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Burak Can, Ali Ihsan Ozkes, Ton Storcken

Working Papers / Documents de travail

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WP 2017 - Nr 34

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Burak Can* Ali I. Ozkes† Ton Storcken‡

September 28, 2017

Abstract

We provide axiomatic characterizations for measures of polarization in profiles of preferences that are represented as rankings of alternatives. Polarization is seen as the extent to which opinions are opposed. We provide characterizations for an extension of this simple intuition on the pairs of alternatives to the cases with more than two alternatives. Our primary generalization allows for different treatment among issues, *i.e.,* pairs of alternatives. Secondly, we show that the characterization result continues to hold when preferences are allowed to attain indifferences. Finally, we show that we can also impose a domain restriction that only allows for single-peaked preferences and retain our characterization. Our results point to a fundamental feature of measures on profile of preferences that are based on pairwise comparisons of alternatives.

Keywords: Polarization · Measurement · Social choice · Single-peaked preferences

JEL codes: D63 · D71 · D72 · D74

*Department of Economics, School of Business and Economics, Maastricht University, P.O. Box 616, 6200 MD Maastricht, The Netherlands; E-mail: b.can@maastrichtuniversity.nl. This research was partially financed by the Netherlands Organization for Scientific Research (NWO) with project nr. 400-09-354.

†Aix-Marseille Univ., CNRS, EHESS, Centrale Marseille, AMSE, 5 Bd Maurice Bourdet, 13001, Marseille, France. E-mail: ali-ihsan.ozkes@univ-amu.fr.

‡Department of Quantitative Economics, Maastricht University, P.O. Box 616, 6200 MD Maastricht, the Netherlands; E-mail: t.storcken@maastrichtuniversity.nl.

1 Introduction

The concept of polarization is prevalent in literatures across social sciences. Although there is no unique definition of polarization even within certain formally well-defined contexts, a common understanding of it can be stated as *the extent to which opinions are opposed*. And this feature of distributions of opinions in our societies matter for the success of collective decision-making processes.

The greater the extent to which opinions move toward opposing modes, the more likely it is that social conflict will arise and the less likely it is that a political system maintains a healthy representation (see DiMaggio et al. (1996)). Negotiation failures and policy gridlocks in governments are argued to stem from polarization (see Barber and McCarty (2015)) and voters are discouraged to vote in high polarization (see Hetherington (2008))[1]. In social choice framework, as argued by Lepelley and Valognes (2003), manipulability of voting rules might be dependent on how homogenous, and hence on how polarized, preferences are. In matching theory, Hałaburda (2010) looks at the affects of similarity among preferences on the stability of matchings in a two-period model of two-sided matching markets.

In search of a first formal approach to the measurement of polarization in social choice settings with ordinal preferences, Can et al. (2015) study measuring polarization in profiles of preferences that are represented as rankings of alternatives. The major premise of their analysis is to extend a basic intuition in case of a pair of alternatives to the cases with more than two alternatives, while keeping within the limits of ordinality of preferences. Namely, when there are two alternatives, the most polarized case is when half of the society prefer one alternative to the other, and the other half have the opposite preference. The least polarized case is when everyone has the same preference ordering, whatever it happens to be. If the straightforward application of this notion to social choice problems with more than two alternatives is pursued, a measure that normalizes the summation of the extents of disagreements over all pairs is obtained. This measure can be characterized by a set of axioms.

In this paper, we generalize this approach in several directions. In certain contexts of collective decision-making, a strict impartiality regarding alternatives is not necessarily desired. Hence our primary generalization is towards allowing different issues (comparisons within pairs of alternatives) to have different weights in the measurement of polarization. We show that this generalization of

[1] For a review of the studies on the measurement, causes, and consequences of polarization (in American politics), see Hetherington (2009).

our previous measure can be characterized by two axioms, one being a regularity axiom and the other a rather technical one emphasizing additivity.

Furthermore, in our initial approach we restricted attention to the cases where individuals can clearly distinguish any two alternatives in their preferences. We show that allowing indifferences in preferences does not harm the applicability of our generalized measure, in the sense that in this wider domain of preferences, we still have the same characterization result hold. Finally, we consider a domain restriction. Single-peaked preferences are long thought to be the most relevant type of preferences when one thinks about a society composed of rational individuals facing a clearly distinguishable set of alternatives. Following his discussion on deliberation-induced single-peakedness in preferences, List (Forthcoming) argues that preferences can be both single-peaked and polarized. We show that our characterization continues to hold under this domain restriction.

1.1 Related literature

In social choice literature, similar concepts can be found. *Consensus* as in Herrera-Viedma et al. (2011), *assent* as in (Baldiga and Green, 2013), and *cohesiveness* as in Alcalde-Unzu and Vorsatz (2013) are immediate examples. Consensus, as formulated firstly in Bosch (2006), can be measured with mappings that assign to any profile of preferences a value in unit interval, which has the following two properties necessarily: first, the value given to a profile is highest, namely 1, if and only if all individuals agree on how to rank alternatives and second, the same value given to any two profiles if the only difference in between them is the names of either the alternatives or individuals. Karpov (2016) surveys research in this line, Alcalde-Unzu and Vorsatz (2008) deliver axiomatic characterizations, and García-Lapresta and Pérez-Román (2011) analyze properties of a class of consensus measures that are based on the distances in between preferences.

Building on an understanding of conflict between two individuals as the disagreement in their top choices, Baldiga and Green (2013) use an aggregate-assent maximizing approach to the selection of the choice rule, where the assent between preferences is the probability that these preferences would be conflictual on a random feasible set. In another work, the level of similarity among preferences in a profile is taken by Alcalde-Unzu and Vorsatz (2013) as cohesiveness, which is then shown to be measured by classes of functions that are characterized by a set of plausible axioms.

Hashemi and Endriss (2014) study measuring the degree of diversity in the preferences by formalizing it in three different ways. In first, diversity is seen as the range of distinct views held, in second, it is the aggregate distance between individual views, and in thirds, it is the distance of the society's views to a single compromise view. Compromise based preference diversity indices, as defined in Hashemi and Endriss (2014), are closely related to the characterized measure in Can et al. (2015). Finally, Costantini et al. (2016) introduces a measure of polarization, in a similar vein, for a setup where individuals report ballots that are binary responses to a set of issues. This measure takes into account the correlation between issues and uncertainty regarding whether individuals tend to take opposite views on issues.

In the following section we introduce basic notations and formal definitions regarding our axiomatic model. Section 3 provides our major result that is related to the weighted polarization measure for linear preferences. Section 4 is devoted to preferences with indifferences, while Section 5 delivers our results regarding single-peaked preferences. We conclude in Section 6.

2 Model

Let A be a finite and nonempty set of m alternatives. For any finite and nonempty set of individuals N, and for any individual i in N, let $p(i)$ denote the preference of i in terms of a weak order, i.e., a complete and transitive binary relation on A. Furthermore, p indicates a *profile*, a combination of such individual preferences. $\mathcal{L}$ denotes the set of all preferences that are linear orders on A, whereas $\mathcal{W}$ denotes the set of all weak orders. So $\mathcal{L}$ is the set of antisymmetric weak orders, hence we have $\mathcal{L} \subset \mathcal{W}$. Individual i's weak preference of alternative a above alternative b is indicated by the ordered pair (a, b) being in relation $p(i)$. We will mostly write ordered pairs as ab instead of (a, b) from now on.

Given a profile p, the following basic notations are used throughout the paper. The weak pairwise comparisons matrix $(m \times m)$ related to p is denoted by $\overline{p}$ and defined at cell ab by

$$\overline{p}_{ab} = \#\{i : 1 \leq i \leq n \text{ and } ab \in p(i)\},$$

the number of agents weakly preferring a to b. The strict pairwise comparisons matrix related to p is denoted by $\overline{\overline{p}}$ and defined at cell ab by

$$\overline{\overline{p}}_{ab} = \#\{i : 1 \leq i \leq n \text{ and } ba \notin p(i)\},$$

the number of agents strictly preferring a to b. For an ordered pair of alternatives ab, $ab_p = \max\{\overline{p}_{ab} - \overline{p}_{ba}, 0\}$ denotes the size of the pairwise majority for a above b at profile p, in case a is prefered by a majority to b. It is zero when b is preferred to a by a majority.

For two profiles p and q of two disjoint sets of individuals, say N_1 and N_2 respectively, let (p, q) denoted the united profile, say r, such that $r(i) = p(i)$ if i is in N_1 and $r(i) = q(i)$ if i is in N_2. This naturally extends to more than two profiles and we write (p, q) and $p+q$ interchangeably. Similarly define $p^2 = (p', p'')$, where $p' \in \mathbb{L}^{N'}$ and $p'' \in \mathbb{L}^{N''}$, to be a *replication* of p if there are bijections $\sigma' : N \leftrightarrow N'$ and $\sigma'' : N \leftrightarrow N''$ such that $p(i) = p'(\sigma'(i))$ and $p(i) = p''(\sigma''(i))$ for all $i \in N$. This naturally extends to define $p^3, p^4, ..$ accordingly.

For a preference R, let R^N denote the unanimous profile where all individuals in N have preference R. Let $-R = \{yx : xy \in R\}$ be the preference where all pairs in R are reversed. If π denotes a permutation on A, then the permuted preference of R is $\pi R = \{(\pi(a), \pi(b)) : ab \in R\}$ which naturally defines the permuted profile πp in a coordinate-wise manner, i.e., $(\pi p)(i) = \pi(p(i))$.

Two profiles p and q are said to be *conflict free* if there is no $ab \in A \times A$, where $a \neq b$, such that $ab_p > 0$ and $ba_q > 0$. That is there are no two different alternatives a and b such at p there is a strict majority for a against b and at q there is a strict majority for b against a. So, per pair of alternatives between p and q there are no conflicting majority comparison outcomes.

Let p and q be two profiles in $\mathbb{L}^N$. We say that p and q form an *elementary change from ab to ba* whenever there is an individual i in N who ranks a and b consecutively in p and furthermore $q(i) = \big(p(i) \cup ba\big) \setminus ab$ and for all j in $N \setminus \{i\}$, $p(j) = q(j)$. This means that $q(i)$ can be obtained from $p(i)$ by only reversing the ordered pair ab.

A *polarization measure* Ψ assigns to each profile p a non-negative real number $\Psi(p)$. Can et al. (2015) characterize with a set of axioms the polarization measure Ψ^* that is defined for a profile p of linear preferences by

$$\Psi^*(p) = \sum_{\substack{(a,b)\in A\times A \\ a\neq b}} \frac{\min\{\overline{\overline{p}}_{ab}, \overline{\overline{p}}_{ba}\}}{2n\binom{m}{2}}.$$

As seen in its axiomatic characterization as well, under Ψ^*, polarization between alternatives are normalized equally over all pairs. For a more inclusive approach, consider a *weight function* ω assigning weights $\omega(ab) \geq 0$ to an ordered pair ab of alternatives, such that $\omega(ab) = \omega(ba)$ and $\omega(aa) = 0$. Then the pairwise weighted polarization measure Ψ^ω is defined by

$$\Psi^\omega(p) = \sum_{(a,b)\in A\times A} \frac{\omega(ab) \times \min\{\overline{\overline{p}}_{ab}, \overline{\overline{p}}_{ba}\}}{n}.$$

Hence, Ψ^* can be written as Ψ^{ω^*}, where $\omega(ab) = \frac{1}{2\binom{m}{2}}$ for all $a, b \in A$ with $a \neq b$. An important axiom for a polarization measure is its regularity, which is defined as follows.

Regularity : $\Psi(p) \in [0,1]$ for all $p \in \mathbb{L}^N$ and furthermore $\Psi(R^N) = 0$ and $\Psi(R^{N_1}, (-R)^{N_2}) = 1$ for all preferences R and all finite and nonempty sets N_1 and N_2 of individuals such that N_1 and N_2 are disjoint and equal in size, i.e., $\#N_1 = \#N_2$.

Lemma 1 *Ψ^ω satisfies regularity if and only if* $\sum\limits_{ab \in A\times A} \omega(ab) = 2.$

Proof. Consider a profile $p = (L, -L)$ for some linear order L, hence $\overline{p}_{ab} = 1$ for all different alternatives a and b. It follows that for weighed polarization measures regularity Ψ^ω implies that we have $\sum\limits_{ab\in A\times A} \omega(ab) = 2$. And reversibly in case these weights add up to 2 it follows that for profiles $(n \cdot L, n \cdot (-L))$ polarization

$$\Psi(n \cdot L, n \cdot (-L)) = \sum_{ab\in A\times A} \frac{\omega(ab) \cdot \min\{n_{ab}(p), n_{ba}(p)\}}{n} = \sum_{ab\in A\times A} \frac{\omega(ab)\cdot n}{2\cdot n} = \sum_{ab\in A\times A} \frac{\omega(ab)}{2} = 1.$$

So the polarization measure is regular. ■

Given any profile p and any permutation on individuals $s : N \leftrightarrow N$, let $sp \in \mathbb{L}^N$ denote the permuted profile, i.e., $sp(i) = p(s(i))$ for all $i \in N$. Then, a rule Ψ satisfies *anonymity* whenever $\Psi(sp) = \Psi(p)$ for all profiles p and all permutations s. Another important axiom is of replication invariance, which is defined as follows.

Replication invariance : $\Psi(2 \cdot p) = \Psi(p)$ for all profiles p.

In this paper, we define the major property through the additivity axiom as follows.

Additivity : For any two conflict free profiles p and q,

$$\Psi(p, q) = \frac{n_p}{n_p + n_q}\Psi(p) + \frac{n_q}{n_p + n_q}\Psi(q).$$

Following remark is an immediate observation.

Remark 1 *Additivity implies replication invariance and therewith anonymity.*

That replication invariance implies anonymity is proven in Proposition 1 of Can et al. (2015) and the first part of the claim is straightforward as

$$\Psi(p, p) = \frac{1}{2}\Psi(p) + \frac{1}{2}\Psi(p) = \Psi(p).$$

3 Linear Orders

Here we consider polarization measures on linear orders. In fact we show that both not the strong variety of regularity and global pairwiseness are redundant conditions in previous characterizations.

Theorem 2 *Let Ψ be a polarization measure for profiles on linear orders. Then Ψ is regular and additive if and only if Ψ is a weighed polarization measure, say Ψ^{ω}, such that $\sum_{ab \in A \times A} \omega(ab) = 2$.*

Proof. The proof consist of 6 major steps. Step 1 is on a fundamental betweenness result. In step 2 we determine the weights. Step 3 shows that the theorem holds for two agents profiles and step 4 that it holds for three agents profiles. By induction on the number of agents we show that the argument holds for profiles on any number of agents. Step 5 deals with the induction basis and step 6 with the induction step.

Step 1 *Betweenness*

Let R_1, R_2, and R_3 be three linear orders such that R_3 is in between R_1 and R_2, *i.e.,*

$$R_1 \cap R_2 \subseteq R_3 \subseteq R_1 \cup R_2.$$

Claim 1 $\Psi(R_1, R_2) = \Psi(R_1, R_3) + \Psi(R_3, R_2)$.

Proof of Claim 1 Because R_3 is in between R_1 and R_2, we have that the profiles (R_3, R_3) and (R_1, R_2) are conflict free as well as the profiles (R_1, R_3) and (R_3, R_2). So, additivity implies

$$\begin{aligned} \Psi(R_1, R_3, R_3, R_2) &= \frac{1}{2}\Psi(R_1, R_3) + \frac{1}{2}\Psi(R_3, R_2) \text{ and} \\ \Psi(R_1, R_3, R_3, R_2) &= \frac{1}{2}\Psi(R_1, R_2) + \frac{1}{2}\Psi(R_3, R_3). \end{aligned}$$

By regularity the latter equation implies

$$\Psi(R_1, R_3, R_3, R_2) = \frac{1}{2}\Psi(R_1, R_2).$$

Combining this and the former yields the desired result.

End of Proof of Claim 1

Step 2 *The Weights*

For two alternatives a and b consider four preferences R^{ab}, Q^{ab}, R^{ba} and Q^{ba} in $\mathcal{L}$ such that $R^{ab} \backslash R^{ba} = Q^{ab} \backslash Q^{ba} = \{ab\}$ and $R^{ba} \backslash R^{ab} = Q^{ba} \backslash Q^{ab} = \{ba\}$. That is both R^{ab} with R^{ba} and

Q^{ab} with Q^{ba} form an elementary change from ab to ba. The following claim essentially defines the weights.

Claim 2 $\Psi(R^{ab}+R^{ba}) = \Psi(Q^{ab}+Q^{ba})$. By applying Claim 1 we have

$$\begin{aligned}
\Psi(R^{ab}, Q^{ba}) &= \Psi(R^{ab}, R^{ba}) + \Psi(R^{ba}, Q^{ba}) \text{ and} \\
\Psi(R^{ab}, Q^{ba}) &= \Psi(R^{ab}, Q^{ab}) + \Psi(Q^{ab}, Q^{ba}).
\end{aligned}$$

So,

$$\Psi(R^{ab}, R^{ba}) + \Psi(R^{ba}, Q^{ba}) = \Psi(R^{ab}, Q^{ab}) + \Psi(Q^{ab}, Q^{ba}).$$

Similarly we have

$$\Psi(Q^{ab}, R^{ab}) + \Psi(R^{ab}, R^{ba}) = \Psi(Q^{ab}, Q^{ba}) + \Psi(Q^{ba}, R^{ba}).$$

Adding anonymity and simplifying now yields

$$\Psi(R^{ab}, R^{ba}) = \Psi(Q^{ab}, Q^{ba}).$$

End of Proof of Claim 2

We define the weights $\omega(ab) = \Psi(R^{ab}, R^{ba})$. For a linear order R we now have by Claim 1

$$\Psi(R, -R) = \sum_{ab\in R} \omega(ab).$$

Regularity implies

$$\sum_{ab\in A\times A} \omega(ab) = 2.$$

Hence we have that $\Psi(R+(-R)) = 1$. Note that by anonymity, weights are symmetric, *i.e.*, $\omega(ab) = \omega(ba)$.

Step 3 *The Two Agents Case*

Let R and Q be two linear orders. It is sufficient to prove that

$$\begin{aligned}
\Psi(R, Q) &= \Psi^{\omega}(R, Q) \\
&= \sum_{ab\in A\times A} \frac{\omega(ab)}{2},
\end{aligned}$$

where $\omega(ab)$ is defined as the polarization of a pair of linear orders forming an elementary change from ab to ba. By Claim 1 it follows that

$$\begin{aligned}\Psi(R,Q) &= \sum_{ab\in R\setminus Q}\omega(ab)\\ &= \sum_{ab\in A\times A}\frac{\omega(ab)}{2}.\end{aligned}$$

Step 4 *The three Agents case*

For a profile p on an arbitrary number of agents let $T_p = \{ab \in A\times A : \overline{\overline{p}}_{ab} > 0\}$, which is antisymmetric by definition. Let p^{-S} denote the profile p restricted to $N\setminus S$ for any $S \subseteq N$.

Claim 3 Let p be a profile such that T_p is complete. Then $T_{p^{-i}} \subseteq T_p$ for all agents i. Therewith p^{-i} and p^{-j} are conflict free for all agents i and j.

Proof of Claim 3 If T_p is complete, for all distinct alternatives a and b with $ab \notin T_p$ we have $ba \in T_p$. This implies $\overline{(p^{-i})}_{ab} \leq \overline{(p^{-i})}_{ba}$ for all agents i. So, $ab \notin T_{p^{-i}}$ which proves the first inclusion. The statement on p^{-i} and p^{-j} being conflict free for all agents i and j now follows readily as $ab \in T_{p^{-i}} \subseteq T_p$, $T_{p^{-j}} \subseteq T_p$ and T_p is anti-symmetric imply that $ba \notin T_{p^{-j}}$. So, $\overline{\overline{(p^{-i})}}_{ab} > 0$ implies $\overline{\overline{(p^{-j})}}_{ab} \geq 0$.

End of Proof of Claim 3

To prove step 4 consider a profile p on three agents say $1, 2$ and 3. We want to prove that $\Psi(p) = \Psi^{\omega}(p)$. As the number of agents is odd we have that T_p is complete. So, by additivity and Claim 3 we have that

$$\begin{aligned}\Psi(p) &= \Psi(2\cdot p)\\ &= \frac{1}{3}\Psi(p^{-1}) + \frac{1}{3}\Psi(p^{-2}) + \frac{1}{3}\Psi(p^{-3}).\end{aligned}$$

As p^{-i} is a profile on two agents for all agents i in $\{1, 2, 3\}$, we have by step 3 that

$$\begin{aligned}\Psi(p) &= \frac{1}{3}\Psi(p^{-1}) + \frac{1}{3}\Psi(p^{-2}) + \frac{1}{3}\Psi(p^{-3})\\ &= \frac{1}{3}\Psi^{\omega}(p^{-1}) + \frac{1}{3}\Psi^{\omega}(p^{-2}) + \frac{1}{3}\Psi^{\omega}(p^{-3})\\ &= \Psi^{\omega}(p).\end{aligned}$$

This proves step 4.

Step 5 *Basis of the induction*

Let $\widetilde{R}$ be a linear order. For alternatives a and b let r^{ab} denote a profile such that $ab... = r^{ab}(1)$ and $...ab = r^{ab}(2)$, where $xy \in r^{ab}(1)$ iff $yx \in r^{ab}(2)$ iff $xy \in \widetilde{R}$ for all $xy \in (A \times A)\backslash\{ab, ba\}$. Let $\sum = \{\sum_{xy \in A\times A} k^{xy} \cdot r^{xy} : k^{xy}$ is an non-negative integer and $k^{xx} = 0$ for all different alternatives x and $y\}$. With induction on n_p we prove $\Psi(p, \Sigma) = \Psi^{\omega}(p, \Sigma)$ for profiles p on $\mathcal{L}$ and profiles $\Sigma \in \sum$. Taking $k^{ab} = 0$ for all $ab \in A \times A$, then yields the desired result. Take $\Sigma = \sum_{ab\in A\times A} k^{ab} \cdot r^{ab}$. Profile Σ can be split into two parts $\Sigma = \Sigma^1 + \Sigma^2$, such that $\Sigma^1 = \sum_{ab\in A\times A} \max\{k^{ab} - k^{ba}, 0\} \cdot r^{ab}$ and $\Sigma^2 = \sum_{ab\in A\times A} \min\{k^{ab}, k^{ba}\} \cdot r^{ab}$. Call $k_1^{ab} = \max\{k^{ab} - k^{ba}, 0\}$ and $k_2^{ab} = \min\{k^{ab}, k^{ba}\}$. Then we have that $k_1^{ab} = 0$ or $k_1^{ba} = 0$. Also we have $k_2^{ab} = k_2^{ba}$ for all ab in $A \times A$. So, Σ^1 and Σ^2 are conflict free and additivity implies $\Psi(\Sigma) = \Psi(\Sigma^1) + \Psi(\Sigma^2)$ and $\Psi^{\omega}(\Sigma) = \Psi^{\omega}(\Sigma^1) + \Psi^{\omega}(\Sigma^2)$. Regularity implies that $\Psi(\Sigma^2) = 1 = \Psi^{\omega}(\Sigma^2)$. As all r^{ab} in Σ^1 having a positive k_1^{ab} are conflict free it follows by the equality of Ψ and Ψ^{ω} on two agents profiles and additivity that $\Psi(\Sigma^1) = \Psi^{\omega}(\Sigma^1)$. So, we have that $\Psi(\Sigma) = \Psi^{\omega}(\Sigma)$. This proves the induction statement for $n_p = 0$.

Next we prove the induction statement for $n_p = 1$. So, let R be a linear order on A and consider $R+\ \Sigma$ for some $\Sigma \in \sum$. It is sufficient to show that $\Psi(R + \Sigma) = \Psi^{\omega}(R + \Sigma)$. Let $T = \{ab \in A \times A : k^{ab} > k^{ba}\}$. Take $h = \sum_{ab\in T} r^{ab}$ and $g = \sum_{ab\in A\times A} k_3^{ab} \cdot r^{ab}$, where $k_3^{ab} = k^{ab}$ if $ab \notin T$ and $k_3^{ab} = k^{ab} - 1$ else. Note that by construction $h + g = \Sigma$ and also $R+\ h$ and g are conflict free. To see this latter statement note that if $ab \in T$, then $\overline{\overline{(R+h)}}_{ab} \geq 0$ and $k_3^{ab} = k^{ab} - 1 \geq k^{ba} = k_3^{ba}$ which means that $\overline{\overline{g}}_{ab} \geq 0$. Similarly, if $ba \in T$, then $\overline{\overline{(R+h)}}_{ba} \geq 0$ and $\overline{\overline{g}}_{ba} \geq 0$. Finally, if $ab \notin T$ and $ba \notin T$, then $\overline{g}_{ab} = \overline{g}_{ba}$. So, additivity implies $\Psi(R + \Sigma) = \Psi(R+h) + \Psi(g)$ and $\Psi^{\omega}(R+\Sigma) = \Psi^{\omega}(R+h) + \Psi^{\omega}(g)$. As $g \in \sum$ it is therefore sufficient to prove that $\Psi(R + h) = \Psi^{\omega}(R + h)$. Let $h_1 = \sum_{ab\in V} r^{ab}$ and $h_2 = \sum_{ab\in T\backslash V} r^{ab}$, where $V = \{ab \in T : ab \notin R$ and $ba \in R\}$. Then $(R + h_1)$ and h_2 are obviously conflict free. So, by additivity it is sufficient to prove that $\Psi(R + h_1) = \Psi^{\omega}(R + h_1)$. Now $\Psi(R + h_1) = \Psi(2 \cdot (R + h_1))$. As $\overline{(2 \cdot R + h_1)}_{ab} = 2 = \overline{(2 \cdot R + h_1)}_{ba}$, $\overline{(h_1)}_{ab} = 2$ and $\overline{(h_1)}_{ba} = 0$ for $ab \in V$ and as $\overline{(h_1)}_{xy} = \overline{(h_1)}_{yx} = 1$ for x and y such that $xy \notin V$ and $yx \notin V$, we have $2 \cdot R + h_1$ and h_1 are conflict free. So, there are α_1 and α_2 such that

$$\begin{aligned} \Psi(R + h_1) &= \Psi(2 \cdot (R + h_1)) \\ &= \alpha_1 \cdot \Psi(2 \cdot R + h_1) + \alpha_2 \cdot \Psi(h_1). \end{aligned}$$

Similarly we have

$$\Psi^{\omega}(R + h_1) = \alpha_1 \cdot \Psi^{\omega}(2 \cdot R + h_1) + \alpha_2 \cdot \Psi^{\omega}(h_1).$$

Therefore, as $\Psi(h_1) = \Psi^\omega(h_1)$, it is sufficient to prove that $\Psi(2 \cdot R + h_1) = \Psi^\omega(2 \cdot R + h_1)$. Let $v = \#V$. Note that $2 \cdot R + h_1$ and $2(v-1) \cdot R$ are conflict free. So, by additivity there are some β_1 and β_2 such that

$$\Psi(2v \cdot R + h_1) = \beta_1 \cdot \Psi(2(v-1) \cdot R) + \beta_2 \cdot \Psi(2 \cdot R + h_1).$$

Further, note that $2 \cdot R + r^{ab}$ and $2 \cdot R + r^{xy}$ are conflict free for all ab and xy in V. So, there are γ_{xy}, such that

$$\Psi(2v \cdot R + h_1) = \sum_{xy \in V} \gamma_{xy} \Psi(2 \cdot R + r^{ab}).$$

Note that r^{ab} and $2 \cdot R + r^{ab}$ are conflict free so $\tau_1 \cdot \Psi(2 \cdot R + r^{ab}) + \tau_2 \cdot \Psi(r^{ab}) = \Psi(2 \cdot (R + r^{ab})) = \Psi(R + r^{ab})$ for some τ_1 and τ_2. As $R + r^{ab}$ is a three agent profile we have that $\Psi(R + r^{ab}) = \Psi^\omega(R + r^{ab})$. But then as $\Psi(r^{ab}) = \Psi^\omega(r^{ab})$ it follows that $\Psi(2 \cdot R + r^{ab}) = \Psi^\omega(2 \cdot R + r^{ab})$. Because

$$\Psi^\omega(2v \cdot R + h_1) = \sum_{xy \in V} \gamma_{xy} \Psi^\omega(2 \cdot R + r^{ab}),$$

we have that $\Psi^\omega(2v \cdot R + h_1) = \Psi(2v \cdot R + h_1)$. Now because $\Psi(2(v-1) \cdot R) = 0 = \Psi^\omega(2(v-1) \cdot R)$ and

$$\Psi^\omega(2v \cdot R + h_1) = \beta_1 \cdot \Psi^\omega(2(v-1) \cdot R) + \beta_2 \cdot \Psi^\omega(2 \cdot R + h_1),$$

it follows that $\Psi(2 \cdot R + h_1) = \Psi^\omega(2 \cdot R + h_1)$.

Step 6 *Induction step*

Let p be an N agents profile with $n_p \geq 2$. Let $\Sigma \in \sum$. It is sufficient to prove that $\Psi(p + \Sigma) = \Psi^\omega(p + \Sigma)$. We choose $\Sigma_1 \in \sum$ such that on the one hand $(n-1) \cdot (p + \Sigma)$ and $\Sigma + n \cdot \Sigma_1$ are conflict free and on the other hand all $p^{-i} + \Sigma + \Sigma_1$ are conflict free for all i in N. Take V an asymmetric and weakly complete relation on A, such that $ab \in V$ if $\overline{\overline{(p + \Sigma)}}_{ab} > 0$ or if $\overline{(p + \Sigma)}_{ab} = \overline{(p + \Sigma)}_{ba}$ and $\overline{\overline{\Sigma}}_{ab} > 0$. So, incase $\overline{(p + \Sigma)}_{ab} = \overline{(p + \Sigma)}_{ba}$ and $\overline{\Sigma}_{ab} = \overline{\Sigma}_{ba}$ we just choose freely either ab or ba in V, but in that case not both. Now take $\Sigma_1 = \sum_{ab \in V} k^{ab} \cdot r^{ab}$. Where $k^{ab} = \overline{\Sigma}_{ba} - \overline{\Sigma}_{ab}$ if $\overline{\overline{(p + \Sigma)}}_{ab} > 0$ and $\overline{\overline{\Sigma}}_{ba} > 0$ and in all other cases $k_{ab} = 1$.

First we prove that $(n-1) \cdot (p + \Sigma)$ and $\Sigma + n \cdot \Sigma_1$ are conflict free. In case $\overline{\overline{(n-1) \cdot (p + \Sigma)}}_{ab} > 0$ then $\overline{\overline{(p + \Sigma)}}_{ab} > 0$ and by construction $\overline{\overline{(\Sigma + n \cdot \Sigma_1)}}_{ab} > 0$. In case $\overline{\overline{(\Sigma + n \cdot \Sigma_1)}}_{ab} > 0$ by construction $\overline{\overline{(p + \Sigma)}}_{ab} \geq 0$ which proves that these two profiles are conflict free.

Next we prove that $p^{-i} + \Sigma + \Sigma_1$ and $p^{-j} + \Sigma + \Sigma_1$ are conflict free for all $i, j \leq n_p$. In case $\overline{\overline{(p+\Sigma)}}_{ab} > 0$ we have that both $\overline{\overline{(p^{-i}+\Sigma+\Sigma_1)}}_{ab} \geq 0$ and $\overline{\overline{(p^{-j}+\Sigma+\Sigma_1)}}_{ab} \geq 0$. So, it cannot happen that $\overline{(p^{-i}+\Sigma+\Sigma_1)}_{ab} > 0$ and $\overline{(p^{-j}+\Sigma+\Sigma_1)}_{ab} < 0$. In case $\overline{(p+\Sigma)}_{ab} = \overline{(p+\Sigma)}_{ba}$ and $\overline{\Sigma}_{ab} > \overline{\Sigma}_{ba}$ then $\overline{\overline{(\Sigma_1)}}_{ab} > 0$ and $\overline{\overline{(p^{-i}+\Sigma+\Sigma_1)}}_{ab} \geq 0$ and $\overline{\overline{(p^{-j}+\Sigma+\Sigma_1)}}_{ab} \geq 0$. In case $\overline{(p+\Sigma)}_{ab} = \overline{(p+\Sigma)}_{ba}$ and $\overline{\Sigma}_{ab} = \overline{\Sigma}_{ba}$ we may without loss of generality assume that $\overline{\overline{(\Sigma_1)}}_{ab} > 0$ and again this yields both $\overline{\overline{(p^{-i}+\Sigma+\Sigma_1)}}_{ab} \geq 0$ and $\overline{\overline{(p^{-j}+\Sigma+\Sigma_1)}}_{ab} \geq 0$.

As $(n-1)\cdot(p+\Sigma)$ and $\Sigma + n\cdot\Sigma_1$ are conflict free, additivity implies that for some α_1 and α_2

$$\begin{aligned} \Psi((n-1)\cdot(p+\Sigma)+\Sigma+n\cdot\Sigma_1) &= \alpha_1\cdot\Psi(p+\Sigma)+\alpha_2\cdot\Psi(\Sigma+n\cdot\Sigma_1) \text{ and} \\ \Psi^{\omega}((n-1)\cdot(p+\Sigma)+\Sigma+n\cdot\Sigma_1) &= \alpha_1\cdot\Psi^{\omega}(p+\Sigma)+\alpha_2\cdot\Psi^{\omega}(\Sigma+n\cdot\Sigma_1). \end{aligned}$$

As $(p^{-i}+\Sigma+\Sigma_1)$ and $(p^{-j}+\Sigma+\Sigma_1)$ are conflict free for all i and j in N, additivity implies that there are β_i such that

$$\begin{aligned} \Psi((n-1)\cdot(p+\Sigma)+\Sigma+n\cdot\Sigma_1) &= \sum_{i\in N}\beta_i\cdot\Psi(p^{-i}+\Sigma+\Sigma_1) \text{ and} \\ \Psi^{\omega}((n-1)\cdot(p+\Sigma)+\Sigma+n\cdot\Sigma_1) &= \sum_{i\in N}\beta_i\cdot\Psi^{\omega}(p^{-i}+\Sigma+\Sigma_1). \end{aligned}$$

The induction hypothesis now implies that $\Psi(p+\Sigma)=\Psi^{\omega}(p+\Sigma)$. This concludes the proof of the theorem. ■

4 Weak Orders

In this section we consider profiles on weak orders.

Theorem 3 *Let Ψ be a polarization measure for profiles on weak orders. Then Ψ is regular and additive if and only if Ψ is a weighed polarization measure, say Ψ^{ω}, such that* $\sum_{ab\in AxA}\omega(ab)=2$.

Proof. If part is left to the reader and we start with the two agents case.

Two agents case : Let R_1 and R_2 be two weak orders. We want to show that $\Psi(R_1+R_2) = \Psi^{\omega}(R_1+R_2)$. Let $L_t \subseteq R_t$ be linear orders, such that $R_1\cap R_2 \subseteq L_t$ for $t\in\{1,2\}$. Then R_1+R_2

and $2 \cdot L_1$ are conflict free. So additivity and regularity imply

$$\begin{aligned}\Psi(R_1 + R_2 + 2 \cdot L_1) &= \frac{1}{2}\Psi(R_1 + R_2) + \frac{1}{2}\Psi(2 \cdot L_1) \\ &= \frac{1}{2}\Psi(R_1 + R_2).\end{aligned}$$

Also $R_1 + L_1$ and $R_2 + L_1$ are conflict free. So by additivity we have

$$\Psi(R_1 + R_2 + 2 \cdot L_1) = \frac{1}{2}\Psi(R_1 + L_1) + \frac{1}{2}\Psi(R_2 + L_1).$$

As R_1 and L_1 are conflict free we may obviously by regularity and additivity conclude that $\Psi(R_1 + L_1) = 0$. Hence all in all we have $\Psi(R_1 + R_2) = \Psi(R_2 + L_1)$. But similarly we have $\Psi(R_2 + L_1) = \Psi(L_2 + L_1)$. So, $\Psi(R_1 + R_2) = \Psi(L_2 + L_1)$ and the previous Theorem then implies $\Psi(R_1 + R_2) = \Psi^\omega(L_2 + L_1) = \Psi^\omega(R_1 + R_2)$.

The case of three or more agents : Like in the proof of the previous theorem it is sufficient to prove by induction on n_p that $\Psi(p+\Sigma) = \Psi^\omega(p+\Sigma)$ for profiles p on $\mathcal{L}$ and profiles $\Sigma \in \sum$. First we prove the basis. Let R be a weak order and $\Sigma \in \sum$ it is sufficient to prove that $\Psi(R + \Sigma) = \Psi^\omega(R + \Sigma)$. In view of the previous Theorem and additivity we may assume that $\Sigma = \sum_{ab \in V} r^{ab}$ where $V \subseteq \{ba : ab \notin R\}$. Take L a linear order such that $L \subseteq R$. Now $R + L + \Sigma$ and Σ are conflict free and also $R + \Sigma$ and $L + \Sigma$ are conflict free. So, by additivity there are α_1 and α_2 such that

$$\begin{aligned}\Psi(R + L + 2 \cdot \Sigma) &= \alpha_1 \cdot \Psi(R + L + \Sigma) + \alpha_2 \cdot \Psi(\Sigma) \\ &= \frac{1}{2}\Psi(R + \Sigma) + \frac{1}{2}\Psi(L + \Sigma).\end{aligned}$$

Also $R + L + \Sigma$ and R are conflict free as well as $R + L + \Sigma$ and L. So, by additivity there are β_1 and β_2 such that

$$\begin{aligned}\Psi(R + L + L + \Sigma) &= \beta_1 \cdot \Psi(R) + \beta_2 \cdot \Psi(L + L + \Sigma) \\ &= \beta_1 \cdot \Psi(L) + \beta_2 \cdot \Psi(R + L + \Sigma).\end{aligned}$$

Because of regularity we may conclude from the latter that $\Psi(L + L + \Sigma) = \Psi(R + L + \Sigma)$. By the previous Theorem we have now

$$\begin{aligned}
\alpha_1 \cdot \Psi(R+L+\Sigma) + \alpha_2 \cdot \Psi^{\omega}(\Sigma) &= \frac{1}{2}\Psi(R+\Sigma) + \frac{1}{2}\Psi^{\omega}(L+\Sigma) \\
\alpha_1 \cdot \Psi^{\omega}(L+L+\Sigma) + \alpha_2 \cdot \Psi^{\omega}(\Sigma) &=
\end{aligned}$$

and as similarly

$$\alpha_1 \cdot \Psi^{\omega}(L+L+\Sigma) + \alpha_2 \cdot \Psi^{\omega}(\Sigma) = \frac{1}{2}\Psi^{\omega}(R+\Sigma) + \frac{1}{2}\Psi^{\omega}(L+\Sigma)$$

we have $\Psi(R+\Sigma) = \Psi^{\omega}(R+\Sigma)$. This proves the basis. With respect to the induction step we remark that the proof of induction step at the previous theorem is valid for profiles on weak orders because no specific information on linear orders is used. ■

5 Single Peaked Preferences

Let $a_1, a_2, ..., a_m$ be a numbering of the alternatives in A. Given this fixed numbering, we call a linear order L *single peaked* whenever for all $1 \leq i < j < k \leq m$ either $a_j a_k$ is in L or $a_j a_i$ is in L. That is, for any triple of alternatives, the one in the middle according to the fixed order is not ordered worst among these three by L. This condition is also known as *value restriction.* By this it follows straightforwardly that there is an alternative, say a_l, which is ordered best at L, where

$$\begin{aligned}
1 \leq i < j \leq l &\implies a_j a_i \in L \text{ and} \\
l \leq j < i \leq m &\implies a_j a_i \in L.
\end{aligned}$$

Denote the set of all single peaked linear orders by $\mathcal{S}$. It is clear that $a_1 a_2 ... a_m = \widetilde{R}$ is in $\mathcal{S}$. Given any single peaked linear order, we may, by elementary changes, move other alternatives to the top and obtain a new linear order in $\mathcal{S}$. This essentially means that $\mathcal{S}$ is connected by elementary changes.

Lemma 4 *Let R and $\widehat{R}$ be two single peaked linear orders in $\mathcal{S}$. Then there is a path in $\mathcal{S}$ of elementary changes from R to $\widehat{R}$ of length $|R \backslash \widehat{R}|$.*

Proof. Let B be the greatest set of alternatives by inclusion such that $R_B = \widehat{R}_B$, $B \times (A-B) \subseteq R$, and $B \times (A-B) \subseteq \widehat{R}$. Let a_i be the top alternative of R_{A-B} and a_j that of $\widehat{R}_{A-B}$. Without loss of generality we may assume that $i < j$. Note that for any $k > j$ we have $a_k \notin B$, as otherwise the value restriction condition at R is violated. So, $Ba_j...a_k... = \widehat{R}$ and we know that the path of elementary changes at which a_i is moved one position to the front is in $\mathcal{S}$. This yields $Ba_ia_j...a_k... = \overline{R}$. Actually $\overline{R}$ is on a shortest path from $\widehat{R}$ to R. Evidently we are now done by an induction argument on the $\#B$. ■

We will prove that on profiles of such single peaked linear orders only weighed polarization measures are regular and additive.

Theorem 5 *Let Ψ be a polarization measure for profiles on single peaked linear orders. Then Ψ is regular and additive if and only if Ψ is a weighed polarization measure, say Ψ^ω, such that* $\displaystyle\sum_{ab \in AxA} \omega(ab) = 2$.

Proof. Here we prove the *only-if-part* and leave the proof of the *if-part* to the reader. Consider the two agent case of the proof of Theorem 1. Claim 1 on betweenness holds also for this special case of single peaked linear orders. Claim 2 holds as well. Therefore like in the proof of Theorem 1 we can define weights. By Lemma 4 we have therewith proved the Theorem for the two agents case.

Let p be a profile of single peaked linear orders. It is well known that there exists a single peaked linear order, say L_p, such that $ab \in L_p$ whenever $ab_p > 0$. That is the pairwise majority decision at p can be extended to a linear order which belongs to the class of single peaked linear orders where all the individual preferences $p(i)$ also belong to. Now $n_p \cdot L_p$ and p are conflict free as well as $p(i) + L_p$ and $p(j) + L_p$ for all agents i and j. So, by additivity

$$\begin{aligned} \Psi(p + n_p \cdot L_p) &= \frac{1}{2}\Psi(p) + \frac{1}{2}\Psi(n_p \cdot L_p) \text{ and} \\ \Psi(p + n_p \cdot L_p) &= \sum_{i=1}^{n_p} \frac{1}{n_p}\Psi(p(i) + L_p). \end{aligned}$$

So, by regularity and the previous

$$\Psi(p) = \sum_{i=1}^{n_p} \frac{2}{n_p}\Psi^\omega(p(i) + L_p).$$

As we similarly have that $\Psi^\omega(p) = \sum_{i=1}^{n_p} \frac{2}{n_p}\Psi^\omega(p(i) + L_p)$, it follows that $\Psi = \Psi^\omega$. ■

6 Conclusion

In this paper, we have considered generalizations of the polarization measure that is characterized as the summation of disagreements over all pairs of alternatives. This measure can be characterized with regularity, neutrality, replication invariance, and support independence. Here, we retain regularity and replace the latters with a single axiom of additivity. This change allows to assign different weights to different pairs of alternatives in the aggregate polarization. Furthermore, we show that the characterization holds when we allow preferences to attain indifferences as well. Finally, we show that it also continues to hold in cases where preferences are restricted to be single-peaked.

We believe the major driving aspect of our results is the pairwiseness of the measures we consider. For instance, our measures cannot distinguish between uniform profiles where all possible orders appear in equal numbers and bipolar profiles where half of the society have the exact opposite of the preference of the other half. Any measure, let it be of consensus, cohesiveness, polarization, diversity, and the like, that bases its treatment on numbers of preferences over pairs of alternatives is determined to regard uniform and bipolar profiles equally. Departures from pairwiseness might have appealing directions depending on the context. For instance, in measuring polarization, one might consider a political context where parties are represented by rankings and a distance notion for rankings that is not necessarily pair-based is provided. In this case, one aspect of polarization is how parties are distant from each other, while another aspect is how homogeneous the supporters of each party are (as in the alienation-identification framework of Esteban and Ray (1994)). We leave out this and other possible approaches for future research.

Another direction for future research is to relate the level of polarization in a society to social choice outcomes. One might wonder how the societal outcomes of different preference aggregation procedures are related to the level of polarization in preferences. Likewise, is it easier or harder to individually manipulate a social choice function if polarization is higher? Finally, studying domain restrictions that are defined by the highest possible polarization levels they allow can be considered as an interesting inquiry.

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