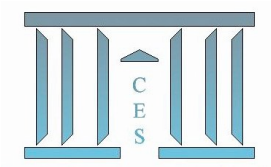
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Maison des Sciences Économiques, 106-112 boulevard de L'Hôpital, 75647 Paris Cedex 13
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Axiomatization of an importance index for Generalized Additive Independence models

Mustapha Ridaoui[1], Michel Grabisch[1], Christophe Labreuche[2]

[1] Paris School of Economics, Université Paris I - Panthéon-Sorbonne, Paris, France
{mustapha.ridaoui,michel.grabisch}@univ-paris1.fr
[2] Thales Research & Technology, Palaiseau, France
christophe.labreuche@thalesgroup.com

Abstract. We consider MultiCriteria Decision Analysis models which are defined over discrete attributes, taking a finite number of values. We do not assume that the model is monotonically increasing with respect to the attributes values. Our aim is to define an importance index for such general models, encompassing Generalized-Additive Independence models as particular cases. They can be seen as being equivalent to k-ary games (multichoice games). We show that classical solutions like the Shapley value are not suitable for such models, essentially because of the efficiency axiom which does not make sense in this context. We propose an importance index which is a kind of average variation of the model along the attributes. We give an axiomatic characterization of it.

Keywords: MultiCriteria Decision Analysis · k-ary game · Shapley value.

1 Introduction

In MultiCriteria Decision Analysis (MCDA), a central question is to determine the importance of attributes or criteria. Suppose the preference of a decision maker has been represented by a numerical model. For interpretation and explanation purpose of the model, a basic requirement is to be able to assess the importance of each attribute. If this is easy for a number of elementary models (essentially additive ones), it becomes more challenging with complex models.

For models based on the Choquet integral w.r.t. a capacity or fuzzy measure (see a survey in [8]), it has been recognized since a long time ago that the Shapley value [15], a concept borrowed from game theory, is the adequate tool to quantify the importance of attributes.

Choquet integral-based models belong to the category of decomposable models, that is, where utility functions are defined on each attribute, and then are aggregated by some increasing function. In this paper, we depart from this kind of models and focus on models where there is no such separation of utilities among the attributes. Typically, the Generalized Additive Independence (GAI) model proposed by Fishburn [4, 5] is of this type, since of the form $U(x) = \sum_{S \in \mathcal{S}} u_S(x_S)$, where $\mathcal{S}$ is a collection of subsets of N, the index set of all attributes. In this paper, however, we do not take advantage of this peculiar form, and consider a numerical model without particular properties, except that the underlying attributes are discrete, and thus take a finite number of values. Note

that in many applications, especially in the AI field, this is the case, in particular for GAI models [1, 6, 2].

As far as we know, the question of the definition of an importance index for such a general case remains open. As we will explain, discrete models can be seen as k-ary capacities or more generally k-ary games [9] (also called multichoice games [11]), and thus it seems natural to take as importance index the various definitions of Shapley-like values for multichoice games existing in the literature. There is however a major drawback inherent to these values: they all satisfy the efficiency axiom, that is, the sum of the importance indices over all attributes is equal to $v(k_N)$, the value of the game when all attributes take the highest value. If this axiom is natural in a context of cooperative game, where the Shapley value defines a rational way to share among the players the total benefit $v(k_N)$ of the game, it has no justification in MCDA, especially if the model v is not monotone increasing.

The approach we propose here is inspired by the calculus of variations: we define the importance index of an attribute as the average variation of v (depicting the satisfaction of the decision maker) when the value of attribute i is increased by one unit. We propose an axiomatic definition, where the chosen axioms are close to those of the original Shapley value.

Section 2 recalls the basic concepts. Section 3 informally defines what is the aim of our importance index. The axiomatic characterization is presented in Section 4. The new index is then interpreted (Section 5).

2 Preliminaries

Throughout the paper, $N = \{1, \ldots, n\}$ is a finite set which can be thought as the set of attributes (in MCDA), players (in cooperative game theory), etc., depending on the application. In this paper, we will mainly focus on MCDA applications. Cardinality of sets will be often denoted by corresponding lower case letters, e.g., n for $|N|$, s for $|S|$, etc.

The set of all possible values taken by attribute $i \in N$ is denoted by L_i. As it is often the case in MCDA, we assume that these sets are finite, and we represent them by integer values, i.e., $L_i = \{0, 1, \ldots, k^i\}$. Alternatives are thus elements of the Cartesian product $L = \times_{i \in N} L_i$ and take the form $x = (x_1, x_2, \ldots, x_n)$ with $x_i \in L_i$, $i = 1, \ldots, n$. For $x, y \in L$, we write $x \leq y$ if $x_i \leq y_i$ for every $i \in N$. For $S \subseteq N$ and $x \in L$, x_S is the restriction of x to S. L_{-i} is a shorthand for $\times_{j \neq i} L_j$. For each $y_{-i} \in L_{-i}$, and any $\ell \in L_i$, (y_{-i}, ℓ_i) denotes the combined alternative x such that $x_i = \ell_i$ and $x_j = y_j, \forall j \neq i$. The vector $0_N = (0, \ldots, 0)$ is the null alternative of L, and $k_N = (k_1^1, \ldots, k_n^n)$ is the top element of L. 0_{-i} denotes the element of L_{-i} in which all coordinates are zero. We call vertex of L any element $x \in L$ such that x_i is either 0 or k^i, for each $i \in N$. We denote by $\Gamma(L) = \times_{i \in N} \{0, k^i\}$ the set of vertices of L. For each $x \in L$, we denote by $S(x) = \{i \in N \mid x_i > 0\}$ the support of x, and by $K(x) = \{i \in N \mid x_i = k^i\}$ the kernel of x. Their cardinalities are respectively denoted by $s(x)$ and $k(x)$. We call *elementary cell* of L a unit cell $\times_{i \in N} \{x_i, x_i + 1\}$, for $x_i \in L_i \setminus k^i$.

We suppose to have a numerical representation $v : L \to \mathbb{R}$ of the preference of the decision maker (DM) over the set of alternatives in L. For the sake of generality, we do not make any assumption on v, except that $v(0_N) = 0$ (this is not a restriction, as most of numerical representations are unique up to a positive affine transformation). In particular, there is no assumption of monotonicity, that is, we do not assume that $x_1 \geq x'_1, \ldots, x_n \geq x'_n$ implies $v(x_1, \ldots, x_n) \geq v(x'_1, \ldots, x'_n)$. Example 1 below illustrates that it is quite common to observe this lack of monotonicity.

Example 1. The level of comfort of humans depends on three main attributes: temperature of the air (X_1), humidity of the air (X_2) and velocity of the air (X_3). Then $v(x_1, x_2, x_3)$ measures the comfort level. One can readily see that v is not monotone in its three arguments. For x_2 and x_3 fixed, v is maximal for intermediate values of the temperature (typically around 23°C). Similarly, the value of humidity maximizing v is neither too low nor too high. Finally, for x_1 relatively large, some wind is well appreciated, but not too much. Hence for any i, and supposing the other two attributes being fixed, there exists an optimal value $\widehat{\ell_i} \in L_i$ such that v is increasing in x_i below $\widehat{\ell_i}$, and then decreasing in x_i above $\widehat{\ell_i}$.

Although we will not use this specific form for v in the sequel, we mention as typical example of a model not necessarily satisfying monotonicity the Generalized Additive Independence (GAI) model, i.e., v is written as $v(x) = \sum_{S \in \mathcal{S}} v_S(x_S)$, where $\mathcal{S}$ is a collection of subsets of N [4, 5]. This model has been widely used in AI [1, 6, 2].

For convenience, we assume from now on that all attributes have the same number of elements, i.e., $k^i = k$ for every $i \in N$ ($k \in \mathbb{N}$). Note that if this is not the case, k is set to $\max_{i \in N} k^i$, and we duplicate some elements of L_i when $k^i < k$. A fundamental observation is that when $k = 1$, v is nothing other than a *pseudo-Boolean function* $v : \{0, 1\}^N \to \mathbb{R}$ vanishing on 0_N, or put otherwise via the identity between sets and their characteristic functions, a *(cooperative) game* (in characteristic form) $\mu : 2^N \to \mathbb{R}$, with $\mu(\varnothing) = 0$. A game v is *monotone* if $v(A) \leq v(B)$ whenever $A \subseteq B$. A monotonic game is called a *capacity* [3] or *fuzzy measure* [16]. For the general case $k \geq 1$, $v : L \to \mathbb{R}$ is called a *multichoice game* or *k-ary game* [11], and the numbers $0, 1, \ldots, k$ in L_i are seen as the level of activity of the players. By analogy with the classical case $k = 1$, a *k-ary capacity* is a monotone k-ary game, i.e., satisfying $v(x) \leq v(y)$ whenever $x \leq y$, for each $x, y \in L$ [9]. Hence, a k-ary capacity represents a preference on L which is increasing with the value of the attributes. We denote by $\mathcal{G}(L) = \{v : L \to \mathbb{R}, v(0_N) = 0\}$ the set of functions defined on L vanishing on 0_N. A function is called *single-peaked* if it has only one local maximum and no troughs.

By analogy with classical games, a unanimity game for k-ary game denoted u_x, for each $x \in L$ with $x \neq 0_N$ is defined by

$$u_x(y) = \begin{cases} 1, \text{ if } y \geq x \\ 0, \text{ otherwise} \end{cases}$$

Note that the set of unanimity games forms a basis of the vector space of k-ary games. One advantage of unanimity games is that they are monotone. Hence this basis is relevant for k-ary capacities. In order to obtain a basis of k-ary games not necessarily made

of monotone functions, we define for each $x \in L$ such that $x \neq 0_N$, the game δ_x by

$$\delta_x(y) = \begin{cases} 1, \text{ if } y = x \\ 0, \text{ otherwise} \end{cases}$$

It is obvious that any k-ary game v can be written as

$$v = \sum_{\substack{x \in L \\ x \neq 0_N}} v(x)\delta_x. \tag{1}$$

3 Definition of an importance index: what do we aim at doing?

We restrict ourselves to the MCDA setting and interpretation in this paper. When dealing with numerical representations of preference in MCDA, one of the primary concerns is to give an interpretation of the model in terms of importance of the attributes. When v is a capacity or a game ($k = 1$), or with continuous models extending capacities and games like the Choquet integral, the standard solution is to take the Shapley value, introduced by Shapley in the context of cooperative games [15]. A *value* is a function $\phi : \mathcal{G}(2^N) \to \mathbb{R}^N$ that assigns to every game μ a payoff vector $\phi(\mu)$. It is interpreted in the MCDA context as the vector of importance of the attributes. The value introduced by Shapley is one of the most popular, and is defined by:

$$\phi_i^{Sh}(\mu) = \sum_{S \subseteq N \setminus i} \frac{(n-s-1)!s!}{n!} \big(\mu(S \cup i) - \mu(S)\big), \forall i \in N. \tag{2}$$

A standard property shared by many values in the literature is *efficiency*: $\sum_{i \in N} \phi_i(\mu) = \mu(N)$. This property is very natural in game theory, as $\mu(N)$ is the total benefit obtained from the cooperation of all players in N, and by efficiency the payoff vector $\phi(v)$ represents a sharing of this total benefit.

If the Shapley value has been widely used in MCDA with great success (see, e.g., [8]), it must be stressed that it was only in the case of monotonically increasing models, i.e., based on a capacity μ. In such cases, $\mu(N)$ is set to 1, the value of the best possible alternative, and the importance index of an attribute could be seen as a kind of contribution of that attribute to the best possible alternative. However if the model is not monotone increasing, such an interpretation fails. Hence, we are facing here a double difficulty: to propose a "value" both valid for $k \geq 1$ and for nonincreasing models. There have been many proposed values for multichoice games, e.g., Hsiao and Raghavan [11], van den Nouweland et al. [17], Klijn et al. [12], Peters and Zank [14] and Grabisch and Lange [10], etc. All of them satisfy the classical efficiency axiom.

We wish to capture in our importance index the impact of each attribute on the overall utility. Let us consider for illustration function δ_y with $k = 2$, $n = 3$ and $y = (2, 1, 1)$. Attribute 1 is non-decreasing and has a positive impact on the overall utility. Attribute 2 has neither a positive nor a negative impact on the overall utility, since $\delta_y(x)$ is going from value 0 (at $x_2 = 0$) to 1 (at $x_2 = 1$), and then decreasing to value 0 again (at $x_2 = 2$). Hence attribute 2 has globally neither a positive nor a negative

impact. The same holds with attribute 3. Hence, denoting by $\phi(\delta_y)$ our importance index for that function, one shall have $\phi_1(\delta_y) > 0$, $\phi_2(\delta_y) = \phi_3(\delta_y) = 0$, so that the sum $\sum_{i \in N} \phi_i(v) > 0$ cannot be equal to $v(2,2,2) - v(0,0,0) = 0$, and hence ϕ does not satisfy efficiency. Rather, the index $\phi_i(v)$ shall measure the impact of attribute i on v, as the total variation on v if we increase the value of attribute i of one unit (going from value x_i to $x_i + 1$), when x is varying over the domain.

4 Axiomatization

We define in this section an importance index according to the ideas explained above, by using an axiomatic description. Our first three axioms are the same as those used by Shapley when characterizing his value in [15]: linearity, null player and symmetry. Our approach will follow Weber [18], who introduces the axioms one by one and at each step gives a characterization. Throughout this section, we consider a value as a mapping $\phi : \mathcal{G}(L) \to \mathbb{R}$.

We expect our importance index to have an exponential complexity of computation, as for the Shapley value. In order to reduce the complexity burden for some functions v having some particular properties, we assume that for GAI models, the value can be decomposed over each utility function v_S. Hence ϕ shall be additive: $\phi_i(v + v') = \phi_i(v) + \phi_i(v')$. Moreover, utilities are in MCDA invariant to positive linear transformations. Hence ϕ_i shall be homogeneous: $\phi_i(\alpha v) = \alpha \phi_i(v)$ for every $\alpha > 0$. Lastly the value shall be symmetric: $\phi_i(-v) = -\phi_i(v)$. The previous three properties yield linearity axiom **L**.

Linearity axiom (L) : ϕ is linear on $\mathcal{G}(L)$, i.e., $\forall v, w \in \mathcal{G}(L), \forall \alpha \in \mathbb{R}$,

$$\phi_i(v + \alpha w) = \phi(v) + \alpha \phi(w).$$

Proposition 1. *Under axiom **(L)**, for all $i \in N$, there exists constants $a_x^i \in \mathbb{R}$, for all $x \in L$, such that $\forall v \in \mathcal{G}(L)$,*

$$\phi_i(v) = \sum_{x \in L} a_x^i v(x). \tag{3}$$

The proof of this result and the other ones are omitted due to space limitation.

The second axiom that characterizes the Shapley value in [18] is called the null player axiom. It says that a player $i \in N$ who brings no contribution (i.e., $\mu(S \cup i) = \mu(S), \forall S \subseteq N \backslash \{i\}$) should receive a zero payoff. This definition can be easily extended to $v \in \mathcal{G}(L)$ and to a MCDA setting.

Definition 1. *A criterion $i \in N$ is said to be null for $v \in \mathcal{G}(L)$ if*

$$v(x + 1_i) = v(x), \forall x \in L, x_i < k.$$

Remark 1. Let $i \in N$ be a null criterion for $v \in \mathcal{G}(L)$. we have,

$$\forall x \in L, v(x_{-i}, x_i) = v(x_{-i}, 0_i).$$

If an attribute is null w.r.t. a function $v \in \mathcal{G}(L)$, then this attribute has no influence on v, and hence the importance of this attribute shall be zero. We propose the following axiom.

Null axiom (N): If a criterion i is null for $v \in \mathcal{G}(L)$, then $\phi_i(v) = 0$.

Proposition 2. *Under axioms (N) and (L), for all $i \in N$, there exists $b_x^i \in \mathbb{R}$, for all $x \in L$ with $x_i < k$, such that $\forall v \in \mathcal{G}(L)$,*

$$\phi_i(v) = \sum_{\substack{x \in L \\ x_i < k}} b_x^i \big(v(x + 1_i) - v(x)\big). \tag{4}$$

This proposition shows that ϕ_i is a linear combination of the added-values on v, going from value x_i to $x_i + 1$, over all x.

The classical symmetry axiom says that the numbering of the attributes has no influence on the value. It means that the computation of value should not depend on the numbering of the attributes.

Let σ be a permutation on N. For all $x \in L$, we denote $\sigma(x)_{\sigma(i)} = x_i$. For all $v \in \mathcal{G}(L)$, The function $\sigma \circ v$ is defined by $\sigma \circ v(\sigma(x)) = v(x)$.

Symmetry axiom (S): For any permutation σ of N,

$$\phi_{\sigma(i)}(\sigma \circ v) = \phi_i(v), \forall i \in N.$$

Proposition 3. *Under axioms (N), (L) and (S), $\forall v \in \mathcal{G}(L), \forall i \in N$,*

$$\phi_i(v) = \sum_{\substack{x \in L \\ x_i < k}} b_{x_i; n_0, \ldots, n_k} \big(v(x + 1_i) - v(x)\big),$$

where $b_{x_i; n_0, \ldots, n_k} \in \mathbb{R}$, and n_j is the number of components of x_{-i} being equal to j.

This result means that the coefficients in front of the added-values on v, going from value x_i to $x_i + 1$, do not depend on the precise value of x, but only on the number of terms of x_{-i} taking values $0, 1, \ldots, k$.

Let us take for example $N = \{1, 2, 3\}$, $k = 2$ and $x = (0, 2, 2)$. We have,

$$b_x^1 = b_{0;0,0,2}, b_x^2 = b_{2;1,0,1}, \text{ and } b_x^3 = b_{2;1,0,1}.$$

The next axiom enables an easier computation of coefficients b_x^i while reducing their number.

Invariance axiom (I): Let us consider two functions $v, w \in \mathcal{G}(L)$ such that, for all $i \in N$,

$$v(x + 1_i) - v(x) = w(x) - w(x - 1_i), \forall x \in L, x_i \notin \{0, k\}$$
$$v(x_{-i}, 1_i) - v(x_{-i}, 0_i) = w(x_{-i}, k_i) - w(x_{-i}, k_i - 1), \forall x_{-i} \in L_{-i}.$$

Then $\phi_i(v) = \phi_i(w)$.

Taking two functions v and w for which the differences $v(x+1_i)-v(x)$ (measuring the added value of improving x of one unit on attribute i) can be deduced from that of w just by shifting of one unit, then the mean importance of attribute i shall be the same for v and w. In other words, what is essential is the absolute value of the differences $v(x+1_i)-v(x)$ and not the value x at which it occurs.

Proposition 4. *Under axioms **(L)**, **(N)** and **(I)**, $\forall v \in \mathcal{G}(L), \forall i \in N$,*

$$\phi_i(v) = \sum_{x_{-i} \in L_{-i}} b^i_{x_{-i}}\big(v(x_{-i}, k_i) - v(x_{-i}, 0_i)\big).$$

Axiom **(I)** implies that we only need to look at the difference of v between the extreme value 0 and k. The evaluation on the intermediate elements of L_i do not count.

Proposition 5. *Under axioms **(L)**, **(N)**, **(I)** and **(S)**, $\forall v \in \mathcal{G}(L), \forall i \in N$,*

$$\phi_i(v) = \sum_{x_{-i} \in L_{-i}} b_{n(x_{-i})}\big(v(x_{-i}, k_i) - v(x_{-i}, 0_i)\big),$$

where $n(x_{-i}) = (n_0, n_1, \ldots, n_k)$ with n_j the number of components of x_{-i} being equal to j.

As explained in Section 3, we do not require that ϕ satisfies efficiency. In the context of game theory, $\phi_i^{Sh}(\mu)$ is the amount of money alloted to player i, so that relation $\sum_{i\in N} \phi_i^{Sh}(\mu) = \mu(N)$ means that all players share among themselves the total worth $\mu(N)$. We have no such interpretation in MCDA. By contrast, we interpret $\phi_i(v)$ as an overall added value when increasing the value of attribute i of one unit – thereby going from any point x to $(x_i + 1, x_{-i})$. Hence $\sum_{i\in N} \phi_i(v)$ can be interpreted as the overall added value when increasing simultaneously the value of all attributes of one unit – thereby going from any point x to $x + 1 = (x_1 + 1, \ldots, x_n + 1)$. For an arbitrary function v, there is a priori no particular property for the previous sum. We thus consider a very special case of functions following Example 1. These functions are single peaked. The simplest version of these functions is the family of functions δ_y. For those functions, we immediately see from Proposition 5 that $\phi_i(\delta_y) = 0$ for every i such that $y_i \neq 0, k$, as already mentioned in Section 3. Based on this remark, we should only bother on attributes which are equal to either 0 or k in y. We have therefore three cases (recall that $s(y), k(y)$ are the cardinalities of the support and kernel of y):

- $k(y) \neq 0$ and $s(y) = n$. Then $y - 1 \in L$ because no component of y is equal to 0, and we have $\delta_y(y) - \delta_y(y-1) = 1$. Note that $\delta_y(x+1) - \delta_y(x) = 0$ for any $x \neq y - 1$ and $x, x + 1 \in L$. Therefore, by the above argument, we have $\sum_{i\in N} \phi_i(\delta_y) = 1$.
- $k(y) = 0$ and $s(y) < n$. This is the dual situation: $y+1 \in L$ because no component is equal to k, and we have $\delta_y(y+1) - \delta_y(y) = -1$. Since $\delta_y(x+1) - \delta_y(x) = 0$ for any other possible x, we get $\sum_{i\in N} \phi_i(\delta_y) = -1$.
- $k(y) \neq 0$ and $s(y) < n$. This time there are both components equal to 0 and to k in y. Therefore, neither $y + 1$ nor $y - 1$ belong to L, and for any possible $x \in L$ s.t. $x + 1 \in L$, we have $\delta_y(x+1) - \delta_y(x) = 0$. Therefore, $\sum_{i\in N} \phi_i(\delta_y) = 0$.

To summarize, we shall write

$$\sum_{i\in N}\phi_i(\delta_y)=\begin{cases}+1 & \text{if } k(y)\neq 0 \text{ and } s(y)=n\\ -1 & \text{if } k(y)=0 \text{ and } s(y)<n\\ 0 & \text{else}\end{cases}$$

This can be written in the following compact form.

Restricted Efficiency axiom (RE): For all $x\in L\setminus\{0_N\}$,

$$\sum_{i\in N}\phi_i(\delta_x)=\delta_x(x_{-i},k_i)-\delta_x(x_{-j},0_j)$$

where, $i=argmax\ x$ and $j=argmin\ x$

Note that the previous formula takes the form of standard efficiency $\sum_{i\in N}\phi_i^{Sh}(\mu)=\mu(N)-\mu(\emptyset)$. The final result is the following.

Theorem 1. *Under axioms **(L), (N), (I), (S)** and **(RE)**, for all $v\in\mathcal{G}(L)$*

$$\phi_i(v)=\sum_{x_{-i}\in L_{-i}}\frac{(n-s(x_{-i})-1)!k(x_{-i})!}{(n+k(x_{-i})-s(x_{-i}))!}\big(v(x_{-i},k_i)-v(x_{-i},0_i)\big),\forall i\in N$$

Our axiomatic characterization has in common with the Shapley value the satisfaction of axioms **(L)**, **(N)** and **(S)**. The other axioms **(I)** and **(RE)** are different.

We note that we have the following relation, for every $v\in\mathcal{G}(L)$

$$\sum_{i\in N}\phi_i(v)=\sum_{\substack{x\in L\\ x_j<k}}\big(v(x+1)-v(x)\big).$$

The right-hand side of this expression corresponds exactly to the interpretation provided above saying that $\sum_{i\in N}\phi_i(v)$ is the overall impact of going from any point x to $x+1$.

We apply Theorem 1 with the following example.

Example 2. Let $x\in L\setminus\{0_N\}$ and $i\in N$.

• The computation of ϕ_i w.r.t. δ_x gives,

$$\phi_i(\delta_{x_{-i},0})=-\phi_i(\delta_{x_{-i},k})=\frac{(n-s(x_{-i})-1)!k(x_{-i})!}{(n+k(x_{-i})-s(x_{-i}))!},\text{ and }\phi_i(\delta_{x_{-i},x_i})=0,\forall x_i\notin\{0,k\}.$$

• The computation of ϕ_i w.r.t. u_x gives,

$$\phi_i(u_x)=\sum_{\substack{y_{-i}\in L_{-i}\\ y_{-i}\geq x_{-i}}}\frac{(n-s(x_{-i})-1)!k(x_{-i})!}{(n+k(x_{-i})-s(x_{-i}))!}.$$

Let us take for exemple $N=\{1,2,3\}, k=3$, and $x=(2,0,3)$, we have,

$$\phi_1(\delta_x)=0,\phi_2(\delta_x)=-\frac{1}{2},\phi_3(\delta_x)=\frac{1}{2},$$
$$\phi_1(u_x)=\frac{3}{2},\phi_2(u_x)=0,\phi_3(u_x)=\frac{9}{2}.$$

We note that $\phi_i(\delta_x)$ can be nul when the peak is attained inside the domain ($i = 1$), is strictly positive if the peak is attained at the maximal value k ($i = 3$), and is strictly negative if the peak is attained at 0 ($i = 2$). On the other hand, $\phi_i(u_x)$ is always non-negative as the unanimity function is non-decreasing.

5 Interpretation

We propose here an interpretation of ϕ in continuous spaces, that is, after extending v to the continuous domain $[0, k]^N$. We consider thus a function $U : [0, k]^N \to \mathbb{R}$ which extends v: $U(x) = v(x)$ for every $x \in L$. The importance of attribute i can be defined as (see [13, proposition 5.3.3 page 141])

$$Imp_i(U) = \int_{[0,k]^{n-1}} \Big(U(k_i, z_{-i}) - U(0_i, z_{-i})\Big) dz_{-i} = \int_{[0,k]^n} \frac{\partial U}{\partial z_i}(z)\, dz.$$

In this formula, the local importance of attribute i for function U at point z is equal to $\frac{\partial U}{\partial z_i}(z)$. The index $Imp_i(U)$ appears as the mean of relative amplitude of the range of U w.r.t. attribute i, when the remaining variables take uniformly random values.

The most usual extension of v on $[0, k]^N$ is the Choquet integral with respect to k-ary capacities [7]. Let us compute Imp_i in this case. We write $Imp_i(U) = \sum_{x \in \{0,\ldots,k-1\}^N} \int_{[x,x+1]^n} \frac{\partial U}{\partial z_i}(z)\, dz$. In $[x, x + 1]^n$, U is equal to $v(x)$ plus the Choquet integral C_{μ_x} w.r.t. capacity μ_x defined by $\mu_x(S) = v((x + 1)_S, x_{-S}) - v(x)$ for every $S \subseteq N$. By [13], $\int_{[0,1]^n} \frac{\partial C_{\mu_x}}{\partial z_i}(z)\, dz = \phi_i^{Sh}(\mu_x)$. Hence

$$Imp_i(U) = \sum_{x \in \{0,\ldots,k-1\}^N} \phi_i^{Sh}(\mu_x). \tag{5}$$

We then obtain the following result.

Lemma 1. *If U is the Choquet integral w.r.t. k-ary capacity v, then $Imp_i(U) = \phi_i(v)$.*

Hence the counterpart of ϕ_i on continuous domains is the integrated local importance.

6 Conclusion and related works

We have proposed a new importance index for GAI models. It quantifies the impact of each attribute on the overall utility. According to the linearity, null criterion, symmetry and invariance properties, $\phi_i(v)$ takes the form of the sum over $x \in \{0, \ldots, k - 1\}^N$ of a value over the restriction of functions v on $\times_{i \in N}\{x_i, x_i + 1\}$ (see (5)). In our construction, the value at an elementary cell $\times_{i \in N}\{x_i, x_i + 1\}$ corresponds to the usual Shapley value.

We will explore in future work the possibility of the use of other values such as the Banzhaf value. We will also investigate other indices ϕ_i which measure the impact in absolute value of attribute i. In this case, $\phi_i(\delta_y)$ is not equal to zero when $0 < y_i < k$.

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