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CES Working Papers

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2017.41R

Version révisée

Maison des Sciences Économiques, 106-112 boulevard de L'Hôpital, 75647 Paris Cedex 13
https://centredeconomiesorbonne.univ-paris1.fr/

ISSN : 1955-611X

Rank-discounting as a resolution to a dilemma in population ethics*

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Draft: 15 November 2018

Abstract

When evaluating well-being distributions in an anonymous (and replication invariant) manner, one faces a dilemma between (i) assigning dictatorship to a single worst-off person, thus succumbing to a tyranny of non-aggregation and (ii) assigning dictatorship to (unboundedly) many better-off persons, thus succumbing to a tyranny of aggregation. We show how this corresponds to a population-ethical dilemma in the variable population setting between, on the one hand, a reversed repugnant conclusion (preferring a very small population with high well-being) and, on the other hand, a repugnant conclusion (preferring a sufficiently large population with lives barely worth living to a population with good lives) or very sadistic conclusion (not preferring a large population with lives worth living to a population with terrible lives).

The dilemma can be resolved by relaxing replication invariance and thus allowing that evaluation in the fixed population setting might change with population size even though the relative distributions of well-being remain unchanged. Rank-dependent criteria are evaluation criteria that resolve this dilemma but fails replication invariance. We provide conditions under which rank-dependent criteria are the only way out of the dilemma. Furthermore, we discuss the following consequence of relaxing replication invariance: It becomes essential to take into account the existence and utility of non-affected people when evaluating population policies with limited scope.

Keywords: Social evaluation, population ethics, rank-dependent welfare.

JEL Classification numbers: D63, D71.

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* We thank Gustaf Arrhenius, Walter Bossert, Orri Stefánsson and Dean Spears for helpful comments. This chapter is part of the research activities at the Centre for the Study of Equality, Social Organization, and Performance (ESOP) at the Department of Economics at the University of Oslo. ESOP has been supported by the Research Council of Norway through its Centres of Excellence funding scheme (project number 179552). Asheim's research has also been supported by l'IMéRA (Institut d'études avancées d'Aix Marseille Université) and CIREQ (Centre interuniversitaire de recherche en économie quantitative – Montréal). Zuber's research has been supported by Agence nationale de la recherche through the Fair-ClimPop project (ANR-16-CE03-0001-01) and the Investissements d'Avenir program (ANR-10-LABX-93).

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1 Introduction

One prominent population ethics view is that there exists a critical level of lifetime well-being which, if experienced by an added individual without changing the well-being levels of the existing population, leads to an alternative which is as good as the original. Combined with a prioritarian criterion where, for fixed population size, the undiscounted sum of the individuals' transformed utilities is maximized, this leads to critical-level prioritarianism (also known in the economics literature as critical-level generalized utilitarianism; Blackorby and Donaldson, 1984; Broome, 2004; Blackorby, Bossert and Donaldson, 2005).

The standard objection to critical-level generalized utilitarianism is that

- if we set the critical level at or below the individually neutral well-being level (above which life is worth living, and below which it is not), then we get the repugnant conclusion (Parfit, 1976, 1982, 1984) where, for any population with excellent lives, there is a population with lives barely worth living that is better, provided that the latter includes sufficiently many people,

- if we set the critical level strictly above the individually neutral well-being level, then we get the very sadistic conclusion (Arrhenius, 2000a, forthcoming) where, for any population with terrible lives not worth living, there is a population with lives worth living that is worse, provided that the latter includes sufficiently many people.

There are other criteria that avoid the repugnant and very sadistic conclusions, but they all have their own serious shortcomings. According to average utilitarianism, adding a life not worth living to a population may be socially desirable. According to critical-level leximin, as defined by Blackorby, Bossert and Donaldson (1996), any population with n persons living excellent lives is worse than a population with $n+1$ persons whose well-being is barely above a critical level. According to leximin, as suggested by Arrhenius (forthcoming, Sect. 6.8), any population is worse than a population consisting of one individual, provided that the worst-off individual of the former has lower well-being than the single individual of the latter.

Even in a fixed population framework, where there is no difference between the various prioritarian criteria discussed above, and also no difference between the two kinds of leximin criteria, both the prioritarian and the egalitarian (viz. leximin) approaches have shortcomings if there are many present and future people. If one considers a completely egalitarian well-being stream in an intergenerational setting with many future generations, and seeks to evaluate whether the present generation should make a sacrifice leading to a uniform benefit to future generations, then prioritarian and leximin approaches reach opposite and extreme conclusions: According to prioritarianism, the sacrifice should always be made provided that there are sufficiently many future generations, while according to leximin, it should never be made. This has been labelled by Fleurbaey and Tungodden (2010) respectively as the “tyranny of aggregation” and the “tyranny of non-aggregation”.

In their paper, Fleurbaey and Tungodden (2010) suggest a class of rank-dependent welfare orderings that do not suffer from the tyranny of aggregation and the tyranny of non-aggregation. Rank-dependent criteria generalize prioritarian ones by adding a weight on transformed utilities that depend on the rank of the individual in the distribution of well-being. In the context of population ethics, Asheim and Zuber (2014) have proposed and axiomatized a family of such rank-dependent criteria, namely the rank-discounted critical-level generalized utilitarianism criteria. They showed that they can avoid shortcomings such as the repugnant and very sadistic conclusions. This chapter studies to what extent other rank-dependent criteria solve dilemmas in population ethics.

In Sections 2 and 3 we introduce the framework, present some general desirable properties of social welfare orderings in the variable population framework and discuss three families of criteria: prioritarian ones, egalitarian ones, and proper rank-dependent criteria.

In Sections 4 and 5, we discuss the tyranny of aggregation and the tyranny of non-aggregation. We show that rank-dependent criteria may escape both tyrannies, at the cost of not satisfying a replication-invariance property. We argue that this property may not be so appealing in the variable population context.

In Section 6 we consider a related dilemma in population ethics between repugnant and sadistic conclusions, adding new results to those already established by, e.g., Blackorby, Bossert and Donaldson (2004) and Asheim and Zuber (2014, Section 4). We show that rank-dependent criteria may simultaneously avoid the repugnant conclusion, the very sadistic conclusion and the reversed repugnant conclusions. A key condition is that the cumulative effect of adding individuals at a given level of lifetime well-being is bounded. In Section 7 we show how negative results extend to a larger class of prioritarian and egalitarian criteria.

Finally, in Section 8 we conclude, while all proofs are contained in an appendix.

2 Framework and basic properties

Let $\mathbb{N}$ denote the set of positive integers (natural numbers), let $\mathbb{R}$ denote the real numbers, and let respectively $\mathbb{R}_+/\mathbb{R}_{++}/\mathbb{R}_-/\mathbb{R}_{--}$ denote the non-negative/positive/non-positive/negative real numbers.

We consider distributions of (lifetime) well-being within finite populations of variable sizes. We favor the interpretation where the population consists of people that either are alive now or will exist in the future. The set of such distributions is $\mathbf{X} = \cup_{k\in\mathbb{N}}\mathbb{R}^k$, with typical element $\mathbf{x} = (x_1, \ldots, x_i, \ldots, x_k)$, and where x_i is the well-being of individual i in this population. Thus a distribution $\mathbf{x}$ of three people with well-being 5, 10 and 20 respectively will be represented by $\mathbf{x} = (5, 10, 20)$, while distribution $\mathbf{y}$ of five people with well-being 1, 17, -4, 15 and 33 respectively will be represented by $\mathbf{y} = (1, 17, -4, 15, 33)$.

Well-being is assumed to be at least level comparable.[1] For every $k \in \mathbb{N}$, and each distribution $\mathbf{x} \in \mathbb{R}^k$, the finite population size in $\mathbf{x}$ is $n(\mathbf{x}) = k$. Following the usual convention in population ethics, a well-being level equal to 0 represents *neutrality*. Hence, well-being is normalized so that above neutrality, a life, as a whole, is worth living; below neutrality, it is not.

For every $k \in \mathbb{N}$ and all $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^k$, we write $\mathbf{x} \gg \mathbf{y}$ whenever $x_i > y_i$ for all $i \in \{1, \ldots, n(\mathbf{x})\}$. Similarly, $\mathbf{x} \geq \mathbf{y}$ means that $x_i \geq y_i$ for all $i \in \{1, \ldots, n(\mathbf{x})\}$, and $\mathbf{x} = \mathbf{y}$ means that $x_i = y_i$ for all $i \in \{1, \ldots, n(\mathbf{x})\}$. We write $\mathbf{x} > \mathbf{y}$ whenever $\mathbf{x} \geq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$.

We assume that distributions of well-being can be normatively ranked by a *social welfare ordering* (SWO) $\succsim$. Such a social welfare ordering is a binary relation, which is complete and transitive. For all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, $\mathbf{x} \succsim \mathbf{y}$ means that the distribution $\mathbf{x}$ is deemed socially at least as good as $\mathbf{y}$. The fact that $\succsim$ is complete means that, for all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, either $\mathbf{x} \succsim \mathbf{y}$ or $\mathbf{y} \succsim \mathbf{x}$ (or both). The fact that $\succsim$ is transitive means that, for all $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z} \in \mathbf{X}$, if $\mathbf{x} \succsim \mathbf{y}$ and $\mathbf{y} \succsim \mathbf{z}$ then $\mathbf{x} \succsim \mathbf{z}$. We let $\sim$ and $\succ$ denote the symmetric and asymmetric parts of $\succsim$. We say that a *social welfare function* (SWF) $W : \mathbf{X} \to \mathbb{R}$ represents a SWO $\succsim$ on $\mathbf{X}$ if, for all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, $\mathbf{x} \succsim \mathbf{y}$ if and only if $W(\mathbf{x}) \geq W(\mathbf{y})$.

For each $\mathbf{x} \in \mathbf{X}$, we let $\mathbf{x}_{[]} = (x_{[1]}, \ldots, x_{[r]}, \ldots, x_{[n(\mathbf{x})]})$ denote the non-decreasing distribution that is obtained by reordering $\mathbf{x}$; i.e., for each rank $r \in \{1, \ldots, n(\mathbf{x}) - 1\}$, $x_{[r]} \leq x_{[r+1]}$. For any real number $z \in \mathbb{R}$ and any positive integer $k \in \mathbb{N}$, we let $(z)_k \in \mathbb{R}^k$ denote the *egalitarian* distribution where all k individual have a well-being level equal to z. For any $\mathbf{x} \in \mathbf{X}$ and $z \in \mathbb{R}$, $(\mathbf{x}, (z)_k)$ denotes the distribution $\mathbf{x}$ with k added individuals, all with lifetime well-being equal to $z \in \mathbb{R}$.

We now introduce through axioms some basic requirements that seem appealing for a SWO. The first axiom is a principle of impartiality that states all individuals are treated in the same way, so that permuting well-being levels (or permuting the order of people in the list of well-being) leaves the distribution equally good.

Anonymity. A SWO $\succsim$ on $\mathbf{X}$ satisfies *anonymity* if, for all $k \in \mathbb{N}$ and all $\mathbf{x} \in \mathbb{R}^k$,

[1] We actually need stronger measurement conditions except for maximin, a population-size independent version of the egalitarian criterion introduced in Definition 2.

$\mathbf{x} \sim \mathbf{y}$ if $\mathbf{y}$ is obtained by reordering the elements of $\mathbf{x}$.

The second principle represent two ideas. First, if we make small mistakes in measuring the well-being of people, this should not radically modify the social assessment so that small changes in distributions does not yield big changes in the social value of the distributions. Second, we want to be able to represent the SWO by a SWF that is defined across different population sizes. This can be done by endorsing the following principle (Blackorby, Bossert and Donaldson, 2005)

Extended Continuity. A SWO $\succsim$ on $\mathbf{X}$ satisfies *extended continuity* if, for all k, $\ell \in \mathbb{N}$ and all $\mathbf{x} \in \mathbb{R}^k$, the sets $\{\mathbf{y} \in \mathbb{R}^\ell \mid \mathbf{y} \succsim \mathbf{x}\}$ and $\{\mathbf{y} \in \mathbb{R}^\ell \mid \mathbf{x} \succsim \mathbf{y}\}$ are closed in $\mathbb{R}^\ell$.

The two next principles can be viewed as version of the Pareto principle, namely that a distribution $\mathbf{x}$ should be preferred to a same-population distribution $\mathbf{y}$ whenever all individuals prefer $\mathbf{x}$ to $\mathbf{y}$. **Monotonicity** combines a Pareto indifference with a weak version of the Pareto principle. **Restricted Dominance** requires minimal sensitivity to individual well-being: starting with an egalitarian distribution, if a single individual is made worse off, then the distribution becomes worse.

Monotonicity. A SWO $\succsim$ on $\mathbf{X}$ satisfies *monotonicity* if, for all $k \in \mathbb{N}$ and all $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^k$, $\mathbf{x} \geq \mathbf{y}$ implies $\mathbf{x} \succsim \mathbf{y}$.

Restricted Dominance. A SWO $\succsim$ on $\mathbf{X}$ satisfies *restricted dominance* if, for all $k \in \mathbb{N}$ and all z, $z' \in \mathbb{R}$ with $z > z'$, $(z)_k \succ (z', (z)_{k-1})$.

The **Pigou-Dalton** transfer principle requires that we prefer more equal distributions. If well-being is transferred from a better-off person to a worse-off person without reversing their relative ranks, so that the resulting distribution is more equal, then the new distribution is at least as good.

Pigou-Dalton. A SWO $\succsim$ on $\mathbf{X}$ satisfies *Pigou-Dalton* if, for all $k \in \mathbb{N}$, all $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^k$, all $\delta > 0$ and all $i, j \in \{1, \ldots, k\}$, if (i) $x_h = y_h$ for all $h \in \{1, \ldots, k\}$, $h \neq i, j$, and (ii) $y_i - \delta = x_i \geq x_j = y_j + \delta$, then $\mathbf{x} \succsim \mathbf{y}$.

In this chapter, we will consider both absolute and relative rank-dependent criteria, where absolute criteria require absolute, rather than relative, population size to be significant. To rule out any role for absolute population size, we can use a **Replication Invariance** property (see Blackorby, Bossert and Donaldson, 2005). This is also a consistency property across population sizes: if distribution $(1, 1)$ is preferred to distribution $(0, 2)$ then $(1, 1, 1, 1)$ is also preferred to distribution $(0, 0, 2, 2)$ because if we only add a 'twin' to each person in the initial distributions.

To formally state this property, we need to introduce the notion of replicas. For all $\ell \in \mathbb{N}$ and all $\mathbf{x} \in \mathbb{R}^k$, let $\ell * \mathbf{x}$ denote the ℓ–replica of $\mathbf{x}$, i.e. a distribution giving $\mathbf{x}$ to ℓ disjoint population of size k.

Replication Invariance. A SWO $\succsim$ on $\mathbf{X}$ satisfies *replication invariance* if, for all $k \in \mathbb{N}$, all $\ell \in \mathbb{N}$, all $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^k$, $\mathbf{x} \sim \mathbf{y}$ implies $\ell * \mathbf{x} \sim \ell * \mathbf{y}$

Our last basic axiom is an independence or consistency axiom that holds on ordered ("comonotonic") distributions. A natural independence condition would be that the social ranking of two same-population distributions $\mathbf{x}$ and $\mathbf{y}$ does not depend on the well-being level of individuals who have the same level in the two distributions, provided that we do not change individual ranks. This principle of Independence with respect to Ordered Vectors was introduced by Ebert (1988).[2]

Wakker (1989) and Köbberling and Wakker (2003) have shown that, under mild additional conditions satisfied by all SWOs discussed in this paper, Independence with respect to Ordered Vectors is implied by a broader condition representing the idea that we have a consistent way to measure transformed well-being. This condition, named Trade-Off Consistency for Ordered Vectors, states that the cardinal measurement of how changes in well-being affect social evaluation should be consistent: it does not depend on the rank of the individuals (provided we compare people at the same rank). The condition also adds that social evaluation of two same-population distributions is independent of the well-being of individuals who have the same rank and well-being in the two distributions (which is exactly Independence with respect to Ordered Vectors).

To introduce Trade-Off Consistency for Ordered Vectors, we need further notation. Let $\mathbf{x}$ be any distribution, and r be an integer in $\{1, \ldots, n(\mathbf{x})\}$ and $z \in \mathbb{R}$ a real number. The distribution $\mathbf{x}_{[-r]}z$, if it exists, is the distribution $\tilde{\mathbf{x}}$ such that $n(\mathbf{x}) = n(\tilde{\mathbf{x}})$, $\tilde{x}_i = x_{[i]}$ for all $i \neq r$ and $\tilde{x}_r = z$. Hence $\mathbf{x}_{[-r]}z$ is an ordered vector that is a permutation of $\mathbf{x}$ except that the well-being of individual with rank r is now z instead of $x_{[r]}$. Such a vector is *well-defined* only if $x_{[r-1]} \leq z \leq x_{[r+1]}$ (or $r = 1$ and $z \leq x_{[2]}$, or $r = n(\mathbf{x})$ and $z \geq x_{[n(\mathbf{x})-1]}$). Formally, Trade-off consistency for ordered vectors is written as follows:

Trade-Off Consistency for Ordered Vectors. A SWO $\succsim$ on $\mathbf{X}$ satisfies *trade-off consistency for ordered vectors* if, for all $k \in \mathbb{N}$, all $\mathbf{x}$, $\mathbf{y}$, $\tilde{\mathbf{x}}$, $\tilde{\mathbf{y}} \in \mathbb{R}^k$, all r $r' \in \{1, \ldots, k\}$ and all z_1, z_2, z_3, $z_4 \in \mathbb{R}$ and $z_{3'} > z_3$, if $\mathbf{x}_{[-r]}z_1$, $\mathbf{y}_{[-r]}z_2$, $\mathbf{x}_{[-r]}z_3$,

[2]The formal statement of the principle is as follows:
Independence wrt. Ordered Vectors. A SWO $\succsim$ on $\mathbf{X}$ satisfies *independence wrt. ordered vectors* if, for all $k \in \mathbb{N}$, all $M \subseteq \{1, \ldots, k\}$ and all $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^k$, $\mathbf{x} \sim \mathbf{y}$ implies $\mathbf{x}' \sim \mathbf{y}'$ whenever $x_{[r]} = y_{[r]}$ and $x'_{[r]} = y'_{[r]}$ for all $r \in M$ and $x'_{[r]} = x_{[r]}$ and $y'_{[r]} = y_{[r]}$ for all $r \in \{1, \ldots, k\} \backslash M$.

$\mathbf{y}_{[-r]}z_4$, $\tilde{\mathbf{x}}_{[-r']}z_1$, $\tilde{\mathbf{y}}_{[-r']}z_2$, $\tilde{\mathbf{x}}_{[-r']}z_3'$ and $\tilde{\mathbf{y}}_{[-r']}z_4$ are all well-defined and $\mathbf{x}_{[-r]}z_1 \sim \mathbf{y}_{[-r]}z_2$, $\mathbf{x}_{[-r]}z_3 \sim \mathbf{y}_{[-r]}z_4$, $\tilde{\mathbf{x}}_{[-r']}z_1 \sim \tilde{\mathbf{y}}_{[-r']}z_2$, then $\tilde{\mathbf{x}}_{[-r']}z_3' \succeq \tilde{\mathbf{y}}_{[-r']}z_4$.[3]

In the axiom, $\mathbf{x}_{[-r]}z_1 \sim \mathbf{y}_{[-r]}z_2$ and $\mathbf{x}_{[-r]}z_3 \sim \mathbf{y}_{[-r]}z_4$ means that the difference in well-being brought by z_1 and z_2 has the same impact on social evaluation as the difference in well-being brought by z_3 and z_4. The axiom says that this difference should not depend on the rank of the individual.

3 Basic representation results

With the basic properties introduced in Section 2, we are able to present three representation results. The proofs of these and other propositions are contained in an appendix.

Proposition 1 *Assume that a* SWO $\succsim$ *on* $\mathbf{X}$ *satisfies* ***Anonymity***, ***Extended Continuity***, ***Montonicity*** *and* ***Restricted Dominance***. *Then there exist,*

- *for each* $k \in \mathbb{N}$, *a continuous, symmetric and non-decreasing function* $e_k : \mathbb{R}^k \to \mathbb{R}$ *satisfying* $e_k((z)_k) = z$ *for all* $z \in \mathbb{R}$, *and*
- *a continuous function* $V : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$, *which is strictly increasing in its second variable,*

such that $W(\mathbf{x}) = V(n(\mathbf{x}), e_{n(\mathbf{x})}(\mathbf{x}))$ *represents* $\succsim$.

The SWOs obtained in Proposition 1 decompose the evaluation of well-being distributions in two steps. Functions e_k are the equally-distributed equivalent function: for all $\mathbf{x} \in \mathbb{R}^k$, the well-being level $z \in \mathbb{R}$ determined by $e_k(\mathbf{x}) = z$ has the property that an egalitarian distribution where k individuals have well-being z is socially as good as $\mathbf{x}$.[4] For each k, the function $e_k : \mathbb{R}^k \to \mathbb{R}$ is a SWF that represents the SWO $\succsim$ on $\mathbb{R}^k$ — the restricted domain of populations of size k — such that for all $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^k$, $\mathbf{x} \succsim \mathbf{y}$ if and only if $e_k(\mathbf{x}) \geq e_k(\mathbf{y})$. Hence, the functions e_k embody the principles of justice within populations of the same size. On the other hand, the function V embody the trade-off between population size and equally-distributed equivalent well-being in the population.

The following are three examples of SWOs that are consistent with this decomposition:

[3]With $\tilde{\mathbf{x}}_{[-r']}z_3' \succ \tilde{\mathbf{y}}_{[-r']}z_4$ for any essential rank r' such that there exists an ordered vector for which increasing the well-being of the individual with r'th level of well-being yields a strict social welfare improvement.

[4]The axioms of **Extended Continuity**, **Montonicity** and **Restricted Dominance** imply that such a level exists and is unique

- *Critical-level generalized utilitarianism* (CLU) has $V(k,e) = k\,(\phi(e) - \phi(c))$ for some *critical level* $c \geq 0$ and
$$e_k(\mathbf{x}) = \phi^{-1}\left(\frac{1}{k}\sum\nolimits_{r=1}^{k}\phi(x_{[r]})\right) \tag{1}$$
for some increasing and continuous function $\phi : \mathbb{R} \to \mathbb{R}$ that transforms well-being into transformed well-being, so that the CLU SWO $\succsim$ is represented by
$$W(\mathbf{x}) = \sum\nolimits_{r=1}^{k}\left(\phi(x_{[r]}) - \phi(c)\right) .$$
For consistency with our discussion below we let the index sum according to rank r, but this is clearly immaterial for the CLU SWO. The CLU SWO has a prominent place in the literature on population ethics (Blackorby and Donaldson, 1984; Broome, 2004; Blackorby, Bossert and Donaldson, 2005).

- *Average generalized utilitarianism* (AU) has $V(k,e) = \phi(e)$ with $e_k(\mathbf{x})$ given by (1), so that the AU SWO $\succsim$ is represented by
$$W(\mathbf{x}) = \frac{1}{k}\sum\nolimits_{r=1}^{k}\phi(x_{[r]}) .$$

- *Maximin* has $V(k,e) = \phi(e)$ and $e_k(\mathbf{x}) = x_{[1]}$, so that the maximin SWO is represented by
$$W(\mathbf{x}) = \phi(x_{[1]}) . \tag{2}$$
In the presentation of maximin we have chosen (2) instead of the equivalent representation $W(\mathbf{x}) = x_{[1]}$ in order to be consistent with our discussion below.

By imposing that the SWO $\succsim$ on $\mathbf{X}$ satisfies trade-off consistency for ordered vectors and Pigou-Dalton, we obtain a smaller class of SWOs, which contains the set of CLU, AU and maximin SWOs as special cases. To introduce the general class, we need the following index of non-concavity of a continuous and increasing function ϕ (see Chateauneuf, Cohen and Meilijson, 2005):
$$G_\phi = \sup_{z_1 < z_2 \leq z_3 < z_4}\left[\frac{\frac{\phi(z_4)-\phi(z_3)}{z_4 - z_3}}{\frac{\phi(z_2)-\phi(z_1)}{z_2 - z_1}}\right] .$$

Proposition 2 *Assume that a* SWO $\succsim$ *on* $\mathbf{X}$*, in addition to the axioms of Proposition 1, satisfies* ***Trade-Off Consistency for Ordered Vectors*** *and* ***Pigou-Dalton****. Then, for each* $k \in N$*, the equally distributed equivalent of* $\mathbf{x} \in \mathbb{R}^k$ *is* *given by*

$$e_k(\mathbf{x}) = \phi_k^{-1}\left(\sum\nolimits_{r=1}^{k} w_r^k \phi_k(x_{[r]})\right) \,, \tag{3}$$

where the weights w_r^k, for $r \in \{1, \ldots, k\}$, are non-negative, non-increasing and sum up to 1, and the function ϕ_k is continuous, increasing and such that

$$G_{\phi_k} \leq \min_{r\in\{1,\ldots,k-1\}} \frac{w_r^k}{w_{r+1}^k} \,.$$

A SWO satisfying the axioms of Propositions 1 and 2 is called a *rank-dependent generalized utilitarian* SWO. We can consider different subclasses of rank-dependent generalized utilitarian SWOs depending on the principles of justice applied for populations with the same size. We will pay particular attention to two subclasses, specified in Definitions 1 and 2.

Definition 1 A SWO $\succsim$ on $\mathbf{X}$ is a *prioritarian* (or *generalized utilitarian*) SWO if it satisfies the axioms of Propositions 1 and 2 with the weights in (3) given by $w_r^k = \frac{1}{k}$ for all $k \in \mathbb{N}$ and $r \in \{1, \ldots, k\}$ and a function $\phi_k = \phi$ for all $k \in \mathbb{N}$.

The prioritarian class includes the CLU and AU SWOs. However, this class is much richer than these few examples, and can encompass many different ways of trading-off population size and average transformed well-being.

For the prioritarian class, Proposition 2 implies that $G_\phi \leq 1$ so that ϕ is weakly concave. If ϕ is affine, then we obtain (non-generalized) utilitarianism. The utilitarian class is the one that has been most studied in population ethics. A famous example is the *total utilitarian* SWO, which is the member of the utilitarian class such that $V(k, z) = kz$ for all $k \in \mathbb{N}$ and $e_k(\mathbf{x}) = \frac{1}{k}\sum_{r=1}^{k} x_{[r]}$. This is a CLU SWO with $\phi(z) = z$ and $c = 0$; thus, it is also prioritarian.

The term "prioritarian" suggests that ϕ is a strictly concave transformation of well-being, thereby giving preference to redistribution of well-being from a better-off to a worse-off person. An extreme case of such preference for redistribution is given by equalitarian SWOs.

Definition 2 A SWO $\succsim$ on $\mathbf{X}$ is an *egalitarian* SWO if it satisfies the axioms of Propositions 1 and 2 with the weights in (3) given by $w_1^k = 1$ for all $k \in \mathbb{N}$.

To the best of our knowledge, only a few papers have considered this version of egalitarianism in the population ethics literature.[5] There have been discussions of other versions of egalitarianism that use a leximin rather than a maximin

[5]Only Bossert (1990a), Blackorby, Bossert and Donaldson (2005, Chap. 5) and Arrhenius (forthcoming) mention versions of egalitarianism.

criterion to define e_k.[6] For instance, Blackorby, Bossert and Donaldson (1996) have argued in favor of Critical-Level Leximin, while Arrhenius (forthcoming, Sect. 6.8) discusses a version of leximin related to the Positional-Extension Leximin of Blackorby, Bossert and Donaldson (1996). One issue though is that leximin is not continuous so that we cannot apply Proposition 1 to define a family of leximin criteria. Given that leximin gives priority to the worst-off, like maximin, the conclusions that we would obtain would not be very different from the ones for maximin in the context of the present chapter.

Proposition 3 *Assume that a* SWO $\succsim$ *on* $\mathbf{X}$*, in addition to the axioms of Propositions 1 and 2, satisfies* ***Replication Invariance****. Then* $\phi_k = \phi$ *for all* $k \in \mathbb{N}$ *and there exists a non-decreasing and concave weighting function* $F : [0,1] \to [0,1]$*, which is continuous on* $(0,1]$ *and satisfies* $F(0) = 0$ *and* $F(1) = 1$*, such that the weights in (3) are given by* $w_r^k = F\left(\frac{r}{k}\right) - F\left(\frac{r-1}{k}\right)$ *for each* $k \in \mathbb{N}$ *and* $r \in \{1, \ldots, k\}$*.*

A SWO satisfying the axioms of Propositions 1, 2 and 3 is called a *relative rank-dependent generalized utilitarian* SWO. Both prioritarian and egalitarian SWOs are in this class. In the case of prioritarianism, we have that the weighting function F is given by $F(\rho) = \rho$ for all $\rho \in [0,1]$. In the case of egalitarianism, we have that the weighting function F is given by $F(0) = 0$ and $F(\rho) = 1$ for all $\rho \in (0,1]$.

4 Tyranny of aggregation, tyranny of non-aggregation and replication invariance

Fleurbaey and Tungodden (2010) have highlighted a dilemma in social ethics between what they call the "tyranny of aggregation" and the "tyranny of non-aggregation". The tyranny of aggregation means that a tiny gain to sufficiently many well-off people might justify imposing a much larger sacrifice on the worst-off. On the other hand, the tyranny of non-aggregation means that a small gain by the worst-off might be sufficient to justify a sacrifice by a great number of people.

To avoid these two tyrannies, Fleurbaey and Tungodden (2010) have proposed the principles of minimal aggregation and minimal non-aggregation.

Minimal Aggregation. A SWO $\succsim$ satisfies *minimal aggregation* if for some $k \in \mathbb{N}$ and for all $\mathbf{x} \in \mathbb{R}^k$, there exist $\delta > \varepsilon > 0$ such that for all $\mathbf{y} \in \mathbb{R}^k$, if (i) for

[6]The leximin ordering $\succsim^k_{\text{lex}}$ over $\mathbb{R}^k$ is defined as follows. For all $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^k$: $\mathbf{x} \sim^k_{\text{lex}} \mathbf{y}$ if and only if $\mathbf{x}_{[\,]} = \mathbf{y}_{[\,]}$; $\mathbf{x} \succ^k_{\text{lex}} \mathbf{y}$ if and only if there exists $r \in \{1, \ldots, k\}$ such $x_{[r]} > y_{[r]}$ and $x_{[\ell]} = y_{[\ell]}$ for all $\ell \in \{1, \ldots, r-1\}$.

some $i \in \{1, \dots, k\}$, $0 \leq x_i - y_i \leq \varepsilon$ and (ii) for all $j \neq i$, $y_j - x_j \geq \delta$, then $\mathbf{y} \succsim \mathbf{x}$.

Minimal Aggregation means that for some population, if all individuals except individual i gain sufficiently, then it is tolerable to impose a loss on individual i if the loss is sufficiently small. Note that the loss may be arbitrary low, so that the principle does not seem very demanding.

To introduce minimal non-aggregation, let us introduce for any distribution $\mathbf{x} \in \mathbf{X}$ the set of worst-off and best-off individuals in this distribution:

$$I_W(\mathbf{x}) = \left\{ i \in \{1, \dots, n(\mathbf{x})\} \mid x_i = \min_{j \in \{1, \dots, n(\mathbf{x})\}} x_j \right\}$$

and

$$I_B(\mathbf{x}) = \left\{ i \in \{1, \dots, n(\mathbf{x})\} \mid x_i = \max_{j \in \{1, \dots, n(\mathbf{x})\}} x_j \right\}.$$

Minimal Non-Aggregation. A SWO $\succsim$ satisfies *minimal non-aggregation* if there exist $0 < \underline{z} < \bar{z}$ and $\delta > \varepsilon > 0$ such that for all $k \in \mathbb{N}$, all $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^k$, if (i) for some $i \in \{1, \dots, k\}$, $i \in I_W(\mathbf{y})$, $y_i \leq \underline{z}$ and $x_i - y_i \geq \delta$ and (ii) for all $j \neq i$ with $x_j \neq y_j$, it holds that $j \in I_B(\mathbf{x}) \cap I_B(\mathbf{y})$, $y_j \geq \bar{z}$ and $y_j - x_j \leq \varepsilon$, then $\mathbf{x} \succsim \mathbf{y}$.

Minimal Non-Aggregation means that if a worst-off individual is sufficiently badly-off and gains enough, there is a small loss that is tolerable for all the best-off, no matter how numerous they are, provided they are sufficiently well-off. Again the principle does not seem very demanding given that the loss by the best-off may be arbitrary small.

The key remark by Fleurbaey and Tungodden (2010) is that prioritarian SWOs do not satisfy **Minimal Non-Aggregation**, while egalitarian SWOs do not satisfy **Minimal Aggregation**. Furthermore, in their Proposition 2(i) they establish conditions under which **Minimal Aggregation** and **Minimal Non-Aggregation** cannot be combined with **Replication Invariance**. Our next proposition illustrate this result under conditions that are stronger than those imposed by Fleurbaey and Tungodden (2010, Proposition 2(i)).

Proposition 4 *Assume that a* SWO $\succsim$ *on* $\mathbf{X}$ *satisfies the axioms of Propositions 1, 2 and 3. Then* $\succsim$ *fails either* ***Minimal Aggregation*** *or* ***Minimal Non-Aggregation****.*

The proof of Proposition 4 shows that the failure of minimal aggregation or minimal non-aggregation depends on the properties of the weighting function F which determines the weights in (3).

(a) The SWO $\succsim$ fails minimal aggregation if and only if $\lim_{\rho\to 0^+} F(\rho) = 1$ (so that $\succsim$ is egalitarian).

(b) The SWO $\succsim$ fails minimal non-aggregation if and only if $\lim_{\rho\to 0^+} F(\rho) < 1$ (this includes any prioritarian SWO).

The upshot of this result is that if we want to satisfy minimal aggregation and minimal non-aggregation, and we want to keep the axioms of Propositions 1 and 2, thus keeping the rank-dependent generalized utilitarian form, then replication invariance must go.

This discussion can be related to Arrhenius' (forthcoming) *general non-extreme priority condition*, capturing the idea that there is some number of people such that a very large loss for them cannot be compensated for by a slight gain for just one person. A SWO $\succsim$ satisfying the axioms of Propositions 1, 2 and 3 fails this condition if $\lim_{\rho\to 0^+} F(\rho) > 0$, since then a large loss for many better-off people can be compensated by a slight gain for a single worst-off person if sufficient many people are inserted at intermediate ranks. Hence, by observation (a) above, **Minimal Aggregation** is weaker than Arrhenius' condition in a setting where these axioms are satisfied.

5 Escaping tyrannies through absolute rank-dependence

A particular class of SWO satisfying the axioms of Propositions 1 and 2, but not necessarily satisfying replication invariance, is obtained by letting weights depend on absolute rather than relative ranks.

Definition 3 A SWO $\succsim$ on $\mathbf{X}$ satisfying the axioms of Propositions 1 and 2 is an *absolute rank-dependent generalized utilitarian* SWO if there exist

(i) non-negative and non-increasing *absolute weights* $(a_r)_{r\in\mathbb{N}}$ with $a_1 = 1$ such that, for all $k \in \mathbb{N}$, the weights in (3) are obtained by letting

$$w_r^k = \frac{a_r}{\sum_{r'=1}^{k} a_{r'}}$$

for all $r \in \{1, \ldots, k\}$, and

(ii) a function ϕ such that $\phi_k = \phi$ for all $k \in \mathbb{N}$.

Prioritarian and egalitarian SWOs are also absolute rank-dependent. In the case of prioritarianism, this is obtained by choosing $a_r = 1$ for all $r \in \mathbb{N}$ as absolute weights. This includes CLU and AU. In the case of egalitarianism, this is obtained by choosing $a_r = 0$ for all $r > 1$. This includes maximin. Hence, prioritarian and egalitarian SWOs are both relative and absolute rank-dependent generalized utilitarian SWOs.

Any absolute rank-dependent generalized utilitarian SWO satisfies the axioms of Propositions 1 and 2, but is not characterized by these axioms. To obtain absolute rank-dependent SWOs, we could add an additional independence property (which corresponds to the High Income Group Aggregation Property as defined by Bossert, 1990b, and to Top Independence as defined by Pivato, 2018)

Same population independence of the existence of the best-off. A SWO $\succsim$ satisfies *same population independence of the existence of the best-off* if, for any $k \in \mathbb{N}$, and for all $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^k$ and $z > \max\{\mathbf{x}_{[k]}, \mathbf{y}_{[k]}\}$, $\mathbf{x} \succsim \mathbf{y}$ if and only if $\big(\mathbf{x}, (z)_1\big) \succsim \big(\mathbf{y}, (z)_1\big)$.

Definition 4 An absolute rank-dependent generalized utilitarian SWO $\succsim$ on $\mathbf{X}$ is a *proper* rank-dependent generalized utilitarian SWO if the absolute weights $(a_r)_{r\in\mathbb{N}}$ satisfy $a_2 > 0$ and $\sum_{r\in\mathbb{N}} a_r < \infty$.

The restriction that a_2 be strictly positive means that no egalitarian SWOs is proper rank-dependent, and the restriction that the absolute weights be summable means that no prioritarian SWO is proper rank-dependent. If there exists a real number $\beta \in (0, 1)$ such that the absolute weights are $a_r = \beta^{r-1}$ for all $r \in \mathbb{N}$, then the SWO is proper rank-dependent and called a *rank-discounted generalized utilitarian* (RDU) SWO. We will return to this particular subclass in the next section.

One drawback of proper rank-dependent generalized utilitarian SWOs is that they allow a large loss for many better-off people to be compensated by a slight gain for a single worst-off person. The reason is that the loss of transformed well-being of the many better-off people is discounted below any positive level by inserting sufficiently many people at intermediate ranks. Hence, SWOs in this class fail Arrhenius' (forthcoming) general non-extreme priority condition (discussed at the end of Section 4).

Definition 5 An absolute rank-dependent generalized utilitarian SWO $\succsim$ on $\mathbf{X}$ is a *regular* rank-dependent generalized utilitarian SWO if there exists a positive number $\kappa \in \mathbb{R}_{++}$ such that, for all $k \in \mathbb{N}$, $a_k \geq \kappa \cdot \big(\sum_{r=k+1}^{+\infty} a_r\big)$.

The restriction imposed by regularity implies in particular that the absolute weights are summable (i.e. $\sum_{r\in\mathbb{N}} a_r < \infty$), but it is stronger than that. Hence no prioritarian SWO is regular rank-dependent. On the other hand, the restriction allows weights to be equal to zero from a certain rank r on. Hence egalitarian SWOs are regular rank-dependent.

As shown by the following result, the absolute rank-dependent generalized utilitarian SWO being proper *and* regular is the key to solving the dilemma posed by the requirements of minimal aggregation and minimal non-aggregation.

Proposition 5 *Assume that a* SWO $\succsim$ *on* $\mathbf{X}$ *is an absolute rank-dependent generalized utilitarian* SWO*. Then* $\succsim$ *satisfies both **Minimal Aggregation** and **Minimal Non-Aggregation** if and only if* $\succsim$ *is proper and regular rank-dependent.*

The proof of Proposition 5 shows that:

(a) If the SWO is egalitarian, then $\succsim$ fails minimal aggregation.

(b) If the SWO is not regular rank-dependent, then $\succsim$ fails minimal non-aggregation.

The proposition implies that, within the class of absolute rank-dependent generalized utilitarian SWOs, there must be positive weights on more than the worst-off person, and the sum of the weights must be converging sufficiently fast. This gives a route out of the dilemma posed by Fleurbaey and Tungodden (2010), but at the cost of failing the requirement of replication invariance. Indeed, Fleurbaey and Tungodden (2010, p. 404) show existence of a SWO satisfying both minimal aggregation and minimal non-aggregation by providing an example of an RDU SWO in a form that is actually in the subclass considered in the subsequent section.

6 Simple rank-dependence and population ethics

Up to now, the general classes of SWOs that we have considered in our propositions and definitions have not specified the trade-off between population size and equally-distributed equivalent well-being in the population, as embodied by the function V of Proposition 1. We turn now to one particularly natural manner to make this trade-off.

Definition 6 An absolute rank-dependent generalized utilitarian SWO $\succsim$ on $\mathbf{X}$ with absolute weights $(a_r)_{r\in\mathbb{N}}$ is a *simple rank-dependent generalized utilitarian* SWO if there exists a critical level $c \in \mathbb{R}_+$ such that the function V of Proposition 1 is given by:

$$V(k,e) = \sum\nolimits_{r=1}^{k} a_r \left(\phi(e) - \phi(c)\right) .$$

The term 'simple' is appropriate since, any rank-dependent generalized utilitarian SWO $\succsim$ is represented by a SWF $W : \mathbf{X} \to \mathbb{R}$ of the form:

$$W(\mathbf{x}) = \sum\nolimits_{r=1}^{k} a_r \left(\phi(x_{[r]}) - \phi(c)\right) . \tag{4}$$

The class of simple prioritarian SWO contains only CLU. In particular, the AU SWO is prioritarian, but not simple. The class of simple egalitarian SWO contains only maximin. To the best of our knowledge, only few papers have considered SWFs similar to the one defined by (4) and that are neither simple prioritarian nor simple egalitarian.

Sider (1991) has proposed principle GV, with similar ideas already discussed by Hurka (1983). Principle GV is related to but incompatible with the form specified by Equation (4). It first divides a population into two ordered sets: one set with the distribution of the people with positive well-being, in order of descending well-being; and another set with the distribution of the people with negative well-being, in order of ascending well-being. For people with negative well-being, principle GV is similar to (4) with decreasing weights a_r, an affine function ϕ and $c = 0$. For people with positive well-being, principle GV applies something like (4) with decreasing weights a_r an affine function ϕ and $c = 0$, but it applies it to the distribution of people ordered in decreasing (and not increasing) order of well-being. Principle GV has been discussed by several papers (Arrhenius, 2000a,b, forthcoming; Pivato, 2018, the latter paper contains also an axiomatization of a generalized version of the principle). These papers point out that Principle GV implies that, if everyone has positive well-being, it is always socially better to make a (non-leaky) well-being transfer from a poor to a rich. Hence, principle GV does not satisfy **Pigou-Dalton**.

Asheim and Zuber (2014) have proposed and axiomatized the class of *rank-discounted critical-level generalized utilitarian* (RDCLU) SWOs. They are a special case of the form specified by (4), where there exists a real number $\beta \in (0, 1)$ such that the weights are $a_r = \beta^{r-1}$, for all $r \in \mathbb{N}$. The fact that the absolute weights a_r are strictly decreasing implies that a RDCLU SWO may satisfy **Pigou-Dalton** even if ϕ is not concave: A RDCLU SWO satisfies **Pigou-Dalton** if and only if $G_\phi \leq \frac{a_r}{a_{r+1}} = \beta^{-1}$. Generalizations of rank-discounted critical-level generalized utilitarian SWOs have been discussed by Zuber (2017) and axiomatized by Pivato (2018).

Any simple rank-dependent generalized utilitarian SWO satisfies the axioms of Propositions 1 and 2, but is not characterized by these axioms. To obtain simple rank-dependent SWOs from absolute rank-dependent SWOs, we could add yet another independence property (which is implied by the Low Income Group Aggregation Property as defined by Bossert, 1990b):

Independence of the Well-being of the Worst-off. A SWO $\succsim$ satisfies the *independence of the well-being of the worst-off* if for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and any $z', z'' \in \mathbb{R}$ such that $\max\{z', z''\} < \min\{x_{[1]}, y_{[1]}\}$, $\big(\mathbf{x}, (z')_1\big) \succsim \big(\mathbf{y}, (z')_1\big)$ if and only if $\big(\mathbf{x}, (z'')_1\big) \succsim \big(\mathbf{y}, (z'')_1\big)$.

According to Parfit (1984), a social welfare ordering leads to the *repugnant conclusion* if:

> "For any possible population of at least ten billion people, all with a very high quality of life, there must be some much larger imaginable population whose existence, if other things are equal, would be better even though its members have lives that are barely worth living."

The repugnant conclusion has caught much attention in the literature on population ethics as most of the literature have discussed ways to avoid such a conclusion. Formally, one can formulate the avoidance of the repugnant conclusion as follows:[7]

Avoidance of the Repugnant Conclusion. A SWO $\succsim$ on $\mathbf{X}$ *avoids the repugnant conclusion* if there exist $k \in \mathbb{N}$, $y \in \mathbb{R}_{++}$ and $z \in (0, y)$ such that, for all $m \geq \ell \geq k$, $(y)_\ell \succsim (z)_m$.

Although the CLU SWO does not avoid the repugnant conclusion if the critical level c equals 0, other generalized utilitarian SWO, e.g. CLU with positive c and AU do so. However, they are subject to other difficulties.

On the one hand, CLU with positive c imply the sadistic conclusion is that, for any distribution with negative well-being, there is an egalitarian distribution with low, but positive well-being that is worse. This is what Arrhenius (2000a, forthcoming) refers to as the *very sadistic conclusion*. Formally, one can formulate the avoidance of the very sadistic conclusion as follows:

Avoidance of the Very Sadistic Conclusion. A SWO $\succsim$ on $\mathbf{X}$ *avoids the very sadistic conclusion* if there exists $\mathbf{x} \in \mathbf{X}$ such that $(z)_k \succsim \mathbf{x}$ for all $k \in \mathbb{N}$ and $z \in \mathbb{R}_{++}$.

One the other hand, AU and also maximin leads to the problematic conclusion that for any egalitarian distribution with very high positive well-being, there is a better one-individual distribution with slightly higher well-being. This might be referred to as the *reversed repugnant conclusion* (Arrhenius, forthcoming). Hence, one can formulate the avoidance of the reversed repugnant conclusion as follows:

Avoidance of the Reversed Repugnant Conclusion. A SWO $\succsim$ on $\mathbf{X}$ *avoids the reversed repugnant conclusion* if there exist $k \in \mathbb{N}$, $y \in \mathbb{R}_{++}$ and $z \in (0, y)$ such that $(z)_k \succsim (y)_1$.

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[7] Other formalization have been proposed, for instance by Blackorby, Bossert and Donaldson (2005). The formulation is slightly stronger than the one they have, but we think it is in the spirit of Parfit's initial formulation.

The next result shows that in the class of simple rank-dependent generalized utilitarian SWOs, being proper is the key to avoiding all these three problematic conclusion.

Proposition 6 *Assume that a* SWO $\succsim$ *on* $\mathbf{X}$ *is a simple rank-dependent generalized utilitarian* SWO*. Then* $\succsim$ *satisfies* ***Avoidance of the Repugnant Conclusion****,* ***Avoidance of the Very Sadistic Conclusion*** *and* ***Avoidance of the Reversed Repugnant Conclusion*** *if and only if* $\succsim$ *is proper rank-dependent.*

The proof of Proposition 6 shows that:

(a) If the SWO is egalitarian, then $\succsim$ entails the reversed repugnant conclusion.

(b) If the SWO has non-summable absolute weights, then $\succsim$ entails either the repugnant conclusion (when $c = 0$) or the very sadistic conclusion (when $c > 0$).

Proposition 6 implies that, within the class of simple rank-dependent generalized utilitarian SWOs, there must be positive weights on more than the worst-off person, and the sum of the weights must be converging, in order to avoid all of the repugnant, very sadistic and reversed repugnant conclusions.

It follows from Proposition 6 and our remark just prior to Definition 5 that any simple rank-dependent generalized utilitarian SWO $\succsim$ avoiding the repugnant conclusion, the very sadistic conclusion and the reversed repugnant conclusion must fail Arrhenius' (forthcoming) general non-extreme priority condition.

Remark that, given that proper rank-dependent generalized utilitarian SWOs satisfy the Pigou-Dalton principle and avoid the repugnant conclusion, they cannot satisfy the famous mere addition principle introduced by Parfit (1984), namely that addition of people with positive well-being is always (at least weakly) socially desirable.

Mere Addition Principle. A SWO $\succsim$ on $\mathbf{X}$ satisfies the *mere addition principle* if for all $z \in \mathbb{R}_{++}$ and all $\mathbf{x} \in \mathbf{X}$ $(z, \mathbf{x}) \succsim \mathbf{x}$.

Indeed, Carlson (1998) proved that the mere addition principle and a non-anti-egalitarianism principle (which follows from the Pigou-Dalton principle) imply a violation of our formulation of the avoidance of the repugnant conclusion.

7 Additional population ethics results

Proposition 6 singles out proper rank-dependent SWOs in the class of simple rank-dependent generalized utilitarian SWOs. We can more generally compare proper and regular rank-dependent SWOs to a large class of prioritarian and egalitarian criteria, including those that are not simple. For this purpose we first state a principle generalizing **Avoidance of the Very Sadistic Conclusion** (see Arrhenius, forthcoming, for a discussion of the principle):

Weak Non-Sadism Condition. A SWO $\succsim$ on $\mathbf{X}$ *satisfies the weak non-sadism condition* if there exist $k \in \mathbb{N}$ and $z \in \mathbb{R}_{--}$ such that for all $\mathbf{x} \in X$, $y \in \mathbb{R}_{++}$ and $\ell \in \mathbb{N}$ $\big(\mathbf{x}, (y)_\ell\big) \succsim \big(\mathbf{x}, (z)_k\big)$.

Any prioritarian social welfare ordering will be subject to one of the two problems: either it implies the repugnant conclusion or it cannot satisfy the weak non-sadism condition.

Proposition 7 *A prioritarian* SWO $\succsim$ *on* $\mathbf{X}$ *cannot satisfy both* ***Avoidance of the Repugnant Conclusion*** *and* ***Weak Non-Sadism Condition****.*

Proposition 7 seems to reject prioritarian criteria for population ethics. One could therefore consider egalitarian criteria. However, they are also subject to a population ethics dilemma. They cannot satisfy the weak non-sadism condition if we want to insist on the intuitive principle that we should not create lives with a negative level of well-being.

Negative Mere Addition Principle. A SWO $\succsim$ on $\mathbf{X}$ *satisfies the negative mere addition principle* if for all $\mathbf{x} \in \mathbf{X}$ and all $z \in \mathbb{R}_{--}$, $\mathbf{x} \succ \big(\mathbf{x}, (z)_1\big)$.

Proposition 8 *An egalitarian* SWO $\succsim$ *on* $\mathbf{X}$ *cannot satisfy both* ***Weak Non-Sadism Condition*** *and* ***Negative Mere Addition Principle****.*

In contrast with these negative results, proper and regular simple rank-dependent SWOs may escape all the population dilemma we have discussed up to now provided function ϕ is concave and bounded above.

Proposition 9 *Assume that a* SWO $\succsim$ *on* $\mathbf{X}$ *is a simple rank-dependent generalized utilitarian* SWO*. Then* $\succsim$ *satisfies* ***Weak Non-Sadism Condition*** *and* ***Negative Mere Addition Principle*** *if* $\succsim$ *is proper and regular rank-dependent and the function* ϕ *is concave and bounded above.*

Note that, by Proposition 6, we already know that the simple rank-dependent generalized utilitarian SWO described in Proposition 9 satisfies **Avoidance of the Repugnant Conclusion**, **Avoidance of the Very Sadistic Conclusion** and **Avoidance of the Reversed Repugnant Conclusion**. So we obtain a family of SWOs that comply with many population ethics requirements. This family deserves receiving further attention.

8 Concluding remarks

In this chapter we have argued that the escape from the tyrannies of aggregation and non-aggregation and the resolution of an important population-ethical dilemma require that we relax the requirement of replication invariance. Thus the normative evaluation of populations and their well-beings may change as populations and their well-beings are replicated. The RDCLU SWO (Asheim and Zuber, 2014) is one example of a population-ethical criterion that offers a solution by relaxing replication invariance. However, as we have shown, a larger class of simple rank-dependent generalized utilitarian SWOs satisfy all the requirements that we have imposed, save replication invariance. In particular, the simple rank-dependent generalized utilitarian SWOs with absolute weights given by $a_r = \frac{R+1-r}{R}$ for $1 \leq r \leq R$ and $a_r = 0$ for $r > R$, where $R \geq 2$, are in this larger class.

It is a standard approach in population ethics to associate with the notion of total population not only the people that are alive at any one time, but also people that will ever live and those that have ever lived; in particular, this is the position taken by Blackorby, Bossert and Donaldson (2005). However, Broome (2004, chap. 13) and Blackorby, Bossert and Donaldson (2005, pp. 131–133) argue that it is reasonable to assume existence independence, requiring that the ranking of any two alternatives be independent of the existence of individuals who ever live and have the same well-being levels in both. Existence independence implies that populations principles can be applied to the affected people only.

The relaxation of replication invariance that we promote here is inconsistent with existence independence, and thus it becomes essential to take into account the existence and utility of non-affected people when evaluating population policies with limited scope. In particular, we must consider today the population policies that should be and will be implemented in the far future, and also take into the utilities of people that are dead if we require that the population-ethical criterion be time consistent. We must also depart from the generation-relative population ethics that Dasgupta (2005, pp. 430–431) supports, because the reproductive choices made by future generations will affect the number and well-being of future people, therefore modifying the goodness of changing the population today. This departure from existence independence might have important implications for the conclusions that follow from population ethics.

Appendix: Proofs

Proof of Proposition 1. If the SWO $\succsim$ on $\mathbf{X}$ satisfies **Montonicity** and **Restricted Dominance**, then it satisfies Minimal Increasingness in Blackorby, Bossert and Donaldson (2005, chap. 5). To see this, consider the distributions $(z)_k$ and $(z')_k$ with z, $z' \in \mathbb{R}$ satisfying $z > z'$. Construct $\mathbf{x} \in \mathbb{R}^k$ by letting $x_i = z'$ for some $i \in \{1, \ldots, k\}$ and $x_j = z$ for all $j \in \{1, \ldots, k\} \backslash \{i\}$. Then $(z)_k \succ \mathbf{x}$ by **Restricted Dominance** and $\mathbf{x} \succsim (z')_k$ by **Montonicity**, so $(z)_k \succ (z')_k$ by transitivity. Thus, the result follows from Blackorby, Bossert and Donaldson (2005, Theorem 5.2). ■

Proof of Proposition 2. From Köbberling and Wakker (2003, Theorem 8), for each population size $k \in \mathbb{N}$, there must exist non-negative weights w_r^k, for $r \in \{1, \ldots, k\}$ and a continuous increasing function ϕ_k such that, for all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ with $n(\mathbf{x}) = n(\mathbf{y}) = k$,

$$\mathbf{x} \succsim \mathbf{y} \Longleftrightarrow \sum\nolimits_{r=1}^{k} w_r^k \phi_k(x_{[r]}) \geq \sum\nolimits_{r=1}^{k} w_r^k \phi_k(y_{[r]}).$$

Using the definition of equally distributed equivalent, we can take the weights w_r^k summing to one. By the uniqueness result in Köbberling and Wakker (2003, Theorem 8), both the weights and the function ϕ_k are unique after normalization, provided that $w_r^k > 0$ for at least two ranks r.

By Chateauneuf, Cohen and Meilijson (2005, Theorem 1) (or a similar result in Asheim and Zuber, 2014), for each $k \in \mathbb{N}$ such social orderings satisfy **Pigou-Dalton** if and only if

$$G_{\phi_k} \leq \min_{r \in \{1,\ldots,k-1\}} \frac{w_r^k}{w_{r+1}^k}.$$

Note that, by Chateauneuf, Cohen and Meilijson (2005, Proposition 3), $G_{\phi_k} \geq 1$, so that this implies $w_r^k \geq w_{r+1}^k$ for all $r \in \{1, \ldots, k-1\}$: the weights are non-increasing. ■

Proof of Proposition 3. By **Replication Invariance**, and by the uniqueness result of functions ϕ_k in Proposition 2, it is easy to show that there must exist a continuous and increasing function ϕ such that $\phi_k = \phi$ for all $k \in \mathbb{N}$, provided that there exist at least two ranks r such that $w_r^k > 0$, for $k \geq 2$ (otherwise, we have $w_1^k = 1$ for all $k \in \mathbb{N}$, and function ϕ_k does not matter: we can take it identical without loss of generality).

Then we can mimic Donaldson and Weymark (1980, Theorem 4) to show that there exists a non-decreasing and concave weighting function F satisfying all the required properties. ■

Proof of Proposition 4. *Case 1:* $\lim_{\rho \to 0^+} F(\rho) = 1$. In this case, the weighting function F is given by $F(0) = 0$ and $F(\rho) = 1$ for all $\rho \in (0, 1]$, so that the SWO $\succsim$ on $\mathbf{X}$ is egalitarian. Any egalitarian SWO clearly contradicts minimal aggregation since no weight is given any person but the worst-off.

Case 2: $\lim_{\rho \to 0^+} F(\rho) < 1$. To show failure of minimal non-aggregation, we must show that, for all $0 < \underline{z} < \bar{z}$ and $\delta > \varepsilon > 0$ there exist $k \in \mathbb{N}$ and $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^k$ with $\mathbf{x} \prec \mathbf{y}$, even though $y_i \leq \underline{z}$ and $x_i - y_i \geq \delta$ for some $i \in I_W(\mathbf{y})$, and it holds that $j \in I_B(\mathbf{x}) \cap I_B(\mathbf{y})$, $y_j \geq \bar{z}$ and $y_j - x_j \leq \varepsilon$ for all $j \in \{1, \ldots, k\}$ with $j \neq i$ and $x_j \neq y_j$.

Choose any $0 < \underline{z} < \bar{z}$ and $\delta > \varepsilon > 0$. Since $\lim_{\rho \to 0^+} F(\rho) < 1$ and $F(\rho)$ is continuous for $\rho \in (0, 1]$, we can choose $m \in \mathbb{N}$ sufficiently large so that $F\left(\frac{1}{m+1}\right) < 1$, and write $\Delta := 1 - F\left(\frac{1}{m+1}\right)$. Construct, for each $\ell \in \mathbb{N}$ with $\ell \geq 2$, $\mathbf{x}^\ell$ and $\mathbf{y}^\ell$ such that $n(\mathbf{x}^\ell) = n(\mathbf{y}^\ell) = \ell(m+1)$, $x_1^\ell = y_1^\ell = y_2^\ell = \underline{z}$, $x_2^\ell = \underline{z} + \delta$, $x_h^\ell = y_h^\ell = \underline{z} + \delta$ for $h = 3, \cdots, \ell$ and $x_{h'}^\ell = \bar{z} - \varepsilon$, $y_{h'}^\ell = \bar{z}$ for $h' = \ell + 1, \cdots, \ell(m+1)$. For each $\ell \geq 2$, let

$$\Gamma(\ell) := F\left(\tfrac{2}{\ell(m+1)}\right) - F\left(\tfrac{1}{\ell(m+1)}\right),$$

where it follows from our assumptions that $\Gamma(\ell) > 0$ for all $\ell \geq 2$ and $\lim_{\ell \to \infty} \Gamma(\ell) = 0$. Hence, we can choose $\ell' \geq 2$ such that

$$\Gamma(\ell')\big(\phi(\underline{z} + \delta) - \phi(\underline{z})\big) < \Delta\big(\phi(\bar{z}) - \phi(\bar{z} - \varepsilon)\big).$$

By definition of the SWOs obtained in Propositions 1, 2 and 3, $\mathbf{y}^{\ell'} \succ \mathbf{x}^{\ell'}$, which contradicts minimal non-aggregation.

Hence, either minimal aggregation or minimal non-aggregation fails. ■

Proof of Proposition 5. *Case 1: The* SWO $\succsim$ *on* $\mathbf{X}$ *is egalitarian.* Clearly minimal aggregation is contradicted since no weight is given any person but the worst-off.

Case 2: The SWO $\succsim$ *on* $\mathbf{X}$ *is not regular rank-dependent.* Assume that there exist real numbers $0 < \underline{z} < \bar{z}$ and $\delta > \epsilon > 0$ as required in the statement of **Minimal Non-Aggregation**. Consider any $k \in \mathbb{R}$, $m \in \mathbb{R}$, with $k < m$. Let $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^l$ be such that $x_i = y_i = \underline{z}$ for all $i < k$, $y_k = \underline{z}$, $x_k = \underline{z} + \delta$, $x_j = z$ and $y_j = z + \epsilon$ for all $j > k$, where $z > \max\{\underline{z} + \delta, \bar{z}\}$. By definition of an absolute rank-dependent SWO, we have:

$$\mathbf{x} \succsim \mathbf{y} \Longleftrightarrow a_k \phi(\underline{z} + \delta) + \left(\sum_{r=k+1}^{m} a_r\right) \phi(z) \geq a_k \phi(\underline{z}) + \left(\sum_{r=k+1}^{m} a_r\right) \phi(z + \epsilon).$$

This can be written:

$$\mathbf{x} \succsim \mathbf{y} \Longleftrightarrow a_k \geq \frac{\phi(z+\varepsilon)-\phi(z)}{\phi(\underline{z}+\delta)-\phi(\underline{z})}\left(\sum_{r=k+1}^{m} a_r\right).$$

But denoting

$$\kappa = \frac{\phi(z+\varepsilon)-\phi(z)}{\phi(\underline{z}+\delta)-\phi(\underline{z})},$$

and given that $\succsim$ is not regular rank-dependent, there must exist $k' \in \mathbb{N}$ such that $a_{k'} < \kappa \cdot \left(\sum_{r=k'+1}^{+\infty} a_r\right)$. This means that there must also exist $m' > k'$ such that $a_{k'} < \kappa \cdot \left(\sum_{r=k'+1}^{m'} a_r\right)$. So, **Minimal Non-Aggregation** must sometimes be violated.

Case 3: The SWO $\succsim$ *on* $\mathbf{X}$ *is proper and regular rank-dependent.* Let $\lambda \in (0,1)$ be such that $\kappa > \lambda \cdot \frac{a_1}{a_2}$. By Chateauneuf, Cohen and Meilijson (2005, Lemma 1), for any x, $y \in \mathbb{R}$, with $x \geq y$:

$$\lambda \cdot G_\phi \geq \frac{\phi(x+\lambda)-\phi(x)}{\phi(y)-\phi(y-1)}.$$

Because the SWO $\succsim$ satisfies **Pigou-Dalton**, we have

$$G_\phi \leq \inf_{k\in\mathbb{N}} \tfrac{a_r}{a_{r+1}} \leq \tfrac{a_1}{a_2}.$$

Therefore, for any x, $y \in \mathbb{R}$, with $x \geq y$:

$$\kappa > \lambda \cdot \tfrac{a_1}{a_2} \geq \frac{\phi(x+\lambda)-\phi(x)}{\phi(y)-\phi(y-1)}.$$

If the SWO $\succsim$ is proper and regular rank-dependent, than it clearly satisfies **Minimal Aggregation**. Let us show that it also satisfies **Minimal Non-Aggregation**, where in the statement of this property we take $\underline{z} = 0$, $\bar{z} > 1$, $\delta = 1$ and $\epsilon = \lambda$. Consider any $k \in \mathbb{N}$, and any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$ and $i \in \{1,\ldots,k\}$, such that (i) $i \in I_W(\mathbf{y})$, $y_i \leq 0$ and $x_i - y_i \geq 1$; and (ii) for all $j \in \{1,\ldots,k\}$ with $j \neq i$ and $x_j \neq y_j$, it holds that $j \in I_B(\mathbf{x}) \cap I_B(\mathbf{y})$, $y_j \geq \bar{z}$ and $y_j - x_j \leq \lambda$. We need to show that $\mathbf{x} \succsim \mathbf{y}$. Let $R = |\{\ell \in \{1,\ldots,k\} : x_\ell \leq x_i\}|$ the number of individuals with well-being lower than x_i in distribution $\mathbf{x}$; $R' = |\{\ell \in \{1,\ldots,k\} : y_\ell \leq y_i\}|$ the number of individuals with well-being lower than y_i in distribution $\mathbf{y}$; $K = |\{j \in \{1,\ldots,k\}\colon\ j \neq i,\ x_j \neq y_j\}|$ the number of individuals other than i with different well-being in $\mathbf{x}$ and $\mathbf{y}$. By definition of a proper rank-dependent SWO:[8]

$$\begin{aligned}\mathbf{x} \succsim \mathbf{y} \iff & \sum_{r=1}^{R'-1} a_r\phi(x_{[r]}) + \sum_{r=R'}^{R} a_r\phi(x_{[r]}) + \sum_{r=R+1}^{k-K} a_r\phi(x_{[r]}) + \sum_{r=k-K+1}^{k} a_r\phi(x_{[r]}) \\ \geq & \sum_{r=1}^{R'-1} a_r\phi(y_{[r]}) + \sum_{r=R'}^{R} a_r\phi(y_{[r]}) + \sum_{r=R+1}^{k-K} a_r\phi(y_{[r]}) + \sum_{r=k-K+1}^{k} a_r\phi(y_{[r]})\end{aligned}$$

By definition of $\mathbf{x}$ and $\mathbf{y}$, this can be written:

$$\mathbf{x} \succsim \mathbf{y} \iff \sum_{r=R'}^{R} a_r\phi(x_{[r]}) + \sum_{r=k-K+1}^{k} a_r\phi(x_{[r]}) \geq \sum_{r=R'}^{R} a_r\phi(y_{[r]}) + \sum_{r=k-K+1}^{k} a_r\phi(y_{[r]})$$

Given that $y_j - x_j \leq \lambda$ for all $j \in \{1, \ldots, k\}$ with $j \neq i$ and $x_j \neq y_j$ (and $x_j = x_{[k]}$ for all such j), a sufficient condition to have $\mathbf{x} \succsim \mathbf{y}$ is:

$$\sum_{r=R'}^{R} a_r \Big[\phi(x_{[r]}) - \phi(y_{[r]})\Big] \geq \left[\sum_{r=k-K+1}^{k} a_r\right] \Big(\phi(x_{[k]} + \lambda) - \phi(x_{[k]})\Big)$$

But $\sum_{r=R'}^{R} a_r \Big[\phi(x_{[r]}) - \phi(y_{[r]})\Big] \geq a_R \Big[\phi(x_i) - \phi(y_i)\Big] \geq a_R \Big[\phi(x_i) - \phi(x_i - 1)\Big]$, and

$$\frac{a_R}{\sum_{r=k-K+1}^{k} a_r} \geq \frac{a_R}{\sum_{r=R}^{+\infty} a_r} \geq \kappa > \frac{\phi(x_{[k]} + \lambda) - \phi(x_{[k]})}{\phi(x_i) - \phi(x_i - 1)}.$$

It is then the case that $\mathbf{x} \succsim \mathbf{y}$. ■

Proof of Proposition 6. *Case 1: The* SWO $\succsim$ *on* $\mathbf{X}$ *is egalitarian.* Clearly the reversed repugnant conclusion is not avoided.

Case 2: The SWO $\succsim$ *on* $\mathbf{X}$ *has non-summable absolute weights and* $c = 0$. Consider any $k \in \mathbb{N}$, $y \in \mathbb{R}_{++}$ and $z \in (0, y)$. By definition of a simple rank-dependent generalized utilitarian SWO:

$$(y)_k \succsim (z)_m \iff \left(\sum_{r=1}^{k} a_r\right) \phi(y) \geq \left(\sum_{r=1}^{m} a_r\right) \phi(z).$$

But given that the absolute weights are non-summable, there must exist m' such

[8] We use the convention that $\sum_{r=k}^{k-1} u_r = 0$.

that

$$\frac{\left(\sum_{r=1}^{m'} a_r\right)}{\left(\sum_{r=1}^{k} a_r\right)} > \frac{\phi(y)}{\phi(z)}$$

so that $(z)_{m'} \succ (y)_k$. This contradicts **Avoidance of the Repugnant Conclusion**.

Case 3: The SWO $\succsim$ *on* $\mathbf{X}$ *has non-summable absolute weights and* $c > 0$. Assume that there exists $\mathbf{x} \in \mathbf{X}$ such that $(z)_k \succsim \mathbf{x}$ for all $k \in \mathbb{N}$ and $z \in \mathbb{R}_{++}$. Let $\ell = n(\mathbf{x})$. By definition of a simple rank-dependent generalized utilitarian SWO, $\mathbf{x} \succsim (x_{[1]})_n$. Take any $z' \in \mathbb{R}_{++}$ such that $z < c$. By definition of a simple rank-dependent generalized utilitarian SWO:

$$(x_{[1]})_n \succ (z')_k \Longleftrightarrow \left(\sum_{r=1}^{n} a_r\right)\big(\phi(x_{[1]}) - \phi(c)\big) > \left(\sum_{r=1}^{k} a_r\right)\big(\phi(z') - \phi(c)\big).$$

If $x_{[1]} > z'$ then $(x_{[1]})_n \succ (z')_n$, so that by transitivity $\mathbf{x} \succ (z')_n$, which is a contradiction. If $x_{[1]} \leq z' < c$ then $\big(\phi(x_{[1]}) - \phi(c)\big) \leq \big(\phi(z) - \phi(c)\big) < 0$ and

$$(x_{[1]})_n \succ (z')_k \Longleftrightarrow \frac{\left(\sum_{r=1}^{k} a_r\right)}{\left(\sum_{r=1}^{n} a_r\right)} > \frac{\phi(c) - \phi(x_{[1]})}{\phi(c) - \phi(z)}.$$

But given that the absolute weights are non-summable, there must exist k' such that

$$\frac{\left(\sum_{r=1}^{k'} a_r\right)}{\left(\sum_{r=1}^{n} a_r\right)} > \frac{\phi(c) - \phi(x_{[1]})}{\phi(c) - \phi(z)}$$

so that $(x_{[1]})_n \succ (z)_{k'}$ and therefore $\mathbf{x} \succ (z^{k'})_n$. This contradicts **Avoidance of the Very Sadistic Conclusion**.

Case 4: The SWO $\succsim$ *on* $\mathbf{X}$ *is proper rank-dependent.* Let us check that $\succsim$ satisfies all the required properties.

Because $\succsim$ is proper rank-dependent $a_2 > 0$ and $(a_1 + a_2)/a_1 > 1$. By continuity of ϕ, for any $y \in \mathbb{R}_{++}$ such that $y > c$ there exists $z \in \mathbb{R}_{++}$ such that

$$1 < \frac{\phi(y) - \phi(c)}{\phi(z) - \phi(c)} < \frac{a_1 + a_2}{a_1}$$

so that $(a_1 + a_2)\big(\phi(z) - \phi(c)\big) > a_1\big(\phi(y) - \phi(c)\big)$. By definition of a simple rank-dependent generalized utilitarian SWO, this implies that $(z)_2 \succ (y)_1$. **Avoidance of the Reversed Repugnant Conclusion** is satisfied.

Because $\succsim$ is proper rank-dependent so that $\sum_{r=1}^{+\infty} a_r < +\infty$, it is also the case that for any $\varepsilon \in \mathbb{R}_{++}$ there exists $\ell \in \mathbb{N}$ such that $\left(\sum_{r=1}^{+\infty} a_r\right)/\left(\sum_{r=1}^{\ell} a_r\right) < 1 + \varepsilon$.

Thus consider $\ell \in \mathbb{N}$, $y \in \mathbb{R}_{--}$ such that

$$\frac{\sum_{r=1}^{+\infty} a_r}{\sum_{r=1}^{\ell} a_r} < \frac{\phi(c)-\phi(y)}{\phi(c)-\phi(0)} = 1 + \frac{\phi(0)-\phi(y)}{\phi(c)-\phi(0)}.$$

Let us show that $(0)_k \succsim (y)_\ell$ for any $k \in \mathbb{N}$. For this to be the case, we need that $\big(\sum_{r=1}^{k} a_r\big)\big(\phi(0)-\phi(c)\big) \geq \big(\sum_{r=1}^{\ell} a_r\big)\big(\phi(y)-\phi(c)\big)$ for all $k \in \mathbb{N}$. Given that $0 \leq c$, a sufficient condition is then that $\big(\sum_{r=1}^{+\infty} a_r\big)\big(\phi(c)-\phi(0)\big) \leq \big(\sum_{r=1}^{\ell} a_r\big)\big(\phi(c)-\phi(y)\big)$, which is true by definition of ℓ and y. For any $z \in \mathbb{R}_{++}$ and any $k \in \mathbb{N}$, $(z)_k \succ (0)_k$ and therefore $(z)_k \succ (y)_\ell$. Hence **Avoidance of the Very Sadistic Conclusion** is satisfied.

If $c > 0$, for any $y > c > z > 0$ and for any $m \geq \ell$, by definition of a simple rank-dependent generalized utilitarian SWO, $(y)_\ell \succ (z)_m$ so that **Avoidance of the Repugnant Conclusion** is satisfied. If $c = 0$ then for any $y > z > 0$, because $\succsim$ is proper rank-dependent, there exists $k \in \mathbb{N}$ such that

$$\frac{\sum_{r=1}^{+\infty} a_r}{\sum_{r=1}^{k} a_r} < \frac{\phi(y)-\phi(0)}{\phi(z)-\phi(0)} = 1 + \frac{\phi(y)-\phi(z)}{\phi(z)-\phi(0)}.$$

Hence, for all $m \geq \ell \geq k$:

$$\begin{aligned}\Big(\sum_{r=1}^{\ell} a_r\Big)\big(\phi(y)-\phi(0)\big) \geq \Big(\sum_{r=1}^{k} a_r\Big)\big(\phi(y)-\phi(0)\big) &> \big(\textstyle\sum_{r=1}^{+\infty} a_r\big)\big(\phi(z)-\phi(0)\big) \\ &\geq \big(\textstyle\sum_{r=1}^{m} a_r\big)\big(\phi(z)-\phi(0)\big).\end{aligned}$$

By definition of a simple rank-dependent generalized utilitarian SWO, $(y)_\ell \succ (z)_m$. Hence **Avoidance of the Repugnant Conclusion** is satisfied. ■

Proof of Proposition 7. Consider any $\hat{k} \in \mathbb{N}$, $\hat{y} \in \mathbb{R}_{++}$, $\hat{z} \in [0, y]$ and $\hat{w} \in (0, z)$.

Let $z < 0$ be the well-being level in the statement of **Weak Non-Sadism Condition**, and k the integer in this statement. For $\ell \geq k$, let $\theta_\ell = [(\ell+k)\phi(\hat{y}) - k\phi(z)]/\ell$. When ℓ is large enough, θ_ℓ can become arbitrary close to $\phi(\hat{y})$, so that $\phi^{-1}(\theta_\ell)$ is well-defined (ϕ is continuous and increasing and thus invertible). Let $\bar{\ell}$ be an integer such that $\phi^{-1}(\theta_{\bar{\ell}})$ is well-defined and denote $\zeta = \phi^{-1}(\theta_{\bar{\ell}})$.

By definition of a prioritarian SWO, $\big((\zeta)_{\bar{\ell}}, (z)_k\big) \sim (\hat{y})_{\bar{\ell}+k}$. But, by **Weak Non-Sadism Condition**, for any $m \in \mathbb{N}$, $\big((\zeta)_{\bar{\ell}}, (\hat{w})_m\big) \succsim \big((\zeta)_{\bar{\ell}}, (z)_k\big)$. However, for m large enough we have:

$$\bar{\ell}\phi(\zeta) + m\phi(\hat{w}) < (\bar{\ell}+m)\phi(\hat{z}),$$

because $\hat{w} < \hat{z}$. Thus, by definition of a prioritarian SWO, there exists $\bar{m}$ large enough (and in particular such that $\bar{m} > k$) such that $(\hat{z})_{\bar{\ell}+\bar{m}} \succ \big((\zeta)_{\bar{\ell}}, (z)_{\bar{m}}\big)$. By transitivity of social welfare orderings, $(\hat{z})_{\bar{\ell}+\bar{m}} \succ (\hat{y})_{\bar{\ell}+k}$.

In conclusion, for any $k \in \mathbb{N}$, $\hat{y} \in \mathbb{R}_{++}$ and $\hat{z} \in [0, \hat{y}]$, there exists $\bar{m} > \bar{\ell}$ such that $(\hat{z})_{\bar{\ell}+\bar{m}} \succ (\hat{z})_{\bar{\ell}+k}$. This is a violation of **Avoidance of the Repugnant Conclusion**. ■

Proof of Proposition 8. By **Weak Non-Sadism Condition**, there exist $z < 0$ and $k \in \mathbb{N}$ such that, for any $z' < z$ and $y \in \mathbb{R}_{++}$: $\big((z')_1, (y)_{k+1}\big) \succsim \big((z')_1, (z)_k\big)$. By definition of an egalitarian SWO and transitivity, this implies that $(z')_{k+2} \succsim (z')_{k+1}$. This is a violation of **Negative Mere Addition Principle**. ■

Proof of Proposition 9. Let us (without loss of generality) normalize function ϕ so that $\phi(0) = 0$.

The simple rank-dependent SWO $\succsim$ *satisfies* ***Weak Non-Sadism Condition***. Assume that function ϕ is concave and bounded above and that there exists $\kappa \in \mathbb{R}_{++}$ such that, for all $k \in \mathbb{N}$, $a_k \geq \kappa \cdot \left(\sum_{r=k+1}^{+\infty} a_r\right)$. Let $\bar{u} := \sup\{\phi(z) : z \in \mathbb{R}\}$. Choose $\hat{z} \in \mathbb{R}_{--}$ such that[9]

$$-\phi(\hat{z}) > \tfrac{\bar{u}}{\kappa}. \tag{5}$$

Then, for any initial distribution $\mathbf{x} \in \mathbf{X}$, let r_1 be the highest rank of an individual with well-being strictly lower than $\hat{z}$ and r_2 the highest rank of an individual with well-being strictly lower than 0. By definition of a simple rank-dependent , for any and $\ell \in \mathbb{N}$, we have $\big(\mathbf{x}, (0)_\ell\big) \succsim \big(\mathbf{x}, (\hat{z})_1\big)$ if and only if:

$$\begin{aligned}
&\sum_{r=1}^{r_1} a_r \left(\phi(x_{[r]}) - \phi(c)\right) + a_{r_1+1}\left(\phi(\hat{z}) - \phi(c)\right) + \sum_{r=r_1+2}^{n(x)+1} a_r \left(\phi(x_{[r-1]}) - \phi(c)\right) \\
&\leq \sum_{r=1}^{r_1+r_2} a_r \left(\phi(x_{[r]}) - \phi(c)\right) + \left(\sum_{r=r_1+r_2+1}^{r_1+\ell+r_2} a_r\right)\left(\phi(0) - \phi(c)\right) \\
&\quad + \sum_{r=r_1+\ell+r_2+1}^{\ell+n(x)} a_r \left(\phi(x_{[r-\ell]}) - \phi(c)\right)
\end{aligned} \tag{6}$$

[9] There must exists such a $\hat{z} \in \mathbb{R}_{--}$: given that $\phi(0) = 0$ and ϕ is concave, so that it is not bounded below on $\mathbb{R}$.

Given that $0 = \phi(0) > \phi(x_{[r]}) \geq \phi(\hat{z})$ for all $r_2 \geq r > r_1$, we have

$$\begin{aligned} & a_{r_1+1}\left(\phi(\hat{z}) - \phi(c)\right) + \sum_{r=r_1+2}^{r_1+r_2+1} a_r \left(\phi(x_{[r-k]}) - c\right) \\ & \leq \sum_{r=r_1+1}^{r_1+r_2} a_r \left(\phi(x_{[r]}) - c\right) + a_{r_1+r_2+1}\left(\phi(\hat{z}) - \phi(c)\right), \end{aligned}$$

and,

$$\sum_{r=r_1+r_2+2}^{n(x)+1} a_r \left(\phi(x_{[r-k]}) - \phi(c)\right) \leq \sum_{r=r_1+r_2+2}^{n(x)+1} a_r \left(\bar{u} - \phi(c)\right) < \left(\sum_{r=r_1+r_2+2}^{+\infty} a_r\right)\left(\bar{u} - \phi(c)\right).$$

Hence,

$$\begin{aligned} & \sum_{r=1}^{r_1} a_r \left(\phi(x_{[r]}) - \phi(c)\right) + a_{r_1+1}\left(\phi(\hat{z}) - \phi(c)\right) + \sum_{r=r_1+2}^{n(x)+1} a_r \left(\phi(x_{[r-1]}) - \phi(c)\right) \\ < & \sum_{r=r_1+1}^{r_1+r_2} a_r \left(\phi(x_{[r]}) - c\right) + a_{r_1+r_2+1}\left(\phi(\hat{z}) - \phi(c)\right) + \left(\sum_{r=r_1+r_2+2}^{+\infty} a_r\right)\left(\bar{u} - \phi(c)\right). \end{aligned}$$

Also, given that $\phi(x_{[r]}) \geq \phi(0)$ for all $r \geq r_2+1$ and $0 \leq c$ (so that $\phi(0)-\phi(c) \leq 0$):

$$\begin{aligned} & \left(\sum_{r=r_1+r_2+1}^{r_1+r_2+\ell} a_r\right)\left(\phi(0) - \phi(c)\right) + \sum_{r=r_1+\ell+r_2+1}^{\ell+n(x)} a_r \left(\phi(x_{[r-\ell]}) - \phi(c)\right) \\ > & \left(\sum_{r=r_1+r_2+1}^{+\infty} a_r\right)\left(\phi(0) - c\right). \end{aligned}$$

Hence,

$$\begin{aligned} & \sum_{r=1}^{r_1+r_2} a_r \left(\phi(x_{[r]}) - \phi(c)\right) + \left(\sum_{r=r_1+r_2+1}^{+\infty} a_r\right)\left(\phi(0) - \phi(c)\right) \\ \leq & \sum_{r=1}^{r_1+r_2} a_r \left(\phi(x_{[r]}) - \phi(c)\right) + \left(\sum_{r=r_1+r_2+1}^{r_1+\ell+r_2} a_r\right)\left(\phi(0) - \phi(c)\right) \\ & + \sum_{r=r_1+\ell+r_2+1}^{\ell+n(x)} a_r \left(\phi(x_{[r-\ell]}) - \phi(c)\right) \end{aligned}$$

A sufficient condition for inequality (6) to hold is therefore:

$$a_{r_1+r_2+1}\left(\phi(\hat{z})-\phi(c)\right)+\left(\sum_{r=r_1+r_2+2}^{+\infty} a_r\right)\left(\bar{u}-\phi(c)\right) \leq \left(\sum_{r=r_1+r_2+1}^{+\infty} a_r\right)\left(\phi(0)-\phi(c)\right).$$

Given that $\phi(0)=0$, this sufficient condition can be written

$$a_{r_1+r_2+1}\phi(\hat{z})+\left(\sum_{r=r_1+r_2+2}^{+\infty} a_r\right)\bar{u} \leq 0.$$

But we know that:

$$\frac{1}{\kappa} \geq \frac{\sum_{r=k+1}^{+\infty} a_r}{a_k} \text{ for all } k \in \mathbb{N}, \quad \text{and} \quad -\phi(z) > \frac{\bar{u}}{\kappa}.$$

This sufficient condition is thus satisfied.

Therefore we have shown that there exist $\hat{z} \in \mathbb{R}_{--}$ such that $\big(\mathbf{x},(0)_\ell\big) \succsim \big(\mathbf{x},(\hat{z})_1\big)$ for any $\ell \in \mathbb{N}$. The Suppes-Sen principle implies that, for any $y \in \mathbb{R}_{++}$ $\big(\mathbf{x},(y)_\ell\big) \succ \big(\mathbf{x},(0)_\ell\big)$, and by transitivity $\big(\mathbf{x},(y)_\ell\big) \succ \big(\mathbf{x},(\hat{z})_1\big)$.

The simple rank-dependent SWO $\succsim$ *satisfies* ***Negative Mere Addition Principle***. Consider any $\mathbf{x} \in \mathbf{X}$ and any $z \in \mathbb{R}_{--}$. Let r_1 be the highest rank of an individual with well-being strictly lower than z and r_2 the highest rank of an individual with well-being strictly lower than c. By definition of a simple rank-dependent SWO $\mathbf{x} \succ \big(\mathbf{x},(z)_1\big)$ if and only if:

$$\begin{aligned}
&\sum_{r=1}^{r_1} a_r\left(\phi(x_{[r]})-\phi(c)\right)+a_{r_1+1}\left(\phi(z)-\phi(c)\right)+\sum_{r=r_1+2}^{r_2+1} a_r\left(\phi(x_{[r-1]})-\phi(c)\right)\\
&+\sum_{r=r_2+2}^{n(x)+1} a_r\left(\phi(x_{[r-1]})-\phi(c)\right)\\
<&\sum_{r=1}^{r_1} a_r\left(\phi(x_{[r]})-\phi(c)\right)+\sum_{r=r_1+1}^{r_2} a_r\left(\phi(x_{[r]})-\phi(c)\right)+\sum_{r=r_2+1}^{n(x)} a_r\left(\phi(x_{[r]})-\phi(c)\right)
\end{aligned} \tag{7}$$

Inequality (7) can be written:

$$a_{r_1+1}\left(\phi(z)-\phi(x_{[r_1+1]})\right)+\sum_{r=r_1+2}^{r_2} a_r\left(\phi(x_{[r-1]})-\phi(x_{[r]})\right)+a_{r_2+1}\left(\phi(x_{[r_2]})-\phi(c)\right)$$

$$< \sum_{r=r_2+1}^{n(x)} (a_r-a_{r+1})\left(\phi(x_{[r]})-\phi(c)\right)$$

By definition of r_1 and r_2 the left-hand side of the above inequality is strictly negative, while the right-hand side is non-negative. Hence inequality (7) holds, and therefore $\mathbf{x} \succ \left(\mathbf{x},(z)_1\right)$. ■

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