Inheritance of Convexity for the P min-Restricted Game

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Inheritance of Convexity for the $\mathcal{P}_{\min}$-Restricted Game

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Abstract

We consider restricted games on weighted graphs associated with minimum partitions. We replace in the classical definition of Myerson restricted game the connected components of any subgraph by the sub-components corresponding to a minimum partition. This minimum partition $\mathcal{P}_{\min}$ is induced by the deletion of the minimum weight edges. We provide a characterization of the graphs satisfying inheritance of convexity from the underlying game to the restricted game associated with $\mathcal{P}_{\min}$.

Keywords: cooperative game, convex game, restricted game, partitions.
AMS Classification: 91A12, 91A43, 90C27, 05C75.

1 Introduction

We consider on a given finite set N with $|N| = n$ a weighted network, *i.e.*, a weighted graph $G = (N, E, w)$ where w is a weight function defined on the set E of edges of G. For a given subset A of N, we denote by $E(A)$ the set of edges in E with both end-vertices in A, by $\Sigma(A)$ the subset of edges of minimum weight in $E(A)$, and by $\sigma(A)$ the minimum edge-weight in $E(A)$. Let G_A be the graph induced by A, *i.e.*, $G_A = (A, E(A))$. Then $\tilde{G}_A = (A, E(A) \setminus \Sigma(A))$ is the graph obtained by deleting the minimum weight edges in G_A. In (Skoda, 2016) we introduced the correspondence $\mathcal{P}_{\min}$ on N which associates to every subset $A \subseteq N$, the partition $\mathcal{P}_{\min}(A)$ of A into the connected components of $\tilde{G}_A$. Then for every game (N, v) we defined the restricted game $(N, \overline{v})$ associated with $\mathcal{P}_{\min}$ by:

$$(1) \qquad \overline{v}(A) = \sum_{F \in \mathcal{P}_{\min}(A)} v(F), \quad \text{for all } A \subseteq N.$$

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We more simply refer to this game as the $\mathcal{P}_{\min}$-restricted game. v is the characteristic function of the game, $v : 2^N \to \mathbb{R}$, $A \mapsto v(A)$ and satisfies $v(\emptyset) = 0$. Compared to the initial game (N, v), the $\mathcal{P}_{\min}$-restricted game $(N, \overline{v})$ takes into account the combinatorial structure of the graph and the ability of players to cooperate in a given coalition and therefore differents aspects of cooperation restrictions. In particular, assuming that the edge-weights reflect the strengths of relationships between players, $\mathcal{P}_{\min}(A)$ gives a partition of a coalition A into subgroups where players are in privileged relationships (with respect to the minimum relationship strength in G_A). Many other correspondences have been considered to define restricted games (see, e.g., Myerson (1977); Algaba et al. (2001); Bilbao (2000, 2003); Faigle (1989); Grabisch and Skoda (2012); Grabisch (2013)). For a given correspondence a classical problem is to study the inheritance of basic properties as superadditivity and convexity from the underlying game to the restricted game. Inheritance of convexity is of particular interest as it implies that good properties are inherited, for instance the non-emptiness of the core, and that the Shapley value is in the core. For the correspondence $\mathcal{P}_{\min}$, we proved in (Grabisch and Skoda, 2012) that we always have inheritance of superadditivity from (N, v) to $(N, \overline{v})$. Let us observe that inheritance of convexity is a strong property. Hence it would be useful to consider weaker properties than convexity. Following alternative definitions of convexity in combinatorial optimization and game theory when restricted families of subsets are considered (not necessarily closed under union and intersection), see, *e.g.*, (Edmonds and Giles, 1977; Faigle, 1989; Fujishige, 2005) we introduced in (Grabisch and Skoda, 2012) the $\mathcal{F}$-convexity by restricting convexity to the family $\mathcal{F}$ of connected subsets of G. In (Skoda, 2016), we have characterized inheritance of $\mathcal{F}$-convexity for $\mathcal{P}_{\min}$ by five necessary and sufficient conditions on the edge-weights. Of course the study of inheritance of $\mathcal{F}$-convexity is also a first key step to characterize inheritance of convexity in the general case. Moreover, in (Grabisch and Skoda, 2012) and (Skoda, 2016), simple examples with only two or three different weights point out that inheritance of convexity can only happen for a very restricted family of graphs and therefore that we have to consider $\mathcal{F}$-convexity if we wish to obtain results for a sufficiently large family of graphs. We have also highlighted in (Skoda, 2016) a relation between Myerson's restricted game introduced in (Myerson, 1977) and the $\mathcal{P}_{\min}$-restricted game. Myerson's restricted game (N, v^M) is defined by $v^M(A) = \sum_{F \in \mathcal{P}_M(A)} v(F)$ for all $A \subseteq N$, where $\mathcal{P}_M(A)$ is the set of connected components of G_A. We proved that inheritance of convexity for Myerson's restricted game is equivalent to inheritance of $\mathcal{F}$-convexity for the $\mathcal{P}_{\min}$-restricted game associated with a weighted graph with only two different weights. In the present paper, we consider inheritance of convexity for the correspondence $\mathcal{P}_{\min}$. As convexity implies $\mathcal{F}$-convexity, the conditions established in (Skoda, 2016) are necessary. Now dealing with disconnected subsets of N, we establish supplementary necessary conditions for convexity. As it was foreseeable by taking into account examples of (Grabisch and Skoda, 2012) and (Skoda, 2016), we get very strong restrictions on edge-weights and on the combinatorial structure of the graph G. In particular, we obtain that edge-weights can have at most three different values and that many cycles must be complete or dominated (in some sense) by two specific vertices. In the case of three different values $\sigma_1 < \sigma_2 < \sigma_3$, there exists only one edge e_1 of minimum weight σ_1 and all edges of weight σ_2 are incident to the same end-vertex of e_1. We give a complete characterization of these connected weighted graphs in Theorems 24, 25 and 27. Though these graphs are very particular, they seem quite interesting. For instance, when there are only two values and at least two minimum weight edges we obtain weighted graphs similar to the ones defined in (Skoda, 2016) relating Myerson's restricted game to the $\mathcal{P}_{\min}$-restricted game.

The article is organized as follows. In Section 2, we give preliminary definitions and results established in (Grabisch and Skoda, 2012). In particular, we recall the definition of convexity, $\mathcal{F}$-convexity and general conditions on a correspondence to have inheritance of superadditivity, convexity or $\mathcal{F}$-convexity. In Section 3 we recall necessary and sufficient conditions on the weight vector w established in (Skoda, 2016) for the inheritance of $\mathcal{F}$-convexity from the original communication game (N, v) to the $\mathcal{P}_{\min}$-restricted game $(N, \overline{v})$. In Section 4, we establish new very restrictive conditions on a weighted graph to have inheritance of convexity with $\mathcal{P}_{\min}$. In Section 5, we prove that these conditions are sufficient.

2 Preliminary definitions and results

Let N be a given set. We denote by 2^N the set of all subsets of N. A game (N, v) is *zero-normalized* if $v(i) = 0$ for all $i \in N$. A game (N, v) is *superadditive* if, for all $A, B \in 2^N$ such that $A \cap B = \emptyset$, $v(A \cup B) \geq v(A) + v(B)$. For any given subset $\emptyset \neq S \subseteq N$, the unanimity game (N, u_S) is defined by:

$$u_S(A) = \begin{cases} 1 & \text{if } A \supseteq S, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

We note that u_S is superadditive for all $S \neq \emptyset$.

Let us consider a game (N, v). For arbitrary subsets A and B of N, we define the value:

$$\Delta v(A, B) := v(A \cup B) + v(A \cap B) - v(A) - v(B).$$

A game (N, v) is *convex* if its characteristic function v is supermodular, *i.e.*, $\Delta v(A, B) \geq 0$ for all $A, B \in 2^N$. We note that u_S is supermodular for all

$S \neq \emptyset$. Let $\mathcal{F}$ be a *weakly union-closed family*[1] of subsets of N such that $\emptyset \notin \mathcal{F}$. A game v on 2^N is said to be $\mathcal{F}$-*convex* if $\Delta v(A, B) \geq 0$, for all $A, B \in \mathcal{F}$ such that $A \cap B \in \mathcal{F}$. Let us note that a game (N, v) is convex if and only if it is superadditive and $\mathcal{F}$-convex with $\mathcal{F} = 2^N \setminus \{\emptyset\}$.

For a given graph $G = (N, E)$, we say that a subset $A \subseteq N$ is connected if the induced graph $G_A = (A, E(A))$ is connected.

For a given correspondence $\mathcal{P}$ on N and subsets $A \subseteq B \subseteq N$, we denote by $\mathcal{P}(B)_{|A}$ the restriction of the partition $\mathcal{P}(B)$ to A.

We recall the following results established in (Grabisch and Skoda, 2012).

Theorem 1. *Let N be an arbitrary set and $\mathcal{P}$ a correspondence on N. The following conditions are equivalent:*

1) For all $\emptyset \neq S \subseteq N$, the $\mathcal{P}$-restricted game $(N, \overline{u_S})$ is superadditive.

2) For all subsets $A \subseteq B \subseteq N$, $\mathcal{P}(A)$ is a refinement of $\mathcal{P}(B)_{|A}$.

3) For all superadditive game (N, v) the $\mathcal{P}$-restricted game $(N, \overline{v})$ is superadditive.

As for all $A \subseteq B \subseteq N$, $\mathcal{P}_{\min}(A)$ is a refinement of $\mathcal{P}_{\min}(B)_{|A}$, Theorem 1 implies the following result.

Corollary 2. *Let $G = (N, E, w)$ be an arbitrary weighted graph. Then for every superadditive game (N, v), the $\mathcal{P}_{\min}$-restricted game $(N, \overline{v})$ is superadditive.*

Theorem 3. *Let N be an arbitrary set and $\mathcal{P}$ a correspondence on N. If for all $\emptyset \neq S \subseteq N$, the $\mathcal{P}$-restricted game $(N, \overline{u_S})$ is superadditive, then the following claims are equivalent.*

1) For all $\emptyset \neq S \subseteq N$, the $\mathcal{P}$-restricted game $(N, \overline{u_S})$ is $\mathcal{F}$-convex.

2) For all $i \in N$, for all $A \subseteq B \subseteq N \setminus \{i\}$ such that A, B, and $A \cup \{i\}$ are in $\mathcal{F}$, and for all $A' \in \mathcal{P}(A \cup \{i\})_{|A}$, $\mathcal{P}(A)_{|A'} = \mathcal{P}(B)_{|A'}$.

We also recall the following lemmas proved in (Grabisch and Skoda, 2012). We include the proofs for completeness as these two results are extensively used throughout the paper.

Lemma 4. *Let us consider subsets $A, B \subseteq N$ and a partition $\{B_1, B_2, \ldots, B_p\}$ of B. If A, B_i, and $A \cap B_i \in \mathcal{F}$, for all $i \in \{1, \ldots, p\}$, then for every $\mathcal{F}$-convex game (N, v) we have:*

$$v(A \cup B) + \sum_{i=1}^{p} v(A \cap B_i) \geq v(A) + \sum_{i=1}^{p} v(B_i). \tag{3}$$

[1] $\mathcal{F}$ is weakly union-closed if $A \cup B \in \mathcal{F}$ for all A, $B \in \mathcal{F}$ such that $A \cap B \neq \emptyset$ (Faigle et al., 2010). Weakly union-closed families were introduced and analysed in (Algaba, 1998; Algaba et al., 2000) and called union stable systems.

Proof. We prove the result by induction. (3) is obviously satisfied for $p = 1$. Let us assume it is satisfied for p and let us consider a partition $\{B_1, B_2, \ldots, B_p, B_{p+1}\}$ of B. We set $B' = B_1 \cup B_2 \cup \ldots \cup B_p$. The $\mathcal{F}$-convexity of v applied to $A \cup B'$ and B_{p+1} provides the following inequality:

$$v((A \cup B') \cup B_{p+1}) + v((A \cup B') \cap B_{p+1}) \geq v(A \cup B') + v(B_{p+1}). \tag{4}$$

By induction (3) is valid for B':

$$v(A \cup B') + \sum_{i=1}^{p} v(A \cap B_i) \geq v(A) + \sum_{i=1}^{p} v(B_i). \tag{5}$$

Adding (4) and (5) we obtain the result for $p + 1$. □

Lemma 5. *Let us consider a correspondence $\mathcal{P}$ on N and subsets $A \subseteq B \subseteq N$ such that $\mathcal{P}(A) = \mathcal{P}(B)_{|A}$. If $A \in \mathcal{F}$ and if all elements of $\mathcal{P}(A)$ and $\mathcal{P}(B)$ are in $\mathcal{F}$, then for every $\mathcal{F}$-convex game (N, v) we have:*

$$v(B) - \overline{v}(B) \geq v(A) - \overline{v}(A). \tag{6}$$

Proof. If $\mathcal{P}(B) = \{B_1, B_2, \ldots, B_p\}$ then $\mathcal{P}(A) = \{A \cap B_1, A \cap B_2, \ldots, A \cap B_p\}$, and Lemma 4 implies (6). □

Finally, we recall a characterization of inheritance of convexity for Myerson's correspondence $\mathcal{P}_M$.

Theorem 6. *(van den Nouweland and Borm, 1991). Let $G = (N, E)$ be a graph. For every convex game (N, v), the Myerson restricted game (N, v^M) is convex if and only if G is cycle-complete.*

3 Inheritance of $\mathcal{F}$-convexity

Let $G = (N, E, w)$ be a weighted graph and let $\mathcal{F}$ be the family of connected subsets of N. In this section we recall necessary and sufficient conditions on the weight vector w established in (Skoda, 2016) for the inheritance of $\mathcal{F}$-convexity from the original communication game (N, v) to the $\mathcal{P}_{\min}$-restricted game $(N, \overline{v})$. We assume that all weights are strictly positive and denote by w_k or w_{ij} the weight of an edge $e_k = \{i, j\}$ in E.

A star S_k corresponds to a tree with one internal vertex and k leaves. We consider a star S_3 with vertices $1, 2, 3, 4$ and edges $e_1 = \{1, 2\}$, $e_2 = \{1, 3\}$ and $e_3 = \{1, 4\}$.

Star Condition. *For every star of type S_3 of G, the edge-weights w_1, w_2, w_3 satisfy, after renumbering the edges if necessary:*

$$w_1 \leq w_2 = w_3.$$

Path Condition. *For every elementary path* $\gamma = (1, e_1, 2, e_2, 3, \ldots, m, e_m, m+1)$ *in* G *and for all* i, j, k *such that* $1 \leq i < j < k \leq m$, *the edge-weights satisfy:*

$$w_j \leq \max(w_i, w_k).$$

Proposition 7. *Let* $G = (N, E, w)$ *be a weighted graph. If for all* $\emptyset \neq S \subseteq N$, *the* $\mathcal{P}_{\min}$*-restricted game* $(N, \overline{u_S})$ *is* $\mathcal{F}$*-convex, then the Star and Path Conditions are satisfied.*

For a given cycle in G, $C = \{1, e_1, 2, e_2, \ldots, m, e_m, 1\}$ with $m \geq 3$, we denote by $E(C)$ the set of edges $\{e_1, e_2, \ldots, e_m\}$ of C and by $\hat{E}(C)$ the set composed of $E(C)$ and of the chords of C in G.

Weak Cycle Condition. *For every simple cycle of* G, $C = \{1, e_1, 2, e_2, \ldots, m, e_m, 1\}$ *with* $m \geq 3$, *the edge-weights satisfy, after renumbering the edges if necessary:*

$$w_1 \leq w_2 \leq w_3 = \cdots = w_m = M \qquad (7)$$

where $M = \max_{e \in E(C)} w(e)$.

Intermediary Cycle Condition.

1) Weak Cycle condition.

2) Moreover $w(e) \leq w_2$ *for all chord incident to* 2, *and* $w(e) \leq \hat{M} = \max_{e \in \hat{E}(C)} w(e)$ *for all chord non-incident to* 2. *Moreover:*

- *If* $w_1 \leq w_2 < \hat{M}$ *then* $w(e) = \hat{M}$ *for all* $e \in \hat{E}(C)$ *non-incident to* 2. *If* e *is a chord incident to* 2 *then* $w_1 \leq w_2 = w(e) < \hat{M}$ *or* $w(e) < w_1 = w_2 < \hat{M}$.
- *If* $w_1 < w_2 = \hat{M}$, *then* $w(e) = \hat{M}$ *for all* $e \in \hat{E}(C) \setminus \{e_1\}$.

Cycle Condition.

1) Weak Cycle condition.

2) Moreover $w(e) = w_2$ *for all chord incident to* 2, *and* $w(e) = \hat{M}$ *for all* $e \in \hat{E}(C)$ *non-incident to* 2.

Proposition 8. *Let* $G = (N, E, w)$ *be a weighted graph.*

1) If G *satisfies the Path condition then the Weak Cycle condition is satisfied.*

2) If G *satisfies the Star and Path conditions then the Intermediary Cycle condition is satisfied.*

3) If for all $\emptyset \neq S \subseteq N$, the $\mathcal{P}_{\min}$-restricted game $(N, \overline{u_S})$ is $\mathcal{F}$-convex, then the Cycle Condition is satisfied.

Weak Pan Condition. *For all connected subgraphs corresponding to the union of a simple cycle $C = \{e_1, e_2, \ldots, e_m\}$ with $m \geq 3$, and an elementary path P such that there is an edge e in P with $w(e) \leq \min_{1 \leq k \leq m} w_k$ and $|V(C) \cap V(P)| = 1$, the edge-weights satisfy:*

(a) either $w_1 = w_2 = w_3 = \cdots = w_m = \hat{M}$,

(b) or $w_1 = w_2 < w_3 = \cdots = w_m = \hat{M}$,

where $\hat{M} = \max_{e \in \hat{E}(C)} w(e)$. In this last case $V(C) \cap V(P) = \{2\}$.

Pan Condition.

1) Weak Pan condition.

2) If Claim (b) of the Weak Pan condition is satisfied and if moreover $w(e) < w_1$ then $\{1, 3\}$ is a maximum weight chord of C.

Proposition 9. *Let $G = (N, E, w)$ be a weighted graph.*

1) If G satisfies the Star, and Path conditions, then the Weak Pan condition is satisfied.

2) If for all $\emptyset \neq S \subseteq N$, the $\mathcal{P}_{\min}$-restricted game $(N, \overline{u_S})$ is $\mathcal{F}$-convex, then the Pan Condition is satisfied.

We say that two cycles are adjacent if they share at least one common edge.

Lemma 10. *Let $G = (N, E, w)$ be a weighted graph satisfying the Star and Path conditions. Then for all pairs (C, C') of adjacent simple cycles in G, we have:*

$$(8) \qquad \hat{M} = \max_{e \in \hat{E}(C)} w(e) = \max_{e \in E(C)} w(e) = \max_{e \in E(C')} w(e) = \max_{e \in \hat{E}(C')} w(e) = \hat{M}'.$$

Proof. We first consider $M = \max_{e \in E(C)} w(e)$ and $M' = \max_{e \in E(C')} w(e)$. Let us consider two adjacent cycles C and C' with $M < M'$. There is at least one edge e_1 common to C and C'. Then we have $w_1 \leq M < M'$ and therefore e_1 is a non-maximum weight edge in C'. By Claim 1 of Proposition 8 the Weak Cycle condition is satisfied. It implies that there are at most two non-maximum weight edges in C'. Therefore there exists an edge e'_2 in C' adjacent to e_1 with $w'_2 = M'$. As $M' > M$, e'_2 is not an edge of C. Let e_2 be the edge of C adjacent to e_1 and e'_2. Then we have $w_2 \leq M < M'$ but it contradicts the Star condition applied to $\{e_1, e_2, e'_2\}$. Therefore $M = M'$. Finally by Claim 2 of Proposition 8 the Intermediary Cycle condition is satisfied and we have $\hat{M} = M = M' = \hat{M}'$. □

Adjacent Cycles Condition. *For all pairs (C, C') of adjacent simple cycles in G such that:*

(a) $V(C) \setminus V(C') \neq \emptyset$ and $V(C') \setminus V(C) \neq \emptyset$,

(b) C has at most one non-maximum weight chord,

(c) C and C' have no maximum weight chord,

(d) C and C' have no common chord,

then C and C' cannot have two common non-maximum weight edges. Moreover C and C' have a unique common non-maximum weight edge e_1 if and only if there are non-maximum weight edges $e_2 \in E(C) \setminus E(C')$ and $e'_2 \in E(C') \setminus E(C)$ such that e_1, e_2, e'_2 are adjacent and:

- $w_1 = w_2 = w'_2$ *if* $|E(C)| \geq 4$ *and* $|E(C')| \geq 4$.
- $w_1 = w_2 \geq w'_2$ *or* $w_1 = w'_2 \geq w_2$ *if* $|E(C)| = 3$ *or* $|E(C')| = 3$.

Proposition 11. *Let $G = (N, E, w)$ be a weighted graph. If for all $\emptyset \neq S \subseteq N$, the $\mathcal{P}_{\min}$-restricted game $(N, \overline{u_S})$ is $\mathcal{F}$-convex, then the Adjacent Cycles Condition is satisfied.*

Finally the following characterization of inheritance of $\mathcal{F}$-convexity was established in (Skoda, 2016).

Theorem 12. *Let $G = (N, E, w)$ be a weighted graph. For every superadditive and $\mathcal{F}$-convex game (N, v), the $\mathcal{P}_{\min}$-restricted game $(N, \overline{v})$ is $\mathcal{F}$-convex if and only if the Path, Star, Cycle, Pan, and Adjacent cycles conditions are satisfied.*

4 Inheritance of convexity

We consider in this section inheritance of convexity. As convexity implies superadditivity and $\mathcal{F}$-convexity, the conditions established in Section 3 are necessary. We now have to deal with disconnected subsets of N. We establish supplementary necessary conditions implying strong restrictions on edge-weights. In particular, we obtain that edge-weights can have at most three different values.

Lemma 13. *Let $G = (N, E, w)$ be a weighted graph and let us assume that for all $\emptyset \neq S \subseteq N$ the $\mathcal{P}_{\min}$-restricted game $(N, \overline{u_S})$ is convex. Let $e_1 = \{1, 2\}$ and $e_2 = \{2, 3\}$ be two adjacent edges, and e be an edge such that:*

$$\max(w_1, w_2) < w(e). \tag{9}$$

Then there exists an edge $e' \in E$ linking e to vertex 2. Moreover, if $w_1 \neq w_2$, then we have either $w_1 < w_2 = w(e')$ or $w_2 < w_1 = w(e')$. Otherwise, we have $w(e') \leq w_1 = w_2$.

Proof. We set $e = \{j, k\}$. Star condition implies $j \neq 2$ and $k \neq 2$ (otherwise we get a contradiction with (9)). By contradiction, let us assume that there is no edge linking e to 2. We can assume $w_1 \leq w_2 < w(e)$. Let us define the following subsets of N, $A_1 = \{2\}$, $A_2 = \{j, k\}$, $A = A_1 \cup A_2$, and $B = A \cup \{1\}$ as represented in Figure 52. We set $i = 3$. Hence $A \subseteq B \subseteq N \setminus \{i\}$. As

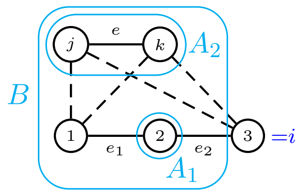

Figure 1: $w_1 \leq w_2 < w(e)$.

there is no edge linking e to 2, we have $\mathcal{P}_{\min}(A) = \{\{2\}, \{j\}, \{k\}\}$. Let us note that, as $w_2 < w(e)$, there is always a component A' of $\mathcal{P}_{\min}(A \cup \{i\})$ containing A_2. As $w_1 < w(e)$, there is also a component B' of $\mathcal{P}_{\min}(B)$ containing A_2. Then $\mathcal{P}_{\min}(B)_{|A' \cap A} \neq \mathcal{P}_{\min}(A)_{|A' \cap A}$ as $\mathcal{P}_{\min}(A)$ corresponds to a singleton partition but $B' \cap A'$ contains A_2. It contradicts Theorem 3 applied with $\mathcal{F} = 2^N \setminus \{\emptyset\}$. Therefore there exists an edge e' linking e to 2. If $e' \neq e_1$ and e_2, then Star condition applied to $\{e_1, e_2, e'\}$ implies $w(e') = w_1$ if $w_2 < w_1$, $w(e') = w_2$ if $w_1 < w_2$, and $w(e') \leq w_1 = w_2$ otherwise. □

We give in Figures 2 and 3 examples of graphs satisfying the necessary condition established in Lemma 13.

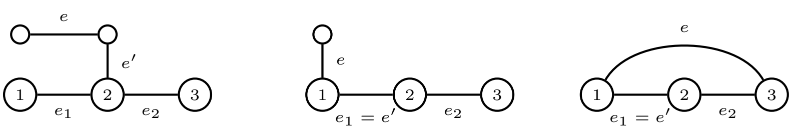

Figure 2: e' linking e to vertex 2.

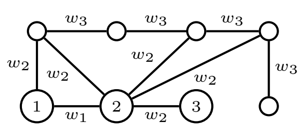

Figure 3: $w_1 < w_2 < w_3$.

Lemma 14. *Let $G = (N, E, w)$ be a weighted graph and let us assume that for all $\emptyset \neq S \subseteq N$ the $\mathcal{P}_{\min}$-restricted game $(N, \overline{u_S})$ is convex. Let*

$e_1 = \{1,2\}$ *and* $e_2 = \{2,3\}$ *be two adjacent edges and let* e *and* e' *be two edges in* E *such that:*

$$\text{(10)} \qquad \max(w_1, w_2) < \min(w(e), w(e')).$$

Then $w(e) = w(e')$.

Proof. We can assume $w_1 \leq w_2$. By contradiction, let us assume $w(e) < w(e')$. Applying Lemma 13 to e (resp. e'), there exists an edge e'_2 (resp. e''_2) linking e (resp. e') to 2 such that $w'_2 \leq \max(w_1, w_2) < w(e)$ (resp. $w''_2 \leq \max(w_1, w_2) < w(e')$) ($e'_2$ (resp. e''_2) may coincide with e_1 or e_2). We set $e'_2 = \{2, 2'\}$ (resp. $e''_2 = \{2, 2''\}$) where $2'$ (resp. $2''$) is an end-vertex of e (resp. e') as represented in Figure 4. If $2' = 2''$, then $e'_2 = e''_2$ and as

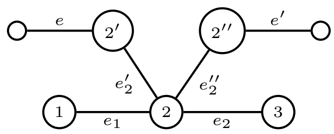

Figure 4: $w'_2 \leq \max(w_1, w_2) < w(e)$ and $w''_2 \leq \max(w_1, w_2) < w(e')$.

$w'_2 < w(e) < w(e')$ we get a contradiction with Star condition applied to $\{e'_2, e, e'\}$. Otherwise, as $w'_2 < w(e) < w(e')$ we can still apply Lemma 13 to e' and the pair of adjacent edges $\{e'_2, e\}$. Hence there exists an edge $e'' \in E$ linking e' to $2'$ (e'' can coincide with e).

Let us first assume $e'' = \{2', 2''\}$ as represented in Figure 5. As $w'_2 < w(e)$

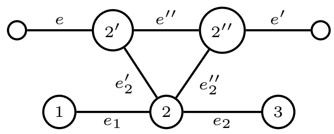

Figure 5: $e'' = \{2', 2''\}$ linking e' to $2'$.

(resp. $w''_2 < w(e')$), Star condition applied to $\{e'_2, e, e''\}$ (resp. $\{e''_2, e', e''\}$) implies $w(e'') = w(e)$ (resp. $w(e'') = w(e')$) and then $w(e) = w(e')$, a contradiction.

Let us now assume $e' = \{1', 2''\}$ and $e'' = \{1', 2'\}$ as represented in Figure 6. Then there is a cycle $C = \{2, e'_2, 2', e'', 1', e', 2'', e''_2, 2\}$. As $w'_2 < w(e)$, Star condition applied to $\{e'_2, e, e''\}$ implies $w(e'') = w(e)$. As $w(e) < w(e')$, we get $w(e'') < w(e')$. Therefore C has three non-maximum weight edges e'_2, e''_2, e'', contradicting the Cycle condition. □

For a given weighted graph $G = (V, E)$, let $\{\sigma_1, \sigma_2, \ldots, \sigma_k\}$ be the set of its edge-weights such that $\sigma_1 < \sigma_2 < \ldots < \sigma_k$ with $1 \leq k \leq n$. We denote by E_i the set of edges in E with weight σ_i.

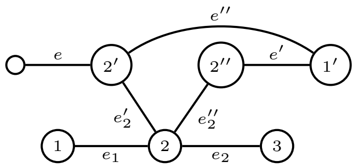

Figure 6: $e'' = \{1', 2'\}$ linking e' to $2'$.

Lemma 15. *Let $G = (N, E, w)$ be a connected weighted graph with at least two different edge-weights $\sigma_1 = \min_{e \in E} w(e)$ and $\sigma_2 = \min_{e \in E, w(e) > \sigma_1} w(e)$, satisfying the Path condition. Then:*

1. *There exists a pair of adjacent edges $\{e_1, e_2\}$ such that $w_1 = \sigma_1$ and $w_2 = \sigma_2$.*
2. *Let N_1 be the set of end-vertices of edges with weight σ_1. The subgraph G_{N_1} is connected.*

Proof. 1. If there is no such pair then we get a contradiction to the Path condition.

2. Let us consider two vertices v' and v'' in N_1. By definition v' and v'' are end-vertices of edges e' and e'' with $w(e') = w(e'') = \sigma_1$. If $e' = e''$ or if e' and e'' are adjacent, then e' or e'' or $e' \cup e''$ corresponds to a path in G_{N_1} linking v' to v''. Otherwise, let γ be a shortest path in G linking e' to e''. Then, the Path condition applied to $\gamma' = e' \cup \gamma \cup e''$ implies $w(e) \leq \max(w(e'), w(e'')) = \sigma_1$ and therefore $w(e) = \sigma_1$ for all edge $e \in \gamma$. Hence γ' is a path from v' to v'' in G_{N_1}. □

The following proposition is a direct consequence of Lemma 14, Proposition 7 and Lemma 15.

Proposition 16. *Let $G = (N, E, w)$ be a connected weighted graph and let us assume that for all $\emptyset \neq S \subseteq N$ the $\mathcal{P}_{\min}$-restricted game $(N, \overline{u_S})$ is convex. Then the edge-weights have at most three different values $\sigma_1 < \sigma_2 < \sigma_3$. Moreover, if $|E_1| \geq 2$, then the edge-weights have at most two different values $\sigma_1 < \sigma_2$.*

Of course, Proposition 16 implies that if the edge-weights have three different values, then there is only one edge with weight σ_1.

Proposition 17. *Let $G = (N, E, w)$ be a connected weighted graph. Let us assume that for all $\emptyset \neq S \subseteq N$, the $\mathcal{P}_{\min}$-restricted game $(N, \overline{u_S})$ is convex. Then for all elementary path $\gamma = \{1, e_1, 2, e_2, \ldots, m, e_m, m+1\}$ in G such that $w_1 < w_m$, we have:*

$$\max(w_1, w_2) \leq w_3 = w_4 = \cdots = w_m. \tag{11}$$

Proof. As convexity implies $\mathcal{F}$-convexity, Propositions 7 and 8 imply that the Star, Path and Cycle conditions are satisfied. The Path condition implies $w_j \leq \max(w_1, w_m) = w_m$ for all j, $1 \leq j \leq m$. Hence (11) is obviously satisfied if $m = 3$. Let us assume $m \geq 4$ and by contradiction $w_{m-1} < w_m$. The Path condition implies $w_2 \leq \max(w_1, w_{m-1}) < w_m$. Therefore $\max(w_1, w_2) < w_m$. Then Lemma 13 implies that there exists an edge e linking e_m to 2 with $w(e) \leq \max(w_1, w_2)$. Hence we have $w(e) < w_m$.

Let us first assume $e = \{2, m\}$ as represented in Figure 7. Then Star

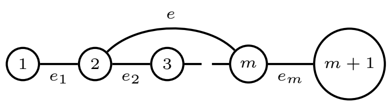

Figure 7: $e = \{2, m\}$.

condition applied to $\{e_{m-1}, e_m, e\}$ implies $w_{m-1} = w_m$, a contradiction.

Let us now assume $e = \{2, m+1\}$ as represented in Figure 8. Then

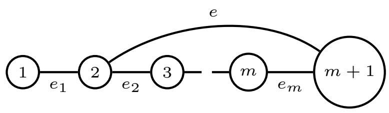

Figure 8: $e = \{2, m+1\}$.

the cycle $C = \{2, e_2, 3 \ldots, m, e_m, m+1, e, 2\}$ contains at least three non-maximum weight edges (e_2, e_{m-1}, e), contradicting the Cycle condition.

Hence we have $w_{m-1} = w_m$ and therefore $w_{m-1} > w_1$. Then we can iterate to get (11). $\square$

A cycle C is said *constant* if all edges in $E(C)$ have the same weight.

Lemma 18 (STRONG PAN CONDITION FOR NON-CONSTANT CYCLE)**.** *Let $G = (N, E, w)$ be a connected weighted graph. Let us assume that for all $\emptyset \neq S \subseteq N$, the $\mathcal{P}_{\min}$-restricted game $(N, \overline{u_S})$ is convex. Let us consider a simple cycle $C_m = \{1, e_1, 2, e_2, \ldots, m, e_m, 1\}$ and an edge $e \in E$ incident to 2, such that:*

$$w(e) < w_1 = w_2 < w_3 = \cdots = w_m = \hat{M} = \max_{e \in \hat{E}(C)} w(e). \quad (12)$$

Then e is not a chord of C_m and C_m is a complete cycle.

Proof. Cycle condition implies that any chord of C_m incident to 2 has weight w_2. Therefore e cannot be a chord of C_m. Let us set $e = \{2, m+1\}$.

Let us first prove that $e'_j = \{2, j\} \in \hat{E}(C_m)$ for all j, $4 \leq j \leq m$. Lemma 13 implies that it is sufficient to prove the existence of such a chord for $m = 4$. Indeed let us assume $e'_j \notin \hat{E}(C_m)$ for a given index j, $4 \leq j \leq m$.

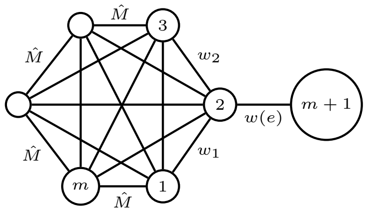

Figure 9: $w(e) < w_1 = w_2 < w_3 = \cdots = w_m = \hat{M}$.

Then Lemma 13 applied to e_{j-1} (resp. e_j) and to the pair of adjacent edges e_1 and e_2 implies that e'_{j-1} (resp. e'_{j+1}) exists in $\hat{E}(C_m)$. Then we have to prove that e'_j is a chord in the cycle $\{2, e'_{j-1}, j-1, e_{j-1}, j, e_j, j+1, e'_{j+1}, 2\}$ as represented in Figure 10.

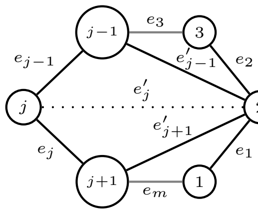

Figure 10: $w_1 = w_2 < w_3 = \cdots = w_m = \hat{M}$

.

By contradiction let us assume $e'_4 \notin \hat{E}(C_4)$. Pan condition implies that $\{1,3\}$ is a maximum weight chord of C_4. Let us define the following subsets of N, $A_1 = \{2,5\}$, $A_2 = \{4\}$, $A = A_1 \cup A_2$, $B = A \cup \{1\}$, and $i = 3$, as represented in Figure 11. If the edge $\{4,5\}$ exists in E, Cycle condition ap-

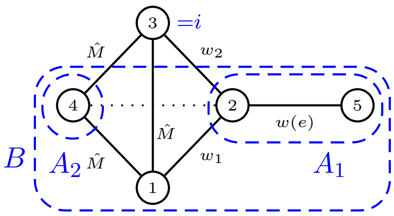

Figure 11: $w(e) < w_1 = w_2 < w_3 = w_4 = \hat{M}$.

plied to the cycle $\{1, e_1, 2, e, 5, \{5,4\}, 4, e_4, 1\}$ implies $w(\{4,5\}) = M > w(e)$. Hence we have either $\mathcal{P}_{\min}(A) = \{\{2\},\{4,5\}\}$ or $\mathcal{P}_{\min}(A) = \{\{2\},\{4\},\{5\}\}$. Moreover $\mathcal{P}_{\min}(A \cup \{i\}) = \{\{2,3,4\},\{5\}\}$ or $\{A \cup \{i\}\}$, and $\mathcal{P}_{\min}(B) = \{\{1,2,4\},\{5\}\}$ or $\{B\}$. If $\mathcal{P}_{\min}(A \cup \{i\}) = \{\{2,3,4\},\{5\}\}$, then taking $A' = \{2,3,4\}$ we get $\mathcal{P}_{\min}(A)_{|A'} = \{\{2\},\{4\}\} \neq \{2,4\} = \mathcal{P}_{\min}(B)_{|A'}$ and it contradicts Theorem 3. Otherwise taking $A' = A \cup \{i\}$, we get

$\mathcal{P}_{\min}(A)_{|A'} = \{\{2\},\{4\},\{5\}\}$ or $\{\{2\},\{4,5\}\}$ and $\mathcal{P}_{\min}(B)_{|A'} = \{\{2,4\},\{5\}\}$ or $\{\{2,4,5\}\}$. Therefore we always have $\mathcal{P}_{\min}(A)_{|A'} \neq \mathcal{P}_{\min}(B)_{|A'}$ and it contradicts Theorem 3.

Let us now prove that $\{j,k\} \in \hat{E}(C_m)$ for all pairs of nodes j, k, with $3 \leq j \leq m-1$ and $k = 1$ or $j+2 \leq k \leq m$. We have $\{2,j\}$ and $\{2,k\}$ in $\hat{E}(C_m)$. Then we can consider the cycle $\tilde{C}_m = \{2, e'_j, j, e_j, j+1, \ldots, k, e'_k, 2\}$ as represented in Figure 12. Then Pan condition applied to $\tilde{C}_m$ and e implies

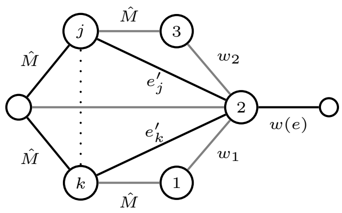

Figure 12: $\tilde{C}_m = \{2, e'_j, j, e_j, j+1, \ldots, k, e'_k, 2\}$ and $w'_j = w'_k = w_1 = w_2$.

that $\{j,k\}$ is a maximum weight chord of $\tilde{C}_m$. □

Remark 1. Let us observe that it results from Propositions 16, 17 and from the Star condition that every pan $\{C_m, P_r\}$ such that C_m has non constant weights and such that P_r has an edge e with $w(e) < \min_{e_j \in E(C_m)} w_j$, satisfies the hypothesis of Lemma 18. (In fact P_r is necessarily limited to one edge).

Lemma 19. *Let $G = (N, E, w)$ be a connected weighted graph. Let us assume that for all $\emptyset \neq S \subseteq N$, the $\mathcal{P}_{\min}$-restricted game $(N, \overline{u_S})$ is convex. Let us moreover assume that the edge-weights have exactly three different values $\sigma_1 < \sigma_2 < \sigma_3$. Then there exist three edges e_1, e_2, e_3 with respective weights σ_1, σ_2, σ_3 such that e_1 and e_2 are incident to a node v and e_3 is adjacent to e_1 or e_2 but not incident to v.*

Three edges e_1, e_2, e_3 satisfying Lemma 19 correspond to three possible situations represented in Figure 13.

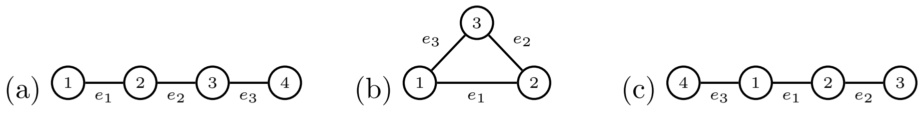

Figure 13: $w_1 = \sigma_1 < w_2 = \sigma_2 < w_3 = \sigma_3$.

Proof. Claim 1 of Lemma 15 implies the existence of e_1 and e_2. Proposition 16 implies the uniqueness of e_1. Let us assume $e_1 = \{1,2\}$ and $e_2 = \{2,3\}$. Let e_3 be an arbitrary edge of weight σ_3. Then Lemma 13 implies that there exists an edge e' linking e_3 to 2 as represented in Figure 14. If $e' \neq e_1$ and $e' \neq e_2$ then Star condition applied to $\{e_1, e_2, e'\}$

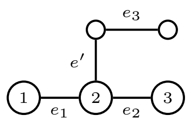

Figure 14: e' linking e_3 to 2 with $w_1 = \sigma_1 < w_2 = \sigma_2 < w_3 = \sigma_3$.

implies $w(e') = \sigma_2$. Therefore we can substitute e' for e_2 and then the 3-tuple $\{e_1, e', e_3\}$ satisfies the conclusion of the lemma. □

Lemma 20. *Let $G = (N, E, w)$ be a connected weighted graph. Let us assume that for all $\emptyset \neq S \subseteq N$, the $\mathcal{P}_{\min}$-restricted game $(N, \overline{u_S})$ is convex. Let us moreover assume that the edge-weights have exactly three different values $\sigma_1 < \sigma_2 < \sigma_3$. Then there is only one edge e_1 with weight σ_1, and all edges with weight σ_2 are incident to the same end-vertex v of e_1. Moreover, all edges with weight σ_3 are not incident to v but connected to v by e_1 or by an edge with weight σ_2.*

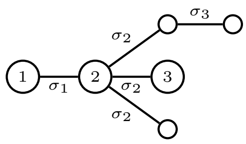

Figure 15: $\sigma_1 < \sigma_2 < \sigma_3$.

Proof. Proposition 16 implies that there is only one edge with weight σ_1.

Lemma 19 implies that there exists e_1, e_2, e_3 in E with $w_i = \sigma_i$ for $i \in \{1, 2, 3\}$ and such that one of the three situations represented in Figure 13 holds. Let e be an edge in $E \setminus \{e_2\}$ with $w(e) = \sigma_2$ and let us assume that e is not incident to 2. Let us first assume $e_3 = \{3, 4\}$ (Case a in Figure 13). Then e cannot be incident to 3 otherwise $\{e, e_2, e_3\}$ contradicts the Star condition. e cannot be incident to 4 otherwise $\{e_2, e_3, e\}$ contradicts the Path condition. Finally, e cannot be incident to 1 otherwise Proposition 17 applied to $\{e, e_1, e_2, e_3\}$ implies $\sigma_2 = \sigma_3$, a contradiction. Let us now assume $e_3 = \{3, 1\}$ (Case b in Figure 13). Then e cannot be incident to 1 (resp. 3) otherwise $\{e, e_1, e_3\}$ (resp. $\{e, e_2, e_3\}$) contradicts the Star condition. Finally, if $e_3 = \{4, 1\}$ (Case c in Figure 13), then we can establish as before with $e_3 = \{3, 4\}$ that e cannot be incident to $1, 3$, and 4.

As G is connected, there exists a shortest path γ linking e to $1, 2$ or 3. The Path condition applied to $\{e\} \cup \gamma \cup \{e_1\}$ or $\{e\} \cup \gamma \cup \{e_2\}$ implies that all edges in γ have weight σ_1 or σ_2. But, as e_1 is the unique edge with weight σ_1, we get $w(e) = \sigma_2$ for any edge e in γ. Let e' be the edge of γ incident to 1, 2, or 3. As $w(e') = \sigma_2$, the first part of the proof implies that e' is necessarily incident to 2. If $e_3 = \{3, 4\}$ (resp. $e_3 = \{4, 1\}$), we consider $\gamma' = \{e\} \cup \gamma \cup \{e_2, e_3\}$ (resp. $\gamma' = \{e\} \cup \gamma \cup \{e_1, e_3\}$) as represented in Figure 16 (resp. Figure 18). Then Proposition 17 applied to γ' implies $\sigma_2 = \sigma_3$ (resp. $\sigma_1 = \sigma_3$), a contradiction. If $e_3 = \{3, 1\}$, then we can still consider $\gamma' = \{e\} \cup \gamma \cup \{e_1, e_3\}$ as represented in Figure 17 and get the same contradiction.

Finally, if an edge with weight σ_3 is incident to v then it contradicts Star condition. Then Lemma 13 implies the result. □

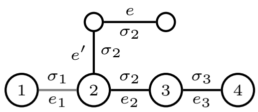

Figure 16: $\gamma' = \{e\} \cup \gamma \cup \{e_2, e_3\}$.

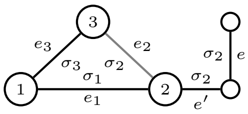

Figure 17: $\gamma' = \{e\} \cup \gamma \cup \{e_1, e_3\}$.

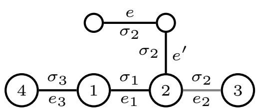

Figure 18: $\gamma' = \{e\} \cup \gamma \cup \{e_1, e_3\}$.

Remark 2. Lemma 20 implies that any chordless cycle containing e_1 has length at most 4. More precisely, let $e_1 = \{1, 2\}$ be the unique edge with weight σ_1. The nodes 1 and 2 are connected in $G \setminus \{e_1\}$ if and only if there exists a cycle $\tilde{C}_m$ in G containing e_1 with $m = 3$ or 4 as represented in Figures 19 and 20 respectively. Let us observe that if $m = 4$, we can

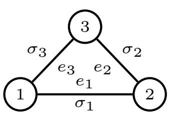

Figure 19: $\tilde{C}_3$.

assume that $\tilde{C}_4$ has no chord otherwise we can replace $\tilde{C}_4$ by a cycle $\tilde{C}_3$. Moreover, the Adjacent cycles condition implies that such a chordless cycle $\tilde{C}_3$ or $\tilde{C}_4$ is necessarily unique (two adjacent chordless cycles cannot have a common minimum weight edge).

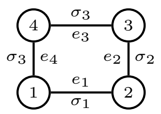

Figure 20: $\tilde{C}_4$.

We now consider the case where the edge-weights have only two different values.

Proposition 21. *Let $G = (N, E, w)$ be a weighted connected graph. Let us assume that for all $\emptyset \neq S \subseteq N$, the $\mathcal{P}_{\min}$-restricted game $(N, \overline{u_S})$ is convex. Let us moreover assume that the edge-weights have exactly two different values $\sigma_1 < \sigma_2$. Then, either there exists only one edge e_1 with weight σ_1 or all edges with weight σ_1 are incident to the same vertex v and no edge with weight σ_2 is incident to v.*

Two examples corresponding to the two possible situations described in Proposition 21 are represented in Figure 21.

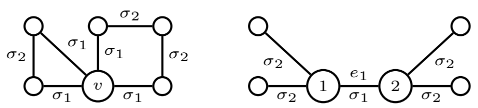

Figure 21: Edge-weights have only two values $\sigma_1 < \sigma_2$.

Proof. As G is connected there is at least one pair of adjacent edges $e_1 = \{1, 2\}$, and $e_2 = \{2, 3\}$ with weights $\sigma_1 < \sigma_2$. Let e be an edge in $E \setminus \{e_1\}$ with weight $w(e) = \sigma_1$ and let us assume that e is not incident to 1. Then e cannot be incident to 2 otherwise it contradicts the Star condition. As G is connected, there exists a shortest path γ linking e to 1 or 2. The Path condition applied to $\{e\} \cup \gamma \cup \{e_1\}$ implies that all edges in γ have weight σ_1. Let e' be the edge of γ incident to 1 or 2. As $w(e') = \sigma_1$, the first part of the proof implies that e' is necessarily incident to 1. Let us consider $\gamma' = \{e\} \cup \gamma \cup \{e_1, e_2\}$ as represented in Figure 22. Then Proposition 17

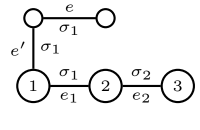

Figure 22: $\gamma' = \{e\} \cup \gamma \cup \{e_1, e_2\}$.

applied to γ' implies $\sigma_1 = \sigma_2$, a contradiction. Therefore e is necessarily incident to 1. Moreover any edge of weight σ_2 incident to 1 would contradict the Star condition. □

In the case of three distinct edge-weights $\sigma_1 < \sigma_2 < \sigma_3$ we establish supplementary necessary conditions on cycles.

Lemma 22. *Let $G = (N, E, w)$ be a weighted connected graph. Let us assume that for all $\emptyset \neq S \subseteq N$, the $\mathcal{P}_{\min}$-restricted game $(N, \overline{u_S})$ is convex. Let us moreover assume that the edge-weights have exactly three different values $\sigma_1 < \sigma_2 < \sigma_3$. Let $\tilde{C}_3 = \{1, e_1, 2, e_2, 3, e_3, 1\}$ be a cycle such that $w_i = \sigma_i$ for $i \in \{1, 2, 3\}$. Then there is at most one cycle C_3 adjacent to $\tilde{C}_3$ containing the node 1 (or equivalently with common edge e_3).*

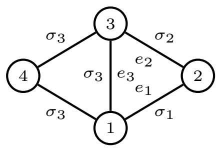

Figure 23: C_3 and $\tilde{C}_3$.

Proof. Proposition 16 implies that e_1 is the unique edge with weight σ_1. Therefore $\tilde{C}_3$ is a unique cycle with three different edge-weights, otherwise there would be two adjacent cycles with e_1 as a common minimum weight edge contradicting the Adjacent cycles condition. By contradiction, let us assume there exist two triangles adjacent to $\tilde{C}_3$ with common edge e_3, *i.e.*, we can find two vertices j and k in N such that $\{1, j\}$, $\{1, k\}$, $\{3, j\}$, and $\{3, k\}$ exist in E. Let us note that these last edges necessarily have weight σ_3 by the Star condition. Let us consider $i = 1$ and the subsets $A_1 = \{2\}$, $A_2 = \{j, k\}$, $A = A_1 \cup A_2$, and $B = A \cup \{3\}$ as represented in Figure 24. Then $\mathcal{P}_{\min}(A) = \{\{2\}, \{j\}, \{k\}\}$, $\mathcal{P}_{\min}(A \cup \{i\}) = \{\{2\}, \{1, j, k\}\}$ (let us

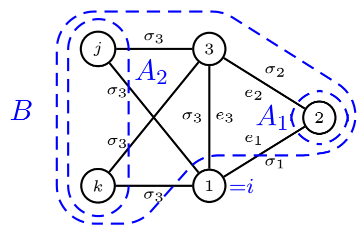

Figure 24: $\tilde{C}_3$ and two other triangles with common edge e_3.

remark that $\{2, j\}$ and $\{2, k\}$ cannot exist in E otherwise it contradicts the Adjacent Cycles condition), and $\mathcal{P}_{\min}(B) = \{\{2\}, \{3, j, k\}\}$. Taking $A' = \{1, j, k\}$ we get $\mathcal{P}_{\min}(A)_{|A'} = \{\{j\}, \{k\}\} \neq \{j, k\} = \mathcal{P}_{\min}(B)_{A'}$ and it contradicts Theorem 3. $\square$

We now establish that if for all $\emptyset \neq S \subseteq N$, the $\mathcal{P}_{\min}$-restricted game $(N, \overline{u_S})$ is convex, and if a graph contains cycles with constant weights, then a Strong Pan condition is necessary as in Lemma 18 for non constant weights.

Lemma 23 (STRONG PAN CONDITION FOR CYCLES WITH CONSTANT WEIGHTS)**.** *Let $G = (N, E, w)$ be a weighted connected graph. Let us assume that for all $\emptyset \neq S \subseteq N$, the $\mathcal{P}_{\min}$-restricted game $(N, \overline{u_S})$ is convex. Let us moreover assume that the edge-weights have at most three different values $\sigma_1 < \sigma_2 \leq \sigma_3$. Then:*

1. *If $|E_1| \geq 2$ (and then $\sigma_2 = \sigma_3$), then every cycle with constant weight σ_2 is complete.*

2. *If $|E_1| = 1$ with $E_1 = \{e_1\}$ and if there exists a cycle C with constant weight σ_2, then there are only two different edge-weights ($\sigma_2 = \sigma_3$). Moreover, if C is not incident to e_1 and not linked to e_1 by an edge, then C is complete.*

3. *If $|E_1| = 1$ with $E_1 = \{e_1\}$ and $e_1 = \{1, 2\}$, then for every cycle C with constant weight σ_2 or σ_3 and incident to 2 (resp. to 1), $\{2, j\} \in E(C)$ for all $j \in V(C) \setminus \{2\}$ (resp. $\{1, j\} \in E(C)$ for all $j \in V(C) \setminus \{1\}$).*

4. *If $|E_1| = 1$ with $E_1 = \{e_1\}$ and $e_1 = \{1, 2\}$, then for every cycle C with constant weight σ_2 or σ_3 and not adjacent to e_1 but linked to e_1 by an edge $e = \{2, k\}$ (of weight σ_2) with $k \in V(C)$, one of the following conditions is satisfied:*

   (a) $\{1, j\} \in E$ for all $j \in V(C)$.

   (b) $\{2, j\} \in E$ for all $j \in V(C)$.

   (c) There is no edge $\{2, j\}$ in E with $j \in V(C) \setminus \{k\}$ and C is complete.

5. *Let us assume that the edge-weights have three different values $\sigma_1 < \sigma_2 < \sigma_3$. Let $e_1 = \{1, 2\}$ be the unique edge with weight σ_1 and let us assume that all edges with weight σ_2 are incident to 2. Then every cycle C_m which does not contain e_1 is complete. Moreover, if $1 \in V(C_m)$ then $m = 3$ and such a cycle is unique, has constant weight σ_3, and is adjacent to a unique triangle $\tilde{C}_3$ which contains the edge e_1.*

Situations corresponding to Claims 3 and 4 in Lemma 23 are represented in Figures 25 to 28.

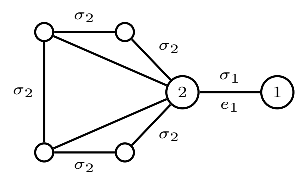

Figure 25: Situation of Claim 3 in Lemma 23.

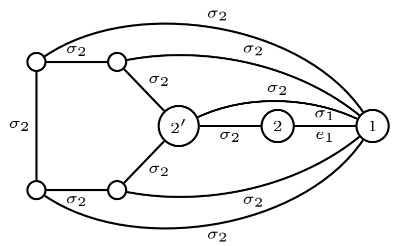

Figure 26: Situation of Claim 4a in Lemma 23.

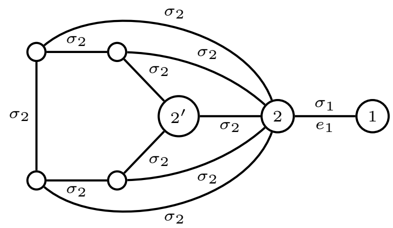

Figure 27: Situation of Claim 4b in Lemma 23.

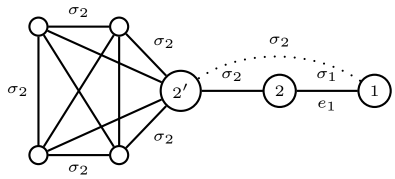

Figure 28: Situation of Claim 4c in Lemma 23.

Remark 3. Let C be a cycle satisfying Claim 3 or Claim 4 in Lemma 23. Two nodes i and j in C cannot be both linked to 1 and 2 (cf. Figure 29), otherwise e_1 is a common minimum weight edge of the cycles induced by $\{1, 2, i\}$ and $\{1, 2, j\}$ and as $\sigma_1 < \sigma_2$, it contradicts the Adjacent Cycles condition.

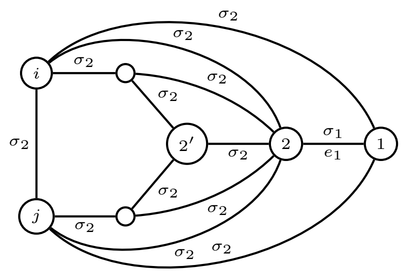

Figure 29: i and j linked to 1 and 2.

Remark 4. If the edge-weights have three different edge-weights $\sigma_1 < \sigma_2 < \sigma_3$, then it follows from Claim 5 in Lemma 23 that the cycle C_m considered in Claim 3 of the same lemma, has to be a triangle ($m = 3$) (of constant weight

σ_3). As a triangle has no chord Claim 3 adds nothing in the particular case of three different edge-weights and of a cycle incident to 1. But, *a priori*, we have to keep this case in Claim 3 to be able to prove Claim 5.

Proof. 1. As $|E_1| \geq 2$, Proposition 16 implies that the edge-weights have at most two different values $\sigma_1 < \sigma_2$. Proposition 21 implies that all edges with weight σ_1 are incident to the same vertex j and no edge with weight σ_2 is incident to j. Let us consider a cycle C with constant weight σ_2. Lemma 13 implies that every edge in E_2 is linked to j by an edge in E_1 as represented in Figure 30. Let us consider the Myerson restricted game

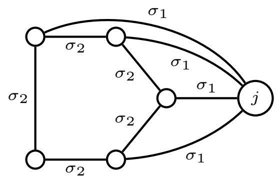

Figure 30: Every edge in E_2 is linked to j by an edge in E_1.

v^M, which corresponds to the partition into connected components. Then we have $\overline{v}(A \cup \{j\}) = v^M(A) + v(j) = v^M(A)$ for all $A \subseteq N \setminus \{j\}$. Hence, for $i \in V(C)$ and for $A \subseteq B \subseteq V(C) \setminus \{i\}$ the inequality:

$$\overline{v}(B \cup \{j\} \cup \{i\}) - \overline{v}(B \cup \{j\}) \geq \overline{v}(A \cup \{j\} \cup \{i\}) - \overline{v}(A \cup \{j\}) \tag{13}$$

is equivalent to:

$$v^M(B \cup \{i\}) - v^M(B) \geq v^M(A \cup \{i\}) - v^M(A). \tag{14}$$

As $(N, \overline{u_S})$ is convex, (13) is satisfied with $v = u_S$. Therefore (14) is also satisfied with $v = u_S$ and then u_S^M is convex if we restrict G to $V(C)$. Then Theorem 6 implies that C has to be complete.

2. If there are three different edge-weights, then Lemma 20 implies that there is no cycle with constant weight σ_2. If $E_1 = \{e_1\}$ with $e_1 = \{1, 2\}$, let us consider a cycle C with constant weight σ_2 not incident to e_1 and not linked by an edge to e_1, as represented in Figure 31. For any game (N, v)

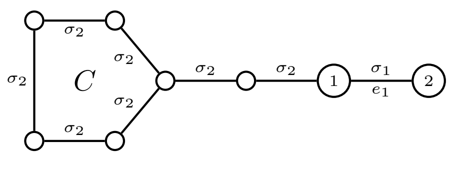

Figure 31: C not linked by an edge to e_1.

and for $A \subseteq N \setminus \{1, 2\}$ such that there is no edge linking A to $\{1, 2\}$ we obviously have:

$$\overline{v}(A \cup \{1, 2\}) = v^M(A) + v(\{1\}) + v(\{2\}) = v^M(A). \tag{15}$$

Hence, for $i \in V(C)$ and for $A \subseteq B \subseteq V(C) \setminus \{i\}$, the subsets A, B, $A \cup \{i\}$, and $B \cup \{i\}$ satisfy (15). Then the inequality

$$\overline{v}((B \cup \{1,2\}) \cup \{i\}) - \overline{v}(B \cup \{1,2\}) \geq \overline{v}((A \cup \{1,2\}) \cup \{i\}) - \overline{v}(A \cup \{1,2\})$$

is equivalent to:

$$v^M(B \cup \{i\}) - v^M(B) \geq v^M(A \cup \{i\}) - v^M(A). \tag{16}$$

Therefore, taking $v = u_S$, we can conclude as in the previous case.
3. Let us consider $E_1 = \{e'_1\}$ with $e'_1 = \{1', 2\}$ and a cycle $C = \{1, e_1, 2, e_2, \ldots, m, e_m, 1\}$ incident to 2. We can assume w.l.o.g. that C has constant weight σ_2. Note that Proposition 8 implies that the Cycle condition is satisfied, and therefore e'_1 cannot be a chord of C. At first we prove that $\{2,4\} \in E$. By contradiction let us assume $\{2,4\} \notin E$. Let us consider $i = 3$ and the subsets $A_1 = \{4\}$, $A_2 = \{1', 2\}$, $A = A_1 \cup A_2$, and $B = (V(C) \setminus \{i\}) \cup \{1'\}$ as represented in Figure 32. Then $\mathcal{P}_{\min}(A) = \{\{1'\}, \{2\}, \{4\}\}$ or

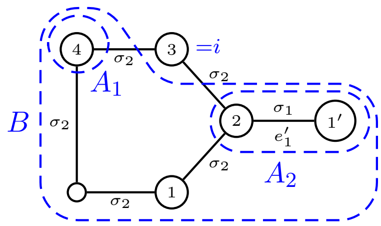

Figure 32: Cycle C incident to 2.

$\{\{1', 4\}, \{2\}\}$, $\mathcal{P}_{\min}(A \cup \{i\}) = \{\{1'\}, \{2, 3, 4\}\}$ or $\{A \cup \{i\}\}$, and $\mathcal{P}_{\min}(B) = \{\{1'\}, V(C) \setminus \{3\}\}$ or $\{B\}$. Then for any $A' \in \mathcal{P}_{\min}(A \cup \{i\})$ containing 2, we have $\{2\} \in \mathcal{P}_{\min}(A)_{|A'}$ but $\{2\} \notin \mathcal{P}_{\min}(B)_{|A'}$ and it contradicts Theorem 3. (If $\mathcal{P}_{\min}(A \cup \{i\}) = \{\{1'\}, \{2, 3, 4\}\}$ then taking $A' = \{2, 3, 4\}$, we get $\mathcal{P}_{\min}(A)_{|A'} = \{\{2\}, \{4\}\} \neq \{\{2, 4\}\} = \mathcal{P}_{\min}(B)_{|A'}$. If $\mathcal{P}_{\min}(A \cup \{i\}) = \{A \cup \{i\}\}$, then taking $A' = A \cup \{i\}$, we get $\mathcal{P}_{\min}(A)_{|A'} = \{\{1'\}, \{2\}, \{4\}\}$ or $\{\{1', 4\}, \{2\}\}$ and $\mathcal{P}_{\min}(B)_{|A'} = \{\{1'\}, \{2, 4\}\}$ or $\{\{1', 2, 4\}\}$. Therefore we always have $\mathcal{P}_{\min}(A)_{|A'} \neq \mathcal{P}_{\min}(B)_{|A'}$, and it contradicts Theorem 3.)
Hence we have $e = \{2, 4\} \in E$ and $w(e) = \sigma_2$, otherwise we have a contradiction with Star condition applied to $\{e, e_1, e_2\}$. Then we can consider the cycle $\{1, e_1, 2, e, 4, e_4, \ldots, m, e_m, 1\}$ and by the same reasoning $\{2, 5\} \in E$. Iterating we get $\{2, j\} \in \hat{E}(C)$ for all $j \in V(C) \setminus \{2\}$.
4. Let us consider $E_1 = \{e'_1\}$ with $e'_1 = \{1', 2'\}$ and a cycle $C = \{1, e_1, 2, e_2, 3, \ldots, m, e_m, 1\}$ with constant weight σ_2 or σ_3 and not incident to e_1 but linked to e_1 by an edge $e = \{2, 2'\}$ of weight σ_2.

If $e' = \{2', j\} \in E$ for some $j \in V(C) \setminus \{2\}$, then we can consider the cycles $C' = \{2', e, 2, e_2, 3, \ldots, e_{j-1}, j, e', 2'\}$ and $C'' = \{2', e', j, e_j, j + 1 \ldots, e_m, 1, e_1, 2, e, 2'\}$ as represented in Figure 33. Then C' and C'' are incident to e'_1 and Claim 3 (applied successively to C' and C'') implies that $\{2', k\} \in E$ for all $k \in V(C)$. Hence Claim 4b is satisfied.

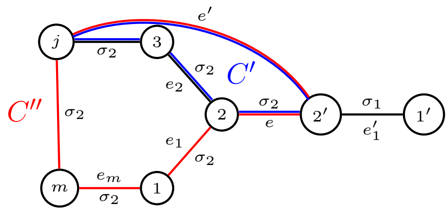

Figure 33: C' and C''.

Let us now assume $\{2', j\} \notin E$ for all $j \in V(C) \setminus \{2\}$. If $\{1', 2\} \in E$, then we can apply the same reasoning as before (interchanging the roles of $1'$ and $2'$). Therefore either $\{1', k\} \in E$ for all $k \in V(C)$ and Claim 4a is satisfied or $\{1', k\} \notin E$ for all $k \in V(C) \setminus \{2\}$. Therefore we assume henceforth that the following condition is satisfied:

(17) There is no edge $\{2', l\}$ with $l \in V(C) \setminus \{2\}$ and if $\{1', 2\} \in E$ there is also no edge $\{1', l\}$ with $l \in V(C) \setminus \{2\}$.

We now have a proof similar to the one of Claim 3. By contradiction let us assume $\{2, 4\} \notin E$. Let us consider $i = 3$, $A_1 = \{4\}$, $A_2 = \{1', 2', 2\}$, $A = A_1 \cup A_2$, and $B = (V(C) \setminus \{i\}) \cup A_2$ as represented in Figure 34. By

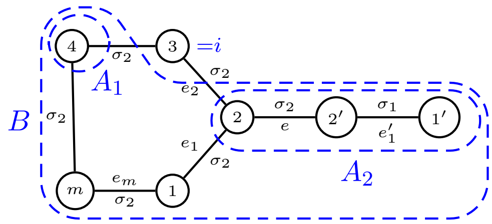

Figure 34: C linked to e'_1 by an edge.

(17), $\{2', 4\}$ does not exist in E. We have to consider several cases:

1. If $\{1', 2\} \in E$ then $\{1', 4\} \notin E$ by (17) and therefore $\mathcal{P}_{\min}(A) = \{\{1', 2, 2'\}, \{4\}\}$.
2. If $\{1', 2\} \notin E$ and $\{1', 4\} \in E$, then $\mathcal{P}_{\min}(A) = \{\{1', 4\}, \{2, 2'\}\}$.
3. If $\{1', 2\} \notin E$ and $\{1', 4\} \notin E$, then $\mathcal{P}_{\min}(A) = \{\{1'\}, \{2, 2'\}, \{4\}\}$.

In every case $\mathcal{P}_{\min}(A \cup \{i\}) = \{A \cup \{i\}\}$ or $\{\{1'\}, A \cup \{i\} \setminus \{1'\}\}$. Indeed we have to delete e'_1 but $1'$ may be linked to 2, 3 or 4. We also have $\mathcal{P}_{\min}(B) = \{B\}$ or $\{\{1'\}, B \setminus \{1'\}\}$. Therefore taking $A' = A \cup \{i\}$ or $A \cup \{i\} \setminus \{1'\}$ we have in every case either $\mathcal{P}_{\min}(B)_{|A'} = \{A\}$ or $\{\{1'\}, A \setminus \{1'\}\}$ or $\{A \setminus \{1'\}\}$. As neither A nor $A \setminus \{1'\}$ is in $\mathcal{P}_{\min}(A)$, we get $\mathcal{P}_{\min}(A)_{|A'} \neq \mathcal{P}_{\min}(B)_{|A'}$, contradicting Theorem 3. Hence $\{2, 4\} \in E$.

Then we can consider the cycle $\{1, e_1, 2, e, 4, e_4, \ldots, m, e_m, 1\}$ and by the same reasoning $\{2, 5\} \in E$. Iterating as in the proof of Claim 3 we get $\{2, j\} \in \hat{E}(C)$ for all $j \in V(C)$.

Let us now prove that $\{j, k\} \in \hat{E}(C)$ for all pairs of nodes j, k, with $3 \leq j \leq m-1$ and $k = 1$ or $j+2 \leq k \leq m$. By contradiction let us assume $\{j, k\} \notin E$ for two nodes j and k with $3 \leq j \leq m-1$ and $k = 1$ or $j+2 \leq k \leq m$. Let us assume that $\{1', j\} \in E$ and $\{1', k\} \in E$. As $e'_j := \{2, j\}$ and $e'_k := \{2, k\}$ are in $\hat{E}(C)$, we obtain two adjacent 4-cycles $\tilde{C}_j = \{2, e'_j, j, \{j, 1'\}, 1', e'_1, 2', e, 2\}$ and $\tilde{C}_k = \{2, e'_k, k, \{k, 1'\}, 1', e'_1, 2', e, 2\}$ without chord (following our assumptions the chords $\{2', j\}$, $\{2', k\}$, $\{1', 2\}$ are not in E) with a common edge e'_1 of minimum weight σ_1 contradicting the Adjacent cycles condition. Hence we can assume that at most one of the edges $\{1', j\}$ or $\{1', k\}$ is in E. We now consider the cycle $\tilde{C} = \{2, e'_j, j, e_j, j+1, \ldots, k, e'_k, 2\}$ and subsets $A_1 = \{j\}$, $A_2 = \{k\}$, $A_3 = \{1', 2'\}$, $A = A_1 \cup A_2 \cup A_3$, $B_1 = V(\tilde{C}) \setminus \{2\}$, $B_2 = A_3$, and $B = B_1 \cup B_2$ as represented in Figure 35 and we choose $i = 2$. Let us observe that to obtain $\mathcal{P}_{\min}(A)$, $\mathcal{P}_{\min}(A \cup \{i\})$

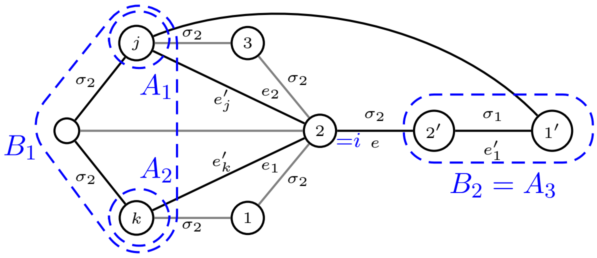

Figure 35: $\tilde{C} = \{2, e'_j, j, e_j, j+1, \ldots, k, e'_k, 2\}$ and $\{1', j\} \in E$.

or $\mathcal{P}_{\min}(B)$ we only have to delete the edge $e'_1 = \{1', 2'\}$ of weight σ_1. As we know that $\{1', j\} \notin E$ or $\{1', k\} \notin E$, $\{j, k\}$ cannot be a subset of any component of $\mathcal{P}_{\min}(A)$, *i.e.*, we cannot have $\mathcal{P}_{\min}(A) = \{\{1', j, k\}, \{2'\}\}$ or $\mathcal{P}_{\min}(A) = \{\{1'\}, \{2'\}, \{j, k\}\}$ but only $\mathcal{P}_{\min}(A) = \{\{1'\}, \{2'\}, \{j\}, \{k\}\}$ or $\mathcal{P}_{\min}(A) = \{\{1', j\}, \{2'\}, \{k\}\}$ or $\mathcal{P}_{\min}(A) = \{\{1', k\}, \{2'\}, \{j\}\}$. As $\{2, j\}$ and $\{2, k\}$ are in $\hat{E}(C)$ and as $i = 2$, j and k are connected in $G_{A \cup \{i\} \setminus \{1', 2'\}}$. Hence there exists $A' \in \mathcal{P}_{\min}(A \cup \{i\})$ such that $\{j, k\} \subseteq A'$. As j and k are in B_1 which is a connected subset of $G_{B \setminus \{1', 2'\}}$ there exists $B' \in \mathcal{P}_{\min}(B)$ such that $\{j, k\} \subseteq B'$. Hence $\{j, k\} \subseteq (B' \cap A') \in \mathcal{P}_{\min}(B)_{|A'}$. But $\{j, k\}$ cannot be a subset of any component of $\mathcal{P}_{\min}(A)$. Hence we have $\mathcal{P}_{\min}(A)_{|A'} \neq \mathcal{P}_{\min}(B)_{|A'}$ contradicting Theorem 3. Therefore $\{j, k\} \in \hat{E}(C)$ and C is complete.

5. Lemma 20 implies that there is a unique edge $e_1 = \{1, 2\}$ with weight σ_1, all edges with weight σ_2 are incident to the same end-vertex 2 of e_1 and all edges with weight σ_3 are not incident to 2 but connected to 2 by e_1 or by an edge with weight σ_2. We have to consider several cases.

1. Let us assume that C_m has non constant weight. As $e_1 \notin E(C_m)$, an edge in $E(C_m)$ has weight σ_2 or σ_3. Edges with weight σ_2 are incident to 2, therefore $2 \in V(C_m)$ and e_1 is adjacent to C_m. Then Lemma 18 implies that e_1 is not a chord of C_m and C_m is complete as represented in Figure 36.

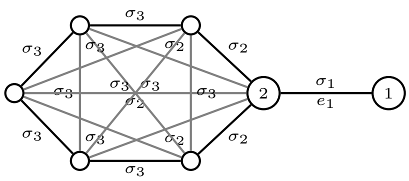

Figure 36: e_1 adjacent to C_m.

2. Let us now assume C_m has constant weight. As all edges with weight σ_2 are incident to 2 they cannot form a cycle. Therefore all edges in C_m have weight σ_3. As all edges with weight σ_3 are not incident to 2, we have $2 \notin V(C_m)$.

   Let us assume $1 \notin V(C_m)$. Then an edge e in $E(C_m)$ cannot be linked to node 2 by e_1, and therefore e is linked to 2 by an edge of weight σ_2. As $m \geq 3$ there exists at least two nodes i and j in $V(C_m)$ such that $\{2, i\}$ and $\{2, j\}$ are in E and have weight σ_2. C_m gives two obvious paths γ and γ' linking i and j. Let us consider the cycles $C'_m = \{2, i\} \cup \gamma \cup \{j, 2\}$ and $C''_m = \{2, i\} \cup \gamma' \cup \{j, 2\}$ as represented in Figure 37. Then we can apply Case 1 (or Lemma 18) to C'_m and C''_m.

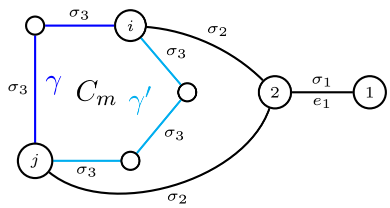

Figure 37: $C'_m = \{2, i\} \cup \gamma \cup \{j, 2\}$ and $C''_m = \{2, i\} \cup \gamma' \cup \{j, 2\}$.

   Therefore C'_m and C''_m are complete. Then $\{i, j\} \in E$ and $\{2, k\} \in E$ for all $k \in V(C_m)$. Hence for any pair of nodes i, j in $V(C_m)$, $\{2, i\}$ and $\{2, j\}$ exist in E and by the same reasoning as before $\{i, j\} \in E$. Therefore C_m is complete.

   Let us now assume $1 \in V(C_m)$. Claim 3 implies that $\{1, i\} \in \hat{E}(C_m)$ for all $i \in V(C_m)$. Only the edges of C_m incident to 1 are linked to 2 by e_1. If $m \geq 5$, then there exist at least two nodes i and j in $V(C_m) \setminus \{1\}$ such that $\{2, i\}$ and $\{2, j\}$ are in E and have weight σ_2 as represented in Figure 38. Then the cycles defined by $\{2, i\}, \{i, 1\}, e_1$ and

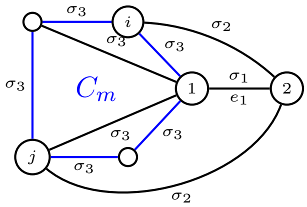

Figure 38: C_m with $m = 5$.

$\{2, j\}, \{j, 1\}, e_1$ are two cycles with a common non-maximum weight edge. They contradict the Adjacent cycles condition. Therefore we have $m \leq 4$. If $m = 4$, let us denote by $e'_1 = \{1, 2'\}$, $e'_2 = \{2', 3'\}$,

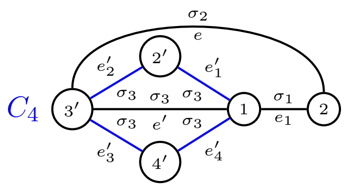

Figure 39: C_4 and $\tilde{C}_3 = \{1, e_1, 2, e, 3', e', 1\}$.

$e'_3 = \{3', 4'\}$, and $e'_4 = \{4', 1\}$ the edges in $E(C_4)$ and by $e' = \{1, 3'\}$ the chord of C_4 incident to 1. Both edges e'_2 and e'_3 have to be linked to node 2. But there can be only one edge e linking 2 to $V(C_4) \setminus \{1\}$, otherwise we get a contradiction to the Adjacent cycles condition as with $m \geq 5$. Hence we necessarily have $e = \{2, 3'\}$ as represented in Figure 39. Then $\tilde{C}_3 = \{1, e_1, 2, e, 3', e', 1\}$ is a triangle containing edges with weights $\sigma_1 < \sigma_2 < \sigma_3$, and $\{1, e'_1, 2', e'_2, 3', e', 1\}$ and $\{1, e', 3', e'_3, 4', e'_4, 1\}$ are two triangles adjacent to $\tilde{C}_3$ containing the edge e' of weight σ_3. It contradicts Lemma 22. Hence $m = 3$. $C_3 = \{1, e'_1, 2', e'_2, 3', e'_3, 1\}$ is a triangle with constant weight σ_3 and e'_2 is linked to 2 by an edge of weight σ_2. There is only one edge e linking e'_2 to 2 otherwise it contradicts the Adjacent cycles condition as with $m \geq 5$. We can assume w.l.o.g. $e = \{2, 2'\}$ as represented in Figure 40. Hence the triangle $\tilde{C}_3 = \{1, e_1, 2, e, 2', e'_1, 1\}$ exists and

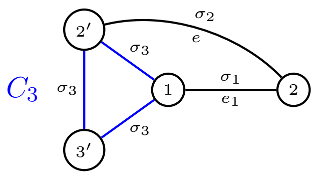

Figure 40: C_3 and $\tilde{C}_3 = \{1, e_1, 2, e, 2', {e_1}', 1\}$.

Lemma 22 implies that C_3 is necessarily unique. $\tilde{C}_3$ is unique as a triangle containing e_1 is necessarily unique (otherwise it contradicts the Adjacent cycles condition). □

Remark 5. If there exists a cycle $\tilde{C}_4$ without chord and containing the edge e_1 of weight σ_1 (such a cycle is unique otherwise it contradicts the Adjacent cycles condition), then Claim 5 of Lemma 23 implies that no cycle of constant weight σ_3 contains vertex 1.

5 Graphs satisfying inheritance of convexity

5.1 Graphs with two edge-weights

Theorem 24. *Let $G = (N, E, w)$ be a weighted connected graph. Let us assume that the edge-weights have only two different values $\sigma_1 < \sigma_2$. Let us consider $E_1 = \{e \in E;\ w(e) = \sigma_1\}$ and $E_2 = \{e \in E;\ w(e) = \sigma_2\}$. Let us assume $|E_1| \geq 2$. Then there is inheritance of convexity for the correspondence $\mathcal{P}_{\min}$ on G if and only if:*

1. *All edges in E_1 are incident to the same vertex 1 and all edges in E_2 are linked to 1 by an edge in E_1.*
2. *One of the two following equivalent conditions is satisfied:*
   - *(a) There is inheritance of convexity for the correspondence $\mathcal{P}_M$ (associated with Myerson's game) on the subgraph $G_1 = (N, E \setminus E_1)$.*
   - *(b) $G_1 = (N, E \setminus E_1)$ is cycle-complete.*

We give in Figure 41 an example of a graph satisfying conditions 1 and 2 of Theorem 24.

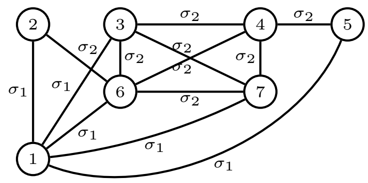

Figure 41: Every edge of weight σ_2 is linked to 1 by an edge of weight σ_1 and the cycle defined by $3, 4, 6, 7$ is complete.

Proof. Conditions 2a and 2b are equivalent by Theorem 6 (van den Nouweland and Borm, 1991). By Proposition 21, Lemma 13 and Lemma 23 (Claim 1), conditions 1 and 2 are necessary. We now prove that there are also sufficient. Let us consider a convex game (N, v). We denote by $(N, \overline{v})$ (resp. (N, v^M)) the restricted game associated with $\mathcal{P}_{\min}$ (resp. $\mathcal{P}_M$) on G (resp. G_1). Let us consider $i \in N$ and subsets $A \subseteq B \subseteq N \setminus \{i\}$. We consider several cases.

1. Let us first assume $i = 1$. As $A \subseteq B \subseteq N \setminus \{1\}$, we have $E(A) \subseteq E(B) \subseteq E_2$ and hence $\mathcal{P}_{\min}(A)$ and $\mathcal{P}_{\min}(B)$ are singleton partitions.

   Then

   $$\overline{v}(B \cup \{i\}) - \overline{v}(B) \geq \overline{v}(A \cup \{i\}) - \overline{v}(A) \tag{18}$$

   is equivalent to $\overline{v}(B \cup \{i\}) \geq \overline{v}(A \cup \{i\})$ and this last inequality is necessarily satisfied as $(N, \overline{v})$ is superadditive (cf. Corollary 2 page 4).

2. We assume now $i \neq 1$ and $A \subseteq N \setminus \{1, i\}$. Then $E(A) \subseteq E(A \cup \{i\}) \subseteq E_2$ and therefore $\mathcal{P}_{\min}(A)$ and $\mathcal{P}_{\min}(A \cup \{i\})$ are singleton partitions. Hence (18) is equivalent to $\overline{v}(B \cup \{i\}) - \overline{v}(B) \geq 0$ and this last inequality is necessarily satisfied as $(N, \overline{v})$ is superadditive.

3. Finally, we assume $i \neq 1$ and $1 \in A$.
   Let us assume $E(A) \neq \emptyset$. An edge in $E(A)$ has weight σ_1 or σ_2. As every edge of weight σ_2 is linked to 1 by an edge of weight σ_1 (by Condition 1) we necessarily have $E(A) \cap E_1 \neq \emptyset$. Therefore $\mathcal{P}_{\min}(A) = \mathcal{P}_M(A)$ and $\overline{v}(A) = v^M(A)$. Now if $E(A) = \emptyset$ then $\mathcal{P}_{\min}(A) = \mathcal{P}_M(A)$ is trivially the singleton partition of A, and we have $\overline{v}(A) = v^M(A) = 0$. By the same reasoning we have $\overline{v}(A \cup \{i\}) = v^M(A \cup \{i\})$, $\overline{v}(B) = v^M(B)$, and $\overline{v}(B \cup \{i\}) = v^M(B \cup \{i\})$. Hence (18) is equivalent to

   $$v^M(B \cup \{i\}) - v^M(B) \geq v^M(A \cup \{i\}) - v^M(A), \tag{19}$$

   and Condition 2 implies that (19) is satisfied. □

Theorem 25. *Let $G = (N, E, w)$ be a weighted connected graph. Let us assume that the edge-weights have only two different values $\sigma_1 < \sigma_2$. Let us assume $|E_1| = 1$. Let $e_1 = \{1, 2\}$ be the unique edge in E_1. Then there is inheritance of convexity for the correspondence $\mathcal{P}_{\min}$ on G if and only if the following conditions are verified:*

1. *There exists at most one chordless cycle containing e_1.*
2. *For every cycle C with constant weight σ_2 either C is complete or all vertices of C are linked to the same end vertex of e_1.*

We give in Figure 42 an example of a graph satisfying conditions 1 and 2 of Theorem 25.

Remark 6. Let us observe that Condition 1 of Theorem 25 implies that e_1 cannot be a chord of any cycle of G.

Proof. The Adjacent cycles condition implies that condition 1 is necessary. Lemma 23 (Claims 2, 3, and 4) implies that condition 2 is necessary.

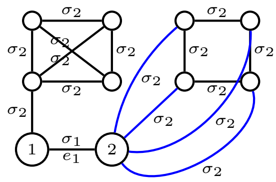

Figure 42: Either a cycle with constant weight is complete or all its vertices are linked to one end-vertex of e_1.

We now prove they are sufficient. Let us consider a convex game (N, v), $i \in N$ and subsets $A \subseteq B \subseteq N \setminus \{i\}$. We have to prove that the following inequality is satisfied:

$$\overline{v}(B \cup \{i\}) - \overline{v}(B) \geq \overline{v}(A \cup \{i\}) - \overline{v}(A). \tag{20}$$

We consider several cases.

1. Let us assume $e_1 \notin E(A)$ and $e_1 \notin E(B)$. Then we have $E(A) \subseteq E(B) \subseteq E_2$ and therefore $\mathcal{P}_{\min}(A)$ and $\mathcal{P}_{\min}(B)$ are singleton partitions. Then (20) is equivalent to $\overline{v}(B \cup \{i\}) \geq \overline{v}(A \cup \{i\})$ and this last inequality is necessarily satisfied as $(N, \overline{v})$ is superadditive (cf. Corollary 2 page 4).

2. Let us assume $e_1 \notin E(A)$ but $e_1 \in E(B)$. Then $i \notin \{1, 2\}$ (otherwise $e_1 \notin E(B)$) and therefore $e_1 \notin E(A \cup \{i\})$. Then $\mathcal{P}_{\min}(A)$ and $\mathcal{P}_{\min}(A \cup \{i\})$ are singleton partitions and (20) is equivalent to $\overline{v}(B \cup \{i\}) - \overline{v}(B) \geq 0$ and this last inequality is necessarily satisfied as $(N, \overline{v})$ is superadditive (cf. Corollary 2 page 4).

3. Let us assume $e_1 \in E(A)$. Then we also have $e_1 \in E(A \cup \{i\})$, $e_1 \in E(B)$, and $e_1 \in E(B \cup \{i\})$. Therefore (20) is equivalent to $v^M(B \cup \{i\}) - v^M(B) \geq v^M(A \cup \{i\}) - v^M(A)$ (where (N, v^M) is the Myerson's game associated with $G_1 = (N, E \setminus \{e_1\})$). Let $\mathcal{P}_M(A) = \{A_1, A_2, \ldots, A_p\}$ be the partition of A into connected components in G_1. If there is no link between i and A then $\mathcal{P}_M(A \cup \{i\}) = \{\mathcal{P}_M(A), \{i\}\}$ and $v^M(A \cup \{i\}) - v^M(A) = v(\{i\}) = 0$. Then (20) is equivalent to $\overline{v}(B \cup \{i\}) - \overline{v}(B) \geq 0$ and this last inequality is necessarily satisfied as $(N, \overline{v})$ is superadditive. Otherwise we can assume that for some integer $r \leq p$, and after reordering if necessary, $A_1, A_2, \ldots, A_r$ are linked to i and the other components of $\mathcal{P}_M(A)$ are not linked to i. Hence $\mathcal{P}_M(A \cup \{i\}) = \{A_1 \cup \ldots \cup A_r \cup \{i\}, A_{r+1}, \ldots, A_p\}$ and we have:

$$v^M(A \cup \{i\}) - v^M(A) = v(A_1 \cup \ldots \cup A_r \cup \{i\}) - \sum_{j=1}^{r} v(A_j). \tag{21}$$

By the same way, we can assume $\mathcal{P}_M(B) = \{B_1, B_2, \ldots, B_q\}$ and $\mathcal{P}_M(B \cup \{i\}) = \{B_1 \cup \ldots \cup B_s \cup \{i\}, B_{s+1}, \ldots, B_q\}$ with $1 \leq s \leq q$. Then we have:

$$v^M(B \cup \{i\}) - v^M(B) = v(B_1 \cup \ldots \cup B_s \cup \{i\}) - \sum_{j=1}^{s} v(B_j). \tag{22}$$

Therefore we have to prove:

$$v(B' \cup \{i\}) - \sum_{j=1}^{s} v(B_j) \geq v(A' \cup \{i\}) - \sum_{j=1}^{r} v(A_j), \tag{23}$$

where $A' = A_1 \cup \ldots \cup A_r$ and $B' = B_1 \cup \ldots \cup B_s$. Let us observe that obviously $\mathcal{P}_M(A)$ is a refinement of $\mathcal{P}_M(B)_{|A}$. To achieve the proof we need the following lemma.

Lemma 26. *Condition 2 on cycles with constant weight σ_2 implies $\mathcal{P}_M(B)_{|A'} = \mathcal{P}_M(A)_{|A'}$,* i.e.*, $A_j \subseteq B_j$, for all j, $1 \leq j \leq r$ and $r \leq s$, after renumbering if necessary.*

Proof. By contradiction, let us assume that two components A_1 and A_2 of $\mathcal{P}_M(A)_{|A'}$ are subsets of the same component $B_1 \in \mathcal{P}_M(B)$, after renumbering if necessary. Let $\tilde{e}_1 = \{i, k_1\}$ (resp. $\tilde{e}_2 = \{i, k_2\}$) be an edge linking i to A_1 (resp. A_2). As $i \notin \{1, 2\}$, we have $\tilde{e}_1 \neq e_1$ and $\tilde{e}_2 \neq e_1$, and therefore $\tilde{e}_1$ and $\tilde{e}_2$ have weight σ_2. As B_1 is connected, there exists an elementary path γ in B_1 linking $k_1 \in A_1$ to $k_2 \in A_2$. Note that $\mathcal{P}_M(B)$ is a partition of B in $G_1 = (N, E \setminus \{e_1\})$, therefore any edge in B_1 has weight σ_2. We obtain a simple cycle $C = \{i, \tilde{e}_1, k_1\} \cup \gamma \cup \{k_2, \tilde{e}_2, i\}$ of constant weight σ_2. If C is complete, the edge $\{k_1, k_2\}$ is a chord of C. Condition 1 and Remark 6 imply $\{k_1, k_2\} \neq e_1$. Then $\{k_1, k_2\}$ links A_1 to A_2 in G_1, a contradiction. If C is not complete, then by Condition 2 all vertices of C are linked to the same end vertex of e_1. We can assume w.l.o.g. that they are linked to vertex 1 as represented in Figure 43. As $e_1 \in E(A)$ and as k_1 and k_2 are in A, we have $\{1, k_1\}$ and $\{1, k_2\}$ in $E(A)$. We also have $\{1, k_1\} \neq e_1$ and $\{1, k_2\} \neq e_1$, otherwise e_1 would be a chord of a cycle contradicting Remark 6. Then A_1 and A_2 are part of a connected component of A in G_1, a contradiction. □

We can now end the proof of Theorem 25. By Lemma 26, $\mathcal{P}_M(B)_{|A'} = \mathcal{P}_M(A)_{|A'}$, and as $A' \subseteq B'$, we have $\mathcal{P}_M(B')_{|A'} = \mathcal{P}_M(A')$. Then Lemma 5 applied to $\mathcal{P}_M$ implies that we have for (N, v):

$$v(B') - \sum_{j=1}^{s} v(B_j) \geq v(A') - \sum_{j=1}^{r} v(A_j), \tag{24}$$

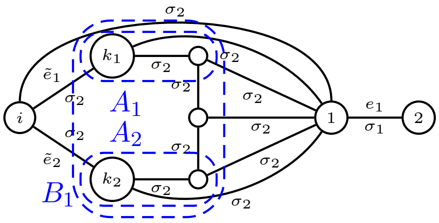

Figure 43: e_1 in $E(A)$ and k_1 in A_1, k_2 in A_2.

with $B' = B_1 \cup \ldots \cup B_s$ and $A' = A_1 \cup \ldots \cup A_r$. The convexity of (N, v) also implies:

$$v(B' \cup \{i\}) - v(B') \geq v(A' \cup \{i\}) - v(A'). \tag{25}$$

Adding (24) and (25) we obtain (23). □

5.2 Graphs with three edge-weights

We now consider the case of a weighted graph with three different edge-weights.

Theorem 27. *Let $G = (N, E, w)$ be a weighted connected graph. Let us assume that the edge-weights have only three different values $\sigma_1 < \sigma_2 < \sigma_3$. Then there is inheritance of convexity for $\mathcal{P}_{\min}$ on G if and only if the following conditions are satisfied:*

1. *There exists only one edge $e_1 = \{1, 2\}$ of weight σ_1.*
2. *An edge in $E \setminus \{e_1\}$ has weight σ_2 if and only if it is incident to the same end-vertex 2 of e_1.*
3. *Every edge of weight σ_3 is connected to 2 by e_1 or by an edge of weight σ_2.*
4. *There exists at most one (chordless) cycle $\tilde{C}_m$ with $m = 3$ or 4 containing e_1.*
5. *Every cycle C_m which does not contain e_1 is complete.*
6. *If a cycle C_m does not contain e_1 and if $1 \in V(C_m)$, then $m = 3$ and such a triangle is unique (if it exists) and is adjacent to a unique triangle $\tilde{C}_3$ which contains the edge e_1.*

Remark 7. Let us observe that Conditions 1 to 5 imply that the graph $G_1 = (N, E \setminus \{e_1\})$ is cycle-complete. Indeed using Condition 5 a cycle C_m of G_1 is a complete cycle in G. By contradiction, let us assume that e_1 is a chord of C_m. Then we can build two adjacent cycles C'_m and C''_m with common edge e_1. Using Conditions 2 and 3 we can replace (if necessary) C'_m and C''_m by two chordless cycles C' and C'' (with e_1 as common edge) of length at most 4. It contradicts Condition 4.

Remark 8. Adjacent cycles condition and Pan condition are straightforward consequences of Conditions 1 to 5 in Theorem 27.

Proof of theorem 27. We only have to prove that Conditions 1 to 6 are sufficient. Let us consider a convex game (N, v), $i \in N$ and subsets $A \subseteq B \subseteq N \setminus \{i\}$. We have to prove that the following inequality is satisfied:

$$\overline{v}(B \cup \{i\}) - \overline{v}(B) \geq \overline{v}(A \cup \{i\}) - \overline{v}(A). \tag{26}$$

Let us note that if i is not linked to A then (26) is trivially satisfied as $(N, \overline{v})$ is superadditive (Corollary 2). We thereafter assume that i is linked to A by at least one edge. We consider several cases.

Case 1 Let us assume $e_1 \in E(A)$.
As $e_1 \in E(A)$, we also have $e_1 \in E(A \cup \{i\})$, $e_1 \in E(B)$, and $e_1 \in E(B \cup \{i\})$. Then (26) is equivalent to:

$$v^M(B \cup \{i\}) - v^M(B) \geq v^M(A \cup \{i\}) - v^M(A) \tag{27}$$

where (N, v^M) is the Myerson's game associated with $G_1 = (N, E \setminus \{e_1\})$. Conditions 1 to 5 imply that G_1 is cycle-complete (cf. Remark 7). Then, (N, v^M) is convex by Theorem 6, and therefore (27) is satisfied.

Case 2 Let us assume $i \neq 1$, $2 \in A$, and $1 \notin B$.
Then we have $i \notin \{1, 2\}$, $e_1 \notin E(A)$, $e_1 \notin E(A \cup \{i\})$, $e_1 \notin E(B)$, and $e_1 \notin E(B \cup \{i\})$. If $E(A) \neq \emptyset$, then Condition 3 implies that every edge $e \in E(A)$ of weight σ_3 is linked to 2 by an edge $e' \in E(A) \cap E_2$ (as $e_1 \notin E(A)$, e_1 cannot link $2 \in A$ to e). Hence $E(A) \cap E_2 \neq \emptyset$ and $\sigma(A) = \sigma_2$. Then $\mathcal{P}_{\min}(A)$ is obtained by deleting edges of E_2 in $E(A)$ and corresponds to the partition $\mathcal{P}_M(A)$ into connected components in the graph $\tilde{G}_3 := (N, E_3)$. Therefore we have $\overline{v}(A) = v^M(A)$ where (N, v^M) is the Myerson's game associated with $\tilde{G}_3$. Now if $E(A) = \emptyset$, then $\mathcal{P}_{\min}(A) = \mathcal{P}_M(A)$ is trivially the singleton partition of A, and we have $\overline{v}(A) = v^M(A)$. By the same reasoning we have $\overline{v}(A \cup \{i\}) = v^M(A \cup \{i\})$, $\overline{v}(B) = v^M(B)$, and $\overline{v}(B \cup \{i\}) = v^M(B \cup \{i\})$. Hence (26) is equivalent to :

$$v^M(B \cup \{i\}) - v^M(B) \geq v^M(A \cup \{i\}) - v^M(A). \tag{28}$$

Let C_m be a cycle of $\tilde{G}_3$. Condition 2 implies that 2 is an isolated vertex in $\tilde{G}_3$. Then $2 \notin C_m$ and Condition 5 claims that C_m is complete. Hence $\tilde{G}_3$ is cycle-complete. Then, (N, v^M) is convex by Theorem 6 and (28) is satisfied.

Case 3 Let us assume $2 \in A$, and $1 \in B \setminus A$.
Then $i \notin \{1, 2\}$ and $e_1 \notin E(A)$ but $e_1 \in E(B)$. As $e_1 \in E(B)$, we are in the same situation as in Case 1 for B and $B \cup \{i\}$: we delete e_1 to get $\mathcal{P}_{\min}(B)$ and $\mathcal{P}_{\min}(B \cup \{i\})$. Then $\overline{v}(B) = v^M(B)$ and $\overline{v}(B \cup \{i\}) = v^M(B \cup \{i\})$ where (N, v^M) is the Myerson's game associated with $G_1 = (N, E \setminus \{e_1\})$. Let us note that Conditions 2 and 3 imply that any edge in G_1 is either incident to 1 or 2 or linked to 2 by an edge of weight σ_2. Therefore the set of connected components of B (resp. $B \cup \{i\}$) in G_1 is only made up of one component containing 1, one component containing 2, and possibly singleton components (the components containing 1 and 2 may also coincide or be reduced to singletons). Let B_1 (resp. B_2) be the connected component of B in G_1 containing vertex 1 (resp. 2). Let $B' \cup \{i\}$ be the connected component of $B \cup \{i\}$ in G_1 containing i. Note that i is necessarily linked by at least one edge to B_1 or B_2 and we have $B_1 \subseteq B'$, or $B_2 \subseteq B'$, or $B_1 \cup B_2 \subseteq B'$.

Let us first assume i not linked to B_2. Then i is necessarily linked to B_1 and Conditions 2 and 3 imply that $\{1, i\}$ is the unique edge linking i to B. As $1 \notin A$, i is not linked to A, a contradiction.

Let us now assume i linked to B_2. Let us first suppose $B_1 = B_2$ (*i.e.*, there exists a chordless cycle C_3 or C_4 in G containing e_1). Then $B' = \bigcup_{j=2}^{p} B_j$ where $B_2, B_3, \ldots, B_p$ are the connected components of B in G_1 linked to i (note that $B_3, \ldots, B_p$ are singletons). We obtain:

$$\overline{v}(B \cup \{i\}) - \overline{v}(B) = v\left(\bigcup_{j=2}^{p} B_j \cup \{i\}\right) - v(B_2). \tag{29}$$

Let us now suppose $B_1 \neq B_2$. If i is not linked to B_1 (but linked to B_2), then we still have $B' = \bigcup_{j=2}^{p} B_j$ where $B_2, B_3, \ldots, B_p$ are the connected components of B in G_1 linked to i and (29) is also satisfied. Finally if i is linked to B_1 (and B_2), then $B' = \bigcup_{j=1}^{p} B_j$ where $B_1, B_2, \ldots, B_p$ are the connected components of B in G_1 linked to i. We get:

$$\overline{v}(B \cup \{i\}) - \overline{v}(B) = v\left(\bigcup_{j=1}^{p} B_j \cup \{i\}\right) - v(B_1) - v(B_2). \tag{30}$$

Then (30) and the superadditivity of (N, v) imply:

$$\overline{v}(B \cup \{i\}) - \overline{v}(B) \geq v\left(\bigcup_{j=2}^{p} B_j \cup \{i\}\right) - v(B_2). \tag{31}$$

As (29) or (31) is satisfied, we get that (31) is always satisfied when i is linked to B_2.

As $e_1 \notin E(A)$ and $e_1 \notin E(A \cup \{i\})$, we are in the same situation as in Case 2 for A and $A \cup \{i\}$: $\mathcal{P}_{\min}(A)$ (resp. $\mathcal{P}_{\min}(A \cup \{i\})$) corresponds to the partition $\mathcal{P}_M(A)$ (resp. $\mathcal{P}_M(A \cup \{i\})$) into connected components in the graph $\tilde{G}_3 = (N, E_3)$. Therefore we have:

$$\overline{v}(A \cup \{i\}) - \overline{v}(A) = v^M(A \cup \{i\}) - v^M(A) \tag{32}$$

where (N, v^M) is the Myerson's game associated with $\tilde{G}_3$. As $1 \notin A$, there is no component of A containing vertex 1. Let A_1 be the component of A in $\tilde{G}_3$ containing vertex 2. By Condition 2, we have $A_1 = \{2\}$ and A_1 cannot be linked to i in $\tilde{G}_3$. Let $\tilde{A}$ be the set of connected components of A in $\tilde{G}_3$ linked to i. If $\tilde{A} = \emptyset$, then $v^M(A \cup \{i\}) - v^M(A) = v(i)$ and (26) is satisfied. Otherwise, let $\tilde{A} = \{A_2, A_3, \ldots, A_q\}$ with $q \geq 2$ be the set of connected components of A in $\tilde{G}_3$ linked to i. Then we have:

$$v^M(A \cup \{i\}) - v^M(A) = v\left(\bigcup_{j=2}^{q} A_j \cup \{i\}\right) - \sum_{j=2}^{q} v(A_j). \tag{33}$$

Let us assume that two components in $\tilde{A}$ are linked to 2 in G (by edges of weight σ_2 by Condition 2). We can suppose that these two components are A_2 and A_3 after renumbering if necessary. Then there exists a simple cycle C containing i, 2, and some vertices $j_2 \in A_2$ and $j_3 \in A_3$ as represented in Figure 44. By Condition 5, C is complete. Then $\{j_2, j_3\} \in E(A) \cap E_3$ and

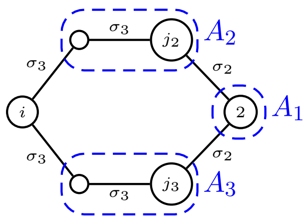

Figure 44: Cycle C.

links A_2 to A_3 in $\tilde{G}_3$, a contradiction. Hence, at most one component A_2 in $\tilde{A}$ is linked to 2. By Condition 3, only A_2 can contain edges and the other potential components in $\tilde{A}$ are singletons. Thus (33) reduces to:

$$v^M(A \cup \{i\}) - v^M(A) = v\left(\bigcup_{j=2}^{q} A_j \cup \{i\}\right) - v(A_2). \tag{34}$$

Following (31) and (34), it is sufficient to prove that the following inequality is satisfied to get (26):

$$v\left(\bigcup_{j=2}^{p} B_j \cup \{i\}\right) - v(B_2) \geq v\left(\bigcup_{j=2}^{q} A_j \cup \{i\}\right) - v(A_2), \tag{35}$$

or equivalently:

$$(36)\qquad v\left(\bigcup_{j=2}^{p} B_j \cup \{i\}\right) + v(A_2) \geq v\left(\bigcup_{j=2}^{q} A_j \cup \{i\}\right) + v(B_2).$$

Each A_j, with $3 \leq j \leq q$, is a singleton $A_j = \{k_j\}$ such that $\{2, k_j\} \notin E$. By contradiction, let us assume $k_j \in B_2$. Then, as $2 \in B_2$ and $\{2, k_j\} \notin E$, there exists a vertex l_2 in B_2 such that the edge $\tilde{e}_2 = \{2, l_2\}$ belongs to E and an elementary path γ in B_2 linking l_2 to k_j. By definition of k_j, we

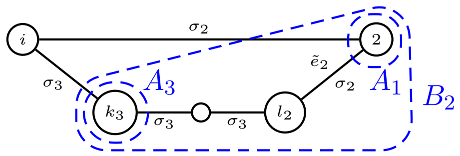

Figure 45: Cycle C.

have $\{i, k_j\} \in E_3$, but $\{2, k_j\} \notin E$. Then Condition 3 implies $\{2, i\} \in E$. We obtain a simple cycle C in G_1 induced by $\{2, i\} \cup \{i, k_j\} \cup \gamma \cup \{l_2, 2\}$ as represented in Figure 45 for $j = 3$. Following Condition 5, C is complete. Hence $\{2, k_j\} \in E$, a contradiction. Therefore $k_j \notin B_2$ for all j, $3 \leq j \leq q$. If $A_2 \cap B_2 \neq \emptyset$, then we have $A_2 \subseteq B_2$ and therefore $(\bigcup_{j=2}^{q} A_j \cup \{i\}) \cap B_2 = A_2$. Then the supermodularity of v applied to $\bigcup_{j=2}^{q} A_j \cup \{i\}$ and B_2 gives:

$$(37)\qquad v\left(\bigcup_{j=2}^{q} A_j \cup \{i\} \cup B_2\right) + v(A_2) \geq v\left(\bigcup_{j=2}^{q} A_j \cup \{i\}\right) + v(B_2).$$

Note that if $A_2 \cap B_2 = \emptyset$, then A_2 is a singleton (otherwise by Condition 3 there is an edge linking A_2 to 2 and then A_2 belongs to B_2), and (37) is still satisfied as $v(A_2) = 0 = v(\emptyset)$. As $\bigcup_{j=2}^{p} B_j \cup \{i\}$ is the connected component of $B \cup \{i\}$ in G_1 containing i, we have $\bigcup_{j=2}^{q} A_j \cup \{i\} \cup B_2 \subseteq \bigcup_{j=2}^{p} B_j \cup \{i\}$, and the superadditivity of v implies:

$$(38)\qquad v\left(\bigcup_{j=2}^{p} B_j \cup \{i\}\right) \geq v\left(\bigcup_{j=2}^{q} A_j \cup \{i\} \cup B_2\right).$$

Finally (38) and (37) imply (36) and therefore (26) is satisfied.

Case 4 Let us assume $2 \notin A$ and $i \neq 2$.
Then $E(A) \subseteq E(A \cup \{i\} \subseteq E_3$, and $\mathcal{P}_{\min}(A)$ and $\mathcal{P}_{\min}(A \cup \{i\})$ are singleton partitions. Hence $\overline{v}(A \cup \{i\}) - \overline{v}(A) = v(i) = 0$ and as $(N, \overline{v})$ is superadditive (cf. Corollary 2 page 4), (26) is satisfied.

Case 5 Let us assume $i = 2$.
Then $E(A) \subseteq E(B) \subseteq E_3$, and $\mathcal{P}_{\min}(A)$ and $\mathcal{P}_{\min}(B)$ are singletons partitions. Then $\overline{v}(A) = \overline{v}(B) = 0$ and (26) is equivalent to $\overline{v}(B \cup \{i\}) \geq \overline{v}(A \cup \{i\})$. This last inequality is satisfied as $(N, \overline{v})$ is superadditive (cf. Corollary 2 page 4).

Case 6 Let us assume $i = 1$ and $2 \in A$.
Then $e_1 \notin E(A)$ and $e_1 \notin E(B)$ but $e_1 \in E(A \cup \{i\})$ and $e_1 \in E(B \cup \{i\})$. As $e_1 \notin E(A)$ (resp. $e_1 \notin E(B)$), we are in the same situation as in Case 2 for A (resp. B): $\mathcal{P}_{\min}(A)$ (resp. $\mathcal{P}_{\min}(B)$) corresponds to the partition $\mathcal{P}_M(A)$ (resp. $\mathcal{P}_M(B)$) into connected components in the graph $\tilde{G}_3 = (N, E_3)$. Let us set:

$$\mathcal{P}_M(A) = \{\{2\}, A_1, A_2, \ldots, A_r, A_{r+1}, \ldots, A_t, A_{t+1}, \ldots, A_p\} \tag{39}$$

and

$$\mathcal{P}_M(B) = \{\{2\}, B_1, B_2, \ldots, B_s, B_{s+1}, \ldots, B_u, B_{u+1}, \ldots, B_q\} \tag{40}$$

where $A_1, A_2, \ldots, A_r$ (resp. $B_1, B_2, \ldots, B_s$) are the connected components of A (resp. B) linked to 2 in G, *i.e.*, for all j, $1 \leq j \leq r$ (resp. $1 \leq j \leq s$), there exists $k_j \in A_j$ (resp. $l_j \in B_j$) such that $\tilde{e}_j = \{2, k_j\}$ (resp. $\{2, l_j\}$) belongs to E. Following Condition 3, the remaining components $A_{r+1}, \ldots, A_p$ (resp. $B_{s+1}, \ldots, B_q$) are necessarily singletons, *i.e.*, $A_j = \{k_j\}$ (resp. $B_j = \{l_j\}$), for all j, $r+1 \leq j \leq p$ (resp. $s+1 \leq j \leq q$) for some vertex k_j (resp. l_j) of G. For $r+1 \leq j \leq t$ (resp. $s+1 \leq j \leq u$), A_j (resp. B_j) is a singleton linked to vertex 1 (*i.e.*, vertex i) by the edge $\{1, k_j\}$ (resp. $\{1, l_j\}$) in G.

Claim 1. *We can assume*

$$A_j \subseteq B_j, \ \forall j, \ 1 \leq j \leq r \leq s, \tag{41}$$

after renumbering if necessary.

Proof of Claim 1. We assume that two components A_1 and A_2 are subsets of the same component B_1. Let γ be a simple path in B_1 linking $k_1 \in A_1$ to $k_2 \in A_2$. Then $\{2, k_1\} \cup \gamma \cup \{k_2, 2\}$ induces a cycle C in G_1 as represented in Figure 46. By Condition 5, C is complete. Hence the chord $\{k_1, k_2\} \in E(A) \cap E_3$ and links A_1 to A_2, a contradiction. □

Claim 2. *We can assume*

$$A_{r+j} = B_{s+j}, \ \forall j, \ 1 \leq j \leq t - r, \tag{42}$$

after renumbering if necessary, except in the two following situations:

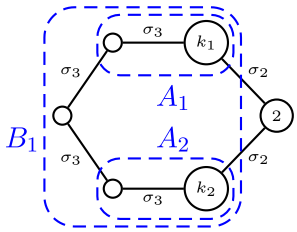

Figure 46: Cycle C.

1. *$\tilde{C}_3$ exists and there exists a unique triangle C_3 adjacent to $\tilde{C}_3$ and containing 1 but not e_1. Assuming $V(\tilde{C}_3) = \{1, 2, 3\}$ then $3 \in B \setminus A$, $C_3 = \{1, A_{r+1}, 3, 1\}$, and we can assume $A_{r+1} \in B_1$ or $A_{r+1} \in B_{r+1}$ and $A_{r+j} = B_{s+j-1}$, for all j, $2 \leq j \leq t - r$.*

2. *$\tilde{C}_4$ exists. Assuming $V(\tilde{C}_4) = \{1, 2, 3, 4\}$ then $3 \in B \setminus A$, $A_{r+1} = \{4\}$, $\tilde{C}_4 = \{1, 2, 3, A_{r+1}, 1\}$ and we can assume $\{3, A_{r+1}\} \subseteq B_1$ or $\{3, A_{r+1}\} \subseteq B_{r+1}$ and $A_{r+j} = B_{s+j-1}$, for all j, $2 \leq j \leq t - r$.*

Proof of Claim 2. By Condition 4, there exists at most one chordless cycle $\tilde{C}_m$ with $m = 3$ or 4 containing e_1. If $m = 3$ (resp. $m = 4$) we assume $V(\tilde{C}_m) = \{1, 2, 3\}$ (resp. $V(\tilde{C}_m) = \{1, 2, 3, 4\}$) after renumbering if necessary and we have $w(\{2, 3\}) = \sigma_2$, and $w(\{3, 1\}) = \sigma_3$ (resp. $w(\{3, 4\}) = w(\{4, 1\}) = \sigma_3$).

To prove (42), we assume by contradiction the existence of a pair (h, j) with $r + 1 \leq h \leq t$ and $1 \leq j \leq s$ such that:

(43) $$A_h \cap B_j \neq \emptyset.$$

As $r + 1 \leq h \leq t$, A_h is a singleton $\{k_h\}$ linked to 1 but not to 2 in G. Let us suppose by contradiction $k_h \in B_j$ for an index $j \in \{1, \ldots, s\}$. Let γ be a shortest path linking k_h to l_j in B_j (we have $k_h \neq l_j$ as A_h is not linked to 2 whereas B_j is linked to 2 in G by $\tilde{e}_j = \{2, l_j\}$). As all edges in γ have weight σ_3, Condition 3 implies that we can always choose l_j such that γ is reduced to the edge $\{k_h, l_j\}$ as represented in Figure 47. Then $\{1, 2, l_j, k_h, 1\}$ induces

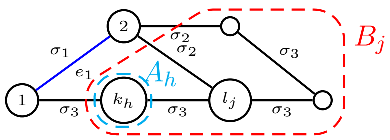

Figure 47: A_h is not linked to 2 but B_j is linked to 2 by $\tilde{e}_j = \{2, l_j\}$.

a cycle of length 4 containing e_1. Note that $\{1, l_j\}$ may exist in G.

If $\{1, l_j\}$ exists in G, then Conditions 4 and 6 imply $l_j = 3$ (*i.e.* $\tilde{C}_3 = \{1, 2, l_j\}$) and $\tilde{C}_3$ is adjacent to the cycle induced by $\{1, k_h, 3, 1\}$, as represented in Figure 48.

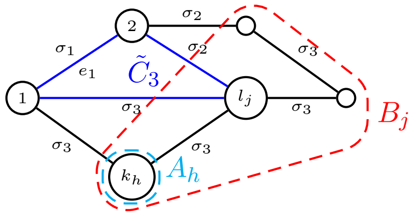

Figure 48: $l_j = 3$ and $\tilde{C}_3$ adjacent to the cycle induced by $\{1, k_h, 3, 1\}$.

As $l_j = 3$ we have $3 \in B \setminus A$. We can assume $h = r+1$ after renumbering if necessary. We get $A_{r+1} \subseteq B_j$ for some j, with $1 \leq j \leq s$. If $1 \leq j \leq r$ (resp. $r+1 \leq j \leq s$), we can assume $j = 1$ (resp. $j = r+1$) after renumbering if necessary (note that B_1 also contains A_1 by Claim 1).

If $\{1, l_j\}$ does not exist in G, then the 4-cycle $\{1, 2, l_j, k_h, 1\}$ has no chord ($\{2, k_h\}$ does not exist in G as k_h is not linked to 2). Then Condition 4 implies $l_j = 3$ and $k_h = 4$, (*i.e.* $\tilde{C}_4 = \{1, 2, l_j, k_h, 1\}$) as represented in Figure 49. Then we have $\{3, 4\} \subseteq B_j$.

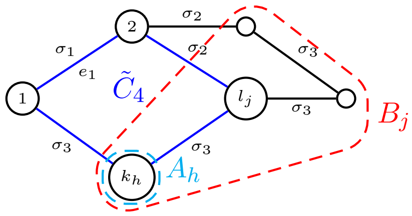

Figure 49: $l_j = 3$ and $k_h = 4$.

As $l_j = 3$ we have $3 \in B \setminus A$. We can assume $h = r+1$ after renumbering if necessary. Then we have $\{3, A_{r+1}\} \subseteq B_j$ for some j, with $1 \leq j \leq s$. If $1 \leq j \leq r$ (resp. $r+1 \leq j \leq s$), we can assume $j = 1$ (resp. $j = r+1$) after renumbering if necessary.

Finally, by Conditions 4 and 6 we cannot have a second pair (h, j) satisfying (43). $\square$

By Condition 4, there exists at most one chordless cycle $\tilde{C}_m$ with $m = 3$ or 4 containing e_1. We have to consider several subcases taking into account the possible existence of such a cycle $\tilde{C}_m$. If $m = 3$ (resp. $m = 4$) we assume $V(\tilde{C}_m) = \{1, 2, 3\}$ (resp. $V(\tilde{C}_m) = \{1, 2, 3, 4\}$) after renumbering if necessary and we have $w(\{2, 3\}) = \sigma_2$, and $w(\{3, 1\}) = \sigma_3$ (resp. $w(\{3, 4\}) = w(\{4, 1\}) = \sigma_3$).

Case 6.1 We assume that there exists a chordless cycle $\tilde{C}_m$ with $m = 3$ or $m = 4$ containing e_1 and that the vertex 3 (resp. the vertices 3 and 4) of $\tilde{C}_3$ (resp. $\tilde{C}_4$) belongs to A. As the two particular situations described in Claim 2 require $3 \in B \setminus A$, (42) is necessarily satisfied, *i.e.*, we have $A_{r+j} = B_{s+j}$ for all j, $1 \leq j \leq t - r$. As $e_1 \in E(A \cup \{i\})$, $\mathcal{P}_{\min}(A \cup \{i\})$ is obtained by deleting e_1 in $E(A \cup \{i\})$ and corresponds to the partition

$\mathcal{P}_M(A \cup \{i\})$ in G_1. Let $A'_1 \cup \{i\}$ be the component of $A \cup \{i\}$ in G_1 containing i. Let us observe that (using $\tilde{C}_m$) 1 and 2 are connected in G_1. Then $A'_1 \cup \{i\} = \bigcup_{j=1}^{r} A_j \cup \bigcup_{j=r+1}^{t} A_j \cup \{2\} \cup \{i\}$ where $A_{r+1}, \ldots, A_t$ are the singletons $A_j = \{k_j\}$ of $\mathcal{P}_{\min}(A)$ linked to 1 in G_1 but not linked to 2. If $m = 3$ (resp. $m = 4$) we can assume 3 in A_1 (resp. 3 and 4 in A_1), after renumbering if necessary, as represented in Figure 50 (resp. Figure 51) with $r = 2$, $t = 3$, and $p = 4$. Then each A_j with $2 \leq j \leq r$ is connected to 2 by

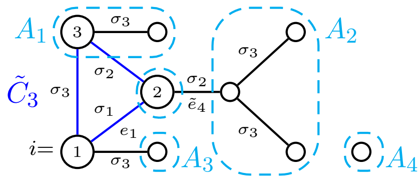

Figure 50: $\tilde{C}_3$ and A_1, A_2 linked to 2, A_3 linked to 1.

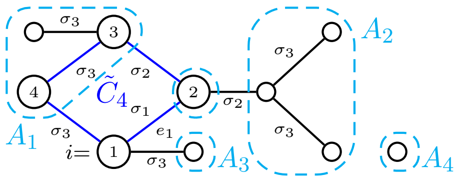

Figure 51: $\tilde{C}_4$ and A_1, A_2 linked to 2, A_3 linked to 1.

$\tilde{e}_j = \{2, k_j\}$ for some $k_j \in A_j$ and connected to 1 by the elementary path $\{1, \{1,3\}, 3, \{3,2\}, 2, \tilde{e}_j, k_j\}$ (resp. $\{1, \{1,4\}, 4, \{4,3\}, 3, \{3,2\}, 2, \tilde{e}_j, k_j\}$) in G_1 if $m = 3$ (resp. $m = 4$). We get $\mathcal{P}_{\min}(A \cup \{i\}) = \{A'_1 \cup \{i\}, A_{t+1}, \ldots, A_p\}$ where A_j, $t+1 \leq j \leq p$, are the singletons of $\mathcal{P}_{\min}(A)$ which are neither linked to 1 nor to 2. In the same way, we have $\mathcal{P}_{\min}(B \cup \{i\}) = \{B'_1 \cup \{i\}, B_{u+1}, \ldots, B_q\}$ with $B'_1 = \bigcup_{j=1}^{s} B_j \cup \bigcup_{j=s+1}^{u} B_j \cup \{2\}$. As A_j for $j \geq r+1$ and B_j for $j \geq s+1$ are singletons, we have:

$$\overline{v}(A \cup \{i\}) - \overline{v}(A) = v(A'_1 \cup \{i\}) - \sum_{j=1}^{r} v(A_j) \tag{44}$$

and

$$\overline{v}(B \cup \{i\}) - \overline{v}(B) = v(B'_1 \cup \{i\}) - \sum_{j=1}^{s} v(B_j). \tag{45}$$

Hence, we have to prove:

$$v(B'_1 \cup \{i\}) - \sum_{j=1}^{s} v(B_j) \geq v(A'_1 \cup \{i\}) - \sum_{j=1}^{r} v(A_j). \tag{46}$$

As $A'_1 \subseteq B'_1$, the convexity of (N, v) implies:

$$v(B'_1 \cup \{i\}) - v(B'_1) \geq v(A'_1 \cup \{i\}) - v(A'_1). \tag{47}$$

We will prove:

$$v(B'_1) - \sum_{j=1}^{s} v(B_j) \geq v(A'_1) - \sum_{j=1}^{r} v(A_j). \tag{48}$$

(48) can also be written:

$$v(B'_1) - \sum_{j=1}^{u} v(B_j) \geq v(A'_1) - \sum_{j=1}^{t} v(A_j). \tag{49}$$

Let us define:

$$B''_1 := \bigcup_{j=1}^{r} B_j \cup \bigcup_{j=s+1}^{s+t-r} B_j \cup \{2\}. \tag{50}$$

so that $B'_1 = B''_1 \cup \bigcup_{j=r+1}^{s} B_j \cup \bigcup_{j=s+t-r+1}^{u} B_j$. By the superadditivity of v, we obtain:

$$v(B'_1) \geq v(B''_1) + \sum_{j=r+1}^{s} v(B_j) + \sum_{j=s+t-r+1}^{u} v(B_j),$$

which is equivalent to:

$$v(B'_1) - \sum_{j=1}^{u} v(B_j) \geq v(B''_1) - \sum_{j=1}^{r} v(B_j) - \sum_{j=s+1}^{s+t-r} v(B_j). \tag{51}$$

By (51), to obtain (48) or (49), it is enough to prove:

$$v(B''_1) - \sum_{j=1}^{r} v(B_j) - \sum_{j=s+1}^{s+t-r} v(B_j) \geq v(A'_1) - \sum_{j=1}^{t} v(A_j),$$

which is equivalent to:

$$v(B''_1) - \sum_{j=1}^{r} v(B_j) - \sum_{j=s+1}^{s+t-r} v(B_j) \geq v(A'_1) - \sum_{j=1}^{r} v(A_j) - \sum_{j=r+1}^{t} v(A_j). \tag{52}$$

As (N, v) is a convex game, we can apply Lemma 4 to A'_1, B''_1 and the partition $\{\{2\}, B_1, \ldots, B_r, B_{s+1}, \ldots, B_{s+t-r}\}$ of B''_1 to get:

$$v(A'_1 \cup B''_1) + \sum_{j=1}^{r} v(A'_1 \cap B_j) + \sum_{j=s+1}^{s+t-r} v(A'_1 \cap B_j) \geq v(A'_1) + \sum_{j=1}^{r} v(B_j) + \sum_{j=s+1}^{s+t-r} v(B_j). \tag{53}$$

By Claims 1 and 2 we have $A'_1 \cap B_j = A_j$ for all j, $1 \leq j \leq r$, and $A'_1 \cap B_j = A_{r+j-s}$ for all j, $s+1 \leq j \leq s+t-r$. Then (53) is equivalent to:

$$(54) \quad v(A'_1 \cup B''_1) + \sum_{j=1}^{r} v(A_j) + \sum_{j=r+1}^{t} v(A_j) \geq \\ v(A'_1) + \sum_{j=1}^{r} v(B_j) + \sum_{j=s+1}^{s+t-r} v(B_j).$$

Finally, Claims 1 and 2 also imply $\bigcup_{j=1}^{r} A_j \subseteq \bigcup_{j=1}^{r} B_j$ and $\bigcup_{j=r+1}^{t} A_j \subseteq \bigcup_{j=s+1}^{s+t-r} B_j$. Therefore we have $A'_1 \subseteq B''_1$, and (54) is equivalent to (52).

Case 6.2

We now assume that there exists a cycle $\tilde{C}_m$ with $m = 3$ or $m = 4$ and that the vertex 3 (resp. either 3 or 4) of $\tilde{C}_3$ (resp. $\tilde{C}_4$) belongs to $B \setminus A$. Moreover, if $m = 4$ we assume 3 and 4 in B. We recall that $A'_1 \cup \{i\}$ denotes the component of $\mathcal{P}_{\min}(A \cup \{i\})$ containing i. Now we have $A'_1 = \bigcup_{j=r+1}^{t} A_j$ where $A_{r+1}, \ldots, A_t$ are the singletons of $\mathcal{P}_{\min}(A)$ linked to 1 but not to 2. Note that if we have $4 \in A$ and $3 \in B \setminus A$, then 4 corresponds to one of the A_j with $r+1 \leq j \leq t$ and therefore belongs to A'_1. The component of $\mathcal{P}_{\min}(A \cup \{i\})$ which contains 2 is $A'_2 \cup \{2\}$ with $A'_2 = \bigcup_{j=1}^{r} A_j$. Hence we have $\mathcal{P}_{\min}(A \cup \{i\}) = \{A'_1 \cup \{i\}, A'_2 \cup \{2\}, A_{t+1}, \ldots, A_p\}$ and this implies:

$$(55) \qquad \overline{v}(A \cup \{i\}) - \overline{v}(A) = v(A'_1 \cup \{i\}) + v(A'_2 \cup \{2\}) - \sum_{j=1}^{t} v(A_j).$$

As 3 (resp. 3 and 4) belong to B if $m = 3$ (resp. $m = 4$), we have $\mathcal{P}_{\min}(B \cup \{i\}) = \{B'_1 \cup \{i\}, B_{u+1}, \ldots, B_q\}$ with $B'_1 = \bigcup_{j=1}^{s} B_j \cup \bigcup_{j=s+1}^{u} B_j \cup \{2\}$ as in Case 6.1. Hence we have:

$$(56) \qquad \overline{v}(B \cup \{i\}) - \overline{v}(B) = v(B'_1 \cup \{i\}) - \sum_{j=1}^{u} v(B_j).$$

By (55) and (56), we have to prove:

$$(57) \quad v(B'_1 \cup \{i\}) - \sum_{j=1}^{u} v(B_j) \geq v(A'_1 \cup \{i\}) + v(A'_2 \cup \{2\}) - \sum_{j=1}^{t} v(A_j).$$

We have to consider subcases.

Subcase 6.2.1

We first assume (42) satisfied in Claim 2, *i.e.*, we have $A_{r+j} = B_{s+j}$ for all j, $1 \leq j \leq t-r$. Then we have $B'_1 = \bigcup_{j=1}^{s} B_j \cup A'_1 \cup \bigcup_{j=s+t-r+1}^{u} B_j \cup \{2\}$. Setting $B'_2 := \bigcup_{j=1}^{s} B_j$, we get:

$$(58) \qquad B'_1 \cup \{i\} = (A'_1 \cup \{i\}) \cup (B'_2 \cup \{2\}) \cup \bigcup_{j=s+t-r+1}^{u} B_j.$$

The superadditivity of (N, v) implies:

$$(59) \qquad v(B_1' \cup \{i\}) \geq v(A_1' \cup \{i\}) + v(B_2' \cup \{2\}) + \sum_{j=s+t-r+1}^{u} v(B_j),$$

which is equivalent to:

$$(60) \quad v(B_1' \cup \{i\}) - \sum_{j=1}^{u} v(B_j) \geq v(A_1' \cup \{i\}) + v(B_2' \cup \{2\}) - \sum_{j=1}^{s+t-r} v(B_j).$$

Hence to prove (57) it is enough to prove:

$$(61) \qquad v(B_2' \cup \{2\}) - \sum_{j=1}^{s+t-r} v(B_j) \geq v(A_2' \cup \{2\}) - \sum_{j=1}^{t} v(A_j).$$

By (42) we have $\sum_{j=r+1}^{t} v(A_j) = \sum_{j=s+1}^{s+t-r} v(B_j)$. Therefore (61) is equivalent to:

$$(62) \qquad v(B_2' \cup \{2\}) - \sum_{j=1}^{s} v(B_j) \geq v(A_2' \cup \{2\}) - \sum_{j=1}^{r} v(A_j).$$

As (N, v) is a convex game, we can apply Lemma 4 to $A_2' \cup \{2\}$, B_2', and the partition $\{B_1, B_2, \ldots, B_s\}$ of B_2' to get:

$$(63) \qquad v(A_2' \cup \{2\} \cup B_2') + \sum_{j=1}^{s} v(A_2' \cap B_j) \geq v(A_2' \cup \{2\}) + \sum_{j=1}^{s} v(B_j).$$

By Claim 1, we have $A_j \subseteq B_j$, for all j, $1 \leq j \leq r \leq s$. Therefore, we have $A_2' \cap B_j = A_j$, for all j, $1 \leq j \leq r$, $A_2' \cap B_j = \emptyset$, for all j, $r+1 \leq j \leq s$, and $A_2' \subseteq B_2'$. Then (63) implies (62).

In the next subcases we assume (42) not satisfied, *i.e.*, we are in one of the two specific situations described in Claim 2.

Subcase 6.2.2

Let us assume $A_{r+1} \in B_1$. We have $A_{r+j} = B_{s+j-1}$ for all j, $2 \leq j \leq t-r$. Let us recall we have:

$$(64) \qquad \overline{v}(B \cup \{i\}) - \overline{v}(B) = v(B_1' \cup \{i\}) - \sum_{j=1}^{u} v(B_j)$$

with $B_1' = \bigcup_{j=1}^{s} B_j \cup \bigcup_{j=s+1}^{u} B_j \cup \{2\}$. Let us set:

$$(65) \qquad \tilde{A} := A_{r+1} \cup \bigcup_{j=1}^{r} A_j \cup \bigcup_{j=r+2}^{t} A_j \cup \{2\}.$$

Applying Lemma 4 to $\tilde{A} \cup \{i\}$, B'_1, and the partition $\{\{2\}, B_1, \ldots, B_s, B_{s+1}, \ldots, B_u\}$ of B'_1 we get:

$$(66) \qquad v(\tilde{A} \cup B'_1 \cup \{i\}) + \sum_{j=1}^{u} v(\tilde{A} \cap B_j) \geq v(\tilde{A} \cup \{i\}) + \sum_{j=1}^{u} v(B_j).$$

By Claim 1 we have $A_j \subseteq B_j$ for all j, $1 \leq j \leq r \leq s$. As $A_{r+1} \in B_1$ and $A_{r+j} = B_{s+j-1}$ for all j, $2 \leq j \leq t-r$, we get $\tilde{A} \cap B_1 = A_1 \cup A_{r+1}$, $\tilde{A} \cap B_j = A_j$ for all j, $2 \leq j \leq r$, $\tilde{A} \cap B_j = \emptyset$ for all j, $r+1 \leq j \leq s$, $\tilde{A} \cap B_j = A_{j-s+r+1}$ for all j, $s+1 \leq j \leq s+t-r-1$, and $\tilde{A} \cap B_j = \emptyset$ for $s+t-r \leq j \leq u$. Moreover, we have $\tilde{A} \subseteq B'_1$. Note that any $A_{j-s+r+1}$ with $s+1 \leq j \leq s+t-r-1$ corresponds to a singleton. Hence (66) becomes:

$$(67) \quad v(B'_1 \cup \{i\}) + v(A_1 \cup A_{r+1}) + \sum_{j=2}^{r} v(A_j) \geq v(\tilde{A} \cup \{i\}) + \sum_{j=1}^{u} v(B_j).$$

From (64) and (67) we obtain:

$$(68) \qquad \overline{v}(B \cup \{i\}) - \overline{v}(B) \geq v(\tilde{A} \cup \{i\}) - v(A_1 \cup A_{r+1}) - \sum_{j=2}^{r} v(A_j).$$

We set $A'_2 := \bigcup_{j=1}^{r} A_j$ and $A'_1 := \bigcup_{j=r+1}^{t} A_j$ so that $\tilde{A} = A'_1 \cup A'_2 \cup \{2\}$. As i and $2 \notin A'_1 \cup A'_2$, the convexity of (N, v) implies:

$$(69) \quad v(\tilde{A} \cup \{i\}) + v(A'_1 \cup A'_2) \geq v(A'_1 \cup A'_2 \cup \{i\}) + v(A'_1 \cup A'_2 \cup \{2\}).$$

Then (68) and (69) imply:

$$(70) \quad \overline{v}(B \cup \{i\}) - \overline{v}(B) \geq v(A'_1 \cup A'_2 \cup \{i\}) + v(A'_1 \cup A'_2 \cup \{2\}) - v(A'_1 \cup A'_2) - v(A_1 \cup A_{r+1}) - \sum_{j=2}^{r} v(A_j).$$

As $i \notin A'_1 \cup A'_2$, the convexity of (N, v) also implies:

$$(71) \qquad v(A'_1 \cup A'_2 \cup \{i\}) + v(A'_1) \geq v(A'_1 \cup \{i\}) + v(A'_1 \cup A'_2).$$

Then (70) and (71) imply:

$$(72) \quad \overline{v}(B \cup \{i\}) - \overline{v}(B) \geq v(A'_1 \cup A'_2 \cup \{2\}) + v(A'_1 \cup \{i\}) - v(A'_1) - v(A_1 \cup A_{r+1}) - \sum_{j=2}^{r} v(A_j).$$

As $A_{r+1} \subseteq A'_1$ and as $A'_1 \cap (A'_2 \cup \{2\}) = \emptyset$, the convexity of (N, v) implies (it is the crucial inequality):

$$(73) \qquad v(A'_1 \cup A'_2 \cup \{2\}) - v(A'_1) \geq v(A_{r+1} \cup A'_2 \cup \{2\}) - v(A_{r+1}).$$

As A_{r+1} is a singleton, we have $v(A_{r+1}) = 0$. (72) and (73) imply:

$$\overline{v}(B \cup \{i\}) - \overline{v}(B) \geq v(A_1' \cup \{i\}) + v(A_{r+1} \cup A_2' \cup \{2\}) - v(A_1 \cup A_{r+1}) - \sum_{j=2}^{r} v(A_j). \tag{74}$$

As $A_1 \subseteq A_2'$ and as $A_{r+1} \cap (A_2' \cup \{2\}) = \emptyset$, the convexity of (N, v) implies:

$$v(A_{r+1} \cup A_2' \cup \{2\}) - v(A_1 \cup A_{r+1}) \geq v(A_2' \cup \{2\}) - v(A_1). \tag{75}$$

(74) and (75) imply:

$$\overline{v}(B \cup \{i\}) - \overline{v}(B) \geq v(A_1' \cup \{i\}) + v(A_2' \cup \{2\}) - \sum_{j=1}^{r} v(A_j). \tag{76}$$

Using (55), we obtain:

$$\overline{v}(B \cup \{i\}) - \overline{v}(B) \geq \overline{v}(A \cup \{i\}) - \overline{v}(A). \tag{77}$$

Subcase 6.2.3
We now assume $A_{r+1} \in \mathcal{B}_{r+1}$. As in the previous case, setting $B_1' = \bigcup_{j=1}^{s} B_j \cup \bigcup_{j=s+1}^{u} B_j \cup \{2\}$ we have:

$$\overline{v}(B \cup \{i\}) - \overline{v}(B) = v(B_1' \cup \{i\}) - \sum_{j=1}^{u} v(B_j). \tag{78}$$

We set:

$$\tilde{A} := \bigcup_{j=1}^{r} A_j \cup \bigcup_{j=r+1}^{t} A_j \cup \{2\}. \tag{79}$$

Applying Lemma 4 to $\tilde{A} \cup \{i\}$, B_1', and the partition $\{\{2\}, B_1, \ldots, B_s, B_{s+1}, \ldots, B_u\}$ of B_1' we get:

$$v(\tilde{A} \cup B_1' \cup \{i\}) + \sum_{j=1}^{u} v(\tilde{A} \cap B_j) \geq v(\tilde{A} \cup \{i\}) + \sum_{j=1}^{u} v(B_j). \tag{80}$$

By Claim 1 we have $A_j \subseteq B_j$ for all j, $1 \leq j \leq r \leq s$. As $A_{r+1} \in \mathcal{B}_{r+1}$ and $A_{r+j} = B_{s+j-1}$ for all j, $2 \leq j \leq t - r$, we get $\tilde{A} \cap B_j = A_j$ for all j, $1 \leq j \leq r+1$, $\tilde{A} \cap B_j = \emptyset$ for all j, $r+2 \leq j \leq s$, $\tilde{A} \cap B_j = A_{j-s+r+1}$ for all j, $s+1 \leq j \leq s+t-r-1$, and $\tilde{A} \cap B_j = \emptyset$ for $s+t-r \leq j \leq u$. Moreover, we have $\tilde{A} \subseteq B_1'$. Note that any $A_{j-s+r+1}$ with $s \leq j \leq s+t-r-1$ corresponds to a singleton. Hence (80) becomes:

$$v(B_1' \cup \{i\}) + \sum_{j=1}^{r} v(A_j) \geq v(\tilde{A} \cup \{i\}) + \sum_{j=1}^{u} v(B_j). \tag{81}$$

From (78) and (81) we obtain:

$$\overline{v}(B \cup \{i\}) - \overline{v}(B) \geq v(\tilde{A} \cup \{i\}) - \sum_{j=1}^{r} v(A_j). \tag{82}$$

We set $A'_2 := \bigcup_{j=1}^{r} A_j$ and $A'_1 := \bigcup_{j=r+1}^{t} A_j$ so that $\tilde{A} = A'_1 \cup A'_2 \cup \{2\}$. Then (82) and the superadditivity of (N, v) imply:

$$\overline{v}(B \cup \{i\}) - \overline{v}(B) \geq v(A'_1 \cup \{i\}) + v(A'_2 \cup \{2\}) - \sum_{j=1}^{r} v(A_j). \tag{83}$$

Using (55), we obtain:

$$\overline{v}(B \cup \{i\}) - \overline{v}(B) \geq \overline{v}(A \cup \{i\}) - \overline{v}(A). \tag{84}$$

Subcase 6.3
We assume either the non-existence of a chordless cycle $\tilde{C}_m$ with $m = 3$ or $m = 4$ containing e_1 or the existence of such a cycle but in this last case we assume that the vertex 3 (resp. 3 or 4) of $\tilde{C}_3$ (resp. of $\tilde{C}_4$) does not belong to B. As the two particular situations described in Claim 2 require the existence of $\tilde{C}_3$ or $\tilde{C}_4$ and $3 \in B \setminus A$, (42) is necessarily satisfied, *i.e.*, we have $A_{r+j} = B_{s+j}$ for all j, $1 \leq j \leq t - r$. As in Case 6.2, we have $\mathcal{P}_{\min}(A \cup \{i\}) = \{A'_1 \cup \{i\}, A'_2 \cup \{2\}, A_{t+1}, \ldots, A_p\}$ with $A'_1 = \bigcup_{j=r+1}^{t} A_j$ and $A'_2 = \bigcup_{j=1}^{r} A_j$, and therefore (55) is still valid:

$$\overline{v}(A \cup \{i\}) - \overline{v}(A) = v(A'_1 \cup \{i\}) + v(A'_2 \cup \{2\}) - \sum_{j=1}^{r} v(A_j). \tag{85}$$

Similarly, for B we have $\mathcal{P}_{\min}(B \cup \{i\}) = \{B'_1 \cup \{i\}, B'_2 \cup \{2\}, B_{u+1}, \ldots, B_q\}$ with $B'_1 = \bigcup_{j=s+1}^{u} B_j$ and $B'_2 = \bigcup_{j=1}^{s} B_j$, and therefore:

$$\overline{v}(B \cup \{i\}) - \overline{v}(B) = v(B'_1 \cup \{i\}) + v(B'_2 \cup \{2\}) - \sum_{j=1}^{s} v(B_j). \tag{86}$$

We have to prove:

$$v(B'_1 \cup \{i\}) + v(B'_2 \cup \{2\}) - \sum_{j=1}^{s} v(B_j) \geq v(A'_1 \cup \{i\}) + v(A'_2 \cup \{2\}) - \sum_{j=1}^{r} v(A_j). \tag{87}$$

As $A'_1 \subseteq B'_1$ the superadditivity of (N, v) implies:

$$v(B'_1 \cup \{i\}) \geq v(A'_1 \cup \{i\}). \tag{88}$$

Applying Lemma 4 to $A'_2 \cup \{2\}$, B'_2, and the partition $\{B_1, B_2, \ldots, B_s\}$ of B'_2, we get:

$$v(A'_2 \cup B'_2 \cup \{2\}) + \sum_{j=1}^{s} v(A'_2 \cap B_j) \geq v(A'_2 \cup \{2\}) + \sum_{j=1}^{s} v(B_j). \tag{89}$$

By Claim 1 we have $A_j \subseteq B_j$ for all j, $1 \leq j \leq r \leq s$. As $A_{r+j} = B_{s+j}$ for all j, $1 \leq j \leq t-r$, we get $A'_2 \cap B_j = A_j$ for all j, $1 \leq j \leq r$ and $A'_2 \cap B_j = \emptyset$ for all j, $r+1 \leq j \leq s$. Moreover, we have $A'_2 \subseteq B'_2$. Hence (89) becomes:

$$v(B'_2 \cup \{2\}) - \sum_{j=1}^{s} v(B_j) \geq v(A'_2 \cup \{2\}) - \sum_{j=1}^{r} v(A_j). \tag{90}$$

Adding (88) and (90), we obtain (87). □

5.3 Disconnected graphs

We finally consider the case of disconnected weighted graphs. We prove that if there is inheritance of convexity for $\mathcal{P}_{\min}$ then the underlying graph G has to be connected or has only one component with non constant weight.

Proposition 28. *Let $G = (N, E, w)$ be a weighted graph and let us assume that for all $\emptyset \neq S \subseteq N$ the $\mathcal{P}_{\min}$-restricted game $(N, \overline{u_S})$ is convex.*

1. *If G has a connected component with three different weights $\sigma_1 < \sigma_2 < \sigma_3$, then all other connected components of G are singletons.*
2. *If G has a connected component with two different weights $\sigma_1 < \sigma_2$ and at least two edges of minimum weight σ_1, then all other connected components of G are singletons.*
3. *If G has a connected component with two different weights $\sigma_1 < \sigma_2$ and only one edge of minimum weight σ_1, then all other connected components of G have constant weight σ_2 or are singletons.*

To prove Proposition 28 we need the following lemma.

Lemma 29. *Let $G = (N, E, w)$ be a weighted graph and let us assume that for all $\emptyset \neq S \subseteq N$ the $\mathcal{P}_{\min}$-restricted game $(N, \overline{u_S})$ is convex. Let $e_1 = \{1, 2\}$ and $e_2 = \{2, 3\}$ be two adjacent edges such that $w_1 < w_2$ and let e be an edge which is not incident to 1. Then we have $w(e) \geq w_2$ and if moreover e is not linked to 2 by an edge, then $w(e) = w_2$.*

Proof. We set $e = \{j, k\}$. If e is incident to 2, Star condition implies $w(e) = w_2$. If e is incident to 3, Path condition implies $w(e) \geq w_2$. Hence we can assume that e is not incident to the vertices 1, 2, and 3. By contradiction, let us assume $w(e) < w_2$. Let us consider $i = 1$ and the subsets $A = \{2, 3\}$ and

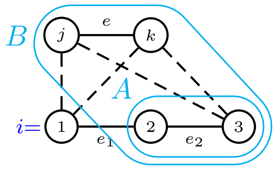

Figure 52: $w_1 < w_2$ and $w(e) < w_2$.

$B = A \cup \{j,k\}$, as represented in Figure 52. Then we have $A \subseteq B \subseteq N \setminus \{i\}$. Let us note that, as $w(e) < w_2$, we have $\sigma(B) < w_2$. Therefore there is a block B' of $\mathcal{P}_{\min}(B)$ such that $A \subseteq B'$. As $w_1 < w_2$, we have $\sigma(A \cup \{i\}) < w_2$ and $\mathcal{P}_{\min}(A \cup \{i\}) = \{A \cup \{i\}\}$ or $\{A, \{i\}\}$. Then taking $A' = A$ we get $\mathcal{P}_{\min}(B)_{|A'} = \{A\} \neq \{\{2\},\{3\}\} = \mathcal{P}_{\min}(A)_{|A'}$ and it contradicts Theorem 3 applied with $\mathcal{F} = 2^N \setminus \{\emptyset\}$. If e is not linked to 2 by an edge, Lemma 13 implies $w(e) \leq \max(w_1, w_2) = w_2$ and therefore $w(e) = w_2$. □

We can now prove Proposition 28.

Proof of Proposition 28. 1. Let us consider edges e_1, e_2, e_3 of weights $\sigma_1 < \sigma_2 < \sigma_3$. By Theorem 27 we can assume $e_1 = \{1,2\}$, $e_2 = \{2,3\}$, and e_3 either incident to 3 or incident to 1. By contradiction, let us consider an edge e which is not connected to e_1. If e_3 is incident to 3 (resp. to 1), then applying Lemma 29 at first to the couple of edges (e_1, e_2) and then to the couple (e_2, e_3) (resp. (e_1, e_3)) we get at first $w(e) = \sigma_2$ and then $w(e) = \sigma_3$, a contradiction.

2. Let us now consider edges e_1, e_2, e_3 of weights $w_1 = w_2 = \sigma_1$ and $w_3 = \sigma_2$. By Theorem 24 we can assume $e_1 = \{1,2\}$, $e_2 = \{1,3\}$, and e_3 incident to 3. By contradiction, let us consider an edge e which is not connected to e_1. At first we apply Lemma 29 to the couple of edges (e_2, e_3) so that we get $w(e) = w_3 = \sigma_2$ and then we apply Lemma 13 to the couple of edges (e_1, e_2) (we have $w(e) = \sigma_2 > \sigma_1 = w_1 = w_2$), so that e has to be linked to 1 by an edge of weight σ_1, a contradiction.

3. Let us now assume there are only two different weights and only one edge $e_1 = \{1,2\}$ of minimum weight σ_1 (in a connected component of G). Let us consider an edge $e_2 = \{2,3\}$ of weight $w_2 = \sigma_2 > \sigma_1$ adjacent to e_1. Let us consider an edge e which is not connected to e_1. At first we apply Lemma 29 to the couple of edges (e_1, e_2) so that we get $w(e) \geq \sigma_2$. Let us assume by contradiction $w(e) > \sigma_2$. Then we apply Lemma 13 to the couple of edges (e_1, e_2) (we have $w(e) > \sigma_2 = \max(w_1, w_2)$), so that e has to be linked to 2 by an edge of weight σ_2, a contradiction. Hence $w(e) = \sigma_2$. □

Remark 9. For a given weighted graph inheritance of convexity can be verified in polynomial time. By Theorem 27 (resp. 24) in the case of three different edge-weights (resp. two different edge-weights with at least two edges of minimal weight), we have to verify the cycle-completeness condition as for Myerson game. More precisely, for two adjacent edges $e_1 = \{1, 2\}$, $e_2 = \{1, 3\}$ we have to see if the vertices 2 and 3 are connected in the graph $(N, E \setminus \{e_1, e_2\})$ and if 2 and 3 are connected in this graph that $\{2, 3\} \in E$. Now by Theorem 25 in the case of two different weights and only one edge $e_1 = \{1, 2\}$ of minimum weight σ_1, for each couple of adjacent edges $e'_1 = \{1', 2'\}$, $e'_2 = \{1', 3'\}$ both of weight σ_2, we have to see if the vertices $2'$ and $3'$ are connected in the graph $(N, E \setminus \{e_1, e'_1, e'_2\})$ and if $2'$ and $3'$ are connected in this graph that either the edge $\{2', 3'\} \in E$ or that the vertices $1'$, $2'$, $3'$ are connected to the same vertex 1 or 2 by an edge. The proof that it is sufficient can be made by induction on the size $|C|$ of a cycle C with constant weight σ_2.

References

Algaba, E. (1998). *Extensión de juegos definidos en sistemas de conjuntos.* PhD thesis, Univ. of Seville.

Algaba, E., Bilbao, J., Borm, P., and Lopez, J. (2000). The position value for union stable systems. *Mathematical Methods of Operations Research*, 52:221–236.

Algaba, E., Bilbao, J., and Lopez, J. (2001). A unified approach to restricted games. *Theory and Decision*, 50(4):333–345.

Bilbao, J. M. (2000). *Cooperative games on combinatorial structures.* Kluwer Academic Publishers, Boston.

Bilbao, J. M. (2003). Cooperative games under augmenting systems. *SIAM Journal on Discrete Mathematics*, 17(1):122–133.

Edmonds, J. and Giles, R. (1977). A min-max relation for submodular functions on graphs. *Annals of Discrete Mathematics*, 1:185–204.

Faigle, U. (1989). Cores of games with restricted cooperation. *ZOR - Methods and Models of Operations Research*, 33(6):405–422.

Faigle, U., Grabisch, M., and Heyne, M. (2010). Monge extensions of cooperation and communication structures. *European Journal of Operational Research*, 206(1):104–110.

Fujishige, S. (2005). *Submodular Functions and Optimization*, volume 58 of *Annals of Discrete Mathematics.* Elsevier, second edition.

Grabisch, M. (2013). The core of games on ordered structures and graphs. *Annals of Operations Research*, 204(1):33–64.

Grabisch, M. and Skoda, A. (2012). Games induced by the partitioning of a graph. *Annals of Operations Research*, 201(1):229–249.

Myerson, R. (1977). Graphs and cooperation in games. *Mathematics of Operations Research*, 2(3):225–229.

Skoda, A. (2016). Convexity of Network Restricted Games Induced by Minimum Partitions. Documents de travail du Centre d'Economie de la Sorbonne 2016.19 - ISSN : 1955-611X.

van den Nouweland, A. and Borm, P. (1991). On the convexity of communication games. *International Journal of Game Theory*, 19(4):421–30.