Placing Sophie Germain within number-theoretical practices of the19th century

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Placing Sophie Germain within number-theoretical practices of the 19th century

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In the history of mathematics, Sophie Germain (1776-1831) is recognized for being a woman who succeeded in producing work on the theory of elasticity and number theory. Her work was appreciated and discussed by some of the most eminent geometers of her time. Several aspects of her life have contributed to her image as an iconic symbol for women in mathematics in the 19th and 20th centuries. She was a self-taught mathematician because she could not attend the new *École polytechnique* and her parents refused, at least initially, to let her learn mathematics. She stayed single her whole life and her mathematical contribution was not linked to a husband or a male member of her family. She explicitly refused the status of “femme savante” and she surrounded herself with male mathematicians only. She was the first woman to obtain the Academy of Sciences prize and attend its sessions as a mathematician. She achieved significant results on Fermat's Last Theorem (FLT), which went almost unnoticed until the early 2000s.

In this paper, I analyse some of these features by focusing on her contribution to number theory in the context of mathematical practices and the social positions of the mathematicians of her time. Even if she published only one short note on number theory in 1831 and one of her contributions — the so-called “Germain theorem” — was integrated in [8], recent analyses of her correspondance with Carl Friedrich Gauss and her manuscripts highlight that her contribution to number theory far exceeds what appears in the publications mentioned above [4, 7].

Let us begin by giving a panorama of number theory from the 1800s to the 1830s.The early nineteenth century was a hinge period for publications. Indeed, in

1805, the available media for mathematical publication for geometers were few and far between. Academic periodicals were difficult to access for non-*académiciens* and scholarly journals such as the *Journal des savants* did not contain any articles on number theory. Books were also expensive to publish and sales depended on limited specialised sellers. For example, the number of publications related to congruences, introduced by Gauss in 1801 in his *Disquisitiones arithmeticae* (*D. A.*) was limited (30 texts between 1801 and 1825). Publications increased from 1825 when new mathematical and scientific journals were created (224 texts between 1826 and 1850, of which only 17 were not included in periodicals) [1]. In fact, Germain was the only woman to publish on number theory in the first half of the 19th century.

At the turn of the 19th century, two treatises on number theory appeared: Adrien-Marie Legendre's *Essai sur la théorie des nombres* [9] and Gauss's *D. A.* [5]. Three points should be highlighted here. Legendre and Gauss had divergent opinions on the definition of number theory. Legendre identified number theory with indeterminate analysis. Gauss explicitly distinguished between these two domains, proposing number theory as being the domain where integer and rational numbers are considered, and not limited to equations. In his book, Gauss gave a coherent presentation of number theory by organizing it around two fundamental objects : congruences and quadratic forms. He gave two different proofs of quadratic reciprocity law and a method to resolve the binomial equation $x^p = 1$ algebraically by reindexing the roots with a primitive root of p, insisting on the links existing between different parts of his work and different mathematical domains, such as algebra and number theory. At the time, French teaching programs were focused on engineering, especially with the *École polytechnique*, and number theory was not taught at all. That is why if someone, male or female, wanted to study this domain, he or she had to read former publications , and especially Legendre's and Gauss's books.

Apart from several memoirs on Gauss sums, reciprocity laws and complex integers published by Gauss after 1801, the *D. A.* were mostly used for the algebraic resolution of binomial equations before 1825. Between the 1820s and the 1860s, new scholars read Gauss' D. A. and published arithmetical papers linked to it. In addition, progress in other mathematical areas, such as the use of complex numbers, Fourier analysis or elliptic functions, were used in number theory. A research domain, called *Arithmetic Algebraic Analysis* [6], was then developed by an international network of scholars. But, at Germain's time, the use of analysis in number theory was marginal and Germain's potential weakness in analysis did not constitute a significant limitation. Analysis was taught at the *École polytechnique*, and Germain's weakness in this domain seemed to have been a cause of the errors contained in her early work on the theory of elasticity [3]. Amoung French number-theoretic production, there was multiform activity based on a strong link between equations and congruences. Specific problems were discussed such as the imaginary roots of congruences (Louis Poinsot, Victor-Amédée Lebesgue, Évariste Galois, Germain), the number of integer roots of a congruence (Guglielmo Libri, Lebesgue) or Fermat's Last Theorem (Legendre, Libri, Germain) [2]. These publications had common roots with Joseph-Louis Lagrange's and Legendre's arithmetical approach and integrated Gauss's objects and methods in a more or less important way.

Germain was precisely one of the first geometers who mastered the contents of Gauss's *D. A.*, as Gauss observed in his correspondance, and who applied the theory of congruences to her number-theoretical work. Moreover, after she impressed Lagrange with her mathematical skills, she became progressively close to geometers such as Gauss, Legendre, Cauchy, Poinsot or Libri, who are authors who published on number theory at her time. So if, as a woman, she could not be taught or attend scientific institutions, the marginal status of number theory at the beginning of the 19th century and the fact that it was not taught in *Polytechnique* for example, meant that her gender stigmatized her less than other mathematical domains. Every geometer wishing to study number theory had to study the same texts as Germain (Gauss' and Legendre's writings mostly) and the few enthusiasts in number theory certainly allowed her to have privileged contact with Gauss.

Gauss admired the arithmetic skills of Germain very much and seemed to consider her a full-fledged colleague, although he never had much time to pursue his arithmetical research or to write to her at greater length. From her manuscripts and her letters to Gauss, Libri and Poinsot, we know that she studied and wrote to Gauss about quadratic, cubic and biquadratic residues, quadratic forms, cyclotomy and FLT. She proposed new proofs of results, tentative generalisations of methods and theorems contained in the *D. A.*. She also developed conjectures and a tentative program to prove FLT, from her first letter to Gauss then in her ninth, in 1819. She tried to construct a proof for whole families of exponents — contrary to Legendre or Dirichlet who obtained proofs for a single exponent — and she imagined a general plan to prove the FLT in general. In her plan, congruences and roots of unity were fundamental — the consideration of the congruence $x^p + y^p \equiv z^p \pmod{\theta}$, where θ is prime, is central for example — and she used her precise knowledge of Gauss's work. In 1819, she also used Poinsot's work to highlight the importance of the ordered way the residues are distributed. But, as Germain soon observed, her plan could not succeed. Nevertheless she showed really impressive skills in calculations in her work, obtained general results on FLT and managed to show that the potential solutions of certain cases should be very big [7]. The only times that Gauss replied to her with some number-theoretical developments was on cyclotomy (to explain one error she made in her previous letter), and on residues (by giving her two theorems to prove). He never made any comment on her proposals for FLT. Gauss was not interested in this "isolated proposition" as he wrote to Olbers in 1816. These different facts underline several interesting points regarding Germain's status in number theory at that time. The content of her work lay both in the line of arithmetic work on indeterminate equations or based on an analogy between equations and congruences published at her time (Legendre, Poinsot, Libri), in keeping with the *D. A.* She also had access to some of the latest arithmetical productions, that were yet to be published.

For example, she received Poinsot's memoir before its publication in the *Journal de l'École polytechnique* [10] and I found notes on one of Gabriel Lamé's paper that were never published in her manuscripts - and that she studied with great attention.

She never directly published her results on FLT. Maybe this was because she knew that she did not succeed in her grand plan. As a woman, she did not have access to some institutions that made it easier to publish mathematical productions. Beyond gender, any male or female mathematician of the time had limited possibilities of publishing an article on number theory in the 1820s, or even in the 1830s. Indeed, another mathematician of Germain's time, Lebesgue, whose work was mainly concerned with number theory but who was neither a polytechnician nor an *académicien*, was only able to regularly publish his arithmetical memoirs from 1836, in the *Comptes-rendus de l'Académie des sciences* and in the *Journal de mathématiques pures et appliquées*, respectively created in 1835 and 1836.

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Emmy Noether, the Thought Space of the Noether School and the Change of Mathematical Thinking: About Thought Styles, Thought Collectives, and Mathematical Productivity

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Much has been written about Emmy Noether (1882-1935), but little about the Noether School - a gap in the history of mathematics that needed to be filled, given that Noether and the school she formed have contributed substantively to the introduction of new approaches and methodological concepts under the heading