Boole Algebra in a Contemporary Setting.

Boole-Operations, Types as Propositions and Immanent Reasoning

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Preliminary words

The work of Souleymane Bachir Diagne has set a landmark in many senses, but perhaps the most striking one is his inexhaustible thrive to build multifarious conceptual links and bridges between traditions and to motivate others to further develop this wonderful realization of unity in diversity. Three main fields of his remarkable work are: history and philosophy of logic (Diagne (1989, 1992)), the renewal of Islamic thinking (Diagne (2001b, 2002, 2008, 2016)) and the specificity of the African philosophy (Diagne (1996, 2001a, 2007).

In the present talk I will focus on philosophy of logic, and more precisely on the algebra of logic of George Boole, that launched Bachir Diagne's (1989) academic carrier. However, the framework has bearings for the other both fields as developed in recent publications in collaboration. I will briefly discuss as an example of application the case of suspensive (muʿallaq) condition (taʿliq) in Islamic law and I might discuss this issue more deeply during the discussion,

More precisely, the main objective of my presentation is to discuss a novel approach to both, the distinction between Boolean operators and inferentially defined connectives, and to bring forward a framework where the interplay of the former with the latter yields *an integrated epistemic and pragmatist conception of reasoning*. The epistemological framework underlying my discussion is the dialogical approach to Per Martin-Löf's (1984) *Constructive Type Theory*.

I will test the fruitfulness of the approach by providing case-studies in the domains of

- Foundations of Mathematics
- Logic
- Epistemology

I Introduction:

Most of the literature differentiating the philosophical perspective underlying the work of Boole and the one of Frege focused on discussing either the different ways both authors understood the relation between logic and psychology and/or the links between mathematics

[1] The paper has been developed in the context of the researches for transversal research axis *Argumentation* (UMR 8163: STL), the research project ADA at the MESHS-Nord-pas-de-Calais and the research projects: ANR-SÊMAINÔ (UMR 8163: STL).

and logic. According to these studies, Boole's framework has been mainly conceived as both a kind of psychologism and a programme for the mathematization of logic in contrast to Frege's radical anti-psychologism and logicist-project for the foundations of mathematics. These comparative studies have also been combined with the contrast between model-theoretical approaches to meaning and the associated notion of *varying domains of discourse*, versus inferentialists approaches to meaning with a *fixed universe of discourse*. While the former, it might be argued, is more naturally understood as an offspring of the algebraic tradition of Boole-Schröder the second can be seen as influenced by Frege's *Begriffsschrift*.[2]

However, from the point of view of contemporary classical logic, and after the meta-mathematical perspective of Gödel, Bernays and Tarski, both Boole's and Frege's view on semantics are subsumed under the same formalization, according to which classical semantics amounts to a function of interpretation between the sentences $\boldsymbol{S}$ of a given language $\boldsymbol{L}$ and the set of truth values $\{\mathbf{0}, \mathbf{1}\}$ – let us call such a set the set **Bool**. This function assumes that the well-formed formulae of $\boldsymbol{S}$ are made dependent upon a domain – either a *local domain of discourse* (in the case of Tarski's-style approach to Boolean-algebra) or a *universal domain* (in the case of Frege). More precisely, this functional approach is based on a separation of cases that simply assumes that the quantifiers and connective take propositional functions into classical propositions – for a lucid insight on its limitations see Sundholm (2004, 2006). In fact, the integration of both views within the same formal semantic closes a gap in Boole's framework already pointed out by Frege: the links between the semantics of propositional and first-order-logic.[3]

*Constructive Type Theor*y includes a third (epistemic) paradigm in the framework that allows at the same time a new way of dividing the waters between Boolean operators and logical connectives, and integrating them in a common inferential system where each of them has specific role to play. The overall paradigm at stake is Brouwer/Heyting/Kolmogorov conception of propositions as sets of proofs embedded in framework, where, thanks to the insight brought forward by the Curry-Howard isomorphism, propositions are read as sets and as types.
In a nutshell: the CTT framework takes judgement rather than proposition as furnishing the minimal unity of knowledge and meaning – as the old philosophical tradition did before the spreading of the metalogical view of Gödel, Tarski and Bernays - see Sundholm (1997, 1998, 2001, 2009, 2012, 2013a,b). Within CTT the simplest form of a judgement (the categorical) is an expression of the form

$$a : B,$$

where "*B"* is a proposition and" *a"* a proof-object on the grounds of which the proposition *B* is asserted to be true , for short it stands for

"*a provides evidence for B".*

[2] Recall jean van Heijenoort's (1967) distinction of a *language as the universal medium* and *language as a calculus*.

[3] Frege points out that within Boole's approach there is no organic link between propositional and first-order logic : " *In Boole the two parts run alongside one another ; so that one is like the mirror image of the other, but for that very reason stands in no organic relation to it*" (Frege [1880/81], the quote stems from the translation Frege (1979, p. 14) " Boole's logical Calculus and the concept Script" In Hermes/Kambartel/Kaulbach (eds.), *Gottlob Frege.Posthumous Writtings*,Oxford : Basil Blackwell, 1979 pp.9-52.

In other words, the expression "*a* : *B*", is the formal notation of the categorical judgment

"*The proposition B is true*",

which is a short-form for

"*There is evidence for B*".

According to this view, a proposition is a set of elements, called proof-objects that make the proposition *B* true. Furthermore we distinguish between c*anonical proof-objects*, those entities that provide a *direct evidence* for the truth the proposition *B*, and *non-canonical proof-objects*, that provide indirect pieces of evidence for *B*.
This generalization also allows another third reading: a proposition is a *type* and its elements are instance of this type. If we follow this reading proof-objects are conceived as instantiations of the type. This type-reading naturally leads to Brouwer/Heyting/Kolmogorov's constructivism mentioned above: If a proposition is understood as the set of its proofs, it might be the case that there is no proof for that proposition at disposition nor do we have proof for its negation (thus, in such a framework, *third excluded* fails). Notice that the constructivist interpretation requires the intensional rather than the extensional constitution of sets – recall the Aristotelian view that no "form" ("type") can be conceived independently of its instances and the vice-versa.

Moreover CTT provides too a novel way to render the meaning of the set {**0**, **1**} as the type **Bool**. More precisely the type **Bool** is characterized as a set the canonical elements of which are **0** and **1**. Thus, each non-canonical element is equal to one of them. But what kind of entities are those (non-canonical) elements that might be equal to **1** or **0**? Since in such a setting **1**, **0** and those equal to them are elements, they are not considered to be of the type proposition, but rather providers of truth or falsity of a proposition (or a set, according to the Curry-Howard isomorphism between propositions-sets-types): they are proof-objects that provide evidence for the assertion **Bool true**).

Let me take the liberty to speak (for the moment) a bit loosely and bring forward an example that is beyond mathematics: Take the s**entence**

Bachir Diagne is from Senegal.

If we take the sentence as expressing the **proposition**

That Bachir Diagne is from Senegal

(i.e., that what Frege called the *sense* or *thought* expressed by that sentence) then, we might be able to bring forward some proof-object, some piece of evidence *a*, such as his passport or birth certificate that renders the proposition true. In such a case we have the assertion that the proposition is true on the grounds of the piece of evidence *a* (the passport)

passport : Bachir Diagne is from Senegal

Or the more general assertion

That Bachir Diagne is from Senegal **true**

(*there is some piece of evidence that Bachir Diagne is from Senegal*)

If we take the sentence *Bachir Diagne is from Senegal* as related to a **Boolean object**, it is then conceived as triggering the outcome of a procedure that yields a non-canonical element, say **X**, of the set **Bool**. In such a case the sentence does not express a proposition, but it can be understood to be the answer to the question

Is Bachir Diagne from Senegal ?

The answer

(**yes**) *Bachir Diagne is from Senegal*

yields the outcome **1**. In other words, the way to determine to which of the canonical elements, **1** or **O** the non-canonical element **X** is equal, requires answering to the question *Is Bachir Diagne from Senegal* ?. Thus In our case, we take it to be equal to **1** [4]

Is Bachir Diagne from Senegal?



yes, *Bachir Diagne is from Senegal*



X = **1** : **Bool**

(The arrows should indicate that determining which of the elements **X** is equal to, is the result of an enquiry (in this case an empirical one)).

Which is not only different from

passport : Bachir Diagne is from Senegal

But from

Bool true

Indeed, while "**X** = **1** : **Bool**", expresses one of the possible outcomes the element **X** can take in **Bool**, "**Bool true**", expresses the fact that the at least one element of the set **Bool** can be brought forward.

Thus, a distinction is drawn between the Boolean object **1** (one of the canonical elements of **Bool**) and the predicate **true** that applies to **Bool**)

Moreover, operations between elements of **Bool** are not then the logical connectives introduced by natural-deduction rules at the right of the colon, but operations between objects occurring at the left of the colon. For example while "**+** " at left of the colon in

[4] For the interpretation of empirical propositions see Martin-Löf (2014)

$$A\mathbf{+}\ B = \mathbf{1} : \mathbf{Bool}\ (\text{given } A = \mathbf{1} : \mathbf{Bool},)$$

stands for an operation between the non-canonical **Boolean** objects *A* and *B*, the disjunction occurring at right of the colon in the assertion

$$b : A \vee B\ (\text{given } b : A),$$

expresses the known logical connective of disjunction that is true because there is a piece of evidence for one of the disjuncts, namely the piece of evidence *b* for *A*.

Since **Bool** is a type, and since according to the Curry-Howard isomorphism, it is itself a proposition, we can certainly have both, propositional connectives as sets of proof-objects, combined with Boolean operations. This allows us, for example, to demonstrate that each canonical element in **Bool** is identical either to **1** or **O**:

$$(\forall x\text{: }\mathbf{Bool})\ \mathbf{Id}(\mathbf{Bool}, x, \mathbf{1}) \vee \mathbf{Id}(\mathbf{Bool}, x, \mathbf{O})\ \textbf{\textit{true}}$$

As already mentioned I will test the approach by discussing some case-studies in the domains of

1. Foundations of Mathematics
2. Logic
3. Epistemology

- Concerning the foundations of mathematics I will discuss in detail how to demonstrate ***within*** the system – that is, without presupposing a metalanguage - that the two canonical elements **yes**, **no** of the set **Bool** are different. This proof yields a straightforward method for developing a demonstration of what is known as the **fourth axiom of Peano's arithmetic** ("0 is identical to no successor of a natural number": $(\forall x : \mathbb{N})\ \neg\mathbf{Id}(\mathbb{N}, 0, \mathbf{s}(x))$). Moreover, such a demonstration gives us the chance to delve into the notion of a universe $\mathcal{U}$ constituted by sets dependent upon the Boolean set {**yes**, **no**}. In other words, while $\mathcal{U}$ is constituted by codes of sets there is no code for $\mathcal{U}$ itself. – universes constitute the constructivist formulation of the mathematical notion of *sets of sets*.

- In relation to the first I will illustrate these issues by showing how to generalize Boolean operators for finite sets within the dialogical setting and I will take the chance to put this dialogical framework by integrating logics tolerant to some contradictions

- More generally; the epistemological background underlying the dialogical framework offers a natural interpretation to the normative account of inferentialism we call i*mmanent reasoning* (see Rahman/Klev/McConaughey/Clerbout 2017-18), which, as briefly sketched above, provides new insights into the way of building empirical propositions out of Boolean sets.
  Indeed, Immanent reasoning, furnishes a formal approach to reasoning that is rooted in the dialogical constitution and "internalization" of content – including empirical one - rather than in the syntactic manipulation of un-interpreted signs (with "internalization" we mean that the relevant content is made part of the setting of the game of giving and

asking for reasons: any relevant content is the content displayed during the interaction.[5] Furthermore, within the framework of immanent reasoning, the internalization of empirical content is obtained by dealing with an "empirical quantity" as the outcome of a procedure triggered by a question specific to that quantity. This, provides a new perspective on Willfried Sellars's (1991, pp. 129-194) notion of *Space of Reasons*. More precisely, the dialogical framework proposed should show how to integrate world-directed thought (that displays empirical content) into an inferentialist approach.

This suggests that the dialogical approach to Constructive Type Theory offers a way to integrate within one epistemological framework the two conflicting readings of the Space of Reasons brought forward by John McDowell (2009, pp. 221-238) on one side, who insists in distinguishing world-direct thought and knowledge gathered by inference and in the other, by Robert Brandom (1997) who interprets Sellars work in a more radical anti-empiricist manner. The point is not only that we can deploy the CTT-distinction between reason as a premise and reason as the piece of evidence justifying a proposition but also that the dialogical framework allows distinguishing between the objective justification level targeted by Brandom (1997, p. 129) but also the subjective level stressed by McDowell. According to our approach the sujective feature corresponds to the play-level, where a concrete player brings forward the statement *It looks red to me*, rather than *It is red*.

The general epistemological upshot from these initial reflections is that, on our view, many of the worries on the interpretation of the space of reasons and on the shortcomings of the standard dialogical approach to meaning (beyond the one of logical constants) have their origin in the neglect of the play level.[6]

II Within and Beyond the set Bool in a Dialogical Setting [7]

II.1 Dialogical Rules for Boolean Operators

In the dialogical framework,[8] the elements of **Bool** are responses to **yes**-**no** questions: so that each element is equal to **yes** or **no**. Responses such as b = **yes** or b = **no** makes explicit one of the possible origins of the answer **yes**(or **no**), namely whether b is or not the case.

Statement	Challenge	Defence
Synthesis **X** ! **Bool**	**Y** ? $^{\textbf{Bool}}$	**X yes** : **Bool** **X no** : **Bool**
Analysis and Equalities **X** p : $C(c)$ (c : **Bool**)	**Y** ?= $c^{\textbf{Bool}}$	**X** c = **yes** : **Bool** **X** c = **no : Bool**

[5] For a discussion on this conception of internalization see Peregrin (2014, pp. 36-42). We will come back to this notion in the last section of the present paper.

[6] For some recent literature on those kind of objections to the approach to meaning of the dialogical conception of logic of Lorenzen/Lorenz (1978) see Duthil Novaes (2015) and Trafford (2017, chapter 4, section 2).

[7] This section is based on previous work in Rahman/M^{c}Conaughey/Klev/Clerbout (2017), Rahman/Redmond/Clerbout (2017) and Rahman/Clerbout/Redmond (2017).

[8] For a short overview see appendix I and II. The first formulation of the dialogical approach to CTT was Clerbout/Rahman (2015).

X c = **yes** : **Bool** … **X** p : $C(c)$ (c : **Bool**)	**Y** ? $_{\text{reason}C(\textbf{yes})}$	**X** p_1 : C(**yes**)
X c = **no** : **Bool** … **X** p : $C(c)$ (c : **Bool**)	**Y** ? $_{\text{reason}C(\textbf{no})}$	**X** p_2 : C(**no**)
		Given the statements **P** p_1 : C(**yes**) (or **P** p_2 : C(**no**)), the play continues by **O** challenging the elementary statement according to the attack prescribed by the general Socratic Rule.

Special Socratic Rule for Bool

P may always bring forwards requests of the form
P ?$^{\textbf{Bool=}}$ a, provided the setting of the play includes the statement **O** a : **Bool**
the responses and further moves are prescribed by the following table

Challenge	Defence
P ?= $a^{\textbf{Bool}}$ (provided **O** a : **Bool)**	**O** a = **yes** : **Bool** or **O** a = **no : Bool**

We can now introduce quite smoothly the rules for the classical truth functional connectives as operations between elements of **Bool.** We leave the description for quantifiers to the diligence of the reader whereby the universal quantifier is understood as a finite sequence of conjunctions and dually, the existential as a finite sequence of disjunctions. .
The dialogical interpretation of the rules below is very close to the usual one: it amounts to the commitments and entitlements specified by the rules of the dialogue: if for instance the response is **yes** to the conjunction, then the speaker is also committed to answer **yes** to further questions on both of the components of that conjunction.

Statement	Challenge	Defence	Strategic Reason
		X (a**x**b) = **yes** : **Bool**	

X a**x**b : **Bool**	**Y** ? = a**x**b	or **X** (a**x**b) = **no** : **Bool**	
X (a**x**b) = **yes** : **Bool**	**Y** $?_L$**x** **yes** or **Y** $?_R$**x** **yes**	**X** a = **yes** : **Bool** respectively **X** b = **yes** : **Bool**	**yes** ⟦<a = **yes**, b = **yes**>⟧^{O} **P** : **Bool** **no** ⟦ a = **no** | a = **no** ⟧^{O}
X (a**x**b) = **no** : **Bool**	**Y**? **x** **no**	**X** a = **yes** : **Bool** or **X** b = **no** : **Bool**	**Glose**: If both of the components of the conjunction are affirmative, the recap-answer is **yes**. If at least one of both component is a denial the recap-answer is **no**.
X a**+** b : **Bool**	? = a**+** b	**X** (a**+** b) = **yes** : **Bool** or **X** (a**+** b) = **no** : **Bool**	**yes** ⟦ a = **yes** | b = **yes** ⟧^{O} **P** : **Bool** **no** ⟦< a = **no** , b = **no**>⟧^{O}
X (a**+** b) = **yes** : **Bool**	**Y**? **+** **yes**	**X** a = **yes** : **Bool** or **X** b= **yes** : **Bool**	
X (a**+** b) = **no** : **Bool**	**Y** $?_L$**+** **no** or **Y** $?_R$**+** **no**	**X** a = **no** : **Bool** respectively **X** b = **no** : **Bool**	
X a→b : **Bool**	**Y** ? = a→b	**X** (a→b) = **yes** : **Bool** or **X** (a→b) = **no** : **Bool**	
X (a→b) = **yes** : **Bool**	**Y** a = **yes** : **Bool** or **Y** b = **no** : **Bool**	**X** b = **yes** : **Bool** respectively **X** a = **no** : **Bool**	**yes** ⟦< a = **yes** , b = **yes** | b = **no** , a = **no** >⟧^{O} **P** : **Bool** **no** ⟦< a = **yes** , b = **no** >⟧^{O}

X $(a \rightarrow b) =$ **no** : **Bool**	**Y** $?_{L \rightarrow \mathbf{no}}$ or **Y** $?_{R \rightarrow \mathbf{no}}$	**X** $a =$ **yes** : **Bool** respectively **X** $b =$ **no** : **Bool**	
X $\sim a$: **Bool**	**Y** ? $\sim a$:	**X** $\sim a =$ **yes** : **Bool** or **X** $\sim a =$ **no** : **Bool**	
X $\sim a =$ **yes** : **Bool**	**Y** ? $\sim_{\mathbf{yes}}$	**X** $a =$ **no** : **Bool**	**P** : **Bool** (**yes** $[\![a = \mathbf{no}]\!]^{\mathbf{O}}$ / **no** $[\![a = \mathbf{yes}]\!]^{\mathbf{O}}$)
X $\sim a =$ **no** : **Bool**		**X** $a =$ **yes** : **Bool**	

II.2 Equality and Identity within the Set Bool

One natural way to combine Boolean operations and elements with propositional connectives is to make use of the identity predicate **Id**, which should be differentiated from the definitional equality, nominal definitions, and equality (or Identity) as a relation that build up a proposition. While the first kind, does not express a proposition but introduces real definitions and establish an equivalence relation between pieces of evidence (proof-objects), the second form produces linguistic abbreviations, the third, is the relation we know from first-order logic and constitutes a proposition.

So we distinguish between

1. the real definition or *judgemental equality* $a = b : A$
2. the *nominal definition*, for example *"1" stands for successor of "s(0)"*
3. the propositional *identity* $c : a =_A b$, or better c : **Id**(A, a, b)

In a dialogical setting

- real definitions express at the object language level the right of the Proponent to state *b* since **O** already stated, both, *a*, and that *a* defines *b*. So **P**'s move *a* = *b* : *A* as a response to request of justifying *b* : *A*, can be read as "you just conceded *a* : *A* and furthermore you conceded that a defines *b*".
- nominal definitions allows **P** to deploy the abbreviations established such kind of definition
- **P** is allowed to state the Identity **Id**(*A*, *a*, *b*) only if he can state that *c* is equal to the local (reflexivity) reason **refl**(*A*, *a*) - that is if he can state **refl**(B, *a*) = *c* : **Id**(*A*, *a*, *b*), and that he can show that the equality *a* = *b* : *B* presupposed by the formation of **Id**(*A*, *a*, *b*) has been fulfilled (see appendix III).
- In fact while winning strategies (dialogical demonstrations) concern the process of bringing forward the piece of evidence that justifies the proposition involved in the judgement, the comitments engaged by asserting that something is one of the pieces of evidence for **Bool**, say, ***a*+ ~*a*** : **Bool**, amounts answering to the question, Which of the canonical elements of **Boo**l is this piece of evidence equal to? – in our case : ***a*+ ~*a*** = **yes** : **Bool.**

Let us see how real definitions and Identity interact in the case of establishing the validity of proposition that every element of the set **Bool** is equal to **yes** or to **no**.

Let us run those plays that together constitute a winning strategy.

Notice that since the set **Bool** contains only two elements universal quantification over **Bool** can be tested by considering each of the elements of the set Each of them triggers a new play

Example:
One interesting application of the use of Booleans is the interpretation and demonstration of the classical third-excluded.

O			**P**		
				! (∀*x* : **Bool**) (**Id**(**Bool**, *x*, **yes**) ∨ **Id**(**Bool**, *x*, **no**)).	0
1	m =1			n = 2	2
3	**yes** : **Bool**	0		**yes** : (**Id**(**Bool**, **yes**, **yes**) ∨ **Id**(**Bool**, **yes**, **no**)).	4
5	? ∨	4		$L^{\vee}$(**yes**) : **Id**(**Bool**, **yes**, **yes**)	6
7	? --- / $L^{\vee}$(**yes**)	6		**refl**(**Bool**, **yes**) : **Id**(**Bool**, **yes**, **yes**)	8
9	? = **refl**(**Bool**, **yes**)	8		**yes** = **refl**(**Bool**, **yes**) : **Bool**	10
11	? $\mathbf{Id}^{(\mathbf{Bool}, \mathbf{yes}, \mathbf{yes})}$	8		**yes** = **yes** : **Bool**	12
				P wins	

While moves 2-4 result from applying the particle rules for the universal quantifier. 5-6 are triggered by applying the particle rules for the disjunction Moves 9-12 are the result of applying the rules of synthesis for **Id** + an application of the general Socratic Rule for local reasons

We leave it to the reader to check the play where the third move is **no** : **Bool**.

Notice that, in this framework, though it is trivial to show

$$\mathbf{P}\ a\mathbf{+}\ {\sim}a = \mathbf{yes} : \mathbf{Bool}$$

we cannot build a winning strategy for:

$$!\ (\forall x : \mathbf{Bool})\ (\mathbf{Id}(\mathbf{Bool}, x, \mathbf{yes}) \vee \neg\mathbf{Id}(\mathbf{Bool}, x, \mathbf{yes})),$$

unless we already presuppose $\neg\mathbf{Id}(\mathbf{Bool}, \mathbf{no}, \mathbf{yes})$. We will come back to this issue further on, but let us first do one of the favourite tasks of logicians, namely generalizing.

II.3 Beyond Bool: Finite Sets and Large Sets of Answers.

A natural extension of the framework is to have a larger set of answers than just the **yes-no** responses of **Bool**. The interpretation scope offered by the generalization is quite broad: it can be interpreted as the different degrees of certainty an answer to a question can take, or, it can also be understood as encoding different possible answers to a question , so that **0** is the answer *a*, **1** is the answer *b* and so on (we will discuss some examples in the following section).

Since the formation rule for a finite set $\mathbb{N}_n$ of n canonical elements (such that *n* stands for some natural number) has in the CTT setting no premisses the dialogical formation rule amounts to the following:

Statement	**Challenge**	**Defence**
X ! $\mathbb{N}^n$	**Y** $?_F\mathbb{N}^n$	**X** ! $\mathbb{N}^n$ **:** *set*

The rules of synthesis and analysis are a straightforward generalization of the set **Bool** (that is the set $\mathbb{N}_2$).

Statement	**Challenge**	**Defence**
Synthesis **X** ! $\mathbb{N}^n$	**Y** ? $\mathbb{N}^n$	**X** $m_1 : \mathbb{N}^n$ ... **X** $m_n : \mathbb{N}^n$
Analysis and Equalities **X** $p : C(c)\ (c : \mathbb{N}^n)$	**Y** $?= c^{\mathbb{N}^n}$	**X** $c = m_1 : \mathbb{N}^n$ **…** **X** $c = m_n : \mathbb{N}^n$

X $c = m_1 : \mathbb{N}^n$ … **X** $c = m_n : \mathbb{N}^n$ … **X** $p : C(c)\ [c : \mathbb{N}^n]$	**Y** $c : \mathbb{N}^n$ … **Y** ? $_{\text{reason}C(c)}$	**X** $p_1 : C(m_1)$ … **X** $p_n : C(m_n)$ Then the play continues by **O** challenging the elementary statement according to the attack prescribed by the general Socratic Rule. This procedure yields the remaining equalities

The case of $\mathbb{N}^0$ and $\mathbb{N}^1$

$\mathbb{N}^0$: If we follow our main interpretation of as statement such as **X** ! $\mathbb{N}^0$ can be understood as stating that there is no local reason that can be adduced for the empty set. From a more dialogical point of view, we can conceive $\mathbb{N}^0$ as the empty set of possible answers to an enquiry. In other words, the player who states it, states that there is no possible answer or solution to the enquiry at stake. In fact, in an analogue way to the Kolmogorov interpretation of a proposition as a problem associated with all what can count as a solution to it, in the dialogical setting one natural reading is to understand a proposition as a solution to a problem or enquiry.
Accordingly, the dialogical rule for $\mathbb{N}^0$ is the same as the one for $\bot$, i.e. the rule for giving up:

- The player who states $\mathbb{N}^0$ (or $p : \mathbb{N}^0$)at move *n* loses the current play. If it is **O** who states it, **P** can adduce the local reason **O**-*gives up*(*n*) in support for any statement that he has not defended before **O** stated $\mathbb{N}^0$ at move *n*.

$\mathbb{N}^1$: If $\mathbb{N}^0$ is in fact the empty set $\bot$, then the unary set is $\top$, inhabited by only one local reason, namely **yesyes** : $\top$

- The player who states $\mathbb{N}^1$, can always adduce **yesyes** as its local reason.

III The set Bool and an application to the foundations of mathematics

III.1 Universes and codes of sets

The main motivation of introducing *universes* is to have a device for dealing with contexts where the use of sets of sets are required. However, cannot have the set of all sets, since we cannot describe all the possible ways of constituting a set. However, since sets of sets are particularly useful in the foundations of mathematics, Martin-Löf (1984, pp. 47-49) introduces the notion of *universe of small sets*. A universe $\mathcal{U}$ =is a set of *codes* of sets, say $\mathbf{n^n}$ is the code of the set $\mathbb{N}^{\mathbf{n}}$. A *small set* is a set with a code. The universe $\mathcal{U}$ has no code in $\mathcal{U}$ (otherwise a paradox follows). The formation of a universe requires a decoding function $\mathcal{T}$ that yields sets from codes, i.e. the evaluation of $\mathcal{T}(\mathbf{n^k})$ yields the set $\mathbb{N}^{\mathbf{k}}$ the code of which is $\mathbf{n^k}$. In the dialogical setting the formation can be formulated in the following way:

Statement	**Challenge**	**Defence**

X ! $\mathcal{U}$	**Y** ? $\mathcal{U}$	**X** $\mathbf{n^0}$: $\mathcal{U}$ … **X** $\mathbf{n^n}$: $\mathcal{U}$
X $\mathbf{n^k}$: $\mathcal{U}$	**Y** ? $\mathcal{J}(\mathbf{n^k})$	**X** $\mathbb{N}^{\mathbf{k}}$: *set*

The notion of universe allows to examine from another angle the difference between the canonical elements of **Bool yes**, **no**, and the expression **true** and **false** as applied to a proposition. As mentioned above in the case of the empty set, the dialogical setting allows reading the statement

X ! A

as expressing that player **X** states that there is a least one possible solution or answer to the enquiry **A.**
In the case

X ! **Bool**

the statement expresses that **X** is committed that there is at least one of two possible answers to the enquiry associated with the set **Bool**·For example

X 0 : **Bool**

which is certainly different of establishing that there is no possible answer to the enquiry

X ! ¬**Bool**

Now, one consequence of this distinction is that in general we cannot demonstrate in such a system (develop a winning strategy) that the elements of **Bool** are different (i.e. it is not the case that they are identical : ¬**Id**(**Bool**, **yes**, **no**), unless we assume that **yes** and **no** are associated to the codes of two disjoint sets, which are elements of a *universe.* In fact it was shown by Jan Smith (1988, pp. 842-843) by means of a metamathematical demonstration, that for any type *A* the demonstration of an inequality of the form ¬**Id**(*A, a, b*) requires *universes* constituted by codes of sets.

In order to develop a winnings strategy for ¬**Id**(**Bool**, **yes**, **no**), i.e., **Id**(**Bool**, **no**, **yes**) ⊃ ⊥., we follow the basic ideas of Martin-Löf's (1984, pp. 51-51) and Nordström/Petersson/Smith, J. (1990, p. 86) demonstration of Peano's fourth axiom.
The main idea is introduce a predicate defined over **Bool**, more precisely the function ***G***(*x*) that evaluates in the universe $\mathcal{U}$.[9] Since it evaluates in $\mathcal{U}$, the function yield codes, namely, if *x* is **no**, then it yields **n_0** and it yields **n_1**, if x is **yes**. The codes **n_0** and **n_1** are codes for the empty set $\mathbb{N}^{\mathbf{0}}$ and the unary set $\mathbb{N}^{\mathbf{1}}$ respectively. So, **t** and **f** are associated to two disjoint sets in $\mathcal{U}$ – thus, since the predicate ***G***(*x*) applies to **yes** but yields the empty set when applied to **no**, then **yes** and **no** cannot be identical. Moreover, the assumption that both of the canonical

[9] Since it is a predicate over **Bool**, it follows the rules for the analysis of these kind of statements (in the CTT its definition stemms from the elimination rules for **Bool**).

elements of **Bool** are identical leads to conclusion that the empty set is inhabited, and this proves its negation. [10]

In the dialogical setting we formulate a Socratic Rule specific to $\boldsymbol{G}(x)$. We also provide the rule of synthesis specific to the unary set $\mathbb{N}^{\mathbf{1}}$.

Statement	**Challenge**	**Defence**
X $\boldsymbol{G}(x)$: $\mathcal{U}$ (x : **Bool**)	**Y no** : **Bool** **Y yes** : **Bool** [notice that if **Y** is **P**, then the challenge assumes that **O** already conceded **yes**, **no** : **Bool**]	**X** ($\boldsymbol{G}$(**no**) = **n_0 :** $\mathcal{U}$ **X** ($\boldsymbol{G}$(**yes**) = **n_1 :** $\mathcal{U}$
X ($\boldsymbol{G}$(**no**) = **n_0 :** $\mathcal{U}$ **X** ($\boldsymbol{G}$(**yes**) = **n_1 :** $\mathcal{U}$	**Y** ? $\mathcal{I}$($\boldsymbol{G}$(**no**)) **Y** ? $\mathcal{I}$($\boldsymbol{G}$(**yes**))	**X** $\mathcal{I}$($\boldsymbol{G}$(**no**) = $\mathbb{N}^{\mathbf{0}}$: *set* **X** $\mathcal{I}$($\boldsymbol{G}$(**yes**) = $\mathbb{N}^{\mathbf{1}}$: *set*
X ! $\mathbb{N}^{\mathbf{1}}$	**Y** ? $\mathbb{N}^{\mathbf{1}}$	**X yesyes** : $\mathbb{N}^{\mathbf{1}}$

Yes and No are not Identical in Bool

We will only display here the relevant play for the determination of the winning strategy (its demonstration). The thesis is stated under the condition that **O** concedes the codes **n_0** and **n_1** are elements of $\mathcal{U}$, the canonical answers (elements) of **Bool** and the special predicate (function) $\boldsymbol{G}(x)$ [x : **Bool**] defined by the specific Socratic rule given above.

O			**P**		
C1 C2 C3 C4	**n_0, n_1 :** $\mathcal{U}$! $\mathbb{N}^{\mathbf{1}}$ **yes, no** : **Bool** $\boldsymbol{G}(x)$: $\mathcal{U}$ [x : **Bool**]			! **Id**(**Bool**, **no**, **yes**) $\supset \bot$.	0

[10] The CTT- demonstration in nutshell is the following:
Define a family of sets $\boldsymbol{G}$: **Bool** $\rightarrow \mathcal{U}$. $\boldsymbol{G}(x) =:_{\mathbf{df}}$ if x then **n_1**, else **n_0** : $\mathcal{U}$ [x : **Bool**].
$\boldsymbol{F}$: **Bool** $\rightarrow$ *set*, by $\boldsymbol{F}(x) =:_{\mathbf{df}} \mathcal{I}(\boldsymbol{G}(x))$: *set* [x : **Bool**].
tt : $\mathcal{I}$($\boldsymbol{G}$(**t**) (given **tt** : $\mathbb{N}^{\mathbf{1}}$, $\boldsymbol{G}$(**t**) = **n_1** : $\mathcal{U}$, and $\mathcal{I}$($\boldsymbol{G}$(**t**)) = $\mathbb{N}^{\mathbf{1}}$: *set*, thus
tt : $\boldsymbol{F}$(**t**).
Assume z : **Id**(**Bool**, **t**, **f**), then
subst(z, **tt**): $\boldsymbol{F}$(**f**)).
Hence, (λ.z)*subst*(z, **tt**) : $\neg$**Id**(**Bool**, **t**, **f**).

1	m =1			n = 2	2
3	p_1 : **Id**(**Bool**, **yes**, **no**)				
5	***G***(**yes**) = **n_1** : $\mathfrak{U}$	0	C4	**yes** : **Bool**.	4
7	$\mathcal{I}$(***G***(**yes**) = $\mathbb{N}^1$: *set*		5	? $\mathcal{I}$(***G***(**yes**))	6
9	**yesyes** : $\mathbb{N}^1$	4	C2	**Y**? $\mathbb{N}^1$	8
11	**yesyes** : $\mathcal{I}$(***G***(**yes**)		9, 7	$\mathcal{I}$(***G***(**yes**)/ $\mathbb{N}^1$	10
13	($\mathcal{I}$(***G***(**no**) = **n_0** : $\mathfrak{U}$		C4	**no** : **Bool**.	12
15	$\mathcal{I}$(***G***(**no**) = $\mathbb{N}^0$: *set*		13	? $\mathcal{I}$(***G***(**no**))	14
17	**Lbz-Id-subst**(p_1, **yesyes**) : $\mathcal{I}$(***G***(**no**)		11, 3	? $_{\textbf{Lbz-Id}}$ *no/yes*	16
19	**Lbz-Id-subst**(p_1, **yesyes**) : $\mathbb{N}^0$		17, 15	$\mathcal{I}$(***G***(**no**) / $\mathbb{N}^0$	18
				O-*give* up-19	20

Description of the play

- After **O**'s challenge (3) on the thesis, **P** counter-attacks(4) the concession C4, following the prescription of the Socratic rules specific to ***G***(*x*). **P** can carry out this challenge because of concession C3. In fact it is justified in the copy-cat rule – we skip here the further challenge of O asking to justifying and P's answer with the reflexivity **yes** = yes : **Bool**).
- Moves 6 and 7 follow from implementing the decoding-prescription for ***G***(*x*).
- Moves 8-11. After **O** provides the local reason **tt** for the unary set $\mathbb{N}^1$, **P** asks **O** to substitute replace $\mathbb{N}^1$ by $\mathcal{I}$(***G***(**yes**), given the equality between both conceded by move 7.
- Moves 12 -15: **P** repeats moves 4,6, but chooses this time **no** : **Bool** instead – we skip here too the moves leading to the reflexivity **no** = **no** : **Bool**.
- Moves 16-20: Move 16 is the crucial move and leads to the victory of **P**: **P** demands **O** to replace **yes** with no within move 11, given the identity conceded in move 3 and given *Leibniz-substitution rule* for **Id**. **O**'s response (17) and her concession (15) that $\mathcal{I}$(***G***(**no**) and $\mathbb{N}^0$ are equal sets, leads her to state the giving up move 19. Indeed, in move 19 **O** is forced to admit that following her own moves the empty set (of answers) is not empty. So, in fact, **P** can , after a *recapitulation* of the possible moves, adduce **O-*give* up**-19 as strategic reason for grounding his thesis and state: **O-*give* up**-19: **Id**(**Bool**, **no**, **yes**) $\supset \perp$ - we did not include this in the play, since we did not develop the whole of the strategy.

The fourth axiom of Peano's arithmetic

The demonstration of the **fourth axiom of Peano's arithmetic** ("0 is identical to no successor of a natural number": $(\forall x : \mathbb{N}) \neg$**Id**($\mathbb{N}$, 0, **s**(*x*))) is very close to the precedent one. Peano's fourth axiom was demonstrated by the first time by Martin-Löf (1984, pp. 51-51) using strong elimination rules for **Id**. Nordström, B., Petersson, K., and Smith, J. (1990, p. 86) provide a demonstration without those rules. Instead of a function defined over **Bool**, what is required is a function ***H***(*x*), defined over the natural numbers such that, the value is the code for the unary set if the *x* is 0 and it is the code for the empty set if *x* is the successor of any natural number – thus there will be a predicate that applies to 0 but not to any other natural number, which contradicts that 0 and the successor of a natural number are identical. We leave to the diligent reader the development of both the dialogical rules for ***H***(*x*) and of the relevant play for building the winning strategy – notice that ***H***(*x*), will be defined following the rules of analysis for predicates defined over $\mathbb{N}$.

The conceptual background underlying these demonstrations is that in order to demonstrate that the canonical element of **Bool** and $\mathbb{N}$ are different, we need to have a look from the outside of the respective sets and assume that there is a universe that such that the Boolean **1** amounts to a code for the **truth**, the unary set; and **0** amounts to a code for the **false**, namely

the empty set. This elucidates George Boole's own use of **1** and **0**, both as selective functions and as the universal domain ⊤ and the empty set ⊥. [11]

Let us now extend the set **Bool** and study some applications for truth-functional non-classical logics

IV Integrating many-valued logics

IV.1 Operations within larger sets

Given, some finite set $\mathbb{N}^n$ as defined above we can define operations over it. For example in the three-elements set $\mathbb{N}^3$ can yield operations that correspond to a three valued-logic, and that are based on the answers, **yes**, **?**, **no**. So $a+^3b$ is equal to **?,** if one of the elements is equal to **no** and the other to **?** or both, *a* and *b*, are equal to **?**.
More generally, for any set $\mathbb{N}^{\mathbf{n}}$ with elements **0**, **1**,…**n**, with minimum **0** and máximum **n**[12]; and with the help of the following definition of "≤" , and its inverse "$x \geq y$".

$$x \leq y = (\exists z : \mathbb{N})\mathbf{Id}(\mathbb{N}, x + z, y) : \mathbf{prop}\ [x : \mathbb{N}, y : \mathbb{N}]$$ [13]
$$x \geq y = (\exists z : \mathbb{N})\mathbf{Id}(\mathbb{N}, x - z, y) : \mathbf{prop}\ [x : \mathbb{N}, y : \mathbb{N}]$$

we obtain the following :

$a\mathbf{x}^{\mathbf{n}}b$ is **equal** to $a = \mathbf{m}$ if $\mathbf{m} \leq \mathbf{m}' = b$, otherwise it is equal to $\mathbf{m}' = b$.
$a+^{\mathbf{n}}b$ is **equal** to $a = \mathbf{m}$ if $\mathbf{m} \geq \mathbf{m}' = b$, otherwise it is equal to $\mathbf{m}' = b$.
$\sim^{\mathbf{n}}a$ is **equal** to $\mathbf{n} - \mathbf{m}$ (where, **n** is the maximum and $\mathbf{m} = a$)
$a\rightarrow^{\mathbf{n}}b$ can be defined as $\neg a+^{\mathbf{n}}b$. Thus,
$a\rightarrow^{\mathbf{n}}b$ is **equal** to $\sim a^{\mathbf{n}} = \mathbf{m}$ if $\mathbf{m} \geq \mathbf{m}' = b$, otherwise it is equal to $\mathbf{m}' = b$.

The dialogical formulation of this generalization is straightforward:

- The defender states that some operation is an element of $\mathbb{N}^{\mathbf{n}}$,

[11] For a discussion on this ambiguity see Prior (1949).
[12] Where **0** can be interpreted as corresponding to lowest truth-value and **n** the highest truth-value of some **n**-valued logic
[13] Within the dialogical framework statements involving ℕ are governed by the following rules
1)

Statement	Challenge	Defence	Description
X $n : \mathbb{N}$	**Y** $?_{s(n)}$	**X** $\mathbf{s}(n) : \mathbb{N}$	If **X** states that *n* is a natural number he is committed to the further statement that its successor is also a natural number.

2)
Given a statement of the form **P** $n : \mathbb{N}\ [0 : \mathbb{N}]$, where "*n*" stands for "1" or "2" …:. . **O** can challenge it by means of the attack $?_n$ / If **P**'s initial statement is $1 : \mathbb{N}\ [0 : \mathbb{N}]$, **P** can respond to the challenge $?_1$ with $\mathbf{s}(0) \equiv_{df} 1 : \mathbb{N}$ only if **O** stated **s**(0): ℕ; similarly for 2 and so on.

- the challenger launches a Socratic-attack on the operation. In other words the challenger requests the defender to show that the operation is equal to some element of $\mathbb{N}^{\mathbf{n}}$
- After the defender chooses one of the elements, the challenger will request him to show that this choice satisfies the $\leq$ (or $\geq$) condition that defines that operation.

For the sake of simplicity we will not display the latter request. The following example should be enough. Assume that the defender stated

$$\mathbf{X}\ (a\mathbf{x}^{\mathbf{n}}b) = \mathbf{m} = a\ : \mathbb{N}^{\mathbf{n}}$$

the challenger can ask then to check if **m** satifies the $\mathbf{m} \leq \mathbf{m}'$ condition required by the operator $\mathbf{x}^{\mathbf{n}}$.[14] Challenge and defence have the following form

$$\mathbf{Y}\ ?\ \mathbf{m} \leq \mathbf{m}'$$

(Does *m* satisfy the condition $\mathbf{m} \leq \mathbf{m}'$?)

$$\mathbf{X}\ \mathbf{m} \leq \mathbf{m}' : \mathbb{N}^{\mathbf{n}}$$

Statement	**Challenge**	**Defence**	**Strategic Reason**
$\mathbf{X}\ a\mathbf{x}^{\mathbf{n}}b\ : \mathbb{N}^{\mathbf{n}}$	$\mathbf{Y}\ ? = a\mathbf{x}^{\mathbf{n}}b$	$\mathbf{X}\ (a\mathbf{x}^{\mathbf{n}}b) = \mathbf{m} = a : \mathbb{N}^{\mathbf{n}}$ if $\mathbf{m} \leq \mathbf{m}'$, where $\mathbf{m}_{\cdot} = \boldsymbol{b} : \mathbb{N}^{\mathbf{n}}$ or $\mathbf{X}\ (a\mathbf{x}^{\mathbf{n}}b) = \mathbf{m}_{\cdot} = \boldsymbol{b} : \mathbb{N}^{\mathbf{n}}$ if $\mathbf{m} > \mathbf{m}'$	$\mathbf{P}\ (a\mathbf{x}^{\mathbf{n}}b) = \mathbf{m}\ \ [\![< \mathbf{m}, \mathbf{m}' >]\!]^{\mathbf{O}} : \mathbb{N}^{\mathbf{n}}$ if $\mathbf{m} \leq \mathbf{m}'$ $\mathbf{P}\ (a\mathbf{x}^{\mathbf{n}}b) = \mathbf{m}'\ \ [\![< \mathbf{m}, \mathbf{m}' >]\!]^{\mathbf{O}} : \mathbb{N}^{\mathbf{n}}$ if $\mathbf{m} > \mathbf{m}'$ **given** $\mathbf{O}\ \mathbf{m} = a : \mathbb{N}^{\mathbf{n}}, \mathbf{m}' = b : \mathbb{N}^{\mathbf{n}}$
$\mathbf{X}\ a+^{\mathbf{n}}b\ : \mathbb{N}^{\mathbf{n}}$	$\mathbf{Y}\ ? = a+^{\mathbf{n}}b$	$\mathbf{X}\ (a+^{\mathbf{n}}b) = \mathbf{m} : \mathbb{N}^{\mathbf{n}}$ if $\mathbf{m} \geq \mathbf{m}'$, where $\mathbf{m}_{\cdot} = \boldsymbol{b} : \mathbb{N}^{\mathbf{n}}$ or $\mathbf{X}\ (a+^{\mathbf{n}}b) = \mathbf{m}' : \mathbb{N}^{\mathbf{n}}$ if $\mathbf{m} < \mathbf{m}'$	$\mathbf{P}\ (a+^{\mathbf{n}}b) = \mathbf{m}\ \ [\![< \mathbf{m}, \mathbf{m}' >]\!]^{\mathbf{O}} : \mathbb{N}^{\mathbf{n}}$ if $\mathbf{m} \geq \mathbf{m}'$ $\mathbf{P}\ (a+^{\mathbf{n}}b) = \mathbf{m}'\ \ [\![< \mathbf{m}, \mathbf{m}' >]\!]^{\mathbf{O}} : \mathbb{N}^{\mathbf{n}}$ if $\mathbf{m} < \mathbf{m}'$ given $\mathbf{O}\ \mathbf{m} = a : \mathbb{N}^{\mathbf{n}}, \mathbf{m}' = b : \mathbb{N}^{\mathbf{n}}$
$\mathbf{X}\ {\sim}^{\mathbf{n}}a : \mathbb{N}^{\mathbf{n}}$	$\mathbf{Y}\ ? = {\sim}^{\mathbf{n}}a$	$\mathbf{X}\ {\sim}^{\mathbf{n}}a = \mathbf{n} - \mathbf{m}, \mathbf{m} = a : \mathbb{N}^{\mathbf{n}}$	$\mathbf{P}\ {\sim}^{\mathbf{n}}a = \mathbf{n} - \mathbf{m}\ [\![< \mathbf{m}, \mathbf{n} >]\!]^{\mathbf{O}} : \mathbb{N}^{\mathbf{n}}$ given $\mathbf{O}\ \mathbf{m} = a : \mathbb{N}^{\mathbf{n}}$ **n**

[14] Again, here we assume the defintion of "≤" and "≥". The dialogical formulation of it deploys a Socratic Rule specific to that relation. Namely, if player **X** states the $n \leq m$, given $x : \mathbb{N}$, $y : \mathbb{N}$, then **Y** can ask **X** to choose a $z : \mathbb{N}$, such that $n+z = m$), similarly for "≥".

X $a\rightarrow^{\mathbf{n}}b : \mathbb{N}^{\mathbf{n}}$	**Y ?** $= a\rightarrow^{\mathbf{n}}b$ **m**, **m'**: $\mathbb{N}^{\mathbf{n}}$ (for **m** $= {\sim}^{\mathbf{n}}a : \mathbb{N}^{\mathbf{n}}$, m' $= b\ \mathbb{N}^{\mathbf{n}}$)	**X** $a\rightarrow^{\mathbf{n}}b_{\mathbf{m}} = \mathbf{m} = {\sim}^{\mathbf{n}}a : \mathbb{N}^{\mathbf{n}}$ if **m** $\geq$ **m'**, where **m**, **= *b*** : $\mathbb{N}^{\mathbf{n}}$ or **X** $a\rightarrow^{\mathbf{n}}b = \mathbf{m'} = b : \mathbb{N}^{\mathbf{n}}$ if **m** $<$ **m'**	**P** $a\rightarrow^{\mathbf{n}}b = \mathbf{m}$: $[\![< \mathbf{m}, \mathbf{m'} >]\!]^{\mathbf{O}} : \mathbb{N}^{\mathbf{n}}$ if **m** $\geq$ **m'** **P** $a\rightarrow^{\mathbf{n}}b = \mathbf{m'}$: $[\![< \mathbf{m}, \mathbf{m'} >]\!]^{\mathbf{O}} : \mathbb{N}^{\mathbf{n}}$ if **m** $\not\geq$ **m'** given **O** **m** $= {\sim}^{\mathbf{n}}a : \mathbb{N}^{\mathbf{n}}$, **m'** $= b : \mathbb{N}^{\mathbf{n}}$

IV.2 The Logics of Formal Inconsistency and the *White Bullet* Operator

In a recent paper by E. A. Barrio, N. Clerbout and S. Rahman (2017), developed a dialogical reconstruction of the so-called *Logics of Formal Inconsistency* (LFI) – see Carnielli/Coniglio/Marcos (2007). The LFI's are logics tolerant to some amount of inconsistency but in which some versions of explosion (ex falso) still hold. Thus, the LFI's are a form of paraconsistent logics, that is, logics where *ex falso sequitur quodlibet* does not generally hold, and so inconsistencies are tolerated. However, the LFI's does not tolerate all forms of inconsistencies but only those considered to be relevant by a context.[15] In fact LFI constitute a whole family of logics distinguished by the kind of inconsistency they allow.
The main result of Barrio/Clerbout/Rahman (2017) is to provide a a formal framework which is applicable to situations in which inconsistent information may appear during certain argumentative interactions, but always within some limits and in particular in a way that there are some "safe" propositions for which inconsistency is not tolerated.
Now this result has been obtained from the dialogical inferentialist point of view. Indeed, what Barrio/Clerbout/Rahman (2017, section 5) did is to reconstruct the many-valued semantics of two of the LFI's into structural rules.
So this is a nice example on how to unify a family of logics to one of the frameworks. We will, as already suggested with the case of Boolean operators, embed the truth-functional semantics of one of the logics studied, namely the *Logic of Pragmatic Truth* or *Quasi-Truth* (MPT) of Coniglio/Silverstrini (2014), within our general framework. However, a generalization for all of the LFI's seem to be straightforward.

The truth-functional semantics for MPT includes the operators of product, addition and negation we described above for $\mathbb{N}^{\mathbf{3}}$ (let us here use the standard three values, **0**, **½**, **1**, where **0** is the minimum and **1** the maximum) and it adds a different negation and a new implication, that we indicate with the superscript MPT, and a consistency operator.

$a\mathbf{x}^{\mathbf{3}}b$ (where $a\mathbf{x}^{\mathbf{3}}b : \mathbb{N}^{\mathrm{MPT}}$) is **equal** to **0** if a = **0** otherwise it is equal to b.
$a\mathbf{+}^{\mathbf{3}}b$ (where $a\mathbf{+}^{\mathbf{3}}b : \mathbb{N}^{\mathrm{MPT}}$) is **equal** to **1** if a = **1** otherwise it is equal to b.

[15] An important closely related logics are the *adaptive logics* of Diderick Batens (1980), where logics are contextually sensitive to different inconsistent situations. Now, those seemed to have a more inferentialist background, than the family of paraconsistent logics that arouse from the work of Newton da Costa by 1970 – see Ottaviano/Da Costa (1970); for an overview of those and their origin see Bobenrieth (1996); for a recent presentation of new developments see Carnielli/Conigilio (2016). M. Beirlaen and M. Fontaine (2016) develop a dialogical reconstruction of some adaptive logics.

$\sim^3 a$ (where $\sim^3 a : \mathbb{N}^{MPT}$) is **equal** to **1** – **m** (where **m** = a)

$\sim^{MPT} a$ (where $\sim^{MPT} a : \mathbb{N}^{MPT}$) is **equal** to **1**, if a = **0**, otherwise it is equal to **0**

$a \rightarrow^{MPT} b$ (where $a \rightarrow^{MPT} b : \mathbb{N}^{MPT}$) is equal to **0** if b = **0** and a = **1** or if b = **0** and a = ½, otherwise it is equal to 1

a° is **equal** to **0**, if a = ½, otherwise it is equal to **1**

The dialogical formulation of these operators in the lines proposed for $\mathbb{N}^{\mathbf{n}}$ is straightforward.

The idea of the *white-bullet* operator "°", called *consistency operator* is to create a fragment where some of the truth-functional objects behave like in classical logic, i.e. in our framework to **Bool**. The dialogical reconstruction of this operator by Barrio/Clerbout/Rahman (2017) deployed the operator "**V°**" which triggers opening a subplay where the rules of the game are classical.

In the present framework we will study it both, as another-truth functional operator that is a non-canonical element of the three-elements set $\mathbb{N}^{MPT}$.

; and as a function that evaluates the elements of some fixed subset C of non-canonical elements of $\mathbb{N}^{\mathbf{MPT}}$ as the codes of the universe $\mathcal{U}$ described above and those codes, the decoding of which yield the empty set *falsum* ($\mathbb{N}^{\mathbf{0}}$, or $\bot$) and the unary set *verum* ($\mathbb{N}^{\mathbf{1}}$, or $\top$). This gives us the insight that "°" triggers a transfer from $\mathbb{N}^{\mathbf{3}}$ to **Bool**.

Thus the insight we win, here is that **V°** should be understood as the following function,

Let $x : \mathrm{C}^{\mathbf{MPT}}$ be an abbreviation of $\{x : \mathbb{N}^{\mathbf{MPT}} \mid \mathrm{C}(x)\}$

Statement	**Challenge**	**Defence**
X V°(x) : $\mathfrak{U}$ ($x : \mathrm{C}^{MPT}$)	**Y** a = **0** : C^{MPT} or **Y** a = **1** : C^{MPT} or **Y** a = **1/2** : C^{MPT}	**X V°** (a) = **n_1 : $\mathfrak{U}$** (if a = **0** : C^{MPT}) **X V°** (a) = **n_1 : $\mathfrak{U}$** (if a = **1** : C^{MPT}) **X V°** (a) = **n_0 : $\mathfrak{U}$** (if a = **1/2** : C^{MPT})
X V°(a) : $\mathfrak{U}$ (if a = **0** : C^{MPT}) **X V°**(a) : $\mathfrak{U}$ (if a = **1** : C^{MPT}) **X V°**(a) : $\mathfrak{U}$ (if a = **1/2** : C^{MPT})	**Y** ? $\mathfrak{I}$(**V°**(a) **Y** ? $\mathfrak{I}$(**V°**(a)) **Y** ? $\mathfrak{I}$(**V°**(a))	**X** $\mathfrak{I}$(**V°**(a) = $\top$: *set* **X** $\mathfrak{I}$(**V°**(a) = $\top$: *set* **X** $\mathfrak{I}$(**V°**(a) = $\bot$: *set*

We can also deploy **Id** within **Bool** for rendering empirical propositions. Moreover, we can even generalize this interpretation it for larger sets than **Bool.** Let us discuss this issue now.

We can also deploy **Id** within **Bool** for rendering empirical propositions. Let us discuss this issue now.

V Empirical Quantities and Finite Sets

V.1 Empirical Quantities as Finite Sets of Answers

As already mentioned in the introduction non-canonical elements of the set **Bool** can be deployed to study the meaning of empirical propositions. More precisely what we need is the notion of empirical quantity. This notion has been introduced by Martin-Löf in applying Constructive Type Theory to the empirical realm (Martin-Löf, 2014). According to this perspective, whereas the quantities of mathematics and logic are determined by computation, empirical quantities are determined by experiment and observation. An example of a quantity of mathematics is 2+2; it is determined by a computation yielding the number 4. An example of an empirical quantity is the colour of some object. This is not determined by computation; rather, one must look at the object under normal conditions.

In the dialogical framework, we can think of empirical quantities as answers to a question. For example, give the question

I
Are Cheryl's eyes blue?

The answer **yes** or **no**, achieved by some kind of empirical procedure accepted in the context, can be defined over the set **Bool,** namely as, being equal to **yes** or **no**. However the question

What is the colour of Cheryl's eyes

might involve many different answers.

If **X** stands for the empirical quantity *Colour of Cheryl's eyes*. We might define the possible answers over some finite set $\mathbb{N}^{n}$ of natural numbers

X = 1 : $\mathbb{N}^{n}$ if Cheryl's eyes are brown
X = 2 : $\mathbb{N}^{n}$ if Cheryl's eyes are green
X = 3 : $\mathbb{N}^{n}$ if Cheryl's eyes are blue
...
X = n : $\mathbb{N}^{n}$ if …

Certainly the question ***Are Cheryl's eyes blue?*** can also be defined over a larger set, if several degrees of colour are to be included as an answer, or alternatively degrees of certainty (definitely blue, quite blue, slightly blue …). Let assume then another set $\mathbb{N}^{j}$ for the degree of colour

Y = 0_1 : $\mathbb{N}^{j}$, if Cheryl's eyes are dark blue.
Y = 0_2 : $\mathbb{N}^{j}$, if Cheryl's eyes are light blue.
Y = 0_3 : $\mathbb{N}^{j}$, if Cheryl's eyes are green-blue.
…
Y = j : $\mathbb{N}^{j}$, if …

Thus the general dialogical rule for an empirical quantity is the following

Statement	Challenge	Defence
X **X** : $\mathbb{N}_n$	**Y** ? = **X**	**X** m_1 = **X** : $\mathbb{N}_n$ ... **X** m_n = **X**: $\mathbb{N}_n$ **The defender chooses**

Notice that determining the value of the empirical quantity is an empirical procedure, specific to that quantity. The result of carrying out such a procedure is determined by the rules for that quantity. Moreover, the value of two different empirical quantities might be the same. The quantities only indicate that the way of determining the answer to the question, might be the same. For example if the underlying set is **Bool**, the enquiry, *Did Jorge Luis Borges compose the poem "Ajedrez"?*, that involves the determination of the value of the empirical quantity **X** might be same as the one of the one involving the enquiry, *Is Ibn al-Haytham the author of Al-Shukūk 'alā Batlamyūs (Doubts Concerning Ptolemy)?;* involving **Y**, namely, **yes**.

This leads to a Socratic Rule specific to statements of the form **X**, **Y**, **Z**: $\mathbb{N}_n$.

For example, given the set $\mathbb{N}_n$

P can defend the challenges

O ? = **X** with the statement **P** m_1 = **X** : $\mathbb{N}_n$
O ? = **Y** with the statement **P** m_1 = **Y** : $\mathbb{N}_n$
O ? = **Z** with the statement **P** m_3 = **Z** : $\mathbb{N}_n$

Incompatibility
A system of rules that targets the development of a more complex meaning network might include incompatibility rules formulated as challenges. Thus instead of establishing the simple use of copy-cat, the game might include more sophisticated rules specific to a particular empirical quantity. For example, if a player responded **yes** to the enquiry *Did the Greek won in 480 BC the sea-battle take of Salamis*? associated with **X**, might not be allowed to respond with **yes** to *Did Xerxes won in 480 BC the sea-battle of Salamis?,* associated with **Z**. The rule might have one of the following forms

Formal incompatibility

If **P** stated
P yes = **Z** : **Bool** **[**gloss: *Xerxes won in 480 BC the sea-battle of Salamis*]
P yes = **X** : **Bool** **[**gloss: *The Greek won in 480 BC the sea-battle of Salamis*]

O can challenge these with

O $\neg$(**Id**(**Bool**, **yes**, **X**) $\wedge$ **Id**(**Bool**, **yes**, **Z**)) **[**gloss: *Both answers cannot be **yes***]
and **P** must give up

Contentual incompatibility

If **P** stated
P yes = **Z** : **Bool** **[**gloss: *Xerxes won in 480 BC the sea-battle of Salamis*]

O can challenge this with
O Id(**Bool**, **yes**, **X**) **[**gloss: *The Greek won in 480 BC the sea-battle of Salamis*]
and **P** must give up

V.2 Dependent Empirical Quantities

Another more sophisticate form of dealing with empirical quantities is to implement a structure where one empirical quantity might be dependent upon a different one. For example let us define the empirical quantity **Y** as the function $b(\mathbf{X}) : \mathbb{N}_j^{\mathbf{n}}$ [$\mathbf{X} : \mathbb{N}^{\mathbf{n}}$]
such that

$\mathbf{Y} =:_{df} b(\mathbf{X}) : \mathbb{N}_j^{\mathbf{n}}$ [$\mathbf{X} : \mathbb{N}^{\mathbf{n}}$]

$b(\mathbf{X}) = j_i : \mathbb{N}^{\mathbf{j}}$, given $\mathbf{X} = n_m : \mathbb{N}^{\mathbf{n}}$,
…
$b(\mathbf{X}) = j_k : \mathbb{N}^{\mathbf{j}}$, given $\mathbf{X} = n_n : \mathbb{N}^{\mathbf{n}}$, if …

Let us take a setting where we are interested in determining the meaning of some empirical propositions. We might establish that for example, that the answer to whether something has a determinate colour, say *red*, presupposes that the player already responded to the question if the object at stake *is coloured or not* at all.
Again in this case the rules of the game might include rules for challenges, like challenging that something is red by denying that the empirical quantity that yields the evaluation **X** has a positive response to the question if the object at stake has a colour.

V.3 Dependent Empirical Quantities and Futures Contingents

Empirical quantities with a special feature are the characteristic quantities of future events, the indicators of whether the event occurs. Following an analogous practice in mathematics, such a quantity **X** may be defined by setting it equal to **yes** if the event occurs and to **no** if the event does not occur. Martin-Löf has employed such a characteristic quantity of a future event in dealing with Aristotle's sea-battle puzzle. According to this interpretation, we can assert the thesis that the answer to the question *Will tomorrow a sea-battle take place*? will have either a positive or negative answer, provided that replace **X** with a variable **x**, and obtain the hypothetical:

$$\mathbf{P}\ !\ (\mathbf{Id}(\mathbf{Bool}, \mathbf{x}, \mathbf{yes}) \vee \mathbf{Id}(\mathbf{Bool}, \mathbf{x}, \mathbf{no}))\ [\mathbf{x} : \mathbf{Bool}].$$

We can assert this, even though for some practical reasons we can't determine yet the value of **x** - recall that there is a winning strategy for ! $(\forall x : \mathbf{Bool})$ $(\mathbf{Id}(\mathbf{Bool}, x, \mathbf{yes}) \vee \mathbf{Id}(\mathbf{Bool}, x, \mathbf{no}))$.

A nice application is the logical analysis of in what the Leibniz called *suspensive conditions*, that he also names *moral condition*,[16] that determine *conditional right* such as

Primus must pay 100 dinar to Secundus, if a ship arrives from Asia
(within some set time frame)

As pointed out Sébastien Magnier (2015, p. 72), traditional legal approaches to conditional right studied in law, suspensive conditions were considered through the notion of existence or legal fiction. According to Leibniz, this problem should be coupled with both a logical and an epistemic analysis: the contracting parties must not have any information yet if the antecedent of the suspensive conditional is either true or false – if the contracting parties know that the antecedent has been satisfied then the right is not of the conditional kind. However the right established by the contract should be considered to be legally binding, despite the fact that the condition has not been yet satisfied.
There are new recent logical reconstructions of conditional right triggered by the work of Matthias Armgardt (2001, 2008, 2010), such as the studies of Thiercelin (2009, 2010), Magnier (2013; 2015), Rahman (2015).

Ansten Klev (2015) deployed Martin-Löf's notion of empirical quantity. According to such an analysis, we let **X** be an empirical quantity that is equal to **yes** if a ship arrives and equal to **no** if within some set time no ship has arrived. This is can be said to be an empirical quantity, since in order to determine it some empirical method is required, like standing on the dock and recording whether a ship arrives within the set time or not. We can now define a function *b* on the set **Bool** =: {**yes**, **no**} by setting

b(**no**) = 0 and b(**yes**) = 100,
where 0 and 100 is understood as amount of money to be paid.

Since **X**, being an element of **Bool,** is equal to either **yes** or **no**, b(**X**) is well defined, since it's evaluation is either 0 or 100. So, b(**X**) is understood as the amount to be payed by Primus to Secundus [**X** : **Bool**]. The suggested analysis is then, when expressed as the thesis of the Proponent

P ! *Primus must pay* b(**X**) *dinar to Secundus*.

On this analysis, the ruling is not hypothetical, but rather categorical, in form. The condition *If a ship arrives* is not given in a hypothesis, but is built into the empirical quantity **X**. The ruling is dependent on the value of: as soon as the value of **X** is determined, then so is b(**X**) and thereby Primus's debt to Secundus. If we can determine that **X** is **no**, then we can assert the debt to be 0; if we can determine that **X** is **yes**, then we can assert the debt to be b(**yes**) = 100. This leaves open the possibility that we shall not be in a position to determine yet **X.**
This form of analysis suggests the name *dependent obligation* rather than *conditional right*
What one is obliged to do depends on the value of an empirical quantity.

Now, also Islamic jurists also have intensive discussions on the issue and they were precursors of Leibniz's rejection of the roman notion of *retroactivity*. As pointed out by Yvon Linant de Bellefonds (1965, pp-425- 430) the Islamic jurists considered that only a restricted set of suspensive (muʿallaq) conditions (taʿliq) yield legally binding contracts. It might be

[16] *Doctrina conditionum* in Leibniz (1964), See too Armgardt (2001)

argued that, from the logical point of view, their rejection was based on hypothetical analysis understanding of conditional right. An indication of this is that transfer of goods are excluded of contracts with suspensive conditions. A suspensive condition – unless there was a clearly defined time frame – might introduce a too hazardous parameter for the establishment of the juridical act. In fact, if the time frame is clearly defined and the condition not absolutely contingent, then it was not considered to fall under suspensive conditions. Thus, contracts stipulating *too vague* conditions such as *If next year I will have a profitable harvest, then .B* , where not considered to be legally binding. However, if the condition is set in a clear time frame then it is not considered to fall under what they understood as suspensive. In fact, only a reduced set of cases were allowed, including those juridic acts that in principle can be revoked, such as a will. Since it can be revoked, the fulfilment of the will might be formulated as in including an explicit suspensive condition – *tacit* conditions have another structure – see Linant de Bellefonds (1965, pp-429- 430).

Perhaps, this might lead to distinguish between *dependent obligations* (rather than susupensive conditions) and *conditional right* (dependent upon suspensive conditions). In relation to the latter a possible reconstruction that stresses the hypothetical character and deploys empirical quantities is the following:

P ! **Id**(**Bool**, **x, yes**) ⊃ **Id**(**N**, **y, yes**) ∧ **Id**(**Bool**, **x, no**) ⊃ **Id**(**N**, **y, no**)) [**x**, **y** : **Bool**]

Where **x** is stands for a variable for the empirical quantity **X** *Ashraf fulfils condition C* [explicitly established as a condition in Zayd's will]
Where **y** is stands for a variable for the empirical quantity **Y** : *Ashraf receives 100 dinar, after Zayd's death* (according to Zayd's will).

The procedure of determining the value of **y** is eminently empirical: it amounts to decide if the contract is or not legally binding (this amounts to verifying if it the condition meets the requirements settled for mu'allaq ta'liq. Similar applies to the determination of **x.**[17]

Notice that the notion of local-reason in general and of empirical quantity in particular care of old (Jaakko Hintikka (1973, pp. 77-82) and the new criticisms (such as the ones brought forwards by James Trafford (2017, pp. 86-88)), that has been raised against the standard dialogical approach to meaning as formulated by Lorenzen/Lorenz (1978).
It is fair to say that the notion of *material dialogues,* seem to be underdeveloped in relation to the formal dialogues that gathered much more attention. However, let us stress that the fathers of dialogical logic where aware of the need of a contentual (*material* was the chose term) basis from the beginning and they tackled the issue with different devices. Lorenz (1970) in particular dedicated to this issue very thorough and deep studies, most of them collected in Lorenz (2010a,b). Moreover, the rules for integrating empirical quantities within the dialogical framework, described above are directly inspired by the *predicator-rules* already discussed in Lorenz/Mittelstrass (1967).[18] Predicator rules, the dialogical counterparts of

[17] See Rahman/Iqbal (2017) for a general dialogical approach to legal reasoning in the context of Islamic Jurisprudence.

[18] In fact *predicator rules* are one part of project called *Orthosprache* proposed by Erlangen Constructivism by 1970, which also challenged the approach of mainstream analytic theory of meaning of their time. The term "*Orthosprache*" was introduced by Paul Lorenzen in 1972, quoted in a footnote in the second edition of the *Logische Propädeutik* (Kamlah and Lorenzen (1972), p. 73, footnote 1) and discussed in the bible of the Erlangen School: *Konstruktive Logik, Ethik und Wissenschaftstheorie* (Lorenzen and Schwemmer (1975)). The

semantic definitions; are part of the play-level and it is the neglect of considering this level of meaning that is partially responsible for the formalistic interpretation of the dialogical framework – we will come back to this neglect further on.

The attentive reader might recall Sellars-Brandom's games. Indeed, as to be (briefly) discussed in the next section this framework opens the path for linking dialogical logic and the games of giving and asking for reasons.

VI Some General Epistemological Consequences:

V.2 On Why the Play Level is Not to be Neglected

The philosophical background of our dialogical approach to Martin-Löf's notion of empirical quantity can be seen as describing how to *internalize* empirical data into the rules of the play (Peregrin, pp. 34-36, 100-104), or to put it in Wilfried Sellars words, placing empirical data *within the space of reasons*. As very well known, Sellars introduces the notion of space of reasons in the context of observational reports such as "This is green". According to Sellars, such a report express a state of knowledge, if the one who brings forward the reports is able to justify his assertion by appeal to some further, and more general, knowledge

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underlying idea is the explicit and constructive development, *by example* (*exemplarisch*), of a contentual language in order to build a specific scientific terminology (Kamlah and Lorenzen (1972), pp. 70–111).
The qualification "by example" refers to one of the major tenets of the overall philosophy of language of the Erlangen School, namely, the idea that we grasp an individual as exemplifying something – type theoreticians will say, as exemplifying a type (see below):

> *Yet even science cannot avoid the fact that things do not proffer themselves everywhere as different of their own accord, more often in important areas (e.g. in the social or historical sciences) science must decide for itself what it wants to regard* ***as of the same kind*** *and what is* ***of different kind****, and address them accordingly.*
> […]
> *As we have said already, the world does not "consist of objects" (of "things in themselves") which are subsequently named by men …*
> […]
> *In the world being disclosed to us all along through language we tend to grasp the individual object as individual at the same time that we grasp it* ***as specimen of*** *… Further, when we say "This is a bassoon" we mean thereby "This instrument is a bassoon" […] or when we say "This is a blackbird", we presuppose that our discussion partner already knows "what kind of an object is meant", that we are talking about birds.* (Kamlah and Lorenzen (1984), p. 37).

Accordingly, the predicators[18] of the *Orthosprache* are introduced via the study of exemplification instances. Now, a scientific terminology does not only consist in a set of predicators or even of sentences expressing propositions: an adequate scientific language constitutes a system of conceptual interrelations. The main logical device of the *Orthosprache* project is to establish the corresponding transitions by *predicator rules* that normalize the passage from one predicator to the other. Moreover, these transition rules are formulated within a dialogical frame, so that given the predicator rule

$$x \varepsilon A \Rightarrow x \varepsilon B$$

(where *x* is a free variable and "*A*" and "*B*" are predicators), we have: if a player brings forward an object of which predicator *A* is said to apply then he is also committed to ascribe the predicator *B* to the same object. The idea is that if, for example, someone claims *k is a bassoon* then he is committed to the further claim *k is a musical instrument* (where *k* is an individual constant: in the *Logische Propädeutik* the application of these norms proceeds by substituting individual constants for free variables). The Constructivists of Erlangen called *material-analytic norms* such transition rules which structure a (fully interpreted) scientific language by setting the *boundaries of a predicator. Material-analytical propositions* (or, more literally, *material-analytical truths*)[18] are defined as those universally quantified propositions which are based on such material-analytic norms (Lorenzen and Schwemmer (1975), p. 215).

about the reliability of such reports. Indeed in "Empiricism and the Philosophy of Mind" (Sellars 1991, pp. 129-194), Sellars writes:

> *The essential point is that in characterizing an episode or a state as that of knowing, we are not giving an empirical description of that episode or state; we are placing it in the logical space of reasons, of justifying and being able to justify what one says*. Sellars (1991, p. 169).

Now, for Brandom (1994; 2000; 2008), relations in the space of reasons are constituted by possibilities of reaching positions of entitlement or commitment *by inference* from prior positions of entitlement or commitment. Brandom's interpretation of the *space of reasons* aims at providing an inferentialist reading to both the internalization and the general knowledge required about the reliability of such reports: inferential rules are what is needed to make language into a vehicle of the game of giving and asking for reasons. To be able to *give* reasons we must be able to make claims that can serve as reasons for other claims; hence our language must provide for sentences that *entail* other sentences. To be able to *ask for* reasons, we must be able to make claims that count as a *challenge* to other claims; hence our language must provide for sentences that are *incompatible* with other sentences. Hence our language must be structured by these entailment and incompatibility relations. Additionally, there is the relation of inheriting commitments and entitlements (by committing myself to *This is a dog* I commit myself also to *This is an animal,* and being entitled to *It is raining* I am entitled also to *The streets are wet*); and also the relation of co-inheritance of incompatibilities (*A* is in this relation to *B* iff whatever is incompatible with *B* is incompatible with *A*). This provides for the inference relation (more precisely, it provides for its several layers).

Laurent Keiff (2007) and Matthieu Marion (2006, 2009, 2009) already pointed out at the relation between dialogical logic and the *games of asking and giving reasons*. To put it in Marion (2010) words

> *My suggestion is simply that dialogical logic is perfectly suited for a precisification of these 'assertion games'. This opens the way to a 'game-semantical' treatment of the 'game of giving and asking for reasons': 'asking for reasons' corresponds to 'attacks' in dialogical logic, while 'giving reasons' corresponds to 'defences'. In the Erlangen School, attacks were indeed described as 'rights' and defences as 'duties',16 so we have the following equivalences:*
>
> *Right to attack ↔ asking for reasons*
> *Duty to defend ↔ giving reasons*
>
> *The point of winning 'assertion games', i.e., successfully defending one's assertion against an opponent, is that one has thus provided a justification or reason for one's assertion.*
> *Referring to the title of the book* [***Making it Explicit***], *one could say that playing games of 'giving and asking for reasons' implicitly presupposes abilities that are made explicit through the introduction of logical vocabulary.* Marion (2010, p. 490) words

Keiff (2007, section 1.2) stresses an important component for linking Brandom's intepretation of Sellar's space of reasons with the dialogical framework, namely the strategic level:

> *Traditionnellement, la logique est présentée comme la science des arguments (ou du raisonnement) préservant la vérité, et les objets de cette théorie sont déterminés par rapport `a cette propriété : les constantes logiques sont les unités syntaxiques dans les énoncés qui constituent un argument que l'on ne peut altérer tout en garantissant la préservation de la vérité. Ce que l'on peut reformuler en termes brandomiens : les constantes logiques sont définies comme les unités syntaxiques qu'on ne peut altérer tout en préservant l'identité des conditions d'assertabilité. Mais l'approche dialogique détermine son objet de façon plus précise : elle définit les conditions d'assertabilité en termes de stratégies de justification.*

Clerbout and Rahman (2015, pp. ix-xi) argued that despite the close links of the dialogical framework to Brandom's inferentialism, there is also an important difference: the play-level. Indeed from dialogical point of view strategies are constituted by plays: if we are prepared to determine meaning from the point of view of dialogical games the constitution of the strategy is a process that cannot be left by side. **To put it other words, not every sequence of moves in games of asking for reasons and providing them is necessarily inferential**, only those plays leading to winning strategies are. To put in the nice of words of Jaroslav Peregrin (2014, pp. 228-29), the prescription for the interaction of questions and answers at the play-level *provides the material by the means the which we reason* not the material that prescribes *how to reason*.[19]:

> *This is a crucial point, because it is often taken for granted that the rules of logic tell us how to reason precisely in the tactical sense of the word. But what I maintain is that this is wrong, the rules do not tell us how to reason, they provide us with things with which, or in terms of which, to reason.* Peregrin (2014, pp. 228-29).

Perhaps, the point that not every move in the space of reasons is inferential can be related to John McDowell's (2004, 2009) worry in relation to Brandom's interpretation of Sellars:

> *Someone can know what colour something is by looking at it only if she knows enough about the effects of different sorts of illumination on colour appearances. The essential thing for our purposes is that the relation of this presupposed knowledge to the knowledge that presupposes it — support in Sellars's second dimension — is not that the presupposing knowledge is inferentially grounded on the presupposed knowledge.* McDowell (2004).

[19] In fact Jaroslav Peregrin (2014) uses the dialogical framework to develop a new approach to the issue on the normativity of logic: he understands the normativity of logic not in the sense of prescriptions on how to reason, but rather as providing the material by the means of which we reason. If we link this proposal with the distinction between the play level and strategic level, we can distinguish prescriptions that aim the development of a play and provide the material for reasoning, from those proper to the tactics, considering the optimal means on how to win. These last prescriptions dictate the design of feasible strategies; Peregrin's suggestion leads to dividing the strategic level with tactics singling out the subset of feasible strategies from the whole set of strategies. While tactical considerations try to find the optimal way to achieve victory, normativity in a more general and fundamental level involves the play level, that is, the level where instruments of reasoning and meaning are forged. Moreover, Peregrin links the normativity of logic with another main conceptual tenet of the dialogical framework, namely, the public feature of the speech-acts underlying an argumentative approach to reasoning. See in particular (Peregrin, 2014, pp. 228-229).

> *We only need to register that it is experience that yields the knowledge expressed in observation reports. Recognizing the second dimension puts us in a position to understand observation reports properly. The knowledge they express is not inferentially grounded on other knowledge of matters of fact, but – in the crucial departure from traditional empiricism – it presupposes other knowledge of fact.* McDowell (2009, p. 223).

Our reconstruction of the controversy between Brandom and McDowell is based on a double articulation:

- the difference between the play and the strategy level;
- and the difference between dependences upon empirical quantities and dependences as structured by premises-conclusion

For short,

> while the dialogical framework leaves room for the interaction of questions and answers that do not reduce to the strategy level, though might have the general aim of constituting them - see Keiff (2004, section 1.1),
> the richer language of the dialogical approach to the CTT allows to analyse empirical reports as constituted by empirical quantities and the propositions that bear them – i.e. as statements involving local reasons adduced in favour of certain proposition

So, we can analyse the report

> *This apple looks green to me*

as the play-level statement of some concrete player, say, *Eloise*

> Eloise $\mathbf{X} = 3 : \mathbb{N}^5$
>
> where $\mathbf{X}$ is the empirical quantity that encodes the response to the enquiry on *the apple being green*

and moreover determining the response to such an empirical quantity might well be dependent upon another empirical quantity, for example

> Eloise $\mathbf{X} =:_{df} b(\mathbf{Y}) = 3 : \mathbb{N}^5\ [\mathbf{Y} : \mathbf{Bool}]$,
>
> where $\mathbf{Y}$ is the empirical quantity that encodes the response to the enquiry on *the apple being coloured*

Notice that we are here like McDowell making one empirical quantity dependent upon another one, by means of a function between those quantities rather that expressing the dependence by means of inferences.[20]

The rules of the play-level *internalize* the empirical features by prescribing the rules specific to the empirical quantity at stake. However this does not mean that we cannot move from the statement *it looks green to me* to the assertion *it is green*, a winning strategy is required that, can be totally rendered by inferential moves: it is sufficient for Eloise to show that she can defend her statement, given the material rules set by the game, against any challenge of her antagonist, *Abelard*, settled by those rules.

VI.3 Further Remarks on the Play Level

[20] In the early stages of the development of the dialogical framework, meaning dependences where normed by means of *transition rules* between *predicators*, at the play level! See our footnote at the end of the precedent section.

The Dialogical Internalization and the Myth of the Given: Let us stress the point that, if our reconstruction of Sellars's observational reports by means of empirical quantities is correct, acknowledging the legitimacy of such reports **does not fall** into the *Myth of the Given.* It suffices to recall that in our approach empirical quantities are non-canonical elements of some set in the context of CTT. In such a context there is no way to approach to some object without apprehending it as determining what it is. Indeed one main tenet of CTT is

1. No entity without type
2. No type without semantical equality

If we recall, the isomorphism between types and propositions we have

Every entity is bearer of a proposition.

This is what the *internalization* of empirical content within a dialogical stance is about: bringing forward local reasons for a proposition.[21] Moreover, the dialogical approach conceives the second point as rendering semantical equality as the result of the interaction of giving and asking for reasons. So this should care of Brandom's (1994, 1997, 2000, 2002) worries about interpreting observational reports in the way suggested by McDowell. .

Interesting is that the discussion between McDowell and Brandom might be paralleled with the opposition between Hintikka's (1973) notion of *outdoor-games* and Lorenz-Lorenzen (1978) indoor-games. Indeed, Hintikka (1973, 77–82), who acknowledges the close links between dialogical logic and game-theoretical semantics, launched an attack against the philosophical foundations of dialogical logic because of its *indoor* or purely formal approach to meaning as use. He argues that *formal proof* games are not much help in accomplishing the task of linking the linguistic rules of meaning with the real world:

> *In contrast to our games of seeking and finding, the games of Lorenzen and Stegmüller are 'dialogical games' which are played 'indoors' by means of verbal 'challenges' and 'responses'.* […]
> […] *If one is merely interested in suitable technical problems in logic, there may not be much to choose between the two types of games. However, from a philosophical point of view, the difference seems to be absolutely crucial. Only considerations which pertain to 'games of exploring the world' can be hoped to throw any light on the role of our logical concepts in the meaningful use of language*. (Hintikka 1973, 81).

Again, the integration of Socratic rules specific to a given predicate and the incorporation of empirical quantities cares about those kind of worries

Cooperation, The in-built Opponent and the Neglect of the Play Level: In recent literature Catarina Duthil Novaes (2015) and James Trafford (2017, pp. 102-105) deploy the term *internalization* not to refer to the internalization of empirical quantities or better of moves involving empirical content that takes place at the play level, but rather the fact that that natural deduction can be seen as having an internalized Opponent, that motivates the inferential steps. This form of internalization is called *built-in Opponent*. The origin of this concept is Göran Sundholm who by 2000, in order to characterize the core of the links

[21] This is the sense of *internalizatio*n discussed by (Peregrin, pp. 34-36, 100-104). However, since he does not use the CTT language, he has not the means to distinguish the empirical quantity from the set (proposition) it instantiates.

between natural deduction and dialogical logic, introduced in his lectures and talks by 2000 the term *implicit interlocutor*. Now, since the notion of *implicit interlocutor* was meant to link the strategy level with natural deduction, the concept of *built-in Opponent*, which is offspring of the former, inherited the same strategic perspective. However, the process that yields the implicit interlocutor is the result of constituting strategies and natural deduction inferences from the play level upwards. Rahman/Clerbout/Keiff (2015), in a paper dedicated to the *Festschrift* for Sundholm, borrowing the term of Jean-Yves Girard, designate the process as *incarnation*. The thorough description of the incarnation process described by Rahman/Clerbout/Keiff (2015) already displays those aspects of the cooperative endeavour, formulated by Duthil Novaes (2015) and quoted by Trafford (2017, p. 102) as a criticism of the dialogical framework. It is fair to say that the standard dialogical framework, not enriched with the language of CTT did not have the means to fully develop the so-called material dialogues, that is dialogues that deal with content. However, if cooperation is to be understood as linked with notion of built-in Opponent, the criticism is simply wrong, and this is because the play level is being neglected: the intersubjective in-built and implicit cooperation of the strategy level (which cares about inferences) grows out of the explicit cooperation of a concrete player at the play level. Moreover as suggested by Rahman/Ibal (2017) if we study cooperation at the play-level, many cases we do not need to endow the notion of inference with non-monotonic features: The play level is the level were cooperative, either destructive or destructive can take place until the definitive answer –given the structural and material conditions of the rules of the game – has been reached. This should provide the answer to Trafford's (2017, p. 86-88) search for an *open- ended* dialogical setting. In other words, open-ended dialogical interaction is a property of the play-level. Certainly, perhaps the point of the objection is that this level is either underdeveloped in the literature – and we acknowledge this fact with the provisos formulated above – or the dialogical approach to meaning does not manage to draw a clean distinction between local and strategic meaning. This takes us to the next further remark.

The Dialogical take on *Tonk*: The notorious case of *tonk* has been several times addressed as a counterargument to inferentialism and also to the indoor-perspective of the dialogical framework. This seems also to be the background of for example Trafford's (2017, p. 86) reproduction of the objection of circularity to the dialogical approach to logical constants. At this point of the discussion Trafford (2017, 86-88) is clearly aware of the distinction between the rules for local meaning and the rules of the strategic level, however he points out that the local meaning is vitiated by the strategic notion of justification. Now, in Rahman/Keiff (2005), Rahman (2012), Rahman/Clerbout/Keiff (2015), and Rahman/Redmond (2016) it has been shown that precisely the case of *tonk* yields a definitive answer to the issue. The argument is as simple as it can be: it can be shown by a straightforward argument that the inferential formulation of rules for *tonk*, correspond to strategic rules that **cannot be constituted** by the formulation of local rules. The player-independence of the local rules – responsible of the branches at the strategic level – do not yield the strategic rules that the inferential rules for *tonk* are purported to prescribe. For short, the dialogical take on *tonk* shows precisely that the rules of local meaning are not circularly dependent upon the strategic ones.

VII Final Words

I tried to honour the work of Bachir Diagne by delving into one of his subjects, namely Boole, not only from the point of view of logic, practiced by him in his early work but

also, from the dynamics that features his epistemological reflections the oral traditions and his insights on Islamic thought. I thought that the best way to honour his work is to practice the dialogical stance he always argued for.

Indeed, perhaps, if you allow me to condense the large work of Souleyman Bachir Diagne, I dare to say that it does both, it conveys the intimate conviction that meaning is something we do together, and it also invites us to participate in the open ended dialogue the human pursue of knowledge and collective understanding is, since the endeavour of reasoning is immanent to the dialogical interaction that makes reason happen.

Acknowledgments:

I am very grateful to Mouhamadou el Hady BA and Oumar Dia (Dakar) for their invitation to participate in the workshop in Dakar 2018 in honour of Souleyman Bachir Diagne., that motivated the composition of the present paper.

Thanks to

the team of doctorates at the university of Lille for their stimulating and intelligent questions: Steephen Eckoubili, Muhammad Iqbal, Hanna Karpenko, Clément Lion, Zoe M^{c}Conaughey and Fachrur Rozie.

Mawusse Kpakpo Akue Adotevi (Lomé), Charles Zacharie Bowao (Brazzaville), Nicolas Clerbout (Valparaíso), Bernadette A. Dango (Bouaké), Nino Guallart (Sevilla), Laurent Keiff (Lille), Gildas Nzokou (Libreville), Marcel Nguimbi (Brazzaville), Auguste Nsonsissa (Brazzaville), and Juan Redmond (Valparaíso) and for enriching discussions also during a series of online seminars.

Special gratitude to

Ansten Klev (Prague, Czech Academy of Sciences) and Johan Georg Granström(Google, Zürich) for helping with technical details and conceptual issues on CTT, particularly so in relation to the demonstrations making use of universes; and to

Moussa Abu Ramadan (Strassbourgh) and Farid Zidani (Alger II) for their teachings and advices concerning the notion of suspensive condition in Islamic Law

John McDowell (Pittsburgh) who indicated me the published paper that condensed some of his criticisms on Brandom's relevant for our brief remarks on the matter.

APPENDIX I : BASIC NOTIONS FOR DIALOGICAL LOGIC[22]

The dialogical approach to logic is not a specific logical system; it is rather a general framework having a rule-based approach to meaning (instead of a truth-functional or a model-theoretical approach) which allows different logics to be developed, combined and compared within it. The main philosophical idea behind this framework is that meaning and rationality are constituted by argumentative interaction between epistemic subjects; it has proved particularly fruitful in history of philosophy and logic. We shall here provide a brief overview of dialogues in a more intuitive approach than what is found in the rest of the book in order to give a feeling of what the dialogical framework can do and what it is aiming at.

Dialogues and interaction

As hinted by its name, this framework studies dialogues; but it also takes the form of dialogues. In a dialogue, two parties (players) argue on a thesis (a certain statement that is the subject of the whole argument) and follow certain fixed rules in their argument. The player who states the thesis is the Proponent, called **P**, and his rival, the player who challenges the thesis, is the Opponent, called **O**. By convention, we refer to **P** as he and to **O** as she. In challenging the Proponent's thesis, the Opponent is requiring of the Proponent that he defends his statement.

The interaction between the two players **P** and **O** is spelled out by challenges and defences, Actions in a dialogue are called moves; they are often understood as speech-acts involving declarative utterances (*statements*) and interrogative utterances (*requests*).[23] The rules for dialogues thus never deal with expressions isolated from the act of uttering them.

The rules in the dialogical framework are divided into two kinds of rules: particle rules, and structural rules.

Particle rules

Particle rules (*Partikelregeln*), or rules for logical constants, determine the legal moves in a play and regulate interaction by establishing the relevant moves constituting *challenges*: moves that are an appropriate attack to a previous move (a statement) and thus require that the challenged player play the appropriate defence to the attack. If the challenged player defends his statement, he has answered the challenge.

Particle rules determine how reasons are asked for and are given for each kind of statement, thus providing the meaning of that statement. In other words, the appropriate attacks and defences—that is, the appropriate ways of asking for and giving reasons—for each statement (or move) gives the meaning of these statements: a conjunction, a disjunction, or a universal quantification, for instance, receive their meaning through the appropriate interaction in a dialogical game, spelled out by the particle rules.

The particle rules provide the meaning of the different logical connectives, which they provide in a dynamic way through appropriate challenges and answers. This feature of dialogues is fundamental for immanent reasoning: the meaning of the moves in a dialogue does not lie in some external semantic, but is immanent to the dialogue itself, that is, in the specific and appropriate way the players interact; we here join the Wittgensteinian conception of meaning as use. The particle rules are spelled out in an anonymous way, that is, without mentioning if it is **P** or **O** who is attacking or defending: the rules are the same for the two players; the meaning of the connectives is therefore independent of who uses them.[24]

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[22] The three appendices are based on the book in progress *Immanent Reasoning* by Rahman/McConaughey/ Klev/Clerbout (2017). The original historical sources of the origins of dialogical logic and their reprintings are to be found in Lorenzen (1969), Lorenzen/Lorenz (1978), Lorenzen/Schwemmer (1975), Lorenz (2010a,b). For an overview see Rahman/Keiff (2005), Krabbe (2006), Keiff (2009), Clerbout (2014a).

[23] Literature pertaining to the dialogical framework also uses the terms posits and assertions to designate what we will here call statements, that is, the act of stating a proposition within a game of giving and asking for reasons; the meaning of a statement is defined by an appropriate challenge and defence, or, in other words, how reasons for this statement can be requested, what constitutes reasons for this statement and how these reasons can be provided (see Keiff (2009)).

[24] In this sense, the particle rules are said to be *symmetric*, see section 0. This is imperative to preserve the dialogical framework from connectives as Prior's (1960) *tonk*. See Redmond/Rahman (2016).

Structural rules

Structural rules (*Rahmenregeln*) on the other hand determine the general course of a dialogue game, such as how a game is initiated, how to play it, how it ends, and so on. The point of these rules is not so much to spell out the meaning of the logical constants by specifying how to act in an appropriate way—this is the role of the particle rules—; it is rather to specify according to what structure interactions will take place. It is one thing to determine the meaning of the logical constants as a set of appropriate challenges and defences, it is another to define whose turn it is to play and when a player is allowed to play a move. One could thus have the same local meaning and change a structural rule, saying for instance that one of the players is allowed to play two moves at a time instead of simply one: this would considerably change the game without changing the local meaning of what is said.

One of the most important structural rules for the present study on immanent reasoning is the Copy-cat rule (or Socratic rule when introducing CTT features in the dialogical context). This rule is not anonymous, it is a restriction on the moves the Proponent is allowed to play: the Proponent is allowed to assert an elementary judgement only if the Opponent has already asserted it. So the Opponent is not concerned by the same exact rules as the Proponent.

The Copy-cat rule accounts for analyticity: the Proponent, who brings forward the thesis, will have to defend it without bringing any element of his own in the play: his defence of the thesis will have to rely only on what the Opponent has conceded, and everything the Opponent concedes comes only from the meaning of the thesis. The Opponent will be challenging the thesis, and challenging and defending the subsequent moves made by the Proponent in reaction to her initial challenge of the thesis; but all these challenges and defences are made according to the particle rules. So everything the Opponent will concede during a play stems from an application of the particle rules starting with the thesis. The only elements whose meaning is left unspecified, in formal plays, are the elementary statements (specifying their meaning is the point of material plays). The Copy-cat rule makes sure that the Proponent is not bringing in any elementary statement to back his thesis that the Opponent might not agree with: the Proponent can only back his thesis with elementary statements that the Opponent herself has already conceded.

Preliminary notions

The language

Let $\mathbb{L}$ be a first-order language built as usual upon the propositional connectives, the quantifiers, a denumerable set of individual variables, a denumerable set of individual constants and a denumerable set of predicate symbols (each with a fixed arity).

We extend the language $\mathbb{L}$ with two labels **O** and **P**, standing for the players of the game, and the two symbols '!' and '?' standing respectively for statements and requests. When the identity of the player does not matter, we use the variables **X** or **Y** (with $\mathbf{X} \neq \mathbf{Y}$).[25]

Plays

A *play* is a legal sequence of moves, that is, a sequence of moves which observes the game rules. Particle rules are not the only rules which must be observed in this respect: the second kind of rules, the *structural rules*, are the rules providing the precise conditions under which a given sequence is a play.

Dialogical games

The *dialogical game* for a statement is the set of all plays from a given *thesis* (initial statement, see below the Starting rule, SR0).

A move in a play

A move $\mathbb{M}$ is an expression of the form '**X**-*e*', where *e* is either

- of the form '! *A*' (read: *the player **X** states A*), for some proposition *A* of $\mathbb{L}$; we say it is an elementary statement, or
- of one of the forms specified by the particle rules (see below).

[25] This aspect (player independence) is fundamental for the symmetry of the rules. See section 0.

Challenges and defences

The words 'attack' and 'defence' are convenient to name certain moves according to their relation to other moves which can be defined in the following way.

- Let σ be a sequence of moves. The function ρ_σ assigns a position to each move in σ, starting with 0.
- The function $\mathbf{F}_\sigma$ assigns a pair $[m, Z]$ to certain moves $\mathbf{M}$ in σ, where m denotes a position smaller than $\rho_\sigma(\mathbf{M})$ and Z is either A or D, standing respectively for 'attack' and 'defence'. That is, the function $\mathbf{F}_\sigma$ keeps track of the relations of attack and defence as they are given by the particle rules.

Let us point that at the local level (the level of the particle rules), this terminology should be bereft of any strategic undertone.

Terminological note: challenge, attack and defence

The standard terminology uses the terms *challenge*, or *attack*, and *defence* (sometimes *answer* in respect of challenges). We shall here make a (subtle) distinction between challenge and attack: a challenge is initiated by an attack and needs this attack to be defended against in order to be answered to. So a challenge requires a defence to be settled, whereas an attack is simply the move that opens the challenge. For instance, using the particle rules exposed below, an attack on an implication will be simply to state the antecedent, and challenging an implication will be to attack it and thus demanding that the player who stated the implication defends her posit by positing the consequent, knowing that the challenger stated the antecedent. As one can see, the difference between challenge and attack is slim, and they may oftentimes be taken as synonymous.

Local meaning of logical constants

Particle rules:

In the dialogical framework, the particle rules state the *local semantics*: only challenges and the corresponding defences for a given logical constant are at stake here, that is, we only take the main logical constant of the proposition into account.

Particle rules provide a decontextualized description of how the game can proceed locally: they specify the way a statement can be challenged and defended according to its main logical constant. In this way the particle rules govern the local level of meaning.

Figure 1 : Particle rules for dialogical games: propositional connectives

	Conjunction	**Disjunction**	**Implication**	**Negation**
Move	$\boldsymbol{X}\,!\,A \wedge B$	$\boldsymbol{X}\,!\,A \vee B$	$\boldsymbol{X}\,!\,A \supset B$	$\boldsymbol{X}\,!\,\neg A$
Challenge	$\boldsymbol{Y}\,?\,L^{\wedge}$ or $\boldsymbol{Y}\,?\,R^{\wedge}$	$\boldsymbol{Y}\,?_{\vee}$	$\boldsymbol{Y}\,!\,A$	$\boldsymbol{Y}\,!\,A$
Defence	$\boldsymbol{X}\,!\,A$ (resp.) $\boldsymbol{X}\,!\,B$	$\boldsymbol{X}\,!\,A$ or $\boldsymbol{X}\,!\,B$	$\boldsymbol{X}\,!\,B$	—

The particle rules for quantifiers has not been introduced, so we will be commenting these rules briefly.

The rules for universal quantification are similar to those for conjunction: stating a universally quantified proposition means that the challenger may choose any individual constant a_i and request of the utterer to make his statement by instantiating every free occurrence of x with a_i. That is, the challenger chooses which proposition he wants the utterer to state.

Properties of universal quantification:

- the challenge is a request;
- the challenger has the choice;
- the defender must state the requested proposition.

The rules for existential quantification are similar to those for disjunction: it is the defender who chooses the proposition he wants to state in response to the challenge.

Properties of existential quantification:

- the challenge is a request;
- the defender has the choice;
- the defender chooses which proposition to state.

Figure 2: particle rules for dialogical games: quantifiers

	Universal quantification	**Existential quantification**
Move	$\boldsymbol{X}\,!\,\forall x B(x)$	$\boldsymbol{X}\,!\,\exists x B(x)$
Challenge	$\boldsymbol{Y}\,?\,[x/a_i]$	$\boldsymbol{X}\,?_{\exists}$
Defence	$\boldsymbol{X}\,!\,B(x/a_i)$	$\boldsymbol{X}\,!\,B(x/a_i)$ with $1 \leq i \leq n$

Symmetry and harmony

In providing the properties of the particle rules, a central feature we have distinguished is who has the choice: is it the challenger or the defender? The meaning of the logical constants is largely determined by who has the choice in the interaction. But notice that in formulating the particle rules, the players' identities are not specified: we do not use **O** and **P** but we use **X** and **Y** instead, thus only specifying who is the challenger and who is the defender *for this particular statement*. That is, we simply provide the appropriate challenge and defence for certain logical constants and determine in this way who has the choice: we provide their meaning in terms of interaction within a dialogue (a game of giving and asking for reasons).

It would not be reasonable to base a game-theoretical approach to the meaning of logical constants in which the meaning differs according to which players utters it: this approach would make interaction senseless, for each player would be meaning something different when uttering the same thing. Equality in action is precisely based on the possibility for a player to say *the same thing* as the other player, and by that to be meaning also the same thing. Equality in action is in this regard the idea that a statement made by a player can be made by another player in a game of giving and asking for reasons (a dialogue) with the exact same meaning as the statement made by the first player, that is with the same particle rules for challenging and defending it. It is thus the interaction based on player-independent rules that allows two different players to be speaking of the same thing: equality between different statements emerges from the interaction itself.

Since the rules for the logical constants are independent of the player's identities—the rules are exactly the same for the two players—we say that these rules are symmetric. This feature captures one of the strengths of the dialogical approach to meaning: the dialogical approach is in this way immune to a wide range of trivializing connectives such as Prior's *tonk*.[26]

Symmetry, or player-independence in the particle rules, must be contrasted with the dialogical rendering of harmony, which concerns the structural rules and the strategy level, not the particle rules. The structural rules, which will be introduced in the next section, are not player independent: the first rule (SR0) specifies who the players are (Proponent or Opponent) according to who plays the first move, that is who states the thesis; that player will be the Proponent. But the rule that matters most in regard to immanent reasoning is the Copy-cat rule (or Socratic rule in a CTT framework); this rule puts a restriction on the Proponent's moves, while those of the Opponent are left unrestricted: which the Proponent cannot play an elementary statement that has not been previously stated by the Opponent.

The purpose of this restriction is to insure that the thesis will be grounded only on what the Opponent has conceded, and thereby secure a form of analyticity that we call immanent reasoning: the Proponent has to ground his thesis on what the Opponent brings forward in the course of their interaction (the dialogue), an interaction that is initiated by the Proponent stating a thesis, which the Opponent challenges with the ensuing series of challenges and defences defined by the particle rules and constituting the dialogue. Thus the Opponent will not

[26] See Rahman/Keiff(2005), Rahman/Clerbout/Keiff (2009) and Rahman/Redmond (2016).

bring forward anything that does not stem from the meaning (defined by the particle rules) of the thesis, and the Proponent will not bring any elementary proposition into the game that he cannot justify within that very same dialogue by referring to the Opponent's own statements ("I am entitled to state this because you have stated it yourself").

Symmetry and harmony are two essential aspects of the dialogical framework and are the principles for immanent reasoning.

Dialogical harmony thus coordinates a player-independent level (the local meaning) and a player-dependent level (the global meaning and the strategy level). This aspect contrasts with the Constructive Type Theory notion of harmony which belongs to proof-theory and stays only at the level of strategies.[27] Immanent reasoning and equality in action emerge from taking the specific aspects of the three levels (local, global and strategic) into account and considering how they intertwine to build these complex and dynamic frameworks that are dialogues.

Global meaning:

The global meaning—as opposed to the local meaning defined by the particle rules—is defined by means of *structural rules* which specify the general way plays unravel by specifying who starts in a play, what moves are allowed and in which order, when a play ends and who wins it.

Preliminary terminology

Terminal plays:

A play is called *terminal* when it cannot be extended by further moves in compliance with the rules.

X-terminal plays:

A play is ***X**-terminal* when the play is terminal and the last move in the play is an **X**-move.

The structural rules

SR0 (Starting rule)

Any dialogue starts with the Opponent stating initial concessions (if any) and the Proponent stating the thesis (labelled move 0). After that, each player chooses in turn a positive integer called the *repetition rank* which determines the upper boundary for the number of attacks and of defences each player can make in reaction to each move during the play.

*Example: if the repetition rank of **O** is* $m \coloneqq 1$*, then **O** may attack or defend against* at most *once each move of **P**. If **P**'s repetition rank is* $n \coloneqq 2$ *, then **P** may attack or defend against* at most *twice each move of **O**.*

SR1: Development rule

The Development rule depends on what kind of logic is chosen: if the game uses classical logic, then it is SR1c that should be used; but if intuitionistic logic is used, then SR1i must be used.

SR1c (Classical Development rule)

Players move alternately. Once the repetition ranks have been chosen, each move is either attacking or defending a move made by the other player, in accordance with the particle rules.

SR1i (Intuitionisitic Development rule)

Players move alternately. Once the repetition ranks have been chosen, each move is either attacking or defending a move made by the other player, in accordance with the particle rules.

Players can respond only to the *last non-answered* challenge of the other player.

[27] Rahman/Redmond (2016) and Rahman/Clerbout/Redmond (2017).

Note: This last clause is known as the Last Duty First *condition, and makes dialogical games suitable for intuitionistic logic (hence this rule's name).*

SR2 (Copy-cat rule)

P may not play an elementary statement unless **O** has stated it first.
Elementary propositions cannot be challenged.

Note: The formulation of this rule has a downside: the thesis of a dialogical game cannot be an elementary statement. For a special version of the Copy-cat rule allowing plays on elementary statements, see below (section 0) where we link this rule to equality.

SR3 (Winning rule)

Player **X** wins the play ζ only if it is **X**-terminal.

Linking the Copy-cat rule (SR2) and equality

The Copy-cat rule[28] is one of the most salient characteristics of dialogical logic. As discussed by Marion/Rückert (2015), it can be traced back to Aristotle's reconstruction of the Platonic dialectics. A purely argumentative point of view can be defined within dialectics as refraining from calling on some authority beyond what has actually been brought forward during the current argumentative interaction, the ultimate authority being the fact that the other person has said it, any other consideration being set aside for the time of the dialectical exchange (in this argumentative perspective). Thus, when an elementary statement is challenged, the challenge can be answered only by invoking the challenger's own concessions. In such a context, the Copy-cat rule can be understood in the following way, when a player plays an elementary statement:

> *my grounds for stating the proposition you are challenging are exactly the same as the ones you brought forward when you yourself stated that very same proposition*.[29]

In this regard, elementary statements actually can be challenged (as opposed to the SR2 formulation above), the answer then being of the form "but you have said it yourself". A special formulation of the Copy-cat rule SR2 addresses this problem.

Special Copy-cat rule

O's elementary statements cannot be challenged. However, **O** can challenge an elementary statement played by **P**. The challenge and corresponding defence is determined by the following table. Notice that this (structural) rule is not player-independent and uses the names of the players.

Copy-cat rule

	Move	**Challenge**	**Defence**
Special Copy-cat rule (Structural Rule 2)	$\boldsymbol{P}\,!\,A$ For elementary A	**O** $?_A$	**P** $!\,sic(n)$ **P** indicates that **O** stated A at move n

The Copy-cat rule, and even more in this special formulation, introduces an asymmetry between the two players (the Proponent's moves are restricted in a way the Opponent's are not).

[28] In previous literature on dialogical logic this rule has been called the *Formal rule* (see Lorenzen/Lorenz (1978)). Since here we will distinguish different formulations of this rule that yield different kind of dialogues we will use the term *Copy-cat rule* when we speak of the rule in standard contexts (such as in the present section)—contexts in which the constitution of the elementary propositions involved in a play is not rendered explicit. When we use the rule in a dialogical framework for CTT, as in the next chapter, we speak of the *Socratic rule*. However, we will continue to use the expression Copy-cat *move* in order to characterize moves of **P** that copies moves of **O**.

[29] For the Platonic origins of this rule see Rahman/Keiff (2010).

Examples of plays

These examples should allow the reader to fully understand the rules given above and their implications, especially the difference between SR1c (classical Development rule) and SR1i (intuitionistic Development rule). In the next chapter (V), strategies will be introduced, which allow to compare different plays (with different choice sequences of the players) and build the best possible way of playing for one of the players.

First example, the third excluded: $A \vee \neg A$

The third excluded (*tertium non datur*) is a principle stating that a proposition either is (A) or is not ($\neg A$), without any third possible option. This principle is much discussed in philosophy and logic, it is a valid principle in classical logic, but is not accepted in intuitionistic logic. If this principle is accepted, the principle of non-contradiction ($\neg(A \wedge \neg A)$) follows, but the reverse is not the case (and intuitionistic logic accepts the principle of non-contradiction but not the principle of third excluded). We will here give a play according to the classical (structural) rules, and then a play according to the intuitionistic (structural) rules.

Play 1: the third excluded—classical rules

O				**P**	
				$!\, A \vee \neg A$	0
1	$n \coloneqq 1$			$m \coloneqq 2$	2
3	$?_{\vee}$	0		$!\, \neg A$	4
5	$!\, A$	2		—	
{3}	$\{?_{\vee}\}$	{0}		$!\, A$	6

P wins (classical rules).

*Note : the curly brackets are inserted to stress the fact that **O** is not actually making a move, but that **P** is using his repetition rank of 2 in order to defend twice **O**'s challenge (move 3). We repeat that challenge (in brackets) in order to know where **P**'s move 6 comes from.*

Notice that **P** would not have won without a repetition rank higher than 1: he would not have been allowed to answer twice to **O**'s challenge (move 3), and thus use her own assertion of A (move 5) triggered by **P**'s first defence to **O**'s challenge (move 3). This example is a good illustration for the Copy-cat rule and for the use of repetition ranks.

Notice also that **P**'s move 6 is an answer to the challenge of move 3, that is a challenge preceeding the last unanswered challenge, which is move 5. This challenge of move 5 will never be answered, because an attack on the negation cannot be defended. So **P** wins because the classical rules for dialogues *do not* restrict **P**'s answers only to the last unanswered challenge. This fact is the key to understand the outcome of the next play, which uses the intuitionistic rules.

Play 2: the third excluded—intuitionistic rules

O				**P**	
				$!\, A \vee \neg A$	0
1	$n \coloneqq 1$			$m \coloneqq 2$	2
3	$?_{\vee}$	0		$!\, \neg A$	4
5	$!\, A$	4		—	

O wins (intuitionistic rules).

Second example, the double negation elimination: $\neg\neg A \supset A$

The elimination of double negation is another example of a principle accepted in classical logic but rejected in intuitionistic logic. This principle is at the core of classical mathematics, for it is what is used in indirect

proofs (concluding A from the demonstration that the negation of A leads to a contradiction, that is from the fact that $\neg\neg A$ holds). This principle is closely linked to the principle of excluded middle. Once again, we give a play with classical rules (**P** wins) and a play with intuitionistic rules (**P** loses).

Play 3: the elimination of double negation—classical rules

O			**P**		
				$!\ \neg\neg A \supset A$	0
1	$m := 1$			$n := 2$	2
3	$!\ \neg\neg A$	0		$!\ A$	6
	—		3	$!\ \neg A$	4
5	$!\ A$	4		—	

P wins (classical rules).

Notice that as for the third excluded, **P** wins here because he does not have to answer only to the last unanswered challenge (which is move 5) but answers a previous challenge (his move 6 is an answer to the challenge of move 3). This move is forbidden by the intuitionistic (structural) rules ("Last Duty First") illustrated in the next play: **P** should play his move 6, but is not allowed to; it is his turn and he cannot play, so he loses.

Play 4: the elimination of double negation—intuitionistic rules

O			**P**		
				$!\ \neg\neg A \supset A$	0
1	$m := 1$			$n := 2$	2
3	$!\ \neg\neg A$	0			
	—		3	$!\ \neg A$	4
5	$!\ A$	4		—	

O wins (intuitionistic rules).

It should be clear from these two examples that the intuitionistic rules for dialogues only concern the *structural* rules, namely *when* (in what conditions) a move (challenge or defence) is allowed, but not the *particle* rules which determine *how* to challenge a move or *how* to answer a challenge.

The intuitionistic rules are only a *restriction* imposed on the classical rules, so if **P** wins a play according to the intuitionistic rules, *a fortiori* he should win according to the classical rules.

Third example, the double negation of the third excluded: $\neg\neg(A \vee \neg A)$

This example is a combination of the previous two. But whereas the principle of third excluded and the principle of double negation elimination are not intuitionistic principles (**P** loses), the double negation of the third excluded $\neg\neg(A \vee \neg A)$ does actually hold with intuitionistic rules. This clearly shows that, for intuitionistic logic, an expression is not equivalent to its double negation (the elimination of the double negation of the third excluded would not yield the third excluded, which would contradict the first example).

Play 5: double negation of third excluded—intuitionistic rules

O			**P**		
				$!\ \neg\neg(A \vee \neg A)$	0
1	$m := 1$			$n := 2$	2
3	$!\ \neg(A \vee \neg A)$	0		—	

	—		3	$!\, A \vee \neg A$	4
5	$?_{\vee}$	4		$!\, \neg A$	6
7	$!\, A$	6		—	
	—		3	$!\, A \vee \neg A$	8
9	$?_{\vee}$	8		$!\, A$	10

P wins (intuitionistic rules).

In this play, **P** also uses his repetition rank of 2 (move 4 and move 8), but this time to challenge move 3 (instead of defending a move). As opposed to the previous examples, he does not need to defend a move preceeding the last unanswered challenge, so this play in winnable by **P** in the intuitionistic and in the classical contexts.

STRATEGY LEVEL

The strategy standpoint is but a generalisation of the procedure which is implemented at the play level; it is a systematic exposition of all the relevant variants of a game—the relevancy of the variants being determined from the viewpoint of one of the two players.

Preliminary notions

Definitions

Extensive form of a dialogical game:

The *extensive form* $\mathbb{E}(\phi)$ of the dialogical game $\mathbb{D}(\phi)$ is simply its tree presentation, also called the game-tree. Nodes are labelled with moves so that the root is labelled with the thesis and paths in $\mathbb{E}(\phi)$ are linear representations of plays and maximal paths represent terminal plays in $\mathbb{D}(\phi)$.

That is, the extensive form of a dialogical game is an infinitely generated tree in which each branch is of a finite length.

Strategy

A *strategy* for player **X** in $\mathbb{D}(\phi)$ is a function which assigns an **X**-move M to every non terminal play ζ having a **Y**-move as last member, such that extending ζ with M results in a play.

X-winning strategy (or X-strategy)

An ***X****-strategy* is *winning* if playing according to it leads to **X**-terminal plays no matter how **Y** moves.

That is, a winning strategy for player **X** defines the situation in which, for any move choice made by player **Y**, **X** has at least one possible move at his disposal allowing him to win.

Extensive form of an X-strategy

Let s*x* be a strategy of player **X** in *D(ϕ)* of extensive form E*(ϕ)*. The extensive form of s*x* is the fragment S*x* of E*(ϕ)* such that:

1. The root of E*(ϕ)* is the root of S*x*,
2. For any node *t* which is associated with an **X**-move in E*(ϕ)*, any immediate successor of *t* in E*(ϕ)* is an immediate successor of *t* in S*x*,
3. For any node *t* which is associated with a **Y**-move in E*(ϕ)*, if *t* has at least one immediate successor in E*(ϕ)* then *t* has exactly one immediate successor in S*x*, namely the one labelled with the **X**-move prescribed by s*x*.

Validity

A proposition is *valid* in a certain dialogical system if and only if **P** has a winning strategy for it.

Some results from existing litterature on the strategy level

The following three results—extracted from existing litterature on the subject—establish the correspondance between the dialogical framework and other frameworks involving classical and intuitionistic logics. We will be using them in order to facilitate the building of dialogical demonstrations: the procedure presented in the next section presupposes them. For more details and recent new proofs of these results, see Clerbout (2014a,b).

P-winning strategies and leaves

Let w be a **P**-winning strategy in $\mathbb{D}(\phi)$. Then every leaf in the extensive form W_ϕ of w is labelled with an

elementary **P**-sentence.

Determinacy

There is an **X**-winning strategy in $\mathbb{D}(\phi)$ if and only if there is no **Y**-winning strategy in $\mathbb{D}(\phi)$.

Soundness and Completeness of Tableaux

Consider first-order tableaux and first-order dialogical games. There is a tableau proof for φ if and only if there is a ***P***-winning strategy in $\mathbb{D}(\phi)$.

The fact that the existence of a ***P***-winning strategy coincides with validity (*there is a **P**-winning strategy in $\mathbb{D}(\phi)$ if and only if φ is valid)* follows from the soundness and completeness of the tableau method with respect to model-theoretical semantics.

This third metalogical result for the standard dialogical framework will be taken here for granted (the proof is given for instance in Clerbout(2014b)..

APPENDIX II: LOCAL REASONS AND DIALOGUES FOR IMMANENT REASONING

Introductory remarks on the choice of CTT

Recent developments in dialogical logic show that the Constructive Type Theory approach to meaning is very natural to the game-theoretical approaches in which (standard) metalogical features are explicitly displayed at the object language-level.[30] This vindicates, albeit in quite a different fashion, Hintikka's plea for the fruitfulness of game-theoretical semantics in the context of epistemic approaches to logic, semantics, and the foundations of mathematics.[31]

From the dialogical point of view, the actions—such as choices—that the particle rules associate with the use of logical constants are crucial elements of their full-fledged (local) meaning: if meaning is conceived as constituted during interaction, then all of the actions involved in the constitution of the meaning of an expression should be made explicit; that is, they should all be part of the object-language.

This perspective roots itself in Wittgenstein's remark according to which one cannot position oneself outside language in order to determine the meaning of something and how it is *linked to syntax*; in other words, language is unavoidable: this is his *Unhintergehbarkeit der Sprache*, one of Wittgenstein's tenets that Hintikka explicitly rejects.[32] According to this perspective of Wittgensteins, language-games are supposed to accomplish the task of studying language from a perspective that acknowledges its *internalized feature*. This is what underlies the approach to meaning and syntax of the dialogical framework in which all the speech-acts that are relevant for rendering the meaning and the "formation" of an expression are made explicit. In this respect, the metalogical perspective which is so crucial for model-theoretic conceptions of meaning does not provide a way out. It is in such a context that Lorenz writes:

> *Also propositions of the metalanguage require the understanding of propositions, […] and thus cannot in a sensible way have this same understanding as their proper object. The thesis that a property of a propositional sentence must always be internal, therefore amounts to articulating the insight that in propositions about a propositional sentence this same propositional sentence does not express a meaningful proposition anymore, since in this case it is not the propositional sentence that is asserted but something about it.*
>
> *Thus, if the original assertion (i.e., the proposition of the ground-level) should not be abrogated, then this same proposition should not be the object of a metaproposition […].*[33]

> *While originally the semantics developed by the picture theory of language aimed at determining unambiguously the rules of "logical syntax" (i.e. the logical form of linguistic expressions) and thus to justify them […]—now language use itself, without the mediation of theoretic constructions, merely via "language games", should be sufficient to introduce the talk about "meanings" in such a way that they supplement the syntactic rules for the use of ordinary language expressions (superficial grammar) with semantic rules that capture the understanding of these expressions (deep grammar).*[34]

Similar criticism to the metalogical approach to meaning has been raised by Göran Sundholm (1997; 2001) who points out that the standard model-theoretical semantic turns semantics into a meta-mathematical formal object in which syntax is linked to meaning by the assignation of truth values to uninterpreted strings of signs (formulae). Language does not express *content* anymore, but it is rather conceived as a system of signs that speak *about* the world—provided a suitable metalogical link between the signs and the world has been fixed. Moreover, Sundholm (2016) shows that the cases of quantifier-dependences motivating Hintikka's IF-logic can be rendered in the CTT framework. What we will here add to Sundholm's observation is that even the interactive features of these dependences can be given a CTT formulation, provided the latter is developed within a dialogical setting.

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[30] See for instance Clerbout/Rahman (2015), Dango (2016), Rahman/Clerboout/Jovanovic (2015).
[31] Cf. Hintikka (1973).
[32] Hintikka (1996) shares this rejection with all those who endorse model-theoretical approaches to meaning.
[33] (Lorenz, 1970, p. 75), translated from the German by Shahid Rahman.
[34] (Lorenz, 1970, p. 109), translated from the German by Shahid Rahman.

Ranta (1988) was the first to link game-theoretical approaches with CTT. Ranta took Hintikka's (1973) Game-Theoretical Semantics (GTS) as a case study, though his point does not depend on that particular framework: in game-based approaches, a proposition is a set of winning strategies for the player stating the proposition.[35] In game-based approaches, the notion of truth is at the level of such winning strategies. Ranta's idea should therefore in principle allow us to apply, safely and directly, instances of game-based methods taken from CTT to the pragmatist approach of the dialogical framework.

From the perspective of a general game-theoretical approach to meaning however, reducing a proposition to a set of winning strategies is quite unsatisfactory. This is particularly clear in the dialogical approach in which different levels of meaning are carefully distinguished: there is indeed the level of strategies, but there is also the level of plays in the analysis of meaning which can be further analysed into local, global and material levels. The constitutive role of the play level for developing a meaning explanation has been stressed by Kuno Lorenz in his (2001) paper:

> *Fully spelled out it means that for an entity to be a proposition there must exist a dialogue game associated with this entity, i.e., the proposition A, such that an individual play of the game where A occupies the initial position, i.e., a dialogue D(A) about A, reaches a final position with either win or loss after a finite number of moves according to definite rules: the dialogue game is defined as a finitary open two-person zero-sum game. Thus, propositions will in general be dialogue-definite, and only in special cases be either proof-definite or refutation-definite or even both which implies their being value-definite.*
>
> *Within this game-theoretic framework […] truth of A is defined as existence of a winning strategy for A in a dialogue game about A; falsehood of A respectively as existence of a winning strategy against A.*[36]

Given the distinction between the play level and the strategy level, and deploying within the dialogical framework the CTT-explicitation program, it seems natural to distinguish between *local* reasons and *strategic* reasons: only the latter correspond to the notion of *proof-object* in CTT and to the notion of *strategic-object* of Ranta. In order to develop such a project we enrich the language of the dialogical framework with statements of the form "$p : A$". In such expressions, what stands on the left-hand side of the colon (here p) is what we call a *local reason*; what stands on the right-hand side of the colon (here A) is a proposition (or set).[37]

The *local* meaning of such statements results from the rules describing how to compose (*synthesis*) within a play the suitable local reasons for the proposition *A* and how to separate (*analysis*) a complex local reason into the elements required by the composition rules for *A*. The synthesis and analysis processes of *A* are built on the formation rules for *A*.

Local reasons and material truth

The most basic contribution of a local reason is its contribution to a material dialogue involving an elementary proposition. Informally, we can say that if the Proponent **P** states the elementary proposition *A*, it is because **P** claims that he can bring forward a reason in defence of his statement. It is the Socratic rule that determines the precise form of that local reason, *specific to A.*[38] Our study focuses on formal—not material—dialogues, but we will still provide some basic elements on material truth in regard to local reasons so as to render in a clearer fashion the limits of our study and its philosophical background, the meaning of formal plays by contrast with what they are not, and the further work that can be carried out from this presentation of dialogues for immanent reasoning.

Approaching material truth

Assume the Proponent states that 1 is an odd number:

$$\mathbf{P}\ !\ 1\ is\ an\ odd\ number$$

the Opponent can then express the following demand, asking **P** for reasons for his statement:

$$\mathbf{O}\ !\ find\ a\ natural\ number\ n\ such\ that\ 1 = 2.n + 1$$

Because of the restriction the Socratic rule imposes on **P**, he can defend his statement by choosing "0", provided that **O** has already endorsed the statement "0 is a natural number" ($0 : \mathbb{N}$). This produces *material truth*.

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[35] That player can be called Player 1, Myself or Proponent.

[36] (Lorenz, 2001, p. 258).

[37] See (Rahman, Redmond, & Clerbout, 2017), (Clerbout & Rahman, 2015) (Rahman & Clerbout, 2013; 2015).

[38] Recall that the Socratic rule does not prohibit the Opponent **O** from challenging an elementary proposition of **P**; the rule only restricts **P**'s authorized moves.

Material truth can then be described in the following way: the statement that a given proposition is materially true requires displaying a local reason *specific to that very proposition*.

Material truth and local reasons

A local reason adduced in defence of a proposition thus prefigures a material dialogue displaying the specific content of that proposition. This constitutes the bottom of the normative approach to meaning of the dialogical framework: *use* (dialogical interaction) is to be understood as *use prescribed by a rule of dialogical interaction*. This applies not only to the meaning of logical constants, but also to the meaning of elementary propositions. This is what Jaroslav Peregrin (2014, pp. 2-3) calls the *role* of a linguistic statement: according to this terminology, and if we place his suggestion in our dialogical setting, we can say that the meaning of an elementary proposition amounts to its *role* in that form of interaction that the Socratic rule for a material dialogue prescribes for that specific proposition. It follows from such a perspective that material dialogues are important not only for the general question of the normativity of logic, but also for the elaboration of a language with content.

Material dialogues and formal dialogues

Summing up, what distinguishes formal dialogues from material dialogues resides in the following:

- The formulation of the Socratic rule of a formal dialogue prescribes a form of interaction based only on the meaning of the logical constant(s) involved, irrespective of the meaning of the elementary propositions in the scope of that constant.
- The choice of the local reason for the elementary propositions involved is left to the authority of the Opponent.

In other words, in a formal dialogue the Socratic rule is *not* specific to any elementary proposition in particular, but it is *general*; definitions that distinguish one proposition from another are introduced during the game according to the local meaning of the logical constant involved: formal dialogues are the purest kind of immanent reasoning.

The synthesis and analysis of local reasons for a proposition A are determined by the actions prescribed by the Socratic rule specific to the kind of play in which A has been stated:

- If the play is *material*, the Socratic rule will describe a kind of action specific to the formation of A.
- If the play is *formal*, as assumed in the main body of our study, the Socratic rule will allow **O** to bring forward the relevant local reasons during the development of the play.

The point is that in formal dialogues, when the Opponent challenges the thesis, the thesis is assumed to be well-formed up to the logical constants, so the formation of the elementary statements is displayed during the development of the dialogue and left to the authority of **O**. So the formation rule for elementary statements does not really take place at the level of local meaning but at level of global meaning.

Since the local reasons for the elementary statements are left to **O**'s authority, what we now need is to describe the process of *synthesis* and *analysis* for local reasons of the logical constants. However, before starting to enrich the language of the standard dialogical framework with local reasons for logical constants let us discuss how to implement a dialogical notion of *formation rules*. The formation rules together with the synthesis and analysis rules settle the local meaning of dialogues for immanent reasoning.

The local meaning of local reasons

Here is an introduction of the formation rules, the synthesis rules, and the analysis rules for local reasons. But we first need to make a clarification on statements and add a piece of notation to the framework:

Statements in dialogues for immanent reasoning

Dialogues are games of giving and asking for reasons; yet in the standard dialogical framework, the reasons for each statement are left implicit and do not appear in the notation of the stament: we have statements of the form $\mathbf{X}\,!\,A$ for instance where A is an elementary proposition. The framework of dialogues for immanent reasoning allows to have explicitly the reason for making a statement, statements then have the form $\mathbf{X}\; a : A$ for

instance where a is the (local) reason **X** has for stating the proposition A. But even in dialogues for immanent reasoning, all reasons are not always provided, and sometimes statements have only implicit reasons for bringing the proposition forward, taking then the same form as in the standard dialogical framework: **X** ! A. Notice that when (local) reasons are not explicit, an exclamation mark is added before the proposition: the statement then has an implicit reason for being made.

A statement is thus both a proposition and its local reason, but this reason may be left implicit, requiring then the use of the exclamation mark.

Adding concessions

In the context of the dialogical conception of CTT we also have statements of the form

$$\mathbf{X} \text{ ! } \pi(x_1, \ldots, x_n)\ [x_i : A_i]$$

where "π" stands for some statement in which $(x_1, \ldots, x_n)$ ocurs, and where $[x_i : A_i]$ stands for some condition under which the statement $\pi(x_1, \ldots, x_n)$ has been brought forward. Thus, the statement reads:

X states that $\pi(x_1, \ldots, x_n)$ under the condition that the antagonist concedes $x_i : A_i$.

We call *required concessions* the statements of the form $[x_i : A_i]$ that condition a claim. When the statement is challenged, the antagonist is accepting, through his own challenge, to bring such concessions forward. The concessions of the thesis, if any, are called *initial concessions*. Initial concessions can include formation statements such as A : *prop*, B : *prop*, for the thesis, $A \supset B$: *prop*.

Formation rules for local reasons: an informal overview

It is presupposed in standard dialogical systems that the players use well-formed formulas (*wff*). The well formation can be checked at will, but only with the usual meta reasoning by which one checks that the formula does indeed observe the definition of a *wff*. We want to enrich our CTT-based dialogical framework by allowing players themselves to first enquire on the formation of the components of a statement within a play. We thus start with dialogical rules explaining the formation of statements involving logical constants (the formation of *elementary* propositions is governed by the Socratic rule, see the discussion above on material truth). In this way, the well formation of the thesis can be examined by the Opponent before running the actual dialogue: as soon as she challenges it, she is *de facto* accepting the thesis to be well formed (the most obvious case being the challenge of the implication, where she has to state the antecedent and thus explicitly endorse it). The Opponent can ask for the formation of the thesis before launching her first challenge; defending the formation of his thesis might for instance bring the Proponent to state that the thesis is a proposition, provided, say, that *A is a* ***set*** is conceded; the Opponent might then concede that *A is a* ***set***, but only after the constitution of *A* has been established, though if this were the case, we would be considering the constitution of an elementary statement, which is a material consideration, not a formal one.

These rules for the *formation* of statements with logical constants are also particle rules which are added to the set of particle rules determining the local meaning of logical constants (called synthesis and analysis of local reasons in the framework of dialogues for immanent reasoning).

These considerations yield the following condensed presentation of the logical constants (plus *falsum*), in which "$\mathfrak{K}$" in "$A \mathfrak{K} B$ " expresses a connective, and "$\mathfrak{Q}$" in "$(\mathfrak{Q}x : A)\ B(x)$ " expresses a quantifier.

Table 1: Formation rules, condensed presentation

	Connective	**Quantifier**	**Falsum**
Move	**X** $A \mathfrak{K} B$: *prop*	**X** $(\mathfrak{Q}x : A)\ B(x)$: *prop*	**X** $\perp$: *prop*
Challenge	**Y** $?_{\mathrm{F}\kappa 1}$ and/or **Y** $?_{\mathrm{F}\kappa 2}$	**Y** $?_{\mathrm{F}q 1}$ and/or **Y** $?_{\mathrm{F}q 2}$	—
Defence	**X** A : *prop* (resp.) **X** B : *prop*	**X** A : *set* (resp.) **X** $B(x)$: *prop* $(x : A)$	—

Because of the *no entity without type* principle, it seems at first glance that we should specify the type of these actions during a dialogue by adding the type "*formation-request*". But as it turns out, we should not: an expression such as "$?_F$: *formation-request*" is a judgement that some action $?_F$ is a formation-request, which should not be confused with the actual act of requesting. We also consider that the force symbol $?_F$ makes the type explicit.

Synthesis of local reasons

The synthesis rules of local reasons determine how to produce a local reason for a statement; they include rules of interaction indicating how to produce the local reason that is required by the proposition (or set) in play, that is, they indicate what kind of dialogic action—what kind of move—must be carried out, by whom (challenger or defender), and what reason must be brought forward.

Implication

For instance, the synthesis rule of a local reason for the implication $A \supset B$ stated by player **X** indicates:

i. that the challenger **Y** must state the antecedent (while providing a local reason for it): **Y** $p_1 : A$.[39]
ii. that the defender **X** must respond to the challenge by stating the consequent (with its corresponding local reason): **X** $p_2 : B$.

In other words, the rules for the synthesis of a local reason for implication are as follows:

Table 2: Synthesis of a local reason for implication

	Implication
Move	**X** ! $A \supset B$
Challenge	**Y** $p_1 : A$
Defence	**X** $p_2 : B$

The general structure for the synthesis of local reasons

More generally, the rules for the synthesis of a local reason for a constant $\mathcal{K}$ is determined by the following triplet:

Table 3: general structure for the synthesis of a local reason for a constant

	A constant $\mathcal{K}$
Move	**X** ! $\varphi[\mathcal{K}]$ ***X** claims that ϕ*
Challenge	***Y** asks for the reason backing such a claim*
Defence	**X** $p : \varphi[\mathcal{K}]$ ***X** states the local reason p for $\varphi[\mathcal{K}]$ according to the rules for the synthesis of local reasons prescribed for $\mathcal{K}$*

[39] This notation is a variant of the one used by Keiff (2004, 2009).

Analysis of local reasons

Apart from the rules for the synthesis of local reasons, we need rules that indicate how to parse a complex local reason into its elements: this is the *analysis* of local reasons. In order to deal with the complexity of these local reasons and formulate general rules for the analysis of local reasons (at the play level), we introduce certain operators that we call *instructions*, such as $L^{\vee}(p)$ or $R^{\wedge}(p)$.

Approaching the analysis rules for local reasons

Let us introduce these instructions and the analysis of local reasons with an example: player **X** states the implication $(A \wedge B) \supset A$. According to the rule for the synthesis of local reasons for an implication, we obtain the following:

Move	**X** ! $(A \wedge B) \supset B$
Challenge	**Y** $p_1 : A \wedge B$

Recall that the synthesis rule prescribes that **X** must now provide a local reason for the consequent; but instead of defending his implication (with **X** $p_2 : B$ for instance), **X** can choose to parse the reason p_1 provided by **Y** in order to force **Y** to provide a local reason for the right-hand side of the conjunction that **X** will then be able to copy; in other words, **X** can force **Y** to provide the local reason for B out of the local reason p_1 for the antecedent $A \wedge B$ of the initial implication. The analysis rules prescribe how to carry out such a parsing of the statement by using *instructions*. The rule for the analysis of a local reason for the conjunction $p_1 : A \wedge B$ will thus indicate that its defence includes expressions such as

- the left instruction for the conjunction, written $L^{\wedge}(p_1)$, and
- the right instruction for the conjunction, written $R^{\wedge}(p_1)$.

These instructions can be informally understood as carrying out the following step: for the defence of the conjunction $p_1 : A \wedge B$ separate the local reason p_1 in its left (or right) component so that this component can be adduced in defence of the left (or right) side of the conjunction.

The general structure for the analysis rules of local reasons

	Move	**Challenge**	**Defence**
Conjunction	**X** $p : A \wedge B$	**Y** ? $L^{\wedge}$ or **Y** ? $R^{\wedge}$	**X** $L^{\wedge}(p)^{X} : A$ (resp.) **X** $R^{\wedge}(p)^{X} : B$
Disjunction	**X** $p : A \vee B$	**Y** $?_{\vee}$	**X** $L^{\vee}(p)^{X} : A$ or **X** $R^{\vee}(p)^{X} : B$
Implication	**X** $p : A \supset B$	**Y** $L^{\supset}(p)^{Y} : A$	**X** $R^{\supset}(p)^{X} : B$

The superscripts with the player label indicate which player is entitled to decide how to resolve the instruction, that is, to decide which local reason to bring forward when carrying out the instruction.

Interaction procedures embedded in instructions

Carrying out the prescriptions indicated by instructions require the following three interaction-procedures:

1. *Resolution of instructions*: this procedure determines how to carry out the instructions prescribed by the rules of analysis and thus provide an actual local reason.
2. *Substitution of instructions*: this procedure ensures the following; once a given instruction has been carried out through the choice of a local reason, say *b*, then every time the same instruction occurs, it will always be substituted by the same local reason *b*.
3. *Application of the Socratic rule*: the Socratic rule prescribes how to constitute equalities out of the resolution and substitution of instructions, linking synthesis and analysis together.

Let us discuss how these rules interact and how they lead to the main thesis of this study, namely that immanent reasoning is equality in action.

From Reasons to Equality

As we have already discussed to some extent one of the most salient features of dialogical logic is the so-called, *Socratic rule* (or Copy-cat rule in the standard—that is, non-CTT—context), establishing that the Proponent can play an elementary proposition only if the Opponent has played it previously.

The Socratic rule is a characteristic feature of the dialogical approach: other game-based approaches do not have it. With this rule the dialogical framework comes with an internal account of elementary propositions: an account in terms of interaction only, without depending on metalogical meaning explanations for the non-logical vocabulary. More prominently, this means that the dialogical account does not rely—contrary to Hintikka's GTS games—on the model-theoretical approach to meaning for elementary propositions.

The rule has a clear Platonist and Aristotelian origin and sets the terms for what it is to carry out a *formal argument*: see for instance Plato's *Gorgias* (472b-c). We can sum up the underlying idea with the following statement:

> there is no better grounding of an assertion within an argument than indicating that it has been already conceded by the Opponent or that it follows from these concessions.[40]

What should be stressed here are the following two points:

1. formality is understood as a kind of *interaction*; and
2. formal reasoning *should not* be understood here as devoid of content and reduced to purely syntactic moves.

Both points are important in order to understand the criticism often raised against formal reasoning in general, and in logic in particular. It is only quite late in the history of philosophy that formal reasoning has been reduced to syntactic manipulation— presumably the first explicit occurrence of the syntactic view of logic is Leibniz's "pensée aveugle" (though Leibniz's notion was not a reductive one). Plato and Aristotle's notion of formal reasoning is neither "static" nor "empty of meaning"—to use Hegel's words quoted in the introduction. In the Ancient Greek tradition logic emerged from an approach of assertions in which meaning and justification result from what has been brought forward during argumentative interaction. According to this view, dialogical interaction is constitutive of meaning.

Some former interpretations of standard dialogical logic did understand formal plays in a purely syntactic manner. The reason for this is that the standard version of the framework is not equipped to express meaning at the object-language level: there is no way of asking and giving reasons for elementary propositions. As a consequence, the standard formulation simply relies on a syntactic understanding of *Copy-cat moves*, that is, moves entitling **P** to copy the elementary propositions brought forward by **O**, regardless of its content.

The dialogical approach to CTT (dialogues for immanent reasoning) however provides a fine-grain study of the contentual aspects involved in formal plays, much finer than the one provided by the standard dialogical framework. In dialogues for immanent reasoning which we are now presenting, a statement is constituted both by a proposition and by the (local) reason brought forward in defence of the claim that the proposition holds. In formal plays not only is the Proponent allowed to copy an elementary proposition stated by the Opponent, as in the standard framework, but he is also allowed to adduce in defence of that proposition the *same* local reason brought forward by the Opponent when she defended that same proposition. Thus immanent reasoning and equality in action are intimately linked. In other words, a formal play displays the *roots of the content* of an elementary proposition, and *not* a syntactic manipulation of that proposition.

Statements of definitional equality emerge precisely at this point. In particular reflexivity statements such as

$$p = p : A$$

express from the dialogical point of view the fact that if **O** states the elementary proposition *A*, then **P** can do the same, that is, play the same move and do it on the same grounds which provide the meaning and justification of *A*, namely *p*.

[40] Recent work (Crubellier, 2014, pp. 11-40) and (Rahman, McConaughey, & Crubellier, 2015) claim that this rule is central to the interpretation of dialectic as the core of Aristotle's logic. Neither Ian Lukasiewicz's (1957) famous reconstruction of Aristotle's syllogistic, nor the Natural Deduction approach of Kurt Ebbinghaus (1964) and John Corcoran (1974) deploy this rule, but Marion and Rückert (2015) showed that this rule displays Aristotle's view on universal quantification.

These remarks provide an insight only on simple forms of equality and barely touch upon the finer-grain distinctions discussed above; we will be moving to these by means of a concrete example in which we show, rather informally, how the combination of the processes of analysis, synthesis, and resolution of instructions lead to equality statements.

THE DIALOGICAL ROOTS OF EQUALITY: DIALOGUES FOR IMMANENT REASONING

In this section we will spell out all the relevant rules for the dialogical framework incorporating features of Constructive Type Theory—that is, a dialogical framework making the players' reasons for asserting a proposition explicit. The rules can be divided, just as in the standard framework, into rules determining local meaning and rules determining global meaning. These include:

1. Concerning *local meaning* (section 0):
    a. formation rules (p. 51);
    b. rules for the synthesis of local reasons (p. 54); and
    c. rules for the analysis of local reasons (p. 54).
2. Concerning *global meaning*, we have the following (structural) rules (section 0):
    a. rules for the resolution of instructions;
    b. rules for the substitution of instructions (p. 57);
    c. equality rules determined by the application of the Socratic rules (p. 57); and
    d. rules for the transmission of equality (p. 60).

We will be presenting these rules in this order in the next two subsections, along with the adaptation of the other structural rules to dialogues of immanent reasoning in the second subsection. The following subsection (0) provides a series of exercises and their solution.

Local meaning in dialogues of immanent reasoning

The formation rules

Formation rules for logical constants and falsum

The formation rules for *logical constants* and for *falsum* are given in the following table. Notice that a statement '$\bot$: **prop**' cannot be challenged; this is the dialogical account for falsum '$\bot$' being by definition a proposition.

Table 4: Formation rules

	Move	Challenge	Defence
Conjunction	**X** $A \wedge B : \boldsymbol{prop}$	**Y** ? $F_{\wedge 1}$ or **Y** ? $F_{\wedge 2}$	**X** $A : \boldsymbol{prop}$ (resp.) **X** $B : \boldsymbol{prop}$
Disjunction	**X** $A \vee B : \boldsymbol{prop}$	**Y** ? $F_{\vee 1}$ or **Y** ? $F_{\vee 2}$	**X** $A : \boldsymbol{prop}$ (resp.) **X** $B : \boldsymbol{prop}$
Implication	**X** $A \supset B : \boldsymbol{prop}$	**Y** ? $F_{\supset 1}$ or **Y** ? $F_{\supset 2}$	**X** $A : \boldsymbol{prop}$ (resp.) **X** $B : \boldsymbol{prop}$
Universal quantification	**X** $(\forall x : A)B(x) : \boldsymbol{prop}$	**Y** ? $F_{\forall 1}$ or **Y** ? $F_{\forall 2}$	**X** $A : \boldsymbol{set}$ (resp.) **X** $B(x) : \boldsymbol{prop}[x : A]$
Existential quantification	**X** $(\exists x : A)B(x) : \boldsymbol{prop}$	**Y** ? $F_{\exists 1}$ or **Y** ? $F_{\exists 2}$	**X** $A : \boldsymbol{set}$ (resp.) **X** $B(x) : \boldsymbol{prop}[x : A]$
Subset separation	**X** $\{x : A \mid B(x)\} : \boldsymbol{prop}$	**Y** ? F_1 or **Y** ? F_2	**X** $A : \boldsymbol{set}$ (resp.) **X** $B(x) : \boldsymbol{prop}[x : A]$

Falsum	**X** $\perp : \boldsymbol{prop}$	—	—

The substitution rule within dependent statements

The following rule is not really a formation-rule but is very useful while applying formation rules where one statement is dependent upon the other such as $B(x) : \boldsymbol{prop}[x : A]$.[41]

Table 5: Substitution rule within dependent statements (subst-D)

	Move	**Challenge**	**Defence**
Subst-D	**X** $\pi(x_1, \dots, x_n)[x_i : A_i]$	**Y** $\tau_1 : A_1, \dots, \tau_n : A_n$	**X** $\pi(\tau_1, \dots, \tau_n)$

In the formulation of this rule, "π" is a statement and "τ_i" is a local reason of the form either $a_i : A_i$ or $x_i : A_i$.

A particular case of the application of Subst-D is when the challenger simply chooses the same local reasons as those occurring in the concession of the initial statement. This is particularly useful in the case of formation plays:

Example of a formation-play

Here is an example of a formation play with some explanation. The standard development rules are enough to understand the following plays.

In this example, the Opponent provides initial concession before the Proponent states his thesis. Thus the Proponent's thesis is

$$(\forall x : A)(B(x) \supset C(x)) : \textbf{prop}$$

given these three provisos that appear as initial concessions by the Opponent:

$$A : \textbf{set},$$
$$B(x) : \textbf{prop}\ [x : A]$$
and $C(x) : \textbf{prop}\ [x : A]$,

This yields the following play:

Play 6: formation-play with initial concessions: first decision-option of **O**

	O			**P**	
0.1	A : **set**				
0.2	$B(x)$: **prop** $[x : A]$				
0.3	$C(x)$: **prop** $[x : A]$			$(\forall x : A)\ B(x) \supset C(x)$: **prop**	0
1	$m \coloneqq 1$			$n \coloneqq 2$	2
3	$?F_{\forall 1}$	0		A : **set**	4

P wins.

Explanation:

- 0.1 to 0.3: **O** concedes that A is a **set** and that $B(x)$ and $C(x)$ are propositions provided x is an element of A.

[41] This rule is an expression at the level of plays of the rule for the substitution of variables in a hypothetical judgement. See (Martin-Löf, 1984, pp. 9-11).

- Move 0: **P** states that the main sentence, universally quantified, is a proposition (under the concessions made by **O**).
- Moves 1 and 2: the players choose their repetition ranks.
- Move 3: **O** challenges the thesis by asking the left-hand part as specified by the formation rule for universal quantification.
- Move 4: **P** responds by stating that *A* is a **set**. This has already been granted with the concession 0.1 so even if **O** were to challenge this statement the Proponent could refer to her initial concession.

This dialogue obviously does not cover all the aspects related to the formation of

$$(\forall x : A)\ B(x)\ \supset\ C(x) : \textbf{prop}.$$

Notice however that the formation rules allow an alternative move for the Opponent's move 3,[42] so that **P** has another possible course of action, dealt with in the following play.

Play 7: formation-play with initial concessions: second decision-option of **O**

O				**P**	
0.1	A : **set**				
0.2	$B(x)$: **prop** $[x : A]$				
0.3	$C(x)$: **prop** $[x : A]$			$(\forall x : A)\ B(x)\ \supset\ C(x)$: **prop**	0
1	$m \coloneqq 1$			$n \coloneqq 2$	2
3	$?\,F_{\forall 2}$	0		$B(x)\ \supset\ C(x)$: **prop** $[x : A]$	4
5	$x : A$	4		$B(x)\ \supset\ C(x)$: **prop**	6
7	$?\,F_{\supset 1}$	6		$B(x)$: **prop**	10
9	$B(x)$: **prop**		0.2	$x : A$	8

P wins.

Explanation:

The second play starts like the first one until move 2. Then:

- Move 3: this time **O** challenges the thesis by asking for the right-hand part.
- Move 4: **P** responds, stating that $B(x) \supset C(x)$ is a proposition, provided that x : A.
- Move 5: **O** challenges the preceding move by granting the proviso and asking **P** to respond (this kind of move is governed by a Subst-D rule).
- Move 6: **P** responds by stating that $B(x) \supset C(x)$ is a proposition.
- Move 7: **O** challenges move 6 by asking the left-hand part, as specified by the formation rule for material implication.
  To defend against this challenge, **P** needs to make an elementary move. But since **O** has not played it yet, **P** cannot defend it at this point. Thus:
- Move 8: **P** launches a counterattack against initial concession 0.2 by granting the proviso $x : A$ (that has already been conceded by **O** in move 5), making use of the same kind of statement-substitution (Subst-D) rule deployed in move 5.
- Move 9: **O** answers to move 8 and states that $B(x)$ is a proposition.
- Move 10: **P** can now defend the challenge initiated with move 7 and win this dialogue.

Once again, there is another possible choice for the Opponent because of her move 7: she could ask the right-hand part. This would yield a dialogue similar to the one above except that the last moves would be about $C(x)$ instead of $B(x)$.

[42] As a matter of fact, increasing her repetition rank would allow **O** to play the two alternatives for move 3 within a single play. But increasing the Opponent's rank usually yields redundancies (Clerbout, 2014a; 2014b) making things harder to understand for readers not familiar with the dialogical approach; hence our choice to divide the example into different simple plays.

Concluding on the formation-play example:

By displaying these various possibilities for the Opponent, we have entered the strategic level. This is the level at which the question of the good formation of the thesis gets a definitive answer, depending on whether the Proponent can always win—that is, whether he has a winning strategy. The basic notions related to this level of strategies are to be found in our presentation of standard dialogical logic.

The rules for local reasons: synthesis and analysis

Now that the dialogical account of formation rules has been clarified, we may further develop our analysis of plays by introducing local reasons. Let us do so by providing the rules that prescribe the synthesis and analysis of local reasons. For more details on each rule, see section 0.

Table 6: synthesis rules for local reasons

	Move	**Challenge**	**Defence**
Conjunction	**X** ! $A \wedge B$	**Y** ? $L^{\wedge}$ or **Y** ? $R^{\wedge}$	**X** $p_1 : A$ (resp.) **X** $p_2 : B$
Existential quantification	**X** ! $(\exists x : A)B(x)$	**Y** ? $L^{\exists}$ or **Y** ? $R^{\exists}$	**X** $p_1 : A$ (resp.) **X** $p_2 : B(p_1)$
Subset separation	**X** ! $\{x : A \mid B(x)\}$	**Y** ? L or **Y** ? R	**X** $p_1 : A$ (resp.) **X** $p_2 : B(p_1)$
Disjunction	**X** ! $A \vee B$	**Y** ? $^{\vee}$	**X** $p_1 : A$ or **X** $p_2 : B$
Implication	**X** ! $A \supset B$	**Y** $p_1 : A$	**X** $p_2 : B$
Universal quantification	**X** ! $(\forall x : A)B(x)$	**Y** $p_1 : A$	**X** $p_2 : B(p_1)$
Negation	**X** ! $\neg A$ Also expressed as **X** ! $A \supset \bot$	**Y** $p_1 : A$	**X** : $\bot$ (**X** gives up[43]).

Table 7: analysis rules for local reasons

	Move	**Challenge**	**Defence**
Conjunction	**X** $p : A \wedge B$	**Y** ? $L^{\wedge}$ or **Y** ? $R^{\wedge}$	**X** $L^{\wedge}(p)^{X} : A$ (resp.) **X** $R^{\wedge}(p)^{X} : B$

[43] The reading of stating bottom as giving up stems from (Keiff, 2007).

Existential quantification	**X** p: $(\exists x\colon A)B(x)$	**Y** ? $L^{\exists}$ or **Y** ? $R^{\exists}$	**X** $L^{\exists}(p)^X\colon A$ (resp.) **X** $R^{\exists}(p)^X\colon B(L^{\exists}(p)^X)$
Subset separation	**X** p: $\{x : A \mid B(x)\}$	**Y** ? L or **Y** ? R	**X** $L^{\{\dots\}}(p)^X\colon A$ (resp.) **X** $R^{\wedge}(p)^X\colon B(L^{\{\dots\}}(p)^X)$
Disjunction	**X** $p\colon A \vee B$	**Y** $?^{\vee}$	**X** $L^{\vee}(p)^X\colon A$ or **X** $R^{\vee}(p)^X\colon B$
Implication	**X** $p\colon A \supset B$	**Y** $L^{\supset}(p)^Y\colon A$	**X** $R^{\supset}(p)^X\colon B$
Universal quantification	**X** p: $(\forall x\colon A)B(x)$	**Y** $L^{\forall}(p)^Y\colon A$	**X** $R^{\forall}(p)^X\colon B(L^{\forall}(p)^Y)$
Negation	**X** $p\colon \neg A$ Also expressed as **X** $p\colon A \supset \bot$	**Y** $L^{\neg}(p)^Y\colon A$ **Y** $L^{\supset}(p)^Y\colon A$	**X** $R^{\neg}(p)^X\colon \bot$ **X** $R^{\supset}(p)^X\colon \bot$

Slim instructions: dealing with cases of anaphora

One of the most salient features of the CTT framework is that it contains the means to deal with cases of anaphora,[44]

Notice that in the formalization of traditional syllogistic form *Barbara*, the projection **fst**(*z*) can be seen as the tail of the anaphora whose head is *z*:

$$\begin{array}{ll} (\forall z : (\exists x : D)A)B[\mathbf{fst}(z)] \text{ true} & \textit{premise 1} \\ (\forall z : (\exists x : D)B)C[\mathbf{fst}(z)] \text{ true} & \textit{premise 2} \\ \hline (\forall z : (\exists x : D)A)C[\mathbf{fst}(z)] \text{ true} & \textit{conclusion} \end{array}$$

In dialogues for immanent reasoning, when a local reason has been made explicit, this kind of anaphoric expression is formalized through *instructions*, which provides a further reason for introducing them. For example if *a* is the local reason for the first premise we have

$$\mathbf{P}\ p : (\forall z : (\exists x : D)A(x))B(L^{\exists}(L^{\forall}(p)^{\mathbf{O}}))$$

However, since the thesis of a play does not bear an explicit local reason (we use the exclamation mark to indicate there is an implicit one), it is possible for a statement to be bereft of an explicit local reason. When there is no explicit local reason for a statement using anaphora, we cannot bind the instruction $L^{\forall}(p)^{\mathbf{O}}$ to a local reason p. We thus have something like this, with a blank space instead of the anaphoric local reason:

$$\mathbf{P}\ !\ (\forall z : (\exists x : D)A(x))B(L^{\exists}(L^{\forall}(\ )^{\mathbf{O}}))$$

But this blank stage can be circumvented: the challenge on the universal quantifier will yield the required local reason: **O** will provide $a\colon (\exists x\colon D)A(x)$, which is the local reason for z. We can therefore bind the instruction on the missing local reason with the corresponding variable—z in this case—and write

$$\mathbf{P}!\ (\forall z : (\exists x : D)A(x))B(L^{\exists}(L^{\forall}(z)^{\mathbf{O}}))$$

We call this kind of instruction, *slim instructions*. For the substitution of slim instructions the following two cases are to be distinguished**:**

[44] See Sundholm (1986, pp. 501-503) and Ranta (1994, pp. 77-99).

Substitution of Slim Instructions 1

Given some slim instruction such as $L^{\forall}(z)^{\mathbf{Y}}$, once the quantifier $(\forall z{:}\, A)B(\ldots)$ has been challenged by the statement $a : A$, the occurrence of $L^{\forall}(z)^{\mathbf{Y}}$ can be substituted by a. The same applies to other instructions.

In our example we obtain:

$$\mathbf{P}!\ (\forall z : (\exists x : D)A(x))B(L^{\exists}(L^{\forall}(z)^{\mathbf{O}}))$$
$$\mathbf{O}\ a : (\exists x : D)A(x)$$
$$\mathbf{P}\ b : B(L^{\exists}(L^{\forall}(z)^{\mathbf{O}}))$$
$$\mathbf{O}\ ?\ a\ /\ L^{\forall}(z)^{\mathbf{O}}$$
$$\mathbf{P}\ b : B(L^{\exists}(a))$$
$$\ldots$$

Substitution of Slim Instructions 2

Given some slim instruction such as $L^{\forall}(z)^{\mathbf{Y}}$, once the instruction $L^{\forall}(c)$—resulting from an attack on the universal $\forall z : \varphi$—has been resolved with $a : \varphi$, then any occurrence of $L^{\forall}(z)^{\mathbf{Y}}$ can be substituted by a. The same applies to other instructions.

Global Meaning in dialogues for immanent reasoning

We here provide the structural rules for dialogues for immanent reasoning, which determine the global meaning in such a framework. They are for the most part similar in principle to the precedent logical framework for dialogues; the rules concerning instructions are an addition for dialogues for immanent reasoning.

Structural Rules

SR0: Starting rule

The start of a *formal dialogue of immanent reasoning* is a move where **P** states the *thesis.* The thesis can be stated under the condition that **O** commits herself to certain other statements called *initial concessions*; in this case the thesis has the form ! $A\,[B_1, \ldots, B_n]$, where A is a statement with implicit local reason and $B_1, \ldots, B_n$ are statements with or without implicit local reasons.

A dialogue with a thesis proposed under some conditions starts if and only if **O** accepts these conditions. **O** accepts the conditions by stating the *initial concessions* in moves numbered 0.1, … 0.*n* before choosing the repetition ranks.

After having stated the thesis (and the initial concessions, if any), each player chooses in turn a positive integer called the *repetition rank* which determines the upper boundary for the number of attacks and of defences each player can make in reaction to each move during the play.

SR1: Development rule

The Development rule depends on what kind of logic is chosen: if the game uses intuitionistic logic, then it is SR1i that should be used; but if classical logic is used, then SR1c must be used.

SR1i: Intuitionistic Development rule, or Last Duty First

Players play one move alternately. Any move after the choice of repetition ranks is either an attack or a defence according to the rules of formation, of synthesis, and of analysis, and in accordance with the rest of the structural rules.

If the logical constant occurring in the thesis is not recorded by the table for local meaning, then either it must be introduced by a nominal definition, or the table for local meaning needs to be enriched with the new expression.[45]

Players can answer only against the *last non-answered* challenge by the adversary.

[45] If the logical constant occurring in the thesis is not recorded by the table for local meaning, then either it must be introduced by a nominal definition based on some logical constant already present in the local rules, or the table for local meaning needs to be enriched with the new expression.

Note: This structural rule is known as the Last Duty First condition, and makes dialogical games suitable for intuitionistic logic, hence the name of this rule.

SR1c: Classical Development rule

Players play one move alternately. Any move after the choice of repetition ranks is either an attack or a defence according to the rules of formation, of synthesis, and of analysis, and in accordance with the rest of the structural rules.

If the logical constant occurring in the thesis is not recorded by the table for local meaning, then either it must be introduced by a nominal definition, or the table for local meaning needs to be enriched with the new expression.

*Note: The structural rules with SR1c (and not SR1i) produce strategies for classical logic. The point is that since players can answer to a list of challenges in any order (which is not the case with the intuitionistic rule), it might happen that the two options of a **P**-defence occur in the same play—this is closely related to the classical development rule in sequent calculus allowing more than one formula at the right of the sequent.*

SR2: Formation rules for formal dialogues

A formation-play starts by challenging the thesis with the formation request **O** $?_{\mathbf{prop}}$; **P** must answer by stating that his thesis is a proposition. The game then proceeds by applying the formation rules up to the elementary constituents of **prop**/**set**.

After that the Opponent is free to use the other particle rules insofar as the other structural rules allow it.

Note: The constituents of the thesis will therefore not be specified before the play but as a result of the structure of the moves (according to the rules recorded by the rules for local meaning).

SR3: Resolution of instructions

1. A player may ask his adversary to carry out the prescribed instruction and thus bring forward a suitable local reason in defence of the proposition at stake. Once the defender has replaced the instruction with the required local reason we say that the instruction has been resolved.
2. The player index of an instruction determines which of the two players has the right to choose the local reason that will resolve the instruction.
    a. If the instruction $\mathcal{I}$ for the logical constant $\mathcal{K}$ has the form $\mathcal{I}^{\mathcal{K}}(p)^{\mathbf{X}}$ and it is **Y** who requests the resolution, then the request has the form **Y** ?…/ $\mathcal{I}^{\mathcal{K}}(p)^{\mathbf{X}}$, and it is **X** who chooses the local reason.
    b. If the instruction $\mathcal{I}$ for the logic constant $\mathcal{K}$ has the form $\mathcal{I}^{\mathcal{K}}(p)^{\mathbf{Y}}$ and it is player **Y** who requests the resolution, then the request has the form **Y** p_i / $\mathcal{I}^{\mathcal{K}}(p)^{\mathbf{Y}}$, and it is **Y** who chooses the local reason.
3. In the case of a sequence of instructions of the form $\pi[\mathcal{I}_i(...(\mathcal{I}_k(p))...)]$, the instructions are resolved from the inside ($\mathcal{I}_k(p)$) to the outside ($\mathcal{I}_i$).

This rule also applies to functions.

SR4: Substitution of instructions

Once the local reason b has been used to resolve the instruction $\mathcal{I}^{\mathcal{K}}(p)^{\mathbf{X}}$, and if the same instruction occurs again, players have the right to require that the instruction be resolved with b. The substitution request has the form $?b/\mathcal{I}_k(p)^{\mathbf{X}}$. Players cannot choose a different substitution term (in our example, not even **X**, once the instruction has been resolved).

This rule also applies to functions.

SR5: Socratic rule and definitional equality

The following points are all parts of the Socratic rule, they all apply.

SR5.1: Restriction of P statements

P cannot make an elementary statement if **O** has not stated it before, except in the thesis.

An elementary statement is either an elementary proposition with implicit local reason, or an elementary proposition and its local reason (not an instruction).

SR5.2: Challenging elementary statements in formal dialogues

Challenges of elementary statements with implicit local reasons take the form:

$$\boldsymbol{X}\ !\ A$$
$$\boldsymbol{Y}\ ?_{reason}$$
$$\boldsymbol{X}\ a : A$$

Where A is an elementary proposition and a is a local reason.

P cannot challenge **O**'s elementary statements, except if **O** provides an elementary initial concession with implicit local reason, in which case **P** can ask for a local reason, or in the context of transmission of equality.

SR5.3: Definitional equality

O may challenge elementary **P**-statements, challenge answered by stating a definitional equality, expressing the equality between a local reason introduced by **O** and an instruction also introduced by **O**.

These rules do not cover cases of transmission of equality. The Socratic rule also applies to the resolution or substitution of functions, even if the formulation mentions only instructions.

We distinguish reflexive and non-reflexive cases of :

SR5.3.1: Non-reflexive cases of the Socratic rule

We are in the presence of a *non-reflexive case* of the Socratic rule when **P** responds to the challenge with the indication that **O** gave the same local reason for the same proposition when she had to resolve or substitute instruction $\mathcal{S}$.

Here are the different challenges and defences determining the meaning of the three following moves:

Table 8: Non-reflexive cases of the Socratic rule

	Move	Challenge	Defence
SR5.3.1a	**P** $a : A$	**O** $? = a$	**P** $\mathcal{S} = a : A$
SR5.3.1b	**P** $a : A(b)$	**O** $? = b^{A(b)}$	**P** $\mathcal{S} = b : D$
SR5.3.1c	**P** $\mathcal{S} = b : D$ (this statement stems from **SR5.3.1b)**	**O** $? \ldots = A(b)$	**P** $A(\mathcal{S}) = A(b)$: ***prop***

Presuppositions:

(i) The response prescribed by SR5.3.1a presupposes that **O** has stated A or $a = b : A$ as the result of the resolution or substitution of instruction $\mathcal{S}$ occurring in $\mathcal{S} : A$ or in $\mathcal{S} = b : A$.

(ii) The response prescribed by SR5.3.1b presupposes that **O** has stated A and $b : D$ as the result of the resolution or substitution of instruction $\mathcal{S}$ occurring in $a : A(\mathcal{S})$.

(iii) SR5.3.1c assumes that **P** $\mathcal{S} = b : D$ is the result of the application of SR5.3.1b. The further challenge seeks to verify that the replacement of the instruction produces an equality in **prop**, that is, that the replacement of the instruction with a local reason yields an equal proposition to the one in which the instruction was not yet replaced. The answer prescribed by this rule presupposes that **O** has already stated $A(b)$: **prop** (or more trivially $A(\mathcal{S}) = A(b)$: **prop**).

The **P**-statements obtained after defending elementary **P**-statements cannot be attacked again with the Socratic rule (with the exception of SR5.3.1c), nor with a rule of resolution or substitution of instructions.

SR5.3.2: Reflexive cases of the Socratic rule

We are in the presence of a *reflexive case* of the Socratic rule when **P** responds to the challenge with the indication that **O** adduced the same local reason for the same proposition, though that local reason in the statement of **O** is not the result of any resolution or substitution.

The attacks have the same form as those prescribed by SR5.3.1. Responses that yield reflexivity presuppose that **O** has previously stated the same statement or even the same equality.

The response obtained cannot be attacked again with the Socratic rule.

SR6: Transmission of definitional equality

Transmission of definitional equality I: Substitution within dependent or independent statements. The expression "type" refers to either **prop** or **set**. For more explanations on this structural rule, see next section (0).

Move	Challenge	Defence
X $b(x) : B(x)\ [x : A]$	**Y** $a = c : A$	**X** $b(a) = b(c) : B(a)$
X $b(x) = d(x) : B(x)\ [x : A]$	**Y** $a : A$	**X** $b(a) = d(a) : B(a)$
X $B(x) : type\ [x : A]$	**Y** $a = c : A$	**X** $B(a)=B(c) : type$
X $B(x)=D(x) : type\ [x : A]$	**Y** $?_{B(x)=D(x)}\ a : A$ or **Y** $?_{B(x)=D(x)}\ a = c : A$	**X** $B(a)=D(a) : type$ or **X** $B(a)=D(c) : type$
X $A = B : type$	**Y** $?_{A=D}\ a : A$ or **Y** $?_{A=D}\ a = c : A$	**X** $a : B$ or **X** $a = c : B$

Transmission of definitional equality II:

Move	Challenge	Defence
X $A : type$	**Y** $?_{type}$- *refl*	**X** $A = A : type$
X $A = B : type$	**Y** $?_{B}$- *symm*	**X** $B = A : type$
X $A = B : type$ **X** $B = C : type$	**Y** $?_{A}$- *trans*	**X** $A = C : type$
X $a : A$	**Y** $?_{A}$- *refl*	**X** $a = a : A$
X $a = b : A$	**Y** $?_{b}$- *symm*	**X** $b = a : A$
X $a = b : A$ **X** $b = c : A$	**Y** $?_{a}$- *trans*	**X** $a = c : A$

SR7: Winning rule for plays

The player who makes the last move in a dialogue wins the dialogue. If **O** stated $\bot$ (or $p : \bot$) , at move n then **P** wins with the move **O**-*gives up*(n). **P** can also adduce **O**-*gives up*(n) as local reason in support for any statement that he has not defended before **O** stated $\bot$ at move n.

Rules for the transmission of definitional equality

As can be expected, definitional equality is transmitted by reflexivity, symmetry[46], and transitivity. Definitional equalities however can also be used in order to carry out a substitution within dependent statements—they can in fact be seen as a special form of application of the substitution rule for dependent statement Subst-D presented in the first section for local meaning, with the formation rules (0, p. 52). We use the expression "type" as encompassing **prop** and **set**.

Table 9: Transmission of definitional equality I: Substitution within dependent or independent statements

Move	Challenge	Defence
X $b(x) : B(x)\ [x : A]$	**Y** $a = c : A$	**X** $b(a) = b(c) : B(a)$
X $b(x) = d(x) : B(x)\ [x : A]$	**Y** $a : A$	**X** $b(a) = d(a) : B(a)$
X $B(x) : type\ [x : A]$	**Y** $a = c : A$	**X** $B(a)=B(c) : type$
X $B(x)=D(x) : type\ [x : A]$	**Y** $?_{B(x)=D(x)}\ a : A$ or **Y** $?_{B(x)=D(x)}\ a = c : A$	**X** $B(a)=D(a) : type$ or **X** $B(a)=D(c) : type$
X $A = B : type$	**Y** $?_{A=D}\ a : A$ or **Y** $?_{A=D}\ a = c : A$	**X** $a : B$ or **X** $a = c : B$

Reading adjuvant for the fourth rule (dependent statements):
If **X** stated that $B(x)$ and $D(x)$ are equal propositional functions, provided that x is an element of the set A—that is, **X** $B(x)=D(x)$: **prop** $[x : A]$—, then **Y** can carry out two kinds of attacks:

1. Stating himself that some local reason, say a, can be adduced for A—***Y*** $a: A$—, and request at the same time of **X** that he replaces x with a in $B(x)=D(x)$, that is stating $B(a)=D(a)$: **prop**.
2. Stating himself an equality such as $a = c$: A, and request at the same time **X** to carry out the corresponding substitutions in $B(x)=D(x)$, that is to state **X** $B(a)=D(c)$: **prop**.

Table 10: Transmission of definitional equality II

Move	Challenge	Defence
X $A : type$	**Y** $?_{type}$- *refl*	**X** $A = A : type$
X $A = B : type$	**Y** $?_B$- *symm*	**X** $B = A : type$
X $A = B : type$		

[46] Symmetry used here is not the same notion as the symmetry of section 0.

X $B = C : type$	**Y** $?_A$- *trans*	**X** $A = C : type$
X $a : A$	**Y** $?_A$- *refl*	**X** $a = a : A$
X $a = b : A$	**Y** $?_b$- *symm*	**X** $b = a : A$
X $a = b : A$ **X** $b = c : A$	**Y** $?_a$- *trans*	**X** $a = c : A$

Reading adjuvant:

In order to trigger reflexivity, transitivity, and symmetry from some equality statements the challenger can attack an equality by asking for each of these properties. For example, if **X** stated $A = B$: **prop/set**, **Y** can ask **X** to state the commutated equality $B = A$: **prop/set** by calling on *symmetry.* The notation of such an attack is as follows: **Y** $?_{B\text{-}symm}$. Similarly, **Y** $?_{A\text{-}refl}$ and **Y** $?_{A\text{-}trans}$ respectively request reflexivity and transitivity.

Appendix II The identity-predicate Id

The dialogical meaning explanation of the identity predicate **Id**(*x*, *y*, *z*) – where *x* is a **set** (or a **prop**) and *y* and *z* are local reasons in support of *A* – is based on the following: **X**'s statement **Id**(*A*, *a*, *b*) presupposes that *a* : *A* and *b* : *A*, and expresses the claim that " *a* and *b* are identical reasons for supporting *A*. The presupposition yields already its formation rule, the second requires a formulation of the Socratic Rule specific to the identity predicate. Let us start with the formation:

Formation of Id

Statement	Challenge	Defence
X ! **Id**(*A*, a_i, a_j) : **prop**	**Y** ?$_{F1}$ **Id**	**X** ! *A* : **set**
	Y?$_{F2}$ **Id**	**X** ! *ai* : *A*
	Y?$_{F3}$ **Id**	**X** ! *aj* : *A*

Socratic Rules for Id

Opponent's statements of identity can only be challenged by means of the rule of global analysis or by Leibniz-substitution rule

The following rules apply to statements of the form **Id**(*A*, *a*, *a*) and the more general statement of identity **Id**(*A*, *a*, *b*). Let us start with the reflexive case.

SR-Id.1 Socratic Rules for Id(*A*, *a*, *a*)

If the Proponent states **P** !**Id**(*A*, *a*, *a*), then he must bring forward the definitional equality that conditions statements of propositional intensional identity (see chapter II.8). Furthermore, the statement **P** !**Id**(*A*, *a*, *a*) commits the proponent to make explicit the local reason behind his statement, namely, the local reason **refl**(*A*, *a*) **specific** of **Id**-statements, the only internal structure of which is its dependence on *a*. Thus; the dialogical meaning of the instruction **refl**(*A*, *a*) amounts to prescribing the definitional equality *a* = **refl**(*A*, *a*) : *A* as defence to the challenge **O** ? =**refl**(*A*, *a*). The following two tables display the rules that implement those prescriptions.

Socratic Rule for the Global Synthesis of the local reason for P ! Id(*A*, *a*, *a*)

Statement	Challenge	Defence
P ! **Id**(*A*, *a*, *a*)	**O** ? $_{reason\mathbf{Id}}$	**P refl**(*A*, *a*) : **Id**(*A*, *a*, *a*)
P refl(*A*, *a*) : **Id**(*A*, *a*, *a*)	**O** ? = **refl**(*A*, *a*)	**P** *a* = *a* : *A*

(This rule presupposes that the well-formation of **Id(*A*, *a*, *a*)** has been established)

The following rule is just applying the general Socratic Rule for local reasons to the specific case of **refl**(*A*, *a*) and shows that the local reason **refl**(*A*, *a*) is in fact equal to *a*.

Socratic Rule for the challenge upon P's use of refl(*A*, *a*)

Statement	Challenge	Defence
P refl(*A*, *a*) : **Id**(*A*, *a*, *a*)	**O** ? =**refl**(*A*, *a*)	**P** *a* = **refl**(*A*, *a*) : *A*

Since in the dialogues of immanent reasoning it is the Opponent who is given the authority to set the local reasons for the relevant sets, **P** can always trigger from **O** the identity statement **O** p : **Id**(A, a, a) for any statement **O** a : A has brought forward during a play. This leads to the next table that *constitutes one of the exceptions* to the interdiction on challenges on **O**'s elementary statements

Socratic Rule
for triggering the reflexivity move O ! Id(*A, a, a*)

Statement	Challenge	Defence
O a : A	**P** $?^{\mathbf{Id}\text{-}a}$	**O refl**(A, a) : **Id**(A, a, a)

Remarks

Notice that it looks as if **P** will not need to use this rule since according to the rule for the synthesis of the local reason for an Identify statement by **P**, he can always state **Id**(A, a, a), provided **O** stated a : A. However, in some case, such as when carrying out a substitution based on identity, **P** might need **O** to make an explicit statement of identity suitable for applying that substitution-law.
This rule

The next rule prescribes how to analyse some local reason p brought forward by **O** in order to support the statement **Id(*A, a, a*)**

Analysis I
The Global Analysis of O *p* : Id(*A, a, a*)

Statement	Challenge	Defence
O p : **Id**(A, a, a)	**P** $?_{\mathbf{Id}} = p$	**O** p = **refl**(A, a) : **Id**(A, a, a)

The second rule for analysis involves statements of the form **Id**(A, a, b), so we need to general rules for statements that are not restricted to reflexivity. In fact the rules for **Id**(A, a, b) can be obtained by re-writing the precedent rules – with the exception of the rule that triggers statements of reflexivity by **O**.
We will not write the rules for **Id**(A, a, b) down but let us stress two important points

(1) the unicity of the local reason **refl**(A, a).
(2) the non-inversibily of the intensional predicate of identity in relation to judgmental equality.

(1) In relation to the first remark, the point is that the local reason produced by a process of synthesis for any identity statement is always **refl**(A, a). In other words, the local reason prescribed by the procedures of synthesis involving the statement ! **Id**(A, a, a) and the statement **Id**(A, a, a) , is the *same one*, namely **refl**(A, a).

(2) In relation to our second point, It is important to remember that the **global synthesis** rule refers to the commitments undertaken by **P** when he affirms the identity between a and b. Such commitment amount to i) providing a local-reason for such identity ii) stating $a = b : A$.
On the contrary the rule of **global analysis of an identity statement by O** prescribes what **P** may require from **O**'s statement. In that case, **P** cannot force **O** to state $a = b : A$ only because she stated **Id**(A, a, b).. This is only possible with the so-called extensional version of propositional identity (see II.8 above and thorough discussion in Nordström et al., 1990, pp. 57-61,). The dialogical view of non-reversibility here is that the r*ule of synthesis set the conditions **P** must fulfil* when he states and identity, not *what follows from his statement of identity*:

Id is transmitted by the rules of reflexivity, symmetry, transitivity and by the **substitution of identicals.**

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