=1}^{n}\eta^j D\mathscr{J}^j(K^*_{t+1})} \leqslant \delta_t \leqslant \sum_{i=1}^{n}\delta_i\frac{\eta^i\beta_i D\mathscr{J}^i(K^*_{t+1})}{\sum_{j=1}^{n}\eta^j\beta_j D\mathscr{J}^j(K^*_{t+1})}$$

*and*

$$\sum_{i=1}^{n}\beta_i\delta_i\frac{\eta^i DJ^i(K^*_{t+1})}{\sum_{j=1}^{n}\eta^j DJ^j(K^*_{t+1})} \leqslant \delta_t \leqslant \sum_{i=1}^{n}\delta_i\frac{\eta^i DJ^i(K^*_{t+1})}{\sum_{j=1}^{n}\eta^j DJ^j(K^*_{t+1})}$$

Proposition 4.4 establishes how it is possible to recover the TCCPO as a solution to a standard planner's program whose discount factor would correspond to a weighted sum of the discount factors and biases of the different agents:

$$\begin{aligned}\delta_t &= \frac{D\mathscr{W}(K^*_{t+1})}{DW(K^*_{t+1})}\\ &= \sum_{i=1}^{n}\beta_i\delta_i\frac{\eta^i D\mathscr{J}^i(K^*_{t+1})}{\sum_{j=1}^{n}\eta^j DJ^j(K^*_{t+1})},\end{aligned}$$

The weighting coefficients $\eta^i D\mathscr{J}^i(K^*_{t+1})/\sum_{j=1}^{n}\eta^j DJ^j(K^*_{t+1})$ are time-dependent in the general case and sum to 1 only in the case $\beta_i = 1, \forall i$. When $\beta_i \leqslant 1$ for every $i \in \{1,\dots,n\}$, the coefficient $\delta_t$ assumes an upper bound defined from an average of $\delta_i$ and a lower bound defined from an average of $\beta_i\delta_i$.

## 5. An Explicit Example & Some Applications

In this part, agents have logarithmic instantaneous utility functions $u^i(c) = \ln c$, for every $i \in \{1,\dots,n\}$ and the production technology is Cobb-Douglas with full depreciation of capital, $F(K,1) = K^\alpha$, with $\alpha \in ]0,1[$. This allows for explicit formulations for the FTCPO and TCCPO solutions. Relying upon distortionary taxation, public debts and initial transfers, it is then proved that both solutions can be recovered as competitive equilibria.

## 5.1 The fully time-consistent pareto optimum (FTCPO) solution

From Proposition 3.2, the maximization program of the social planner in this case is a standard optimal growth problem:

$$\max \sum_{t=0}^{+\infty} \gamma^t \left\{ \sum_{i=1}^{n} \eta^i \ln c_t^i \right\} \tag{5.1}$$
$$\text{s.t. } \mathrm{K}_{t+1} + \sum_{i=1}^{n} c_t^i = \mathrm{K}_t^{\alpha}, \text{ with } \mathrm{K}_0 \text{ given.}$$

Without loss of generality, it is first assumed that $\sum_{i=1}^{n} \eta^i = 1$. The following proposition—the proof is standard and omitted—characterizes the optimal solution:

Proposition 5.1. *Equation (5.1) has an optimal solution characterised by:* $\forall t \geq 0$, $\forall i = 1, \ldots, n$,

$$c_t^{i*} = \eta^i (1 - \gamma\alpha) \mathrm{K}_t^{*\alpha} \tag{5.2a}$$
$$\mathrm{K}_{t+1}^{*} = \gamma\alpha \mathrm{K}_t^{*\alpha} \tag{5.2b}$$

*Along this solution, capital converges to the modified golden rule level:* $\mathrm{K}^* = (\gamma\alpha)^{1/(1-\alpha)}$.

## 5.2 The time-consistent constrained Pareto optimum (TCCPO) solution

The optimal strategies $c_t^i = \vartheta_{\mathrm{C}}^i(\mathrm{K}_t)$ for $i = 1, \ldots, n$, and $\mathrm{K}_{t+1} = \vartheta_{\mathrm{K}}(\mathrm{K}_t)$ are solution of the maximization program :

$$\mathrm{W}(\mathrm{K}_t) = \max_{(c_t^i, \mathrm{K}_{t+1})} \sum_{i=1}^{n} \eta^i \ln c_t^i + \mathscr{W}(\mathrm{K}_{t+1}) \tag{5.3}$$
$$\text{s.t.} \quad \mathrm{K}_{t+1} + \sum_{i=1}^{n} c_t^i = \mathrm{K}_t^{\alpha}$$

The following proposition gives the optimal strategies:

Proposition 5.2. *Under the previous assumptions:*

(i) *The time-consistent constrained optimum is characterised by optimal strategies such that:* $\forall t \geq 0$, $\forall i = 1, \ldots, n$,

$$\tilde{c}_t^i = \left[\eta^i \Big/ \left(1 + \alpha \sum_{i=1}^{n} \frac{\eta^i \beta_i \delta_i}{1 - \alpha\delta_i}\right)\right] \tilde{K}_t^\alpha \tag{5.4a}$$

$$\begin{aligned} \tilde{K}_{t+1} &= \left[\alpha \sum_{i=1}^{n} \frac{\eta^i \beta_i \delta_i}{1 - \alpha\delta_i} \Big/ \left(1 + \alpha \sum_{i=1}^{n} \frac{\eta^i \beta_i \delta_i}{1 - \alpha\delta_i}\right)\right] \tilde{K}_t^\alpha \\ &= (1 - \xi)\tilde{K}_t^\alpha \end{aligned} \tag{5.4b}$$

$$\text{with } \xi = \left[1 \Big/ \left(1 + \alpha \sum_{i=1}^{n} \frac{\eta^i \beta_i \delta_i}{1 - \alpha\delta_i}\right)\right]$$

(ii) *Capital converges to a stationary value* $\tilde{K}$ *such that:*

$$\tilde{K} = \left[\alpha \sum_{i=1}^{n} \frac{\eta^i \beta_i \delta_i}{1 - \alpha\delta_i} \Big/ \left(1 + \alpha \sum_{i=1}^{n} \frac{\eta^i \beta_i \delta_i}{1 - \alpha\delta_i}\right)\right]^{1/(1-\alpha)}$$

The comparison between (5.2b) and (5.4b) shows that the coefficient $\gamma$ in (5.2b) is replaced by

$$1 - \xi = \alpha \sum_{i=1}^{n} \frac{\eta^i \beta_i \delta_i}{1 - \alpha\delta_i} \Big/ \left(1 + \alpha \sum_{i=1}^{n} \frac{\eta^i \beta_i \delta_i}{1 - \alpha\delta_i}\right)$$

in (5.4b). When $\beta_i = 1$ for every $i \in \{1, \ldots, n\}$, this coefficient has a simple interpretation as it can be written:

$$\sum_{i=1}^{n} \left[\frac{\eta^i}{1 - \alpha\delta_i} \Big/ \sum_{h=1}^{n} \frac{\eta^h}{1 - \alpha\delta_h}\right] \delta_i$$

It appears as a weighted average of the $\delta_i$ with weights

$$\frac{\eta^i}{1 - \alpha\delta_i} \Big/ \sum_{h=1}^{n} \frac{\eta^h}{1 - \alpha\delta_h}$$

that increase with $\delta_i$. The social planner now uses a weighted average of the $\delta_i$ to decide the optimal path, and not a discount parameter $\gamma \geq \sup(\delta_i)$. When the coefficients $\beta_i$ may differ from 1, $1 - \xi$ is an increasing function of all $\beta_i$. So, if $\beta_i < 1$ $\forall i$, this tends to reduce capital accumulation.

The optimal strategies (5.4a) can be compared with (5.2a). It is obvious to check that, if $\gamma \geq \sup(\delta_i)$ and $\beta_i \in (0, 1)$, then

$$1 \Big/ \left(1 + \alpha \sum_{i=1}^{n} \frac{\eta^i \beta_i \delta_i}{1 - \alpha\delta_i}\right) > (1 - \gamma\alpha)$$

For the same level of the capital stock, consumptions of all agents are higher through (5.4a) than for the solution (5.2a). This result also implies that for the same level of the capital stock $K_t$, $K_{t+1}$ will be higher with (5.2b) than with (5.4b).

## 5.3 Recovering the FTCPO and TCCPO solutions as competitive equilibria

Consider now a competitive economy populated by $n$ agents characterised by the set of pairs $(\beta_i, \delta_i)$ and a logarithmic utility function. Along the literature, it is assumed that they behave in a sophisticated way. At the initial date, every agent $i$ is endowed with an amount $a_0^i$ of capital units. The payment on the capital units states as $R_t$ at period $t$. At any period, each agent offers a quantity $1/n$ of labour remunerated at $w_t$.

Within such an economy, the government will use three policy instruments: a distortionary taxation on the capital stock $X^i$ that is specific to any of the agents, a public debt that is issued in an amount $b_t$ at period $t$ and finally an initial transfer of wealth $(T_0^i)_{i\in\{1,\dots,n\}}$. For a given agent $i$, the return on capital after taxation is given by $R_t X^i$ at period $t$. For $X^i < 1$, the agent is taxed whereas, for $X^i > 1$, he is subsidized.

Definition 4. *An equilibrium is characterised by a sequence* $(a_t^i, c_t^i, K_t, w_t, R_t)_{t\in\mathbb{N}, i=1,2,\dots,n}$, *a sequence of policy instruments* $(X^i, b_t)_{t\in\mathbb{N}, i=1,2,\dots,n}$ *and initial transfers* $(T_0^i)_{i=1,2,\dots,n}$ *such*

*that, for any* $t \in \mathbb{N}$ *and for any* $i = 1, 2, \ldots, n$*:*

$$c_{t+1}^i = c_t^i \mathrm{R}_{t+1} \mathrm{X}^i (1 - \lambda_i), \tag{5.5a}$$

$$a_{t+1}^i = \mathrm{R}_t \mathrm{X}^i a_t^i + w_t - c_t^i, \ t \geqslant 1, \tag{5.5b}$$

$$a_1^i = \mathrm{R}_0 \mathrm{X}^i (a_0^i + \mathrm{T}_0^i) + w_0 - c_0^i, \tag{5.5c}$$

$$\sum_{t=0}^{+\infty} c_t^i \Big/ \prod_{\tau=0}^{t} (R_\tau \mathrm{X}^i) = a_0^i + \mathrm{T}_0^i + \sum_{t=0}^{+\infty} w_t \Big/ \prod_{\tau=0}^{t} (R_\tau \mathrm{X}^i), \tag{5.5d}$$

$$b_0 = \sum_{i=1}^{n} \mathrm{T}_0^i, \tag{5.5e}$$

$$b_{t+1} = \mathrm{R}_t b_t - \mathrm{R}_t \sum_{i=1}^{n} (1 - \mathrm{X}^i) a_t^i, \ t \geqslant 1, \tag{5.5f}$$

$$b_1 = \mathrm{R}_0 b_0 - \mathrm{R}_0 \sum_{i=1}^{n} (1 - \mathrm{X}^i)(a_0^i + \mathrm{T}_0^i), \tag{5.5g}$$

$$\mathrm{K}_{t+1} = \mathrm{K}_t^\alpha - \sum_{i=1}^{n} c_t^i, \tag{5.5h}$$

$$\sum_{i=1}^{n} a_t^i = \mathrm{K}_t + b_t, \ t \geqslant 1, \tag{5.5i}$$

$$\sum_{i=1}^{n} (a_0^i + \mathrm{T}_0^i) = \mathrm{K}_0 + b_0, \tag{5.5j}$$

$$w_t = \frac{(1 - \alpha)}{n} \mathrm{K}_t^\alpha \ \textit{and} \ \mathrm{R}_t = \alpha \mathrm{K}_t^{\alpha - 1}, \tag{5.5k}$$

*with* $\mathrm{K}_0$, $a_0^i$, $i = 1, 2, \ldots, n$ *given such that* $\mathrm{K}_0 = \sum_{i=1}^{n} a_0^i$*, and*

$$\lambda_i = \frac{1 - \delta_i}{1 - \delta_i + \beta_i \delta_i}. \tag{5.6}$$

Within the above definition, Equation (5.5a) corresponds to the generalized Euler equation of Harris & Laibson [22] and characterizes the consumption behaviour. Equations (5.5b) and (5.5c) feature the budget constraints at dates $t \geqslant 1$ and $t = 0$. Equation (5.5d) depicts the inter-temporal budget constraint of agent $i = 1, 2, \ldots, n$. Equation (5.5e) defines the initial value of the debt of the government. Equations (5.5f) and (5.5g) list the budget constraints of the government at dates $t \geqslant 1$ and $t = 0$. The holding of equation (5.5h) is associated with the equilibrium of the goods market at date $t \in \mathbb{N}$ whilst equations (5.5i) and (5.5j) respectively depict the

equilibrium on the capital market at dates $t \geqslant 1$ and $t = 0$. Finally, equation (5.5k) defines the equilibrium wage rate and the equilibrium return on the capital stock.

Proposition 5.3. *The FTCPO solution can be decentralized with a distortionary taxation on capital* $X^i$*, public debt* $b_t$ *and an initial transfer of wealth* $T_0^i$ *in period* 0*, with the following values:*

$$X^i = \gamma \frac{(1 - \delta_i + \beta_i \delta_i)}{\beta_i \delta_i} \tag{5.7}$$

$$T_0^i = \frac{K_0 \beta_i \delta_i}{\alpha\gamma(1 - \delta_i)} \left[ \eta^i (1 - \gamma\alpha) - \frac{(1 - \alpha)}{n} \right] - a_0^i \tag{5.8}$$

*and* $b_t$ *defined recursively as:*

$$b_0 = \sum_{i=1}^{n} T_0^i, \quad b_1 = R_0^* b_0 - R_0^* \sum_{i=1}^{n} (1 - X^i)(a_0^i + T_0^i),$$

$$b_{t+1} = R_t^* b_t - R_t^* \sum_{i=1}^{n} (1 - X^i) a_t^i, \ t \geqslant 1,$$

*with* $a_t^i$ *defined recursively as:*

$$a_0^i \text{ given}, \quad a_1^i = R_0^* X^i (a_0^i + T_0^i) + w_0^* - c_0^{i*},$$

$$a_{t+1}^i = R_t^* X^i a_t^i + w_t^* - c_t^{i*}, \ t \geqslant 1$$

*and* $w_t^* = (1 - \alpha) K_t^{*\alpha} / n$ *and* $R_t^* = \alpha K_t^{*\alpha - 1}$.

As $\gamma \geq \sup_{i \in \{1, \ldots, n\}} (\delta_i)$, it is clear from (5.7) that $X^i > 1$ for all $i$ if $\beta_i < 1$. Capital accumulation must be subsidized for all agents.

Remark 2. *It is readily checked that* $b_0 < 0$ *if, for every* $i \in \{1, \ldots, n\}$*,*

$$\eta_i \geqslant \frac{1 - \alpha}{(1 - \gamma\alpha) n}. \tag{5.9}$$

*The condition* (5.9) *means that the weights of the different agents should not be too much unequal. As capital accumulation should be subsidized for all agents, the government makes a negative aggregate initial transfer that results in a negative debt. This in turn allows for financing the future subsidies.*

Proposition 5.4. *The TCCPO can be decentralized with a distortionary taxation on capital* $X^i$*, public debt* $b_t$ *and an initial transfer of wealth* $T_0^i$ *in period* 0*, with the following values:*

$$X^i = \left[\sum_{j=1}^{n} \frac{\eta^j \beta_j \delta_j}{1-\alpha\delta_j} \Big/ \left(1+\alpha\sum_{j=1}^{n} \frac{\eta^j \beta_j \delta_j}{1-\alpha\delta_j}\right)\right] \frac{(1-\delta_i+\beta_i\delta_i)}{\beta_i\delta_i} \tag{5.10a}$$

$$T_0^i = \frac{K_0\beta_i\delta_i}{(1-\delta_i)} \frac{\eta^i\xi-(1-\alpha)n^{-1}}{1-\xi} - a_0^i \text{ with } \xi = 1 \Big/ \left(1+\alpha\sum_{j=1}^{n} \frac{\eta^j \beta_j \delta_j}{1-\alpha\delta_j}\right) \tag{5.10b}$$

*and* $b_t$ *defined recursively as:*

$$b_0 = \sum_{i=1}^{n} T_0^i, \quad b_1 = \tilde{R}_0 b_0 - \tilde{R}_0 \sum_{i=1}^{n} (1-X^i)(a_0^i + T_0^i),$$

$$b_{t+1} = \tilde{R}_t b_t - \tilde{R}_t \sum_{i=1}^{n} (1-X^i) a_t^i, \ t \geqslant 1,$$

*with* $a_t^i$ *defined recursively as:*

$$a_0^i \text{ given} \quad a_1^i = \tilde{R}_0 X^i (a_0^i + T_0^i) + \tilde{w}_0 - \tilde{c}_0^i,$$

$$a_{t+1}^i = \tilde{R}_t X^i a_t^i + \tilde{w}_t - \tilde{c}_t^i, \ t \geqslant 1$$

*and* $\tilde{w}_t = (1-\alpha)\tilde{K}_t^\alpha/n$ *and* $\tilde{R}_t = \alpha\tilde{K}_t^{\alpha-1}$.

The argument of the proof of Proposition 5.4 is similar to the one of Proposition 5.3 and omitted. Equation (5.10a) has a simple interpretation in the case $\beta_j = 1, \forall j$, as it can be written:

$$X^i = \sum_{j=1}^{n} \left(\frac{\eta^j \delta_j}{1-\alpha\delta_j} \Big/ \sum_{h=1}^{n} \frac{\eta^h}{1-\alpha\delta_h}\right) \frac{1}{\delta_i}$$

The R.H.S. component that multiplies $1/\delta_i$ appears as a weighted average of the $\delta_j$ with weights given by

$$\frac{\eta^j}{1-\alpha\delta_j} \Big/ \sum_{h=1}^{n} \frac{\eta^h}{1-\alpha\delta_h}$$

and that increase with $\delta_j$. To decentralize the constrained time consistent optimum, the social planner must subsidize agents $i$ with a $\delta_i$ smaller than this weighted average but tax agents $i$ with a $\delta_i$ higher than this weighted average.

When $\beta_j$ differs from 1, $X^i$ is adjusted: the term

$$\sum_{j=1}^{n} \frac{\eta^j \beta_j \delta_j}{1-\alpha\delta_j} \Big/ \left(1+\alpha \sum_{j=1}^{n} \frac{\eta^j \beta_j \delta_j}{1-\alpha\delta_j}\right)$$

increases with the $\beta_j$ when the term $(1-\delta_i+\beta_i\delta_i)/\beta_i\delta_i$ decreases with $\beta_i$.

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# A. Proof of proposition 2.1

(i) A preliminary lemma will first establish the existence of interior consumption allocations.

Lemma **A.1.** *There exists* $(\mathscr{C}^1, \ldots, \mathscr{C}^n) \in \Omega(K_0)$ *such that, for every* $i \in \{1, \ldots, n\}$ *and for every* $t \in \mathbb{N}$, $c_t^i > 0$.

*Proof.* Assume that, for every $i \in \{1, \ldots, n\}$ and for every $t \in \mathbb{N}$,

$$c_t^i = \frac{1}{n}\varepsilon[F(K_t, 1) + (1 - \eta)K_t], \tag{A.1}$$

with $\varepsilon \in ]0, 1[$. The sequence $(K_t)_{t \geqslant 0}$ follows $K_{t+1} = H(K_t)$, for

$$H(K_t) := [F(K_t, 1) + (1 - \mu)K_t](1 - \varepsilon).$$

Under Assumption 4, $DH(0) > 1$ for small enough values of $\varepsilon$. Hence, and as $H(0) = 0$, $H(K_0) > K_0$ for $K_0$ in some right neighbourhood of 0. The function $H(\cdot)$ being however increasing, the dynamics of $(K_t)_{t \geqslant 0}$ is monotonically increasing for $K_0$ close to 0 and the consumption sequence $(c_t^i)_{t \geqslant 0}$ defined from Equation (A.1) is interior and feasible. QED

The *if* part is straightforward.

As for the *only if* part, first define $\eta_0^i$ as $\eta_0^i = \Delta_0^i$. Then assume that (2.4) satisfies Selves-Pareto. Consider $(\mathscr{C}^1, \ldots, \mathscr{C}^n) \in \mathring{\Omega}(K_0)$ such that, $\forall\, i, \forall\, \tau$, $c_\tau^i > 0$. Consider the *transformation*

$$
\begin{aligned}
&{}_t\mathrm{T}^i : \Omega(K_0) \to \Omega(K_0) \\
&{}_t\mathrm{T}^i(\mathscr{C}^1, \ldots, \mathscr{C}^n) = (\mathscr{C}^1, \ldots, \mathscr{C}^{i-1}, {}_t\check{\mathscr{C}}^i, \mathscr{C}^{i+1}, \ldots, \mathscr{C}^n),
\end{aligned}
$$

for ${}_t\check{\mathscr{C}}^i = ({}_tc_\tau^i)_{\tau \in \mathbb{N}}$ such that:

– $u^i\left({}_tc_\tau^i\right) = u^i(c_\tau^i) - {}_t\varepsilon_\tau$ for $\tau \leqslant t - 1$ with $\varepsilon_\tau^t > 0$;

– $u^i\left({}_tc_t^i\right) = u^i(c_t^i) + \phi_t$ with $\phi_t > 0$;

– $u^i\left({}_t c_\tau^i\right) = u^i(c_\tau^i)$ for every $\tau > t$.

The coefficients $\varepsilon_\tau^t$ et $\phi_t$ are selected in order for the level of utility of all selves $\tau \leqslant t-1$ to remain unaffected by the transformation while the level of utility of the self $t$ would be strictly improved by this transformation. Note that the transformed allocation satisfies ${}_t\mathrm{T}^i(\mathscr{C}^1, \ldots, \mathscr{C}^n) \in \Omega(\mathrm{K}_0)$ for arbitrarily small values of $\phi_t$.

First consider the case $t = 1$. Self $\tau = 0$ of agent $i$ would be indifferent if:

$$u^i(c_0^i) = (u^i(c_0^i) - \varepsilon_0^1) + \beta_i \delta_i \phi_1$$

or, $\varepsilon_0^1 = \beta_i \delta_i \phi_1$. Assuming selves Pareto, the value of the collective utility function (2.4) must be higher with the second sequence:

$$\Delta_0^i u^i(c_0^i) < \Delta_0^i (u^i(c_0^i) - \varepsilon_0^1) + \Delta_1^i \phi_1,$$

or $\Delta_0^i \varepsilon_0^1 < \Delta_1^i \phi_1$. Then, $\left(\Delta_1^i - \beta_i \delta_i \Delta_0^i\right) \phi_1 > 0$. But this is only possible if $\Delta_1^i - \beta_i \delta_i \Delta_0^i > 0$. We set $\eta_1^i = \Delta_1^i - \beta_i \delta_i \Delta_0^i$ which gives: $\eta_1^i = \Delta_1^i - \beta_i \delta_i \eta_0^i$. This last equation corresponds to (2.5) for $t = 1$, using $\Delta_0^i = \eta_0^i$.

Secondly, consider the case $t \geq 2$. Any self $\tau \leq t-1$ of agent $i$ must be indifferent between the two sequences $\mathscr{C}^i$ and ${}_t\check{\mathscr{C}}^i$. For self 0:

$$u^i(c_0^i) + \beta_i \sum_{\tau=1}^{t-1} \delta_i^\tau u^i(c_\tau^i) = u^i(c_0^i) - \varepsilon_0^t + \beta_i \sum_{\tau=1}^{t-1} \delta_i^\tau \left(u^i(c_\tau^i) - \varepsilon_\tau^t\right) + \beta_i \delta_i^t \phi_t$$

which implies:

$$\varepsilon_0^t + \beta_i \sum_{\tau=1}^{t-1} \delta_i^\tau \varepsilon_\tau^t = \beta_i \delta_i^t \phi_t$$

By the same way, for self $x$ such that $1 \leq x < t-1$, we get:

$$\varepsilon_x^t + \beta_i \sum_{\tau=x+1}^{t-1} \delta_i^{\tau-x} \varepsilon_\tau^t = \beta_i \delta_i^{t-x} \phi_t \tag{A.2}$$

Finally, for self $t-1$,

$$\varepsilon_{t-1}^t = \beta_i \delta_i \phi_t \tag{A.3}$$

Considering (A.2) written for self $x$ and self $x+1$, we get:

$$\varepsilon_x^t = \delta_i(1-\beta_i)\varepsilon_{x+1}^t \tag{A.4}$$

Developing forward (A.4) and using (A.3), we get:

$$\varepsilon_x^t = [\delta_i(1-\beta_i)]^{t-1-x}\,\beta_i\delta_i\phi_t \tag{A.5}$$

Assuming Selves-Pareto, the collective utility function (2.4) must be higher with the second sequence ${}_t\check{\mathscr{C}}^i$:

$$\sum_{\tau=0}^{t-1}\Delta_\tau^i u^i(c_\tau^i) < \sum_{\tau=0}^{t-1}\Delta_\tau^i\left(u^i(c_\tau^i)-\varepsilon_\tau^t\right)+\Delta_t^i\phi_t$$

or,

$$\sum_{\tau=0}^{t-1}\Delta_\tau^i\varepsilon_\tau^t < \Delta_t^i\phi_t$$

Using (A.5), this condition can be written:

$$\phi_t\left[\Delta_t^i - \sum_{\tau=0}^{t-1}\Delta_\tau^i\,[\delta_i(1-\beta_i)]^{t-1-\tau}\,\beta_i\delta_i\right] > 0$$

which implies:

$$\Delta_t^i - \sum_{\tau=0}^{t-1}\Delta_\tau^i\,[\delta_i(1-\beta_i)]^{t-1-\tau}\,\beta_i\delta_i > 0 \tag{A.6}$$

This condition must be true for all $t \geq 2$.

Let us define recursively $\eta_t^i$ as

$$\eta_t^i = \Delta_t^i - \sum_{\tau=0}^{t-1}\Delta_\tau^i\,[\delta_i(1-\beta_i)]^{t-1-\tau}\,\beta_i\delta_i \tag{A.7}$$

with $\eta_0^i = \Delta_0^i$ and $\eta_1^i = \Delta_1^i - \beta_i\delta_i\Delta_0^i$.

Using (A.7) in $t$ and $t-1$, it is obtained that:

$$\eta_t^i - \delta_i(1-\beta_i)\eta_{t-1}^i = \Delta_t^i - \delta_i(1-\beta_i)\Delta_{t-1}^i - \delta_i\beta_i\Delta_{t-1}^i$$

or

$$\eta_t^i - \delta_i(1-\beta_i)\eta_{t-1}^i = \Delta_t^i - \delta_i\Delta_{t-1}^i$$

which corresponds to (2.5).

But, from (A.7), $\eta_t^i < \Delta_t^i$. As $(\Delta_t^i)_{t\in\mathbb{N}} \in \ell^1$, it is finally obtained that $(\eta_t^i)_{t\in\mathbb{N}} \in \ell^1$.

(ii) Consider the sum

$$\sum_{t=0}^{+\infty} \eta_t^i \left\{ u^i\left(c_t^i\right) + \beta_i \sum_{\tau=1}^{+\infty} (\delta_i)^\tau \, u^i\left(c_{t+\tau}^i\right) \right\}$$

Let $\Gamma_t^i$ denote the coefficient of the component $u^i(c_t^i)$. It then derives that:

$$\Gamma_0^i = \eta_0^i,$$
$$\Gamma_t^i = \eta_t^i + \beta_i\delta_i\eta_{t-1}^i + \cdots + \beta_i(\delta_i)^t\eta_0^i.$$

Then:

$$\Gamma_t^i - \delta_i\Gamma_{t-1}^i = \eta_t^i - \delta_i(1-\beta_i)\eta_{t-1}^i.$$

From Equation (2.5) and $\Gamma_0^i = \eta_0^i = \Delta_0^i$, it is obtained that, for every $t \in \mathbb{N}$, $\Gamma_t^i = \Delta_t^i$. Whence the possibility of reformulating Equation (2.4) as Equation (2.7).

QED

## B. Proof of Proposition 3.1

*If.* It is straightforward to check that (3.1) is equal to (3.3) when $\eta_t$ follows (3.2). And it is well-known that the objective function (3.3) is time consistent.

*Only if.* Assume that (3.1) is time consistent. By definition, for any consumption sequences $(c_t)_{t\geq 1}$ and $(c'_t)_{t\geq 1}$

$$\sum_{t=0}^{+\infty} \eta_t \left[ u(c_{t+1}) + \beta \sum_{\tau=1}^{+\infty} u(c_{t+\tau+1})\delta^\tau \right] = \sum_{t=0}^{+\infty} \eta_t \left[ u(c'_{t+1}) + \beta \sum_{\tau=1}^{+\infty} u(c'_{t+\tau+1})\delta^\tau \right] \Leftrightarrow$$
$$\forall c_0,\ \eta_0 \left[ u(c_0) + \beta \sum_{\tau=1}^{+\infty} u(c_\tau)\delta^\tau \right] + \sum_{t=1}^{+\infty} \eta_t \left[ u(c_t) + \beta \sum_{\tau=1}^{+\infty} u(c_{t+\tau})\delta^\tau \right]$$
$$= \eta_0 \left[ u(c_0) + \beta \sum_{\tau=1}^{+\infty} u(c'_\tau)\delta^\tau \right] + \sum_{t=1}^{+\infty} \eta_t \left[ u(c'_t) + \beta \sum_{\tau=1}^{+\infty} u(c'_{t+\tau})\delta^\tau \right]$$

This definition can be simplified:

$$\sum_{t=0}^{+\infty} \eta_t \left[ u(c_{t+1}) + \beta \sum_{\tau=1}^{+\infty} u(c_{t+\tau+1})\delta^\tau \right] = \sum_{t=0}^{+\infty} \eta_t \left[ u(c'_{t+1}) + \beta \sum_{\tau=1}^{+\infty} u(c'_{t+\tau+1})\delta^\tau \right] \quad \text{(B.1)}$$

$$\iff \forall c_0,\ \eta_0\beta \sum_{\tau=1}^{+\infty} u(c_\tau)\delta^\tau + \sum_{t=1}^{+\infty} \eta_t \left[ u(c_t) + \beta \sum_{\tau=1}^{+\infty} u(c_{t+\tau})\delta^\tau \right]$$

$$= \eta_0\beta \sum_{\tau=1}^{+\infty} u(c'_\tau)\delta^\tau + \sum_{t=1}^{+\infty} \eta_t \left[ u(c'_t) + \beta \sum_{\tau=1}^{+\infty} u(c'_{t+\tau})\delta^\tau \right] \quad \text{(B.2)}$$

Consider $\mathscr{C} = (c_\tau)_{\tau\in\mathbb{N}} \in \mathring{\Omega}(K_0)$ such that, for every $\tau$, $c_\tau > 0$. For every $t \geqslant 2$, consider the transformed sequences ${}_t\check{\mathscr{C}} = ({}_tc_\tau)_{\tau\in\mathbb{N}}$ defined by:

– $u({}_tc_1) = u(c_1) - {}_t\varepsilon$ ;

– $u({}_tc_t) = u(c_t) + {}_t\phi$ ;

– $u({}_tc_\tau) = u(c_\tau)$ for every $\tau \neq 1$ and $\tau \neq t$ ;

where ${}_t\phi$ is defined in such a way that Equation (B.1) is satisfied. Note again that the transformed allocation belongs to $\Omega(K_0)$ for arbitrarily small values of ${}_t\varepsilon$.

Applying (B.1) for $(c_\tau)_{\tau\in\mathbb{N}}$ and $({}_2c_\tau)_{\tau\in\mathbb{N}}$, it is obtained that:

$$\eta_0 {}_2\varepsilon = (\eta_0\beta\delta + \eta_1){}_2\phi.$$

Defining $\gamma = {}_2\varepsilon/{}_2\phi$, it derives that:

$$\gamma\eta_0 = \eta_0\beta\delta + \eta_1 \quad \text{(B.3)}$$

Equation (B.3) determines the value of $\gamma$ with respect to $\eta_0$ and $\eta_1$. It implies a first constraint on $\gamma$ : $\gamma \geq \beta\delta$.

It is clear that it is possible to choose any positive value for $\eta_0$, because if we multiply (3.1) by any positive scalar, we obtain an objective function that corresponds to the same preferences for the social planner. Therefore, we can say that (B.3) determines $\gamma$ with respect to $\eta_1$. Another equivalent interpretation is that we can take any value for $\gamma$ such that $\gamma \geq \beta\delta$, and that (B.3) determines $\eta_1$. This second interpretation will be retained in what follows.

Applying (B.2) for $(c_\tau)_{\tau\in\mathbb{N}}$ and $({}_2c_\tau)_{\tau\in\mathbb{N}}$:

$$ {}_2\varepsilon(\eta_0\beta\delta+\eta_1) = {}_2\phi(\eta_0\beta\delta^2+\eta_1\beta\delta+\eta_2). $$

Using (B.3) and as $\gamma = {}_2\varepsilon/{}_2\phi$, it is obtained that:

$$ \gamma^2\eta_0 = \eta_0\beta\delta^2+\eta_1\beta\delta+\eta_2 \tag{B.4} $$

Applying (B.1) for $(c_\tau)_{\tau\in\mathbb{N}}$ and $({}_tc_\tau)_{\tau\in\mathbb{N}}$, ${}_t\varepsilon$ and ${}_t\phi$ must satisfy:

$$ \eta_0{}_t\varepsilon = (\eta_0\beta\delta^{t-1}+\eta_1\beta\delta^{t-2}+\cdots+\eta_{t-1}){}_t\phi. $$

It is then convenient to set $z_t = {}_t\varepsilon/{}_t\phi$, that gives:

$$ \frac{\eta_0\beta\delta^{t-1}+\eta_1\beta\delta^{t-2}+\cdots+\eta_{t-1}}{\eta_0} = z_t. \tag{B.5} $$

Applying (B.2) for $(c_\tau)_{\tau\in\mathbb{N}}$ and $({}_tc_\tau)_{\tau\in\mathbb{N}}$, it derives:

$$ (\eta_0\beta\delta+\eta_1){}_t\varepsilon = (\eta_0\beta\delta^t+\eta_1\beta\delta^{t-1}+\cdots+\eta_t){}_t\phi $$

or

$$ \frac{\eta_0\beta\delta^t+\eta_1\beta\delta^{t-1}+\cdots+\eta_t}{\eta_0\beta\delta+\eta_1} = z_t \tag{B.6} $$

Taking the ratio of (B.5) at date $t+1$ over (B.6) at date $t$, it is obtained that:

$$ \frac{\eta_0\beta\delta+\eta_1}{\eta_0} = \frac{z_{t+1}}{z_t} $$

and, from (B.3), $z_{t+1}/z_t = \gamma$ with $z_2 = \gamma$. Thus,

$$ z_t = \gamma^{t-1}. $$

Finally, (B.5) considered at date $t+1$ can be reformulated as:

$$ \gamma^t\eta_0 = \eta_0\beta\delta^t+\eta_1\beta\delta^{t-1}+\eta_2\beta\delta^{t-2}+\ldots+\eta_{t-1}\beta\delta+\eta_t \tag{B.7} $$

Rearranging and considering this same expression at date $t-1$:

$$ \eta_t = (\gamma^t-\beta\delta^t)\eta_0-\beta\delta^{t-1}\eta_1-\beta\delta^{t-2}\eta_2-\beta\delta^{t-3}\eta_3-\cdots-\beta\delta\eta_{t-1}, $$

$$ \eta_{t-1} = (\gamma^{t-1}-\beta\delta^{t-1})\eta_0-\beta\delta^{t-2}\eta_1-\beta\delta^{t-3}\eta_2-\beta\delta^{t-4}\eta_3-\cdots-\beta\delta\eta_{t-2}. $$

Then noticing that:

$$\eta_t - \delta\eta_{t-1} = -\beta\delta\eta_{t-1} + (\gamma^t - \delta\gamma^{t-1})\eta_0,$$

or, for every $t \geqslant 1$:

$$\eta_t = \delta(1-\beta)\eta_{t-1} + \gamma^{t-1}(\gamma - \delta)\eta_0.$$

In order to solve this equation, let $\eta_t = \chi_t\gamma^t$, or $\chi_t = \gamma^{-t}\eta_t$. Replacing in the above, it is derived that:

$$\gamma\chi_t = \delta(1-\beta)\chi_{t-1} + (\gamma - \delta)\eta_0.$$

This equation assumes a constant solution given by $\chi_t = \chi$ for every $t$,

$$\chi = \frac{(\beta - \eta)\eta_0}{\gamma - \delta(1-\beta)}.$$

Letting then $v_t = \chi_t - \chi$, $v_t$ emerges as as solution of $\gamma v_t = \delta(1-\beta)v_{t-1}$, whence:

$$v_t = v\left[\frac{\delta(1-\beta)}{\gamma}\right]^t.$$

One infers from this the eventual expression of $\chi_t$:

$$\chi_t = \frac{(\gamma - \delta)}{\gamma - \delta(1-\beta)}\eta_0 + v\left[\frac{\delta(1-\beta)}{\gamma}\right]^t$$

and, finally, the one of $\eta_t$:

$$\eta_t = \frac{(\gamma - \delta)}{\gamma - \delta(1-\beta)}\eta_0\gamma^t + v\left[\delta(1-\beta)\right]^t,$$

the expression of $v$ being in its turn determined by the initial condition $\eta_0$:

$$\eta_0 = \frac{(\gamma - \delta)}{\gamma - \delta(1-\beta)}\eta_0 + v,$$

whence:

$$v = \frac{\beta\delta\eta_0}{\gamma - \delta(1-\beta)}.$$

Finally remembering that $\eta_t$ is to be positive for every $t$, it is first noticed that, for $\gamma \geqslant \delta$, this is always the case. In the remaining configuration and for $\gamma < \delta$, the

argument shall proceed by establishing that such a requirement cannot be fulfilled. Indeed, and for $\gamma < \delta(1-\beta) < \delta$, one obtains $\eta_t < 0$ for every arbitrarily large value of $t$, as a result of the second component. Similarly, and for $\delta(1-\beta) < \gamma < \delta$, $\eta_t < 0$ equally prevails for every arbitrarily large value of $t$, this time as a result of the first component. Remembering that $(\eta_t)_{t\in\mathbb{N}} \in \ell^1$, this in turn implies that $\gamma < 1$. QED

## C. Proof of Proposition 3.2

The *if* part of the statement is first established by noticing that, for weights $\eta_t^i$ given by (3.5) , it is straightforward to show that the Collective Utility Function (3.4) takes the form (3.6), which is time-consistent.

As for the *only if* part of the statement, first let, in order to save on notations, the collective preferences order corresponding to (3.4) be denoted by $\succeq$, the associated indifference relation being denoted by $\sim$. The argument shall first proceed by showing that, if the collective utility function is time-consistent, then time-consistency is established for every individual agent $i \in \{1, \ldots, n\}$.

The proof is straightforward and builds upon applying Definition 1 to feasible sequences such that only the consumption sequences of agent $i$ are modified from $(\mathscr{C}_1, \ldots, \mathscr{C}_{i-1}, \mathscr{C}_i, \mathscr{C}_{i+1}, \ldots, \mathscr{C}_n)$ to $(\mathscr{C}_1, \ldots, \mathscr{C}_{i-1}, \mathscr{C}_i', \mathscr{C}_{i+1}, \ldots, \mathscr{C}_n)$. Consider then the relations

$$(\mathscr{C}_1, \ldots, \mathscr{C}_{i-1}, \mathscr{C}_i, \mathscr{C}_{i+1}, \ldots, \mathscr{C}_n) \sim (\mathscr{C}_1, \ldots, \mathscr{C}_{i-1}, \mathscr{C}_i', \mathscr{C}_{i+1}, \ldots, \mathscr{C}_n),$$

$$((x_1, \mathscr{C}_1), \ldots, (x_{i-1}, \mathscr{C}_{i-1}), (x_i, \mathscr{C}_i), (x_{i+1}, \mathscr{C}_{i+1}), \ldots, (x_n, \mathscr{C}_n))$$
$$\sim ((x_1, \mathscr{C}_1), \ldots, (x_{i-1}, \mathscr{C}_{i-1}), (x_i, \mathscr{C}_i'), (x_{i+1}, \mathscr{C}_{i+1}), \ldots, (x_n, \mathscr{C}_n)).$$

As (3.4) is additively separable, all terms related to agents $1, 2, \ldots, i-1, i+1, \ldots, n$ disappear and the equations that remain are the ones that feature the time-consistency for agent $i$. The results of Proposition 3.1 are therefore directly applicable to this configuration.

As a consequence, Proposition 3.1 can be applied and, $\forall\, i \in \{1, \ldots, n\}$, there exists

$\gamma_i \in [\delta_i, 1[$ such that:

$$\eta_t^i = \eta_0^i \left\{ \frac{(\gamma - \delta_i)}{\gamma - \delta_i(1 - \beta_i)} \gamma^t + \frac{\beta_i \delta_i [\delta_i(1 - \beta_i)]^t}{\gamma - \delta_i(1 - \beta_i)} \right\}. \tag{C.1}$$

The collective utility function (3.4) can be specificed as:

$$\sum_{i=1}^{n} \eta_0^i \sum_{t=0}^{+\infty} (\gamma_i)^t u^i(c_t^i), \ \gamma_i \in [\delta_1, 1[;$$

It remains to check that time-consistency implies that, for every $i, j \in \{1, 2, \ldots, n\}$, $\gamma_i = \gamma_j$. Consider $(\mathscr{C}_1, \ldots, \mathscr{C}_n) \in \mathring{\Omega}(K_0)$ a feasible interior allocation such that, for every $k \in \{1, 2, \ldots, n\}$ and for every $t \in \mathbb{N}$, $c_t^k > 0$. Another feasible allocation $(\mathscr{C}_1', \ldots, \mathscr{C}_n')$ is defined such that

$$\begin{cases} \forall\, k \neq i \text{ and } k \neq j, & \mathscr{C}_k' = \mathscr{C}_k, \\ \text{for } i, & c_1^{i\prime} = c_1^i + \varepsilon, \text{ with } \varepsilon > 0, \\ & \forall\, t \neq 1, c_t^{i\prime} = c_t^i, \\ \text{for } j \neq i, & c_1^{j\prime} = c_1^j - \varphi, \text{ with } \varphi > 0 \text{ and } \varphi < c_1^j, \\ & \forall\, t \neq 1, c_t^{j\prime} = c_t^j, \end{cases}$$

where $\varepsilon$ and $\varphi$ are small enough so that $(\mathscr{C}_1', \ldots, \mathscr{C}_n') \in \mathring{\Omega}(K_0)$. The parameters $\varepsilon$ and $\varphi$ are then understood as marginal variations of the consumption sequence so that the following equivalence holds:

$$({}_1\mathscr{C}_1', \ldots, {}_1\mathscr{C}_n') \sim ({}_1\mathscr{C}_1, \ldots, {}_1\mathscr{C}_n),$$

that implies:

$$\eta_0^i u^i(c_1^i) + \eta_0^j u^j(c_1^j) = \eta_0^i u^i(c_1^{i\prime}) + \eta_0^j u^j(c_1^{j\prime}). \tag{C.2}$$

Now, the holding of time-consistency further implies that:

$$((c_0^1, {}_1\mathscr{C}_1'), \ldots, (c_0^n, {}_1\mathscr{C}_n')) \sim ((c_0^1, {}_1\mathscr{C}_1), \ldots, (c_0^n, {}_1\mathscr{C}_n)),$$

that in its turn results in:

$$\eta_0^i \gamma_i u^i(c_1^i) + \eta_0^j \gamma_j u^j(c_1^j) = \eta_0^i \gamma_i u^i(c_1^{i\prime}) + \eta_0^j \gamma_j u^j(c_1^{j\prime}).$$

Dividing the above expression by $\gamma_i$, it is obtained that:

$$\eta_0^i u^i(c_1^i) + \eta_0^j \frac{\gamma_j}{\gamma_i} u^j(c_1^j) = \eta_0^i u^i(c_1^{i\prime}) + \eta_0^j \frac{\gamma_j}{\gamma_i} u^j(c_1^{j\prime}).$$

Computing the difference with (C.2), it derives that:

$$\eta_0^j \left(1 - \frac{\gamma_j}{\gamma_i}\right)[u^j(c_1^j) - u^j(c_1^{j\prime})] = 0.$$

But $\eta_0^j > 0$ and $c_1^j \neq c_1^{j\prime}$, that implies $\gamma_i = \gamma_j$. Thus, more generally and for every $k \in \{1, \ldots, n\}$, $\gamma_k = \gamma$. Moreover, and as, for every $i \in \{1, \ldots, n\}$, $\gamma_i \in [\delta_i, 1[$, which implies that $\gamma \in [\sup_{k \in \{1,\ldots,n\}} \delta_k, 1[$ and the argument is complete. QED

## D. Proof of Lemma 4.4

*Proof.* The first-order conditions of this program are available, for every $i \in \{1, \ldots, n\}$, as:

$$\eta^i \mathrm{D}u^i(c_t^i) = \nu_t,$$
$$\mathrm{D}\mathscr{W}(\mathrm{K}_{t+1}) = \sum_{i=1}^{n} \eta^i \beta_i \delta_i \mathrm{D}\mathscr{J}^i(\mathrm{K}_t) = \nu_t,$$

for $\nu_t$ that denotes the multiplier associated with the resource constraint (B.2).

The envelope condition states as:

$$\mathrm{DW}(\mathrm{K}_t) = \mathrm{D}\mathscr{W}(\mathrm{K}_{t+1})\big[\mathrm{D}_{\mathrm{K}}\mathrm{F}(\mathrm{K}_t, 1) + (1 - \mu)\big].$$

The statement follows. QED

## E. Proof of Proposition 4.2

*Proof.* (i) and (ii). Consider the $n$ first-order conditions:

$$\eta^i \mathrm{D}u^i[\vartheta_{\mathrm{C}}^i(\mathrm{K}_t)] = \mathrm{D}\mathscr{W}[\mathrm{F}(\mathrm{K}_t, 1) + (1 - \mu)\mathrm{K}_t - \sum_{i=1}^{n} \vartheta_{\mathrm{C}}^i(\mathrm{K}_t)].$$

Let $\vartheta_{\mathrm{C}}(\mathrm{K}_t) := \sum_{i=1}^{n} \vartheta_{\mathrm{C}}^i(\mathrm{K}_t)$ and $\mathrm{G}(\mathrm{K}_t) = \mathrm{F}(\mathrm{K}_t, 1) + (1 - \mu)\mathrm{K}_t$, this reformulates to:

$$\mathrm{D}u^i[\vartheta_{\mathrm{C}}^i(\mathrm{K}_t)] = \frac{\mathrm{D}\mathscr{W}[\mathrm{G}(\mathrm{K}_t) - \vartheta_{\mathrm{C}}(\mathrm{K}_t)]}{\eta^i}.$$

Under the decreasingness of $\mathrm{D}u^i(\cdot)$, this becomes:

$$\vartheta_{\mathrm{C}}^i(\mathrm{K}_t) = (\mathrm{D}u^i)^{-1}\Big[\frac{\mathrm{D}\mathscr{W}[\mathrm{G}(\mathrm{K}_t) - \vartheta_{\mathrm{C}}(\mathrm{K}_t)]}{\eta^i}\Big],$$
$$\vartheta_{\mathrm{C}}(\mathrm{K}_t) = \sum_{i=1}^{n}(\mathrm{D}u^i)^{-1}\Big[\frac{\mathrm{D}\mathscr{W}[\mathrm{G}(\mathrm{K}_t) - \vartheta_{\mathrm{C}}(\mathrm{K}_t)]}{\eta^i}\Big]. \tag{E.1}$$

The RHS member of Equation (E.1) being a decreasing function of $\vartheta_{\mathrm{C}}(\mathrm{K}_t)$, upon existence, it is uniquely defined.

The existence argument *per se* does follow from the fact that:

a) if $\vartheta_{\mathrm{C}}(\mathrm{K}_t) = 0$, the RHS member of Equation (E.1) is positive;

b) if $\vartheta_{\mathrm{C}}(\mathrm{K}_t) \longrightarrow \mathrm{G}(\mathrm{K}_t)$, the RHS member of Equation (E.1) tends to 0.

As a consequence, the existence of $\vartheta_{\mathrm{C}}(\mathrm{K}_t)$ is duly established in the interval $]0, \mathrm{G}(\mathrm{K}_t)[$. Moreover, from Assumption 4, $\mathrm{G}(0) = 0$, and, from Assumptions 1 and 5 (Inada conditions on $u(\cdot)$ and $\mathscr{W}(\cdot)$), it is obtained that $\lim_{\mathrm{K}_t \to 0} \vartheta_{\mathrm{C}}(\mathrm{K}_t) = 0$. Both $u^i(\cdot)$ and $\mathscr{W}(\cdot)$ being of class $\mathrm{C}^2$, $\vartheta^C(\mathrm{K}_t)$ is of class $\mathrm{C}^1$ on $]0, \bar{\mathrm{K}}]$. Further, and from Equation (E.1), $\mathrm{G}(\cdot)$ being increasing as a function of $\mathrm{K}_t$, it is clear that such a property is also satisfied for $\vartheta_{\mathrm{C}}(\cdot)$. Similarly, by definition, $\vartheta_{\mathrm{K}}(\mathrm{K}_t) = \mathrm{G}(\mathrm{K}_t) - \vartheta_{\mathrm{C}}(\mathrm{K}_t)$ and $\vartheta_{\mathrm{K}}(\mathrm{K}_t)$ is of class $\mathrm{C}^1$ on $]0, \bar{\mathrm{K}}]$ such that $\lim_{\mathrm{K}_t \to 0} \vartheta_{\mathrm{K}}(\mathrm{K}_t) = 0$.

Finally remark that Equation (E.1) can accordingly be reformulated to:

$$\vartheta_{\mathrm{C}}(\mathrm{K}_t) = \sum_{i=1}^{n}(\mathrm{D}u^i)^{-1}\Big[\frac{\mathrm{D}\mathscr{W}[\vartheta_{\mathrm{K}}(\mathrm{K}_t)]}{\eta^i}\Big].$$

But $\vartheta_{\mathrm{C}}(\cdot)$ has already been established to be increasing, that in turn implies that $\vartheta_{\mathrm{K}}(\cdot)$ is likewise increasing.

Also note $\vartheta_{\mathrm{C}}^i(\cdot)$ is implicitly defined as a solution to:

$$\vartheta_{\mathrm{C}}^i(\mathrm{K}_t) = (u^i)^{-1}\Big[\frac{\mathrm{D}\mathscr{W}[\vartheta_{\mathrm{K}}(\mathrm{K}_t)]}{\eta^i}\Big],$$

which implies that $\vartheta_{\mathrm{C}}^i(\cdot)$ is an increasing map.

Finally integrating that $\lim_{\mathrm{K}_t \to 0} \vartheta_{\mathrm{C}}(\mathrm{K}_t) = 0$, it derives that $\lim_{\mathrm{K}_t \to 0} \vartheta_{\mathrm{C}}^i(\mathrm{K}_t) = 0$ and the details of the statement follow.

(iii). The result springs from the derivation with respect to $K_t$ of (4.3), (4.2), and (4.5):

$$\begin{aligned}
DJ^i(K_t) &= Du[\vartheta^i(K_t)]D\vartheta_C(K_t) + \beta_i\delta_i D\mathscr{J}^i(K_{t+1})D\vartheta_K(K_t),\\
D\vartheta_K(K_t) &= D_K F(K_t, 1) + (1-\mu) - \sum_{i=1}^{n} D\vartheta_C^i(K_t),\\
DJ^i(K_t) &= (1-\beta_i)Du[\vartheta^i(K_t)]D\vartheta_C^i(K_t) + \beta_i D\mathscr{J}^i(K_t).
\end{aligned}$$

The statement follows. QED

## F. Proof of Proposition 4.3

*Proof.* (i). For a steady state $(K^*, c^{i*})$, $i \in \{1, \dots, n\}$, it is obtained, from (4.8a), (4.8b), (4.9a), (4.9b) and (4.9c), that:

$$\eta^i Du^i(c^{i*}) = \nu = D\mathscr{W}(K^*), \tag{F.1a}$$

$$DW(K^*) = D\mathscr{W}(K^*)\big[D_K F(K^*, 1) + (1-\mu)\big], \tag{F.1b}$$

$$DJ^i(K^*) = Du^i[\vartheta_C^i(K^*)]D\vartheta_C^i(K^*) + \beta_i\delta_i D\mathscr{J}^i(K^*)D\vartheta_K(K^*), \tag{F.1c}$$

$$D\vartheta_K(K^*) = D_K F(K^*, 1) + (1-\mu) - \sum_{i=1}^{n} D\vartheta_C^i(K^*), \tag{F.1d}$$

$$DJ^i(K^*) = (1-\beta_i)Du^i[\vartheta_C^i(K^*)]D\vartheta_C^i(K^*) + \beta_i D\mathscr{J}^i(K^*). \tag{F.1e}$$

Further combining (F.1c) and (F.1e):

$$\begin{aligned}
D\mathscr{J}^i(K^*) &= \frac{Du^i[\vartheta_C^i(K^*)]D\vartheta_C^i(K^*)}{1-\delta_i D\vartheta_K(K^*)},\\
DJ^i(K^*) &= (1-\beta_i)Du^i[\vartheta_C^i(K^*)]D\vartheta_C^i(K^*) + \beta_i\frac{Du^i(\vartheta_C^i(K^*))D\vartheta_C^i(K^*)}{1-\delta_i D\vartheta_K(K^*)}.
\end{aligned}$$

From (4.4), it is obtained that:

$$\begin{aligned}
DW(K^*) &= \sum_{i=1}^{n}\eta^i DJ^i(K^*)\\
&= \sum_{i=1}^{n}\eta^i(1-\beta_i)Du^i[\vartheta_C^i(K^*)]D\vartheta_C^i(K^*) + \sum_{i=1}^{n}\eta^i\beta_i\frac{Du^i(\vartheta_C^i(K^*))D\vartheta_C^i(K^*)}{1-\delta_i D\vartheta_K(K^*)}.
\end{aligned}$$

From (4.7), it is obtained that:

$$\begin{aligned} \mathrm{D}\mathscr{W}(\mathrm{K}^*) &= \sum_{i=1}^{n} \eta^i \beta_i \delta_i \mathrm{D}\mathscr{J}^i(\mathrm{K}^*) \\ &= \sum_{i=1}^{n} \eta^i \beta_i \delta_i \frac{\mathrm{D}u^i(\vartheta_{\mathrm{C}}^i(\mathrm{K}^*))\mathrm{D}\vartheta_{\mathrm{C}}^i(\mathrm{K}^*)}{1 - \delta_i \mathrm{D}\vartheta_{\mathrm{K}}(\mathrm{K}^*)}. \end{aligned}$$

Letting $\zeta = \mathrm{D}_{\mathrm{K}}\mathrm{F}(\mathrm{K}^*, 1) + 1 - \mu$ and replacing the expressions of $\mathrm{DW}(\mathrm{K}^*)$ and $\mathrm{D}\mathscr{W}(\mathrm{K}^*)$ in (F.1b) and simplifying the expressions of $\eta^i \mathrm{D}u^i[\vartheta^i(\mathrm{K}^*)]$ by making use of (F.1a), the details of the statement follow.

(ii) For $\beta_i = 1, \forall\, i \in \{1, \ldots, n\}$, the analysis boils down to the argument of Drugeon & Wigniolle [14]. Equation (4.10) assumes a unique solution $\zeta_1 = \mathrm{D}_{\mathrm{K}}\mathrm{F}(\mathrm{K}^*, 1)+1-\mu \in \left]1/\underline{\delta}, 1/\overline{\delta}\right[$, where $\underline{\delta} = \inf_{i\in\{1,\ldots,n\}}(\delta_i)$ and $\overline{\delta} = \sup_{i\in\{1,\ldots,n\}}(\delta_i)$.

(iii) In the neighbourhood of $\beta_i = 1$, $i \in \{1, \ldots, n\}$ and for $\beta_i < 1$, noticing that the right hand side of Equation (4.10) increases with $\zeta$, one infers the existence of a unique solution. Moreover and as $\beta_i < 1$, $\zeta > \zeta_1$. QED

## G. Proof of Lemma 4.5

*Proof.* Consider the following generalized Lagrangian:

$$\mathscr{L}_t = \Delta_t \sum_{i=1}^{n} \eta^i u^i(c_t^i) + \mu_{t+1}\Big[\mathrm{F}(\mathrm{K}_t, 1) + (1-\mu)\mathrm{K}_t - \sum_{i=1}^{n} c_t^i\Big] - \mu_t \mathrm{K}_t.$$

The optimality conditions derive as:

$$\Delta_t \eta^i \mathrm{D}u^i(c_t^i) = \mu_{t+1},$$

$$\mu_t = \mu_{t+1}\big[\mathrm{D}_{\mathrm{K}}\mathrm{F}(\mathrm{K}_t, 1) + 1 - \mu\big],$$

$$\lim_{t\to+\infty} \mu_t \mathrm{K}_t = 0.$$

Rearranging and eliminating the multiplier $\mu_t$ between these equations, it is obtained that:

$$\eta^i \mathrm{D}u^i(c_t^i) = \eta^j \mathrm{D}u^j(c_t^j),$$
$$\mathrm{D}u^i(c_{t-1}^i) = \delta_{t-1}[\mathrm{D}_\mathrm{K}\mathrm{F}(\mathrm{K}_t, 1) + 1 - \mu]\mathrm{D}u^i(c_t^i),$$
$$\lim_{t\to+\infty} \Delta_t \mathrm{D}u^i(c_t^i)[\mathrm{D}_\mathrm{K}\mathrm{F}(\mathrm{K}_t, 1) + 1 - \mu] = 0,$$
$$\mathrm{K}_{t+1} = \mathrm{F}(\mathrm{K}_t, 1) + (1-\eta)\mathrm{K}_t - \sum_{i=1}^{n} c_t^i.$$

The statement follows.

QED

## H. Proof of Proposition 4.4

*Proof.* (i) The argument of the proof proceeds by establishing that $(\mathrm{K}_t^*, c_t^{i*})_{t\in\mathbb{N}, i=1,\ldots,n}$ corresponds to a solution to the system defined by equations (4.11a), (4.11b), (4.11c), (4.11d) plus the condition (4.12).

Equation (4.11a) holds because

$$\eta^i \mathrm{D}u^i(c_t^{i*}) = \eta^j \mathrm{D}u^j(c_t^{j*})$$

as a result of equation (4.8a). Equation (4.11b) is satisfied because the consideration of (4.8a) at dates $t$ and $t-1$ delivers:

$$\mathrm{D}u^i(c_t^{i*}) = \mathrm{D}\mathscr{W}(\mathrm{K}_{t+1}^*),$$
$$\mathrm{D}u^i(c_{t-1}^{i*}) = \mathrm{D}\mathscr{W}(\mathrm{K}_t^*),$$

whence:

$$\frac{\mathrm{D}u^i(c_{t-1}^{i*})}{\mathrm{D}u^i(c_t^{i*})} = \frac{\mathrm{D}\mathscr{W}(\mathrm{K}_t^*)}{\mathrm{D}\mathscr{W}(\mathrm{K}_{t+1}^*)}.$$

Making then use of the envelope condition (4.8b):

$$\frac{\mathrm{D}u^i(c_{t-1}^{i*})}{\mathrm{D}u^i(c_t^{i*})} = \frac{\mathrm{D}\mathscr{W}(\mathrm{K}_t^*)}{\mathrm{D}W(\mathrm{K}_t^*)}[\mathrm{D}_\mathrm{K}\mathrm{F}(\mathrm{K}_t^*, 1) + 1 - \mu],$$

that results in (4.11b) making use of (4.12) at date $t-1$.

From its definition, $\delta_t$ converges to a limit value $\delta$ such that:

$$\delta = \frac{\mathrm{D}\mathscr{W}(\mathrm{K}^*)}{\mathrm{D}W(\mathrm{K}^*)}$$

Using the definion of $\mathscr{W}$ and $W$, one gets:

$$\delta = \frac{\sum_{i=1}^{n} \eta^i \beta_i \delta_i \mathrm{D}\mathscr{J}^i(\mathrm{K}^*)}{\sum_{i=1}^{n} \eta^i \mathrm{D}\mathrm{J}^i(\mathrm{K}^*)}$$

with:

$$\mathrm{D}J^j(\mathrm{K}^*) = \mathrm{D}\vartheta^i_{\mathrm{C}}(\mathrm{K}^*)\mathrm{D}u(c^{i*}) + \beta_i\delta_i\mathrm{D}\vartheta_{\mathrm{K}}(\mathrm{K}^*)\mathrm{D}\mathscr{J}^i(\mathrm{K}^*)$$
$$\mathrm{D}\mathscr{J}^i(\mathrm{K}^*) = \mathrm{D}\vartheta^i_{\mathrm{C}}(\mathrm{K}^*)\mathrm{D}u(c^{i*}) + \delta_i\mathrm{D}\vartheta_{\mathrm{K}}(\mathrm{K}^*)\mathrm{D}\mathscr{J}^i(\mathrm{K}^*)$$

From these two equations, we deduce that: $\beta_i\delta_i\mathrm{D}\mathscr{J}^i(\mathrm{K}^*) \leq \delta_i\mathrm{D}\mathrm{J}^i(\mathrm{K}^*)$ and

$$\delta \leq \frac{\sum_{i=1}^{n} \eta^i \delta_i \mathrm{D}\mathrm{J}^i(\mathrm{K}^*)}{\sum_{i=1}^{n} \eta^i \mathrm{D}\mathrm{J}^i(\mathrm{K}^*)}$$

This last expression appears as a weigthed average of the $\delta_i$ with coefficients

$$\frac{\eta^i \mathrm{D}\mathrm{J}^i(\mathrm{K}^*)}{\sum_{j=1}^{n} \eta^j \mathrm{D}\mathrm{J}^j(\mathrm{K}^*)}$$

As $\delta < 1, \lim_{t\to+\infty}\Delta_t = 0$. Equation (4.11c) is satisfied as soon as $c_t^{i*}$, $i = 1, \ldots, n$ and $\mathrm{K}_t^*$ converge toward non-zero stationary values. Finally, Equation (4.11d) is satisfied by construction.The statement follows.

(ii) Equation (4.11b) at the steady state gives $\delta = 1/\zeta$, with $\zeta$ that solves (4.10).

(iii) Consider the expression of $\delta_t$:

$$\delta_t = \sum_{i=1}^{n} \beta_i\delta_i \frac{\eta^i \mathrm{D}\mathscr{J}^i(\mathrm{K}^*_{t+1})}{\sum_{j=1}^{n} \eta^j \mathrm{D}\mathrm{J}^j(\mathrm{K}^*_{t+1})} \tag{H.1}$$

From the definition of $\mathrm{J}^i(\cdot)$ and $\mathscr{J}^i(\cdot)$, it is obtained that, $\forall\, i \in \{1, \ldots, n\}$:

$$\mathrm{D}\mathrm{J}^i(\mathrm{K}_t) = \mathrm{D}u^i[\vartheta^i_{\mathrm{C}}(\mathrm{K}_t)]\mathrm{D}\vartheta^i_{\mathrm{C}}(\mathrm{K}_t) + \beta_i\delta_i\mathrm{D}\mathscr{J}^i(\mathrm{K}_{t+1})\mathrm{D}\vartheta_{\mathrm{K}}(\mathrm{K}_t),$$
$$\mathrm{D}\mathscr{J}^i(\mathrm{K}_t) = \mathrm{D}u^i[\vartheta^i_{\mathrm{C}}(\mathrm{K}_t)]\mathrm{D}\vartheta^i_{\mathrm{C}}(\mathrm{K}_t) + \delta_i\mathrm{D}\mathscr{J}^i(\mathrm{K}_{t+1})\mathrm{D}\vartheta_{\mathrm{K}}(\mathrm{K}_t).$$

whence $\mathrm{DJ}^i(\mathrm{K}_t) \leqslant \mathrm{D}\mathscr{J}^i(\mathrm{K}_t)$. Making use of (H.1), it derives that:

$$\delta_t \geqslant \sum_{i=1}^{n} \beta_i \delta_i \frac{\eta^i \mathrm{DJ}^i(\mathrm{K}^*_{t+1})}{\sum_{j=1}^{n} \eta^j \mathrm{DJ}^j(\mathrm{K}^*_{t+1})},$$
$$\delta_t \geqslant \sum_{i=1}^{n} \beta_i \delta_i \frac{\eta^i \mathrm{D}\mathscr{J}^i(\mathrm{K}^*_{t+1})}{\sum_{j=1}^{n} \eta^j \mathrm{D}\mathscr{J}^j(\mathrm{K}^*_{t+1})},$$

*i.e.*, two minorants for $\delta_t$.

It is also obtained that, for every $i \in \{1, \ldots, n\}$, $\mathrm{DJ}^i(\mathrm{K}_t) \geqslant \beta_i \mathrm{D}\mathscr{J}^i(\mathrm{K}_t)$, whence, finally:

$$\delta_t \leqslant \sum_{i=1}^{n} \delta_i \frac{\eta^i \mathrm{DJ}^i(\mathrm{K}^*_{t+1})}{\sum_{j=1}^{n} \eta^j \mathrm{DJ}^j(\mathrm{K}^*_{t+1})},$$
$$\delta_t \leqslant \sum_{i=1}^{n} \delta_i \frac{\eta^i \beta_i \mathrm{D}\mathscr{J}^i(\mathrm{K}^*_{t+1})}{\sum_{j=1}^{n} \eta^j \beta_j \mathrm{D}\mathscr{J}^j(\mathrm{K}^*_{t+1})},$$

and the statement follows. QED

## I. Proof of Proposition 5.2

*Proof.* The solution is obtained starting from the conjecture that: $c^i_t = \xi^i \mathrm{K}^\alpha_t$, which implies $\mathrm{K}_{t+1} = (1 - \xi)\mathrm{K}^\alpha_t$ with $\xi = \sum_{i=1}^{n} \xi^i$. Introducing the variable $x_t = \ln \mathrm{K}_t$, the dynamics of capital becomes:

$$x_{t+1} = \alpha x_t + \ln(1 - \xi) \tag{I.1}$$

The function $\mathscr{J}^i$ is defined as:

$$\mathscr{J}^i(\mathrm{K}_t) = \sum_{\tau=0}^{+\infty} (\delta_i)^\tau \ln\left(\xi^i \mathrm{K}^\alpha_{t+\tau}\right) = \sum_{\tau=0}^{+\infty} (\delta_i)^\tau \left[\ln\left(\xi^i\right) + \alpha x_{t+\tau}\right]$$

Using (I.1), it is obtained:

$$\mathscr{J}^i(\mathrm{K}_t) = \alpha x_t \left[1 + \delta_i\alpha + (\delta_i\alpha)^2 + (\delta_i\alpha)^3 + \cdots\right] + b_i,$$

with $b_i$ some constant term. Then

$$\mathscr{J}^i(\mathrm{K}_t) = \frac{\alpha \ln \mathrm{K}_t}{1 - \alpha\delta_i} + b_i.$$

The program (5.3) becomes

$$W(K_t) = \max_{(c_t^i, K_{t+1})} \sum_{i=1}^{n} \eta^i \ln(c_t^i) + \ln(K_{t+1}) \sum_{i=1}^{n} \frac{\eta^i \beta_i \delta_i \alpha}{1 - \alpha\delta_i}$$
$$\text{s.t. } K_{t+1} + \sum_{i=1}^{n} c_t^i = K_t^\alpha$$

The solution gives (5.4a) and (5.4b). QED

## J. Proof of Proposition 5.3

*Proof.* The pairs $(c_t^{i*}, K_t^*)$ are proved to be solutions of (5.5a),..., (5.5k) for the values of the policy parameters defined in the proposition.

From (5.2a), $c_{t+1}^{*i} = c_t^{*i} \gamma R_{t+1}^*$. Then, (5.5a) will be satisfied with $c_t^i = c_t^{*i}$ if $X^i$ is defined as $X^i = \gamma/(1 - \lambda_i)$, or (5.7).

Equations (5.5b), (5.5c), (5.5e), (5.5f) and (5.5g) are satisfied by the definition of $b_t$ and $a_t^i$. (5.5h) is satisfied by the optimal path.

From (5.5d), $T_0^i$ is defined in such a way that

$$\sum_{t=0}^{+\infty} (c_t^{i*} - w_t^*) \Big/ \prod_{\tau=0}^{t} (R_\tau^* X^i) = a_0^i + T_0^i$$

or

$$T_0^i = -a_0^i + \sum_{t=0}^{+\infty} \left[ \eta^i (1 - \gamma\alpha) - \frac{(1-\alpha)}{n} \right] K_t^{*\alpha} \Big/ \prod_{\tau=0}^{t} \left( \alpha K_\tau^{*\alpha-1} \frac{\gamma}{1 - \lambda_i} \right)$$

From (5.2b),

$$K_t^{*\alpha} \Big/ \prod_{\tau=0}^{t} \left( \alpha K_\tau^{*\alpha-1} \frac{\gamma}{1 - \lambda_i} \right) = \frac{(1 - \lambda_i)^{t+1} K_0}{\alpha\gamma}.$$

Then,

$$K_t^{*\alpha} \Big/ \prod_{\tau=0}^{t} \left( \alpha K_\tau^{*\alpha-1} \frac{\gamma}{1 - \lambda_i} \right) = \frac{(1 - \lambda_i) K_0}{\alpha\gamma} \sum_{t=0}^{+\infty} (1 - \lambda_i)^t = \frac{(1 - \lambda_i) K_0}{\alpha\gamma\lambda_i}$$

and (5.8) is obtained.

Equation (5.5j) results from (5.5e).

From (5.5b) and (5.5f), it is obtained $\forall t \geq 1$ :

$$\sum_{i=1}^{n} a_{t+1}^{i} - b_{t+1} = \mathrm{R}_{t}^{*}\left(\sum_{i=1}^{n} a_{t}^{i} - b_{t}\right) + (1-\alpha)\mathrm{K}_{t}^{*\alpha} - \sum_{i=1}^{n} c_{t}^{i*},$$

and from (5.5c) and (5.5g),

$$\sum_{i=1}^{n} a_{1}^{i} - b_{1} = \mathrm{R}_{0}^{*}\left[\sum_{i=1}^{n}\left(a_{0}^{i} + \mathrm{T}_{0}^{i}\right) - b_{0}\right] + (1-\alpha)\mathrm{K}_{0}^{*\alpha} - \sum_{i=1}^{n} c_{0}^{i*}$$

As (5.5j) holds, a simple recurrence shows that (5.5i) is satisfied. QED