Sequential equilibrium without rational expectations of prices: A theorem of full existence

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Sequential equilibrium without rational expectations of prices: a theorem of full existence

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Abstract

We consider a pure exchange economy, where agents, typically asymmetrically informed, exchange securities, on financial markets, and commodities, on spot markets. Consumers have private characteristics, anticipations and beliefs, and no model to forecast prices. They are dispensed with rational expectation and bounded rationality assumptions, such as Radner's (1972, 1979), Kurz' (1994) or Koutsougeras-Yannelis' (1999). We show that they face an incompressible uncertainty, represented by a so-called "minimum uncertainty set". This uncertainty typically adds to the exogenous one, on the state of nature, an 'endogenous uncertainty' over future spot prices. At equilibrium, all agents expect the 'true' price on every spot market as a possible outcome, and elect optimal strategies, ex ante, which clear on all markets, ex post. We show this sequential equilibrium exists whenever agents' prior anticipations embed the minimum uncertainty set. This outcome differs from the standard generic existence results of Hart (1975), Radner (1979), and Duffie-Shaffer (1985), among others, based on the rational expectations of prices.

Key words: sequential equilibrium, temporary equilibrium, perfect foresight, existence, rational expectations, financial markets, asymmetric information, arbitrage.

JEL Classification: D52

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1 Introduction

When agents' information is incomplete or asymmetric, the issue of how markets may reveal information is essential and, yet, debated. Quoting Ross Starr (1989), "*the theory with asymmetric information is not well understood at all. In short, the exact mechanism by which prices incorporate information is still a mystery and an attendant theory of volume is simply missing.*" A traditional response is given by the REE (rational expectations equilibrium) model by assuming, quoting Radner (1979), that "*agents have a 'model' or 'expectations' of how equilibrium prices are determined*". Under this assumption, agents know the relationship between private information signals and equilibrium prices, along a so-called "*forecast function*".

Cornet-De Boisdeffre (2002) suggests an alternative approach, where agents' asymmetric information is represented by private information signals, which correctly inform each agent that tomorrow's state of nature will be in a subset of the state space. The latter paper extends the classical definitions of equilibrium, prices and no-arbitrage condition to asymmetric information. Generalizing Cass (1984), De Boisdeffre (2007) shows the existence of equilibrium on purely financial markets is characterized, in this setting, by that no-arbitrage condition. This existence result differs from Radner's (1979) REE generic one. Finally, Cornet-De Boisdeffre (2009) shows the above no-arbitrage condition may always be reached by agents, with no price model, from observing exchange opportunities on financial markets.

The above papers may picture the information transmission on actual markets and restore a full existence property of equilibrium. But they still retain Arrow's (1953) and Radner's (1972) rational expectation hypothesis (also called the conditional perfect foresight hypothesis), stating that agents know the map between future realized states and equilibrium prices. In such a setting, the states of nature are exogenous and represent all individual ex ante uncertainty.

Yet, actual states typically encompass unobservable variables. Arrow (1953) acknowleges this by noticing that a complete market of exogenous state-contingent claims does not exist and should be replaced by state-contingent financial transfers. In his setting, Kurz and Wu (1996) notice, "*agents need to know the maps from states at future dates to prices in the future and it is entirely unrealistic to assume that agents can find out what this sequence of maps is.*" Quoting Radner (1982) himself, this condition "*seems to require of the traders a capacity for imagination and computation far beyond what is realistic*". So the question of the possibility and the way to discard rational expectations in the sequential equilibrium model.

Radner's (1972-79) rational expectation assumptions would be justified if agents knew all the primitives of the economy (endowments, preferences, etc...) and their relations to equilibrium prices, and if they had elected one common price anticipation in each state (amongst typically many possibilities and interests), with the common knowledge of game theory. Otherwise, the equilibrium outcome would typically differ from the standard sequential equilibrium. Such conditions are unrealistic.

Probably the first, best known and most radical escape to rational expectations was the temporary equilibrium model, introduced by J. Hicks and later developed by J.-M. Grandmont. It is traditionally presented as dichotomic from the sequential equilibrium model (see Grandmont, 1982). At a temporary equilibrium, agents have exogenous anticipations, which need not be self-fulfilling. Current markets clear at agents' initial plans, which are typically revised, at each period, after observing realized prices and events. Equilibrium allocations need not clear on future spot markets, where agents may face bankruptcy, due to mistaken anticipations. This outcome explains why the temporary equilibrium did not thrive as the perfect foresight's, which lets agents coordinate across periods, on perfectly anticipated prices.

A less radical approach is referred to as bounded rationality. In this line of research, Kurz' (1994) rational belief equilibrium (RBE) allows agents to lack the "*structural knowledge*" of how equilibrium prices are determined. This unawareness may be due to uncertainty about the beliefs, characteristics and actions of other agents. It leads to an additional uncertainty on future variables, which Kurz calls "*endogenous uncertainty*", describes as the major cause of economic fluctuations, and shows to be consistent with heterogenous beliefs.

Bounded rationality models also serve to study learning processes with differential information (alternative to the REE's), and the links between the information structure and equilibrium or core allocations. This is done, in particular, by Koutsougeras and Yannelis (1999), who emphasize "*that the study of cooperative solution concepts (e.g., the core and the (Shapley) value) in differential information economies appears to be a successful alternative to the traditional rational expectations equilibrium, because they provide sensible and reasonable outcomes in situations where any rational expectations equilibrium (REE) notion fails to do so.*"

The current paper departs from both perfect foresight and bounded rationality models, though it resumes endogenous uncertainty in defining the state space. Its asymmetric information concerns the probability assessments over future prices, but also the sets of possible states of nature and anticipations in each state. In our view, bounded rationality still demands inference and computational skills, as well as informations, which typically exceed agents' possibilities. In the real world, their beliefs, actions and characteristics are all private and their observations are limited.

This restricts their reckonning capacities to a bare minimum and, consequently, their ability to construct any model, such as one of consistent beliefs. Kurz' RBE focusses on non-stationnary price solutions, so as to allow for heterogeneity of beliefs and dynamic fluctuations. An asymptotic limit to the probability distributions over price series is assumed to exist and to be approximated on the finite observations that agents can make. Yet, with non-stationnary distributions, the asymptotic limits typically differ from their finite proxies. This is one example of why we think bounded rationality is still too demanding from the layman's reckonning skills. The model we propose requires no structural knowledge, nor computation from agents.

Due to their private characteristics, agents face an incompressible uncertainty over the set of clearing market prices to expect, represented by a so-called and never empty "*minimum uncertainty set*". The set consists of all possible equilibrium prices along agents' private beliefs today and is consistent with Kurz and Wu's (1996) notice that price uncertainty and economic fluctuations are "*primarily endogenous and internally propagated phenomena (...) generated by the actions and beliefs of the agents (...) and by their uncertainty about the actions of other agents*".

That set (or a bigger one) might be inferred, we argue, by a tradehouse or a financial institution from observing and treating past data on long time series, rather than by consumers themselves. Yet, future equilibrium prices cannot be reckonned precisely by any agent or institution, because this would require to know every agent's beliefs and characteristics. Only a set of possible equilibrium prices could be assessed ex ante, or the minimum uncertainty set, but not the precise location of future prices within that set. Locating equilibrium prices obeys an uncertainty principle. The uncertainty over a set of anticipations is assessed by agents privately.

The current model's sequential equilibrium concept of "*correct foresight equilib-*

rium" (CFE) is thus defined as De Boisdeffre's (2007), except for agents' forecasts, which need no longer be unique, but form sets containing the prices to prevail. The CFE, we argue, reconciles into one concept the sequential and temporary equilibria. It is sequential, since anticipations are self-fulfilling ex post. It is also temporary since forecasts are exogenously given. Along our main Theorems, whether the financial stucture be nominal or real, and beliefs be symmetric or asymmetric, a CFE exists whenever agents' anticipation sets include the minimum uncertainty set.

In our view, this approach to information transmission and equilibrium pictures actual behaviours on markets. Endowed with no price model, unaware of the primitives of the economy, and with limited observational and reckonning capacities, consumers have exogenous anticipations and face endogenous uncertainty. They infer, first, the coarsest arbitrage-free refinement of their initial anticipations from observing trade, along De Boisdeffre (2016). Whence reached, they have no means of further refining their anticipation sets. Then, market forces, driven by price and demand correspondences, lead to equilibrium.

The paper is organized as follows: Section 2 presents the model. Section 3 states the existence Theorem for purely financial markets. Section 4 proves this Theorem. Section 5 shows the full existence of equilibria when assets are nominal , or real, or a mix of both. An Appendix proves technical Lemmas.

2 The basic model

We consider, throughout, a two-period economy, with private information signals, a consumption market and a financial market. The sets, I, S, L and J, respectively, of consumers, states of nature, goods and assets are all finite. The first period is also referred to as $t = 0$ and the second, as $t = 1$. At $t = 0$, there is an uncertainty upon which state of nature, $s \in S$, will prevail tomorrow. The non random state at $t = 0$ is denoted by $s = 0$ and, whenever $\Sigma \subset S$, we let $\Sigma' := \{0\} \cup \Sigma$. Similarly, we denote by $l = 0$ the unit of account and let $L' = \{0\} \cup L$.

2.1 Markets, information and beliefs

Agents consume and may exchange the same consumption goods, $l \in L$, on the spot markets of each period. The generic i^{th} agent's welfare is measured, ex post, by a utility index, $u_i : \mathbb{R}_+^{L \times L} \to \mathbb{R}_+$, over her consumptions at both dates.

At the first period $(t = 0)$, each agent, $i \in I$, receives a private information signal, $S_i \subset S$, about which states of the world may occur at $t = 1$. That is, she knows that no state, $s \in S \backslash S_i$, will prevail tomorrow. Each set S_i is assumed to contain the true state. Hence, the pooled information set, denoted by $\underline{\mathbf{S}} := \cap_{i \in I} S_i$, is non-empty and we let, w.l.o.g., $S = \cup_{i \in I} S_i$. Such a collection of $\#I$ finite sets, whose intersection is non-empty, is called an information structure. Agents' information structure, (S_i), is henceforth set as given and always referred to.

Agents are unaware of the primitives of the economy and of other agents' beliefs and actions. They fail to know how market prices are determined and face uncertainty over future spot prices. Thus, at $t = 0$, the generic i^{th} agent elects a private set of anticipations, out of the price set, $P := \{p \in \mathbb{R}_{++}^L : \|p\| = 1\}$, in each state $s \in S_i$. We refer to $\Omega := S \times P$ as the set of forecasts and denote by ω its generic element, and by $\mathcal{B}(\Omega)$ its Borel σ-algebra. A forecast, $\omega := (s, p) \in \Omega$, is thus a pair of a random state, $s \in S$, and a conditional spot price, $p \in P$, expected in that state.

Remark 1 Strictly positive prices in P are related to strictly increasing preferences, as assumed below. For simplicity, but w.l.o.g., the set, P, normalizes all agents' price expectations to one. In each state, this common value of one could be replaced by any other positive value without changing the model's properties.

We now define anticipation structures and beliefs.

Definition 1 *An anticipation set is a closed subset of* $\Omega := S \times P$. *A collection of anticipation sets,* $\Omega_i := \cup_{s \in S_i} \{s\} \times P^i_s$, *for each* $i \in I$, *is an anticipation structure if:*

(a) $P^i_s \neq \varnothing$, *for every* $(i, s) \in I \times S_i$, *and* $\cap_{i \in I} P^i_s \neq \varnothing$, *for every* $s \in \underline{\mathbf{S}}$.

Let (Ω_i) *be a given anticipation structure. An anticipation structure,* (Ω'_i), *which is smaller (for the inclusion relation) than* (Ω_i), *is called a refinement of* (Ω_i), *and denoted by* $(\Omega'_i) \leq (\Omega_i)$. *It is said to be self-attainable if* $\cap_{i \in I} \Omega'_i = \cap_{i \in I} \Omega_i$.

A belief is a probability distibution over $(\Omega, \mathcal{B}(\Omega))$, *whose support is an anticipation set. A collection of beliefs,* (π_i), *whose supports define an anticipation structure,* (Ω_i), *is called a structure of beliefs, said to support* (Ω_i) *and denoted by* $(\pi_i) \in \Pi_{(\Omega_i)}$.

Only spot markets in states $s \in \underline{\mathbf{S}}'$ may open. We therefore restrict admissible commodity prices in states of $s \in \underline{\mathbf{S}}'$ to the set $\mathcal{P} := \{p \in \mathbb{R}^L_{++} : \|p\| \leqslant 1\} \times P^{\underline{\mathbf{S}}}$, which is consistent with consumers' anticipations.

Agents may operate financial transfers across states in S' by exchanging, at $t = 0$, finitely many assets, $j \in J$, which pay off, at $t = 1$, conditionally on the realization of forecasts. According to Sections, these assets may be nominal (i.e., pay in cash) or real (i.e., pay in godds) or a mix of both. All assets' payoffs define a $(S \times L') \times J$ return matrix, V, whose generic row across forecasts, $\omega \in \Omega$, is denoted $V(\omega) \in \mathbb{R}^J$. We let $\mathcal{V}$ be the set of $(S \times L') \times J$ matrices. Since payoffs will face "trembles" in the fifth Section, for every $n \in \mathbb{N}$, we let $\mathcal{V}_n := \{V' \in \mathcal{V} : \|V' - V\| \leqslant 1/n\}$.

The generic payoff of an asset, $j \in J$, in a state, $s \in S$, is a bundle $v^j_s := (v^{jl}_s) \in \mathbb{R}^{L'}$, of the quantities, v^{j0}_s, of cash, and v^{jl}_s, of each good $l \in L$, that the asset delivers if state $s \in S$ obtains.[2] We restrict asset prices to the set $Q := \{q \in \mathbb{R}^J : \|q\| \leqslant 1\}$ w.l.o.g. and let $P_0 := \{p \in \mathbb{R}^L_{++} : \|p\| \leqslant 1\} \times Q$ be the set of first period prices. Along the forecast $\omega = (s, p := (p^l)) \in \Omega$, the generic j^{th} asset is a contract which promises to pay $v_s^{j0} + \sum_{l \in L} p^l v_s^{jl}$ in cash if the forecast, ω, obtains. Thus, at asset price, $q \in Q$, agents may buy or sell unrestrictively portfolios of assets, $z = (z_j) \in \mathbb{R}^J$, for $q \cdot z$ units of account at $t = 0$, against the promise of delivery of a flow, $V(\omega) \cdot z$, of conditional cash payoffs across forecasts, $\omega \in \Omega$.

We now define arbitrage-free anticipation structures.

Definition 2 *Given price $q \in Q$, an anticipation structure, (Ω_i), is said to be q-arbitrage-free if following Condition holds:*

(a) $\nexists (i,z) \in I \times \mathbb{R}^J : -q \cdot z \geqslant 0$ *and* $V(\omega) \cdot z \geqslant 0, \forall \omega \in \Omega_i$, *with one strict inequality.*

An anticipation structure, (Ω_i), is said to be arbitrage-free if it is q-arbitrage-free for some price, $q \in Q$, and we denote by $\mathcal{AS}$ their set. We denote by $\mathcal{SB}$ the set of structures of beliefs, which support an arbitrage-free anticipation structure, $(\Omega_i) \in \mathcal{AS}$.

2.2 The agent's behaviour and the concept of equilibrium

The generic i^{th} agent receives an endowment, $e_i := (e_{is}) \in \mathbb{R}^{L \times S'_i}_{++}$, granting the commodity bundles, $e_{i0} \in \mathbb{R}^L_{++}$ at $t = 0$, and $e_{is} \in \mathbb{R}^L_{++}$, in each state $s \in S_i$, if it prevails. We let $e := (e_i) \in \times_{i \in I} \mathbb{R}^{L \times S'_i}_{++}$ be the bundle of endowments across agents. Since endowments will face "trembles" in the fifth Section, for every $n \in \mathbb{N}$, we let $E_n := \{e' \in \times_{i \in I} \mathbb{R}^{L \times S'_i}_{+} : \|e' - e\| \leqslant 1/n\}$ and assume w.l.o.g. that $E_1 \subset \times_{i \in I} \mathbb{R}^{L \times S'_i}_{++}$, henceforth considered as a fixed set.

[2] if the asset, $j \in J$, is nominal $v_s^{jl} = 0$, for every pair $(s, l) \in S \times L$. If the asset is real, $v_s^{j0} = 0$, for every $s \in S$.

Agents' forecasts are represented by an arbitrage-free anticipation structure, say $(\Omega_i) \in \mathcal{AS}$, which is reached when they elect their strategies at $t = 0$, jointly with beliefs, $(\pi_i) \in \Pi_{(\Omega_i)}$, along Definition 1. The assumption that agents' forecasts are arbitrage-free is proved to be non restrictive in De Boisdeffre (2016), since they may always infer from markets a (unique coarse) self-attainable arbitrage-free refinement of any anticipation structure. Then, the i^{th} the agent's consumption set is that of continuous mappings, $x : \Omega_i' \to \mathbb{R}_+^L$ (where $\Omega_i' := \{0\} \cup \Omega_i$), denoted by $X_{\Omega_i} := \mathcal{C}\ (\Omega_i',\ \mathbb{R}_+^L)$.

Given the observed prices, $\omega_0 := (p_0, q) \in P_0$, at $t = 0$, and her anticipation set, Ω_i, the generic i^{th} agent's consumptions, $x \in X_{\Omega_i}$, are mappings, relating $s = 0$ to a consumption decision, $x_{\omega_0} := x_0 \in \mathbb{R}_+^L$, at $t = 0$, and, continuously on Ω_i, every forecast, $\omega \in \Omega_i$, to a consumption decision, $x_\omega \in \mathbb{R}_+^L$, at $t = 1$, which is conditional on the realization of the forecast ω. Her budget set is defined as follows:

$$B_i(\omega_0, \Omega_i) := \{(x, z) \in X_{\Omega_i} \times \mathbb{R}^J :\ p_0 \cdot (x_0 \text{-} e_{i0}) \leqslant -q \cdot z,\ \ p_s \cdot (x_\omega \text{-} e_{is}) \leqslant V(\omega) \cdot z,\ \forall \omega := (s, p_s) \in \Omega_i\}.$$

Given agents' structure of beliefs at the time of trading, $(\pi_i) \in \Pi_{(\Omega_i)}$, each consumer, $i \in I$, has preferences represented by the V.N.M. utility function:

$$x \in X_{\Omega_i} \mapsto U_i^{\pi_i}(x) := \int_{\omega \in \Omega_i} u_i(x_0, x_\omega) d\pi_i(\omega).$$

The above economy, denoted $\mathcal{E}_{(\pi_i)} = \{(I, S, L, J), V, (S_i), (\Omega_i), (\pi_i), (e_i), (u_i)\}$, retains the small consumer price-taker hypothesis, by which no single agent may, alone, have a significant impact on prices. It is called standard under the following Conditions:

- ***Assumption A1*** *(strong survival): for each* $i \in I$, $e_i \in \mathbb{R}_{++}^{L \times S_i'}$;
- ***Assumption A2****: for each* $i \in I$, u_i *is continuous, strictly concave and increasing:* $[(x, y, x', y') \in \mathbb{R}_+^{4L},\ (x, y) \leqslant (x', y'),\ (x, y) \neq (x', y')] \Rightarrow [u_i(x', y') > u_i(x, y)]$.

Strict concavity is retained to alleviate the proof of a selection amongst optimal strategies (see proof of Lemma 4). The consumer elects an optimal strategy in her budget set. This yields the following concept of sequential equilibrium:

Definition 3 *A collection of prices,* $p := (p_s) \in \mathcal{P}$ *and* $q \in Q$*, of an anticipation structure,* $(\Omega_i) \in \mathcal{AS}$*, beliefs,* $(\pi_i) \in \Pi_{(\Omega_i)}$*, and strategies,* $(x_i, z_i) \in B_i(\omega_0, \Omega_i)$*, defined for each* $i \in I$ *(where* $\omega_0 := (p_0, q)$*) is a sequential equilibrium of the economy,* $\mathcal{E}_{(\pi_i)}$*, or correct foresight equilibrium (C.F.E.), if the following Conditions hold:*

(a) $\forall i \in I, \ \forall s \in \underline{\mathbf{S}}, \ \omega_s := (s, p_s) \in \Omega_i$;

(b) $\forall i \in I, \ (x_i, z_i) \in \arg\max_{(x,z) \in B_i(\omega_0, \Omega_i)} \ U_i^{\pi_i}(x)$;

(c) $\sum_{i \in I} (x_{i0} - e_{i0}) = 0$;

(d) $\sum_{i \in I} (x_{i\omega_s} - e_{is}) = 0, \ \forall s \in \underline{\mathbf{S}}$;

(e) $\sum_{i \in I} z_i = 0$.

Under the above conditions, price $p \in \mathcal{P}$*, and each forecast,* $\omega_s := (s, p_s) \in \Omega$*, for* $s \in \underline{\mathbf{S}}$*, are said to support equilibrium. A collection,* $\{p, q, (\Omega_i), (\pi_i), (x_i), (z_i)\}$*, which meets Conditions* (b)*-*(c)*-*(e) *is called a temporary equilibrium.*

2.3 The model's notations

For convenience, we summarize the model's notations in this single sub-Section:

- $\mathcal{E}_{(\pi_i)} = \{(I, S, L, J), V, (S_i), (\Omega_i), (\pi_i), (e_i), (u_i)\}$ summarizes the economy's characteristics. There are two periods, $t \in \{0, 1\}$, finite sets, $I, \ S, \ L, \ J$, respectively, of consumers, states of nature, goods and assets, a payoff matrix, V, information sets, $S_i \subset S$, and $\underline{\mathbf{S}} := \cap_{i \in I} S_i \neq \varnothing$, an anticipation structure, $(\Omega_i) \in \mathcal{AS}$, and beliefs, $(\pi_i) \in \Pi_{(\Omega_i)}$, along Definition 1, endowments, $e := (e_i) \in \times_{i \in I} \mathbb{R}_{++}^{L \times S'_i}$, and utility indexes, u_i, defining a V.N.M. utility function, $U_i^{\pi_i}$, for each $i \in I$.

- For every $n \in \mathbb{N}$, we let $E_n := \{e' \in \times_{i \in I} \mathbb{R}_+^{L \times S'_i} : \|e' - e\| \leqslant 1/n\}$ and assume w.l.o.g. that $E_1 \subset \times_{i \in I} \mathbb{R}_{++}^{L \times S'_i}$, henceforth considered as fixed.

- We let $s = 0$ be the non-random state at $t = 0$ and denote $\underline{\mathbf{S}}' := \{0\} \cup \underline{\mathbf{S}}$ and $S'_i := \{0\} \cup S_i$, for each $i \in I$. We let $l = 0$ be the account unit and $L' := \{0\} \cup L$.

- $Q := \{q \in \mathbb{R}^J : \|q\| \leqslant 1\}$, $P := \{p := (p^l) \in \mathbb{R}_{++}^L : \|p\| = 1\}$, $\mathcal{P} := \{p \in \mathbb{R}_{++}^L : \|p\| \leqslant 1\} \times P^{\underline{\mathbf{S}}}$ and $\Omega := S \times P$ are the sets, respectively, of asset prices, expected spot prices, market prices (for goods) and *forecasts*.

- $\mathcal{V}$ is the set of all $(S \times L') \times J$ matrices $(V \in \mathcal{V})$. For every $n \in \mathbb{N}$, we let $\mathcal{V}_n := \{V' \in \mathcal{V} : \|V' - V\| \leqslant 1/n\}$.

3 The core existence theorem

With the model's endogenous uncertainty, only the set of possible equilibrium forecasts could be assessed. No agent or institution would know the true forcasts' location within that set, because this would require to know all agents' private beliefs, characteristics and actions.

This set is the "*minimum uncertainty set*", defined below. The following Theorems of Section 3 and 5 show that equilibrium exists, whenever agents' forecasts embed the latter set. This existence result holds whatever the anticipations and beliefs agents have, and the types of assets (nominal or real or a mix of both) they exchange. This full existence result is worth noticing, so it differs from the generic ones of the classical sequential equilibrium models. It builds on a core Theorem 1.

3.1 Endogenous uncertainty and the existence of equilibrium

We start with a definition.

Definition 4 *Let Λ be the set of prices, $p := (p_s) \in \mathcal{P}$, which support the equilibrium of an economy, $\mathcal{E}_{(\pi_i)}$, for some arbitrary structure of beliefs, $(\pi_i) \in \mathcal{SB}$. The set of forecasts, $\Delta := \{\omega \in \Omega : \exists p := (p_s) \in \Lambda,\ \exists s \in \underline{\mathbf{S}},\ \omega = (s, p_s)\}$, which support an equilibrium, is called the minimum uncertainty set.*

Lemma 1 *Under Assumptions A1-A2, the following Assertions hold:*

(i) $\exists \delta > 0 : \Lambda \subset \mathbb{R}^L_+ \times [\delta, 1]^{L \times \underline{\mathbf{S}}}$, *hence,* $\Delta \subset \underline{\mathbf{S}} \times [\delta, 1]^L$;

(ii) *the bound, δ, may be chosen independent of $V \in \mathcal{V}_1$ and $(e_i) \in E_1$.*

Proof See the Appendix. □

Assumption A3 *(correct foresight): for each $i \in I$, the relation $\Delta \subset \Omega_i$ holds, in which $(\Omega_i) \in \mathcal{AS}$ is the given anticipation structure of the economy, $\mathcal{E}_{(\pi_i)}$.*

Theorem 1 *Under Assumptions A1-A2-A3, an economy, $\mathcal{E}_{(\pi_i)}$, with purely financial markets admits an equilibrium (C.F.E.), for any structure of beliefs, $(\pi_i) \in \mathcal{SB}$.*

3.2 Endogenous uncertainty and how to reach correct anticipations

Along Theorem 1, above, as long as agents have correct foresight (i.e., meet Assumption *A3*), a C.F.E. exists whatever their beliefs. Markets clear ex post at one self-fulfilling common anticipation. We now argue why the set of all equilibrium forecasts may be one of "*minimum uncertainty*" and how it could be assessed.

On the first issue, when today's beliefs are private, no equilibrium price should be ruled out *a priori*, given agents' unknown anticipations today. Theoretically, this set is of incompressible uncertainty. Practically, it would be so because no agent knows the beliefs and characteristics of other agents, nor has structural knowledge, along Kurz (1994). Past price series confirm that erratic fluctuations may occur not only in periods of enhanced uncertainty. Yet, if no agent has structural knowledge and access to private data, how can this minimum uncertainty set, or a bigger set, be inferred ? The response may simply be empirical, that is, only require observations.

On this issue, the model specifies *normalized* prices (extended by Remark 1). It is often possible to observe past prices and reckon their *relative* values, in a wide array of situations, or states, which typically replicate over time (hence, embed $\underline{\mathbf{S}}$). Relative prices vary between observable upper and lower bounds.

Along a sensible assumption, markets are mostly at equilibrium and, with long enough series, all equilibrium forecasts would lie within the bounds of the series' convex hulls.[3] Such a statistical method and its iterative verification across periods require no price model and need not be performed by consumers, but by a tradehouse or financial institution, having greater computational facilities. The applications to finance they might infer are obvious. On consumer side, if agents should agree on a minimal span of price risk, they typically keep private their beliefs and have idiosyncratic anticipations, explaining their likely asymmetries.

4 The existence proof

Hereafter, we set as given an arbitrary anticipation structure, $(\Omega_i) \in \mathcal{AS}$, and beliefs, $(\pi_i) \in \Pi_{(\Omega_i)}$, and assume that the economy, $\mathcal{E}_{(\pi_i)}$, meets Assumptions *A1-A2-A3*. In the following sub-Section 4.1, the financial structure is represented by an arbitrary payoff matrix $V \in \mathcal{V}$ (which needs not be nominal), to present results that

[3] e.g., if the future reflects the past, if $\underline{\mathbf{S}}$ is also a set of past states and, for every $s \in \underline{\mathbf{S}}$, the past price serie, $(p_s^t) \in (P)^{T_s}$ (where $T_s \in \mathbb{N}$) is large, then, iteratively, the set $\{(s, y_s) \in \underline{\mathbf{S}} \times P : \ y_s = \sum_{t=1}^{T_s} \alpha_t p_s^t \ / \| \sum_{t=1}^{T_s} \alpha_t p_s^t \|, \ (\alpha_t) \in \mathbb{R}_+^{T_s}, \ \sum_{t=1}^{T_s} \alpha_t = 1\}$, could easily be checked to always contain the self-fulfilling forecasts.

will serve in the following Section 5. In sub-Sections 4.2 and 4.3, the payoff matrix is restricted to be nominal.

The proof proceeds in three steps. Sub-Section 4.1 defines, via finite partitions, a non-decreasing sequence, $\{(\Omega_i^n)\}_{n\in\mathbb{N}}$, of finite refinements of (Ω_i), whose limit is dense in (Ω_i). Sub-Section 4.2 constructs a sequence of finite auxiliary economies, which all admit equilibria along De Boisdeffre (2007). Sub-Section 4.3 derives a CFE of the economy $\mathcal{E}_{(\pi_i)}$ from these auxiliary equilibria.

4.1 Finite partitions of agents' anticipation sets

- Let $(i,n) \in I \times \mathbb{N}$ be given. We define an integer, $K_{(i,n)} \in \mathbb{N}$, and a partition, $\mathcal{P}_i^n = \{\Omega_{(i,n)}^k\}_{1\leqslant k\leqslant K_{(i,n)}}$, of Ω_i, such that $\pi_i(\Omega_{(i,n)}^k) > 0$, for each $k \in \{1, ..., K_{(i,n)}\}$.
- In each set $\Omega_{(i,n)}^k$ (for $k \leqslant K_{(i,n)}$), we select exactly one element, $\omega_{(i,n)}^k$, to form the discrete sub-set, $\Omega_i^n := \{\omega_{(i,n)}^k\}_{1\leqslant k\leqslant K_{(i,n)}}$, of Ω_i.
- We define mappings, $\pi_i^n : \Omega_i^n \to \mathbb{R}_+$, by $\pi_i^n(\omega_{(i,n)}^k) = \pi_i(\Omega_{(i,n)}^k)$ and $\Phi_i^n : \Omega_i \to \Omega_i^n$, by its restrictions, $\Phi^n_{i\ /\ \Omega_{(i,n)}^k}(\omega) = \omega_{(i,n)}^k$, for each $k \leqslant K_{(i,n)}$ and every $\omega \in \Omega_{(i,n)}^k$.

And we henceforth assume that the Assertions of the following Lemma hold.

Lemma 2 *For each $(i,n) \in I \times \mathbb{N}$, we may choose the above $\mathcal{P}_i^n$, Ω_i^n, Φ_i^n, such that:*

(i) $\Omega_i^n \subset \Omega_i^{n+1}$ *and* $\mathcal{P}_i^{n+1}$ *is finer than* $\mathcal{P}_i^n$*;*

(ii) $\cup_{n\in\mathbb{N}}\,\Omega_i^n$ *is everywhere dense in* Ω_i*;*

(iii) for every $\omega \in \Omega_i$, $\omega = \lim_{n\to\infty} \Phi_i^n(\omega)$, *and* $\Phi_i^n(\omega)$ *converges uniformly to* ω*;*

(iv) there exist $N \in \mathbb{N}$, *such that* (Ω_i^n) *is arbitrage-free for every* $n \geqslant N$.

For simplicity, we henceforth assume that $N = 1$.

Proof See the Appendix, which provides one example of such sets and maps. □

4.2 The auxiliary economies, $\mathcal{E}^n$

Given $n \in \mathbb{N}$, we define an economy, $\mathcal{E}^n = \{(I, S, L, J), V, (\Omega_i'^n), (e_i), (u_i^n)\}$, with same periods, sets of agents, goods and endowments as above. The realizable states and the generic i^{th} agent's expectations are artefactual and defined as follows:

- $\Omega_i'^n := \underline{\mathbf{S}} \cup \Omega_i^n$ is the agent's information set, defining the information structure, $(\Omega_i'^n)$, of a formal state space, $\Omega^n := \cup_{i \in I} \Omega_i'^n$, whose set of realizable states is $\underline{\mathbf{S}}$.
- In each state $s \in \underline{\mathbf{S}}$, the i^{th} agent has a perfect foresight of the spot price.
- In each state $(s, p) \in \Omega_i^n$, the i^{th} agent is certain that price $p \in P$ will prevail.

By induction on $n \in \mathbb{N}$, we define a sequence of equilibrium prices, $(p^n, q^n) \in \mathcal{P} \times Q$ in the following way. For all prices, $(p := (p_s), q) \in \mathcal{P} \times Q$, we let the generic i^{th} agent's consumption set, budget set, and utility function in the economy $\mathcal{E}^n$ be:

$X_i^n := \mathbb{R}_+^{L \times \Omega_i'^n}$, whose generic element is denoted by $x := [(x_s)_{s \in \underline{\mathbf{S}}'}, (x_\omega)_{\omega \in \Omega_i^n}]$;

$$B_i^n(p,q) := \{\ (x,z) \in X_i^n \times \mathbb{R}^J :\ p_0 \cdot (x_0 - e_{i0}) \leqslant -q \cdot z, \quad p_s \cdot (x_s - e_{is}) \leqslant V(s, p_s) \cdot z,\ \forall s \in \underline{\mathbf{S}}$$
$$\textit{and}\ \ p \cdot (x_\omega - e_{is}) \leqslant V(\omega) \cdot z,\ \forall \omega := (s,p) \in \Omega_i^n\ \};$$

and $x \in X_i^n \mapsto\ u_i^n(x) := \frac{1}{n \# \underline{\mathbf{S}}} \sum_{s \in \underline{\mathbf{S}}} u_i(x_0, x_s)\ + (1 - \frac{1}{n}) \sum_{\omega \in \Omega_i^n} u_i(x_0, x_\omega) \pi_i^n(\omega)$.

Henceforth, the payoff matrix, V, is assumed to be nominal, so that $V(s) := V(s,p)$, for every $(s,p) \in \Omega$, only depends on $s \in S$. The above economy, $\mathcal{E}^n$, is of the De Boisdeffre's (2007) type. Hence, from its Theorem 1 and proof, it admits an equilibrium, for every $n \in \mathbb{N}$, defined as follows:

Definition 5 *A collection of prices, $(p, q) \in \mathcal{P} \times Q$, and strategies, $(x_i, z_i) \in B_i^n(p,q)$, for each $i \in I$, is an equilibrium of the economy $\mathcal{E}^n$, if the following Conditions hold:*

(a) $\forall i \in I,\ (x_i, z_i) \in \arg\max_{(x,z) \in B_i^n(p,q)}\ u_i^n(x)$;

(b) $\sum_{i\in I}(x_{is}-e_{is})=0,\ \forall s\in \underline{\mathbf{S}}'$;

(c) $\sum_{i\in I} z_i=0$.

We set as given, for every $n\in\mathbb{N}$, such equilibria, $\mathcal{C}^n := \{p^n, q^n, (x_i^n), (z_i^n)\}$, in each economy $\mathcal{E}^n$. From the proof of Theorem 1 in De Boisdeffre (2007), the elected equilibrium satisfies $\|p_0^n\|+\|q^n\|\geqslant 1$, for each $n\in\mathbb{N}$, hence, $\|p_0^*\|+\|q^*\|\geqslant 1$. Moreover, the sequence, $\{\mathcal{C}^n\}:=\{n\in\mathbb{N}\mapsto \mathcal{C}^n\}$, meets the following properties:

Lemma 3 *For each $i\in I$, we let $Z_i:=\{z\in\mathbb{R}^J : V(\omega)\cdot z=0,\ \forall\omega\in\Omega_i\}$, $Z_i^{\perp}$ be its orthogonal complement and $Z:=\sum_{i\in I} Z_i$. Given $\{\mathcal{C}^n\}$, we let $z_i^n = z_i^{on}+z_i^{\perp n}$ be the decomposition of z_i^n on $Z_i\times Z_i^{\perp}$, for each $(i,n)\in I\times\mathbb{N}$. The following Assertions hold:*

(i) the price sequence $\{(p^n,q^n)\}$ may be assumed to converge to $(p^*,q^*)\in\overline{\mathcal{P}}\times Q$, such that $\{(s,p_s^*)\}_{s\in\underline{\mathbf{S}}}\subset\overline{\Delta}\subset(\cap_{i\in I}\Omega_i)$;

(ii) the sequences $\{(x_{is}^n)_{s\in\underline{\mathbf{S}}'}\}$ and $\{(z_i^{\perp n})_{i\in I}\}$ may be assumed to converge, say to $(x_{is}^*)_{s\in\underline{\mathbf{S}}'}$ and $(z_i^{\perp *})\in\mathbb{R}^{J\times I}$, such that $\sum_{i\in I}(x_{is}^*-e_{is})_{s\in\underline{\mathbf{S}}'}=0$ and $\sum_{i\in I} z_i^{\perp *}\in Z$;

(iii) there exists $(z_i^*)\in\mathbb{R}^{J\times I}$, such that $\sum_{i\in I} z_i^*=0$ and $(z_i^*-z_i^{\perp *})\in Z_i$ for every $i\in I$.

Lemma 4 *Let $B_i(\omega,z)=\{x\in\mathbb{R}_+^L : p\cdot(x-e_{is})\leqslant V(\omega)\cdot z\}$, be given sets, for every $z\in\mathbb{R}^J$ and all $\omega:=(s,p)\in\Omega_i$. Along Lemma 3, the following Assertions hold for all $i\in I$:*

(i) the correspondence $\omega\in\Omega_i\mapsto \arg\max\ u_i(x_{i0}^*,x)$, for $x\in B_i(\omega,z_i^*)$, is a continuous map, whose embedding, $x_i^*:\omega\in\Omega_i'\mapsto x_{i\omega}^*$, is a consumption, that is, $x_i^*\in X_{\Omega_i}$;

(ii) $U_i^{\pi_i}(x_i^*)=\lim_{n\to\infty} u_i^n(x_i^n)$.

Proof of the Lemmas See the Appendix. □

4.3 An equilibrium of the initial economy

We now prove Theorem 1, via the following Claim.

Claim 1 *The collection,* $\{p^*, q^*, (\Omega_i), (\pi_i), (x_i^*), (z_i^*)\}$*, of prices, anticipation sets, beliefs, allocation and portfolios of Lemmas 3-4, defines a CFE of the economy* $\mathcal{E}_{(\pi_i)}$.

Proof We let $\mathcal{C}^* := \{p^*, q^*, (\Omega_i), (\pi_i), (x_i^*), (z_i^*)\}$ be defined as in Claim 1. From Lemma 3, $\mathcal{C}^*$ meets Conditions (a)-(c)-(d)-(e) of Definition 3 of equilibrium, above. We now show that $\mathcal{C}^*$ meets Condition (b) of the same Definition 3.

From the definition of $\mathcal{C}^n$, the relations $p_0^n \cdot (x_{i0}^n - e_{i0}) \leqslant -q^n \cdot z_i^n$ hold, for each $(i, n) \in I \times \mathbb{N}$, and yield $p_0^* \cdot (x_{i0}^* - e_{i0}) \leqslant -q^* \cdot z_i^*$, for each $i \in I$, in the limit $(n \to \infty)$. We let $\omega_0^* := (p_0^*, q^*)$. From Lemma 4-$(i)$, the relations $x_i^* \in X_{\Omega_i}$ and $p_s \cdot (x_{i\omega}^* - e_{is}) \leqslant V(\omega) \cdot z_i^*$ also hold, for every $i \in I$ and every $\omega = (s, p_s) \in \Omega_i$, and imply $[(x_i^*, z_i^*)]_{i \in I} \in \times_{i \in I} B_i(\omega_0^*, \Omega_i)$.

Next, we assume, by contraposition, that $\mathcal{C}^*$ fails to meet Condition (b) of Definition 3, that is, there exist $i \in I$, $(x, z) \in B_i(\omega_0^*, \Omega_i)$ and $\varepsilon \in \mathbb{R}_{++}$, such that:

$$(I) \qquad \varepsilon + U_i^{\pi_i}(x_i^*) < U_i^{\pi_i}(x).$$

We may, moreover, assume that $(x, z) \in B_i(\omega_0^*, \Omega_i)$ is such that:

$$(II) \qquad \exists\ (\delta, M) \in \mathbb{R}_{++}^2\colon\ x_\omega \in [\delta, M]^L,\ \forall \omega \in \Omega_i.$$

The existence of an upper bound to consumptions x_ω (for $\omega \in \Omega_i$) results from the relation $(x, z) \in B_i(\omega_0^*, \Omega_i)$, which implies a bound to financial transfers and from the fact that Ω_i is closed in $S \times P$. Moreover, for $\alpha \in]0, 1]$ small enough, the strategy $(x^\alpha, z^\alpha) := ((1 - \alpha)x + \alpha e_i, (1 - \alpha)z) \in B_i(\omega_0^*, \Omega_i)$ meets both relations (I) and (II), from Assumption *A1* and from the uniform continuity (on a compact set) of the mapping $(\alpha, \omega) \in [0, 1] \times \Omega_i \mapsto\ (x_\omega^\alpha,\ u_i(x_0^\alpha, x_\omega^\alpha))$. So, relations (II) may indeed be assumed.

From Lemmas 1-3, $p^* \in \mathbb{R}_+^L \times [\delta, 1]^{L \times \mathbf{S}}$. Then, from the relations (I)-(II) and $(x, z) \in B_i(\omega_0^*, \Omega_i)$, the definition of Ω_i, Assumptions *A1-A2* and uniform continuity arguments, we may also assume there exists $\gamma \in \mathbb{R}_{++}$, such that:

$(III) \quad p_0^*\cdot(x_0 - e_{i0}) \leqslant -q^*\cdot z$ and $p_s\cdot(x_\omega - e_{is}) < -\gamma + V(\omega)\cdot z,\ \forall\omega := (s, p_s) \in \Omega_i$.

From relations (I)-(II)-(III), we may also assume there exists $\gamma' \in]0, \gamma[$, such that:

$(IV) \quad p_0^*\cdot(x_0 - e_{i0}) \leqslant -\gamma' - q^*\cdot z$ and $p_s\cdot(x_\omega - e_{is}) \leqslant -\gamma' + V(\omega)\cdot z,\ \forall\omega := (s, p_s) \in \Omega_i$.

We recall from above that $\|p_0^*\| + \|q^*\| \geqslant 1$. The above assertion is obvious, from relations (III), if $p_0^*\cdot(x_0 - e_{i0}) < -q^*\cdot z$. Assume that $p_0^*\cdot(x_0 - e_{i0}) = -q^*\cdot z$. If $p_0^* = 0$, then, $q^* \neq 0$, and relations (IV) hold if we replace z by $z - q^*/N$, for $N \in \mathbb{N}$ big enough. If $p_0^* \neq 0$ and $x_0 \neq 0$, the desired assertion results from Assumption *A1* and above. Else, $-q^* \cdot z = -p_0^* \cdot e_{i0} < 0$, and a slight change in portfolio insures relations (IV).

From relations (IV), the continuity of the scalar product and Lemmas 1-2-3, there exists $N_1 \in \mathbb{N}$, such that, for every $n \geqslant N_1$:

$$(V) \quad \begin{cases} p_0^n\cdot(x_0 - e_{i0}) \leqslant -q^n\cdot z \\ p_s^n\cdot(x_{(s,p_s^*)} - e_{is}) \leqslant V(s, p_s^n)\cdot z,\ \forall s \in \underline{\mathbf{S}} \\ p_s\cdot(x_\omega - e_{is}) \leqslant V(s, p_s)\cdot z,\ \forall\omega := (s, p_s) \in \Omega_i^n \end{cases}$$

Along relations (V), for each $n \geqslant N_1$, we define, in $\mathcal{E}^n$, the strategy $(x^n, z) \in B_i^n(p^n, q^n)$ by $x_0^n := x_0$, $x_s^n := x_{(s,p_s^*)}$, $x_\omega^n := x_\omega$, for $(s, \omega) \in \underline{\mathbf{S}} \times \Omega_i^n$, and recall that:

- $U_i^{\pi_i}(x) := \int_{\omega\in\Omega_i} u_i(x_0, x_\omega) d\pi_i(\omega)$;
- $u_i^n(x^n) := \frac{1}{n\#\underline{\mathbf{S}}} \sum_{s\in\underline{\mathbf{S}}} u_i(x_0, x_s^n) + (1 - \frac{1}{n}) \sum_{\omega\in\Omega_i^n} u_i(x_0, x_\omega)\pi_i^n(\omega)$.

Then, from above, from relation (II), Lemma 2, and the uniform continuity of $x \in X_{\Omega_i}$ and u_i on compact sets, there exists $N_2 \geqslant N_1$ such that:

$(VI) \quad |U_i^{\pi_i}(x)$-$u_i^n(x^n)| < \int_{\omega\in\Omega_i} |u_i(x_0, x_\omega)$-$u_i(x_0, x_{\Phi_i^n(\omega)})| d\pi_i(\omega) + \frac{\varepsilon}{4} < \frac{\varepsilon}{2}$, for every $n \geqslant N_2$.

From equilibrium conditions and Lemma 4-(ii), there exists $N_3 \geqslant N_2$, such that:

$(VII) \qquad u_i^n(x^n) \leqslant u_i^n(x_i^n) < \frac{\varepsilon}{2} + U_i^{\pi_i}(x_i^*)$, for every $n \geqslant N_3$.

Let $n \geqslant N_3$ be given. The above Conditions (I)-(VI)-(VII) yield, jointly:

$$U_i^{\pi_i}(x) < \frac{\varepsilon}{2} + u_i^n(x^n) \leqslant \frac{\varepsilon}{2} + u_i^n(x_i^n) < \varepsilon + U_i^{\pi_i}(x_i^*) < U_i^{\pi_i}(x).$$

This contradiction proves that $\mathcal{C}^*$ meets Condition (b) of Definition 3, hence, from above, is a C.F.E. of the economy $\mathcal{E}_{(\pi_i)}$. This completes the proof of Theorem 1. □

Theorem 1, above, holds for nominal asset structures when agents have correct foresight. We now examine existence for other financial and anticipation structures.

5 The existence theorems with arbitrary assets

In sub-Section 5.1, we show that temporary equilibria always exist, that is, for arbitrary beliefs and financial structures. In the following sub-Sections, we extend the above Theorem 1 to an economy with smooth preferences and arbitrary assets.

5.1 Temporay equilibria with arbitrary structures of payoffs and beliefs

Theorem 2 *Under Assumptions A1-A2, an economy, $\mathcal{E}_{(\pi_i)}$, with an arbitrary payoff matrix, $V \in \mathcal{V}$, admits a temporary equilibrium, for any structure of beliefs, $(\pi_i) \in \mathcal{SB}$.*

Proof In the definition of auxiliary economies in Section 4, we may assume that the set of realizable states, $\underline{\mathbf{S}}$, is empty. This assumption is purely formal, artefactual. Then, for each $n \in \mathbb{N}$, the economy, $\mathcal{E}^n$, is well defined, anticipation sets, (Ω_i^n), are exogenous, whereas, for $\omega_0 := (p_0, q) \in P_0$, the generic i^{th} agent's budget set is:

$$B_i^n(\omega_0) := \{(x,z) \in X_i^n \times \mathbb{R}^J : \ p_0 \cdot (x_0 - e_{i0}) \leqslant -q \cdot z \ \ and \ \ p \cdot (x_\omega - e_{is}) \leqslant V(\omega) \cdot z, \ \forall \omega := (s,p) \in \Omega_i^n\}.$$

From De Boisdeffre's (2007) existence theorem and proof, for every $n \in \mathbb{N}$, the above economy, $\mathcal{E}^n$, admits a temporary equilibrium, defined as follows:

Definition 6 *A collection of prices,* $\omega_0 := (p_0, q) \in P_0$*, such that* $\|\omega_0\| \geqslant 1$*, and strategies,* $(x_i, z_i) \in B_i^n(\omega_0)$*, defined for each* $i \in I$*, is an equilibrium of the economy* $\mathcal{E}^n$*, if the following Conditions hold:*

(a) $\forall i \in I,\ (x_i, z_i) \in \arg\max_{(x,z) \in B_i^n(\omega_0)}\ u_i^n(x)$*;*

(b) $\sum_{i \in I}\ (x_{i0} - e_{i0}) = 0$*;*

(c) $\sum_{i \in I}\ z_i = 0$.

Indeed, all arguments of the proof of Theorem 1 in De Boisdeffre (2007) apply, mutatis mutandis, with the artefactual assumption that $\underline{\mathbf{S}}$ is empty and yield an equilibrium along the above Definition, say $\mathcal{C}^n := \{\omega_0^n, (x_i^n), (z_i^n)\}$. Similarly, all arguments of Lemmas 3 and 4 and Claim 1 above apply, mutatis mutandis, and yield a temporary equilibrium, $\mathcal{C}^* := \{\omega_0^*, (\Omega_i), (\pi_i), (x_i^*), (z_i^*)\}$, along Definition 3. □

5.2 The economy with arbitrary assets and correct beliefs

We have to change the framework slightly, so as to be able to apply standard generic existence results of the litterature. We will conform to Duffie-Shafer's (1985) setting. Admissible commodity prices are now restricted to the new price set:

$\mathcal{P} := \{p := (p_s^l) \in \mathbb{R}_{++}^{L \times \underline{\mathbf{S}}'} : \sum_{(l,s) \in L \times \underline{\mathbf{S}}'}\ p_s^l = 1\}$.[4]

The set $Q = \{q \in \mathbb{R}^J : \|q\| \leqslant 1\}$ may still be retained for admissible asset prices (with a bound to be one w.l.o.g.). Indeed, asset prices may always be bounded via individual state prices - or price functions along De Boisdeffre's (2016). Since

[4] We keep the same notations as in Section 2, so as to refer to the same Definitions as above, with reference to the new sets. In particular, with Section 5's new definitions of the price and forecast sets, Definitions 1, 2 and 3 may be kept as is.

anticipation structures are arbitrage-free, the asset price weighs the rows of payoffs on every agent's forecasts, and can always be bounded uniformly. Moreover, Duffie-Shafer (1985) proceeds in the same way. It sets as given a state price vector (whose components are all ones, p. 295), instead of an upper bound to asset prices.

Consistently with the latter definition of the price set, $\mathcal{P}$, the set of forecasts is now $\Omega := S \times \{p \in \mathbb{R}^L_{++} : \|p\| < 1\}$. An anticipation set is a closed subset of Ω, and the structures of anticipations and beliefs are defined accordingly, along Definition 1.

We assume that one agent, say $i = 1$, is fully informed upon the true states ($S_1 = \underline{\mathbf{S}}$) and true spot price that can prevail in any state and that she is endowed with exactly one unit of each good in any state, $s \in \underline{\mathbf{S}}'$. Thus, for all price $p \in \mathcal{P}$, one has $p \cdot e_1 = 1$. The latter relation, $p \cdot e_1 = 1$, holds, at so called pseudo-equilibria in Duffie-Shafer (1985), and below. Under the above small consumer hypothesis, the latter assumption is an artefact of no cost, which will permit to normalize price anticipations on the unit simplex. The other characteristics of the current economy are the same as above, in Section 2, to which we add Duffie-Shafer's following smoothness assumptions, the Inada Conditions and an additional assumption on payoffs, which will permit to bound the asset price norm uniformly from below:

- ***Assumption A3****: for each* $i \in I$, u_i *is* C^∞ *on* $\mathbb{R}^{L\times L}_{++}$;
- ***Assumption A4****:* $\forall (i, \overline{x}) \in I \times \mathbb{R}^{L\times L}_{++}$, $\{x \in \mathbb{R}^{L\times L}_{++} : u_i(x) \geqslant u_i(\overline{x})\}$ *is closed in* $\mathbb{R}^{L\times L}_{++}$;
- ***Assumption A5****: for each* $i \in I$, u_i *meets the Inada Conditions*;
- ***Assumption A6****: there exists one asset, with non-negative payoffs in all states, and having at least one positive payoff in one state* $s \in \underline{\mathbf{S}}$.

In the current economy, denoted by $\mathcal{E}_{(\pi_i)} = \{(I, S, L, J), V, (S_i), (\Omega_i), (\pi_i), (e_i), (u_i)\}$, the definition of equilibrium is Definition 3, above. The proof of its full existence builds on auxiliary finite economies, which we now present.

5.2 The auxiliary economies, $\mathcal{E}^n_{(\pi_i)}$

We set as given (a generic) structure of beliefs, $(\pi_i) \in \mathcal{SB}$, whose supports define a structure, $(\Omega_i) \in \mathcal{AS}$. We construct, for all $n \in \mathbb{N}$, an auxiliary economy, denoted $\mathcal{E}^n_{(\pi_i)}$, resuming all Section 4 definitions and notations, in anything but budget sets.

For every tuple, $(i, p, q, V') \in I \times \mathbb{R}_+^{L \times \underline{\mathbf{S}}'} \times Q \times \mathcal{V}$ and endowment bundles, $e' := (e'_i) \in E_1 := \{(e'_i) \in \times_{i \in I} \mathbb{R}_+^{L \times S'_i} : \|(e'_i) - (e_i)\| \leqslant 1\}$, the generic i^{th} agent's budget set is now:

$$B_i^n(p, q, V', e'_i) := \{ \ (x, z) \in X_i^n \times \mathbb{R}^J : \ p_0 \cdot (x_0 - e'_{i0}) \leqslant -q \cdot z, \quad p_s \cdot (x_s - e'_{is}) \leqslant V'(s, p_s) \cdot z, \ \forall s \in \underline{\mathbf{S}}$$
$$\textit{and} \ \ p \cdot (x_\omega - e_{is}) \leqslant V(\omega) \cdot z, \ \forall \omega := (s, p) \in \Omega_i^n \ \}.$$

In the above budget sets, the payoff matrix and endowments may only differ from the original ones (i.e., $V \in \mathcal{V}$ and $(e_i) \in E_1$) in realizable states (i.e., $s \in \underline{\mathbf{S}}'$). This restriction yields the following simple concept of auxiliary equilibrium:

Definition 7 *A collection of prices, $(p, q) \in \mathbb{R}_{++}^{L \times \underline{\mathbf{S}}'} \times Q$, payoff matrix, $V' \in \mathcal{V}$, endowments, $(e'_i) \in E_1$, and strategies, $(x_i, z_i) \in B_i^n(p, q, V', e'_i)$, defined for each $i \in I$, is an equilibrium of the economy $\mathcal{E}^n_{(\pi_i)}$, if the following conditions hold:*

(a) $\forall i \in I, \ (x_i, z_i) \in \arg\max_{(x,z) \in B_i^n(p,q,V',e'_i)} \ u_i^n(x)$;

(b) $\sum_{i \in I} \ (x_{is} - e'_{is}) = 0, \ \forall s \in \underline{\mathbf{S}}'$;

(c) $\sum_{i \in I} \ z_i = 0$;

(d) $p \cdot e'_1 = 1$.

We define a related concept of pseudo-equilibrium, after introducing new sets:

- we let $\mathcal{G}$ be the set of all full column rank $\underline{\mathbf{S}} \times J$ matrices;

- for every $L \in \mathcal{G}$, we denote by $< L >$ its $\#J$-dimensional span in $\mathbb{R}^{\underline{\mathbf{S}}}$;

- for every $V' \in \mathcal{V}$ and $p := (p_s) \in \mathbb{R}^{L \times \underline{\mathbf{S}}'}_{++}$, we let $V'(p)$ be the $\underline{\mathbf{S}} \times J$ matrix, whose generic row is $V'(s, p_s) \in \mathbb{R}^J$ (for $s \in \underline{\mathbf{S}}$) and denote by $< V'(p) >$ its span in $\mathbb{R}^{\underline{\mathbf{S}}}$;

- for every triple $(p, i, x) \in \mathbb{R}^{L \times \underline{\mathbf{S}}'}_{++} \times I \times X^n_i$, we let $p \,\square\, x \in \mathbb{R}^{\Omega'^n_i}$ be the vector, whose first components are the scalar products, $p_s \cdot x_s$, for each $s \in \underline{\mathbf{S}}$, and subsequent components are the products, $p_\omega \cdot x_\omega$, for each $\omega := (s, p_\omega) \in \Omega^n_i$;

- we let $E^*_n := \mathbb{R}^{L \times \Omega'^n_i \times I}_{++}$ be the sets of arbitrary endowment bundles, namely, for each agent, $i \in I$, the conditional bundles, $e'_{is} \in \mathbb{R}^L_{++}$, in each realizable state, $s \in \underline{\mathbf{S}}'$, and $e'_{i\omega} \in \mathbb{R}^L_{++}$, in each idiosyncratic state, $\omega \in \Omega^n_i$;

- for every $L := (L_s)_{s \in \underline{\mathbf{S}}} \in \mathcal{G}$, and every $i \in I$, we let $[\frac{L}{V_i}]$ be the $\Omega'^n_i \times J$ matrix whose first generic rows are the $L_s \in \mathbb{R}^J$, in each state $s \in \underline{\mathbf{S}}$, and subsequent rows are the $V(\omega) \in \mathbb{R}^J$, in each state $\omega \in \Omega^n_i$. We denote $< \frac{L}{V_i} >$ the matrix' span in $\mathbb{R}^{\Omega'^n_i}$.

We now define the following concept of pseudo-equilibrium in the economy $\mathcal{E}^n_{(\pi_i)}$:

Definition 8 *A collection of prices, $p := (p_s) \in \mathbb{R}^{L \times \underline{\mathbf{S}}'}_{++}$, payoff matrices, $L \in \mathcal{G}$ and $V' \in \mathcal{V}$, endowments, $e' := (e'_i) \in E^*_n$, and an allocation, $(x_i) \in \times_{i \in I} X^n_i$ defines a pseudo-equilibrium of the economy $\mathcal{E}^n_{(\pi_i)}$, if the following conditions hold:*

(a) $x_1 \in \arg\max \; u^n_i(x)$, *for* $x \in \{ \; x \in X^n_1 \; : \; p \cdot (x - e'_1) = 0 \; \}$;

(b) *for every* $i \in I \backslash \{1\}$, $x_i \in \arg\max \; u^n_i(x)$,

for $x \in \{ \; x \in X^n_i \; : \; \sum_{s \in \underline{\mathbf{S}}'} \; p_s \cdot (x_s - e'_{is}) + \sum_{\omega \in \Omega^n_i} \; p_\omega \cdot (x_\omega - e'_{i\omega}) = 0$ *and* $p \,\square\, (x - e'_i) \; \in < \frac{L}{V_i} > \}$;

(c) $< V'(p) > \subset < L >$;

(d) $\sum_{i \in I} \; (x_{is} - e'_{is}) = 0, \; \forall s \in \underline{\mathbf{S}}'$;

(e) $\sum_{i \in I} \; z_i = 0$;

(f) $p \cdot e'_1 = 1$.

Given $(e', V') \in E^*_n \times \mathcal{V}$, we say that (p, L) is a pseudo-equilibrium, if there exists $x \in$

$\times_{i\in I} X_i^n$*, such that* (x, p, L) *is a pseudo-equilibrium. We let* $\mathcal{E}$ *be the pseudo-equilibria manifold, that is, the set of collections,* (p, L, e', V')*, such that* (p, L) *is a pseudo-equilibrium. We define the projection,* $\pi : \mathcal{E} \to E_n^* \times \mathcal{V}$*, by* $\pi(p, L, e', V') := (e', V')$*.*

The above definitions extend Duffie-Shafer's (1985, pp. 288-289) to the economy $\mathcal{E}^n_{(\pi_i)}$. The following Claim states the full existence of pseudo-equilibria.

Claim 2 *Given Definition 8, the following Assertions hold:*

(i) $\mathcal{E}$ *is a smooth manifold without boundary of same dimension than* $\pi(\mathcal{E})$*;*

(ii) π *is proper;*

(iii) *there exists a regular value* (e^*, V^*) *of* π*, such that* $\#\pi^{-1}(e^*, V^*) = 1$*;*

(iv) $\pi^{-1}(e', V') \neq \varnothing$*, for every* $(e', V') \in E_n^* \times \mathcal{V}$*;*

(v) *the set of singular values of* π *is closed and null.*

Proof As we let the reader check, no argument in Duffie-Shafer (1985) is altered by the presence of the fixed set of unrealizable states, $\Omega^n \backslash \underline{\mathbf{S}}$, in which payoffs are fixed exogenous, as are anticipations. Only the spans generated by payoffs in realizable states ($s \in \underline{\mathbf{S}}$) matter. No argument is altered by the presence of nominal payoffs.

Assertion (i) results, mutatis mutandis, from Duffie-Shafer's (1985) Section 4. $\square$

Assertion (ii) results, mutatis mutandis, from Duffie-Shafer's Fact 10 (p 295). $\square$

Assertion (iii) A price, $p^* := (p_s^*) \in \mathbb{R}_{++}^{L\times\underline{\mathbf{S}}'}$, and matrix, $V^* \in \mathcal{V}$, are set as given, such that $V^*(p) \in \mathcal{G}$, and we let $L^* := V^*(p)$. The fact that there exist endowments, $(e_i^*) \in E_n^*$, which are optimal for each agent (meet Conditions (a)-(b) of Definition 8) is obvious from Assumption *A5* (align gradients with common and individual prices). That is, the elected price, p^*, and endowments, (e_i^*), yield a pseudo-equilibrium, (p^*, L^*), with no trade. This pseudo-equilibrium is Pareto optimal and affordable at any price, hence, unique, whereas the value $((e_i^*), V^*)$ is regular, as shown, mutatis mutandis, by Duffie-Shafer (see p. 296). □

Assertion *(iv)* From Assertion *(iii)* and mod. 2 degree theory, there is an odd number of pseudo-equilibria at any regular value of π. The value (e', V') is regular from the Definition, if $\pi^{-1}(e', V') = \varnothing$. Hence, $\pi^{-1}(e', V') \neq \varnothing$ for all $(e', V') \in E_n^* \times \mathcal{V}$. □

Assertion (v) is a standard application of Sard' theorem and demonstrated, mutatis mutandis, in Duffie-Shafer (p. 297), to which we refer the reader. □

Remark 2 In Definition 8, payoffs and anticipations in all idiosyncratic states $(\omega \in \Omega^n \backslash \underline{\mathbf{S}})$ were fixed, independently of $n \in \mathbb{N}$. Contrarily, the endowments, $(e'_i) \in E_n^*$, were allowed to vary in these idiosyncratic states. This flexibility was required to find a unique pseudo-equilibrium in Claim 2-*(ii)*. Consequently, the generic set of regular values of π in Claim 2-(v) was a sub-set of $E_n^* \times \mathcal{V}$. To simplify exposition, but w.l.o.g., we henceforth consider this set of regular values of π to be a generic subset of $E_n^{**} \times \mathcal{V}$, where $E_n^{**} := \{(e'_i) \in E_n^* : e'_{i\omega} = e_{is},\ \forall i \in I,\ \forall \omega = (s,p) \in \Omega_i^n\}$. That is, endowments are fixed at their original values, in all idiosyncratic states $(\omega \in \Omega^n \backslash \underline{\mathbf{S}})$. This simplification is formal. It is unnecessary for proving Theorem 3, below, but it avoids heavy notations in defining auxiliary equilibria, and for proving Lemma 4.

Claim 2 and Remark 2 yield the following existence result.

Claim 3 *For every $n \in \mathbb{N}$ and every $(\pi_i) \in \mathcal{SB}$, there exist prices, $(p^n, q^n) \in \mathbb{R}_{++}^{L \times \underline{\mathbf{S}}'} \times Q$, endowments, $(e_i^n) \in E_n$, a payoff matrix, $V^n \in \mathcal{V}_n$, and strategies, $(x_i^n, z_i^n) \in B_i^n(p^n, q^n, V^n, e_i^n)$, for each $i \in I$, which define an equilibrium of the economy, $\mathcal{E}_{(\pi_i)}^n$, along Definition 7.*

Proof Let $n \in \mathbb{N}$ and $(\pi_i) \in \mathcal{SB}$ be given. From Claim 2-(iii)-(v) there exist a regular value, $(e^n, V^n) \in E_n \times \mathcal{V}_n$, and a pseudo-equilibrium, $(p^n, L^n, e^n, V^n) \in \pi^{-1}(e^n, V^n)$. From the definitions of regularity and pseudo-equilibria, the relations $L^n = V^n(p^n) \in \mathcal{G}$ hold. As standard (e.g., Duffie-Shafer, p. 289), the pseudo-equilibrium, (p^n, L^n, e^n, V^n), is equivalent to an equilibrium, $\{p^n, q^n, V^n, (e_i^n), (x_i^n), (z_i^n)\}$, along Definition 7. □

Henceforth, we set as given one equilibrium, $\mathcal{C}^n_{(\pi_i)} := \{p^n, q^n, V^n, (e_i^n), (x_i^n), (z_i^n)\}$, in the economy, $\mathcal{E}^n_{(\pi_i)}$, for each $n \in \mathbb{N}$. The sequences, $\{(s, p_s^n)\}$, for $s \in \underline{\mathbf{S}}$, meet the lower bound condition of Lemma 1, as shown in the Appendix, and admit cluster points, whose set is denoted by $\Delta_{(\pi_i)}$. The above structure of beliefs, $(\pi_i) \in \mathcal{SB}$, was generic. We may proceed in the same way as above for all structures of beliefs, $(\pi_i) \in \mathcal{SB}$. This leads to a well defined set, $\Delta^* := \cup_{(\pi_i) \in \mathcal{SB}} \Delta_{(\pi_i)}$, and to the following Assumption.

Assumption A7 *: for every $i \in I$, the relation $\Delta^* \subset \Omega_i$ holds, in which $(\Omega_i) \in \mathcal{AS}$ is the given anticipation structure of the economy, $\mathcal{E}_{(\pi_i)}$.*

We will show that Δ^* is in fact a subset of Δ. Before, we have to prove Theorem 3.

Theorem 3 *Under Assumptions A1 to A7, the economy with arbitrary assets, $\mathcal{E}_{(\pi_i)}$, admits an equilibrium (C.F.E.), for any structure of beliefs, $(\pi_i) \in \mathcal{SB}$.*

Proof First, we set fixed and given an arbitrary structure of beliefs, $(\pi_i) \in \mathcal{SB}$. For each $n \in \mathbb{N}$, an equilibrium, $\mathcal{C}^n_{(\pi_i)} := \{p^n, q^n, V^n, (e_i^n), (x_i^n), (z_i^n)\}$, is well defined from above. Under Assumptions *A1* to *A7*, their sequence, $\{\mathcal{C}^n_{(\pi_i)}\}$, meets the above Assertions of Lemma 3, upon replacing Δ by Δ^*, and of Lemma 4, as is. Both results are demonstrated in the Appendix. Then, Theorem 3 results from the following Claim 4.

Claim 4 *The collection, $\{p^*, q^*, (\Omega_i), (\pi_i), (x_i^*), (z_i^*)\}$, of prices, anticipation sets, beliefs, allocation and portfolios of Lemmas 3-4 is a CFE of the economy $\mathcal{E}_{(\pi_i)}$.*

Proof The proof is identical to that of Claim 1, which we let the reader check. The only difficulty is for proving the relations (IV) of sub-Section 4.3. From Lemma 1, Lemma 2-(iv) and Assumption *A5*, the sequence $\{\|q^n\|\}$ admits positive lower and upper bounds, for an appropriate choice of individual state prices in the auxiliary economies. Then, $\|q^*\| > 0$ and relations (IV) in sub-Section 4.3 hold. The arguments of Claim 1, which all apply, lead to a price $p^* \in \overline{\mathcal{P}}$ (from Definition 7-(d)), which is shown to be an equilibrium price. From Assumption *A2*, this implies $p^* \in \mathcal{P}$. □

It follows from the above proof that any element in Δ^* is an equilibrium forecast, that is, $\Delta^* \subset \Delta$. Hence, Asssumption *A7* can be replaced by *A3*, in Theorem 3.

Appendix

Lemma 1 *Under Assumptions A1-A2, the following Assertions hold:*

(i) $\exists \delta > 0 : \Lambda \subset \mathbb{R}^L_+ \times [\delta, 1]^{L \times \underline{\mathbf{S}}}$, *hence,* $\Delta \subset \underline{\mathbf{S}} \times [\delta, 1]^L$*;*

(ii) *the bound,* δ*, may be chosen independent of* $V \in \mathcal{V}_1$ *and* $(e_i) \in E_1$*.*

Proof First, we introduce new notations and let, for every $(i, s, x) \in I \times \underline{\mathbf{S}} \times \mathbb{R}^{L \times S'_i}_+$:

- $\widetilde{e} \in \mathbb{R}^L_{++}$ have all components equal to $\alpha = \min e'^l_{is} > 0$ for $(i, s, l, (e'_i)) \in I \times S'_i \times L \times E_1$;
- $y \succ^i_s x$ denote a consumption, s.t. $u_i(y_0, y_s) > u_i(x_0, x_s)$ and $y_{s'} = x_{s'}$, $\forall s' \in S'_i \backslash \{s\}$;
- $\mathcal{A} := \{(x_i) \in \times_{i \in I} \mathbb{R}^{L \times S'_i}_+ : \sum_{i \in I} x_{is} = \sum_{i \in I} e_{is}, \ \forall s \in \underline{\mathbf{S}}'\}$;
- $P_s := \{p \in \overline{\mathcal{P}} : \exists j \in I, \ \exists (x_i) \in \mathcal{A}, \text{ such that } (y \succ^j_s x_j) \Rightarrow (p_s \cdot y_s \geqslant p_s \cdot x_{js} \geqslant p_s \cdot \widetilde{e})\}$.

Since all equilibrium prices belong to $\cap_{s \in \underline{\mathbf{S}}} P_s$, from the definition, it suffices to prove that the following Lemmata 1 holds with an independent bound, δ.

Lemmata 1 *The following Assertions hold:*

(i) $\forall s \in \underline{\mathbf{S}}$, P_s *is a closed, hence, compact set;*

(ii) $\exists\ \delta > 0 : \forall (s,l) \in \underline{\mathbf{S}} \times L,\ \forall p := (p^l_{s'}) \in P_s,\ p^l_s \geqslant \delta.$

Proof of Lemmata 1 Assertion (i) From the definition, for each $(n,s) \in \mathbb{N} \times \underline{\mathbf{S}}$ the set P_s contains p^n. Let $s \in \underline{\mathbf{S}}$ and a converging sequence $\{p^k\}_{k\in\mathbb{N}}$ of elements of P_s be given. Its limit, p, is in $\overline{\mathcal{P}}$, a closed set. We may assume there exist (a same) $j \in I$ and a sequence, $\{x^k\}_{k\in\mathbb{N}} := \{(x^k_i)\}_{k\in\mathbb{N}}$, of elements of $\mathcal{A}$, converging to some $x := (x_i)$ in the closure of $\mathcal{A}$ in $\times_{i\in I}(\mathbb{R}_+ \cup \{+\infty\})^{L\times S'_i}$, such that, for each $k \in \mathbb{N}$, (p^k, j, x^k) satisfies the conditions of the definition of P_s. From the definition of $\mathcal{A}$, $\{(x^k_{is'})\}_{k\in\mathbb{N}}$, is bounded, hence, $x_{s'} := (x_{is'}) \in \mathbb{R}^{L\times I}_+$ is finite, for each $s' \in \underline{\mathbf{S}}'$.

For every $k \in \mathbb{N}$, we let $\widetilde{x}^k := (\widetilde{x}^k_i) \in \mathcal{A}$ be defined by $(\widetilde{x}^k_{i0}) := (x_{i0}) \in \mathbb{R}^{L\times I}_+$ and $(\widetilde{x}^k_{is}) := (x_{is}) \in \mathbb{R}^{L\times I}_+$ and $(\widetilde{x}^k_{is'}) := (x^k_{is'})$, for each $(i,s') \in I \times S'_i \backslash \{s\}$. Then, the relations $p^k_s \cdot (x^k_{js} - \widetilde{e}) \geqslant 0$, which hold for every $k \in \mathbb{N}$, yield, in the limit, $p_s \cdot (\widetilde{x}^k_{js} - \widetilde{e}) = p_s \cdot (x_{js} - \widetilde{e}) \geqslant 0$. We now show that there exists $k \in \mathbb{N}$, such that $(p, j, \widetilde{x}^k)$ satisfies the conditions of the definition of P_s (*i.e.*, $p = \lim p^k \in P_s$ and P_s is closed).

By contraposition, assume that, for each $k \in \mathbb{N}$, there exists $y^k \in \mathbb{R}^{L\times S'_j}_+$, such that $y^k_{s'} = \widetilde{x}^k_{js'}$, for each $s' \in S'_j \backslash \{s\}$, $u_j(x_{j0}, y^k_s) > u_j(x_{j0}, x_{js})$ and $p_s \cdot (y^k_s - x_{js}) < 0$. Then, given $k \in \mathbb{N}$, we show the following relations:

$$(I) \qquad \forall K > k,\ \exists k' > K,\ u_j(x^{k'}_{j0}, y^k_s) > u_j(x^{k'}_{j0}, x^{k'}_{js}).$$

If not, one has $u_j(x^{k'}_{j0}, y^k_s) \leqslant u_j(x^{k'}_{j0}, x^{k'}_{js})$, for k' big enough, which implies, in the limit $(k' \to \infty)$, $u_j(x_{j0}, y^k_s) \leqslant u_j(x_{j0}, x_{js})$, in contradiction with the above assumption that $u_j(x_{j0}, y^k_s) > u_j(x_{j0}, x_{js})$. Hence, relations (I) hold. From the definition of the sequence $\{x^k\}_{k\in\mathbb{N}}$, relations (I) imply $p^{k'}_s \cdot (y^k_s - x^{k'}_{js}) \geqslant 0$, and, in the limit $(k' \to \infty)$,

$p_s \cdot (y_s^k - x_{js}) \geqslant 0$, in contradiction with the inequality, $p_s \cdot (y_s^k - x_{js}) < 0$, assumed above. This contradiction proves that $p := \lim p^k \in P_s$, hence, all P_s are compact. $\square$

Assertion (ii) Let $(s,l) \in \underline{\mathbf{S}} \times L$ and $p := (p_{s'}^l) \in P_s$ be given. Let $e \in \mathbb{R}^L$ have zero components but the l^{th}, equal to 1. We prove that $p_s^l = p_s \cdot e > 0$. Indeed, let $(p, j, (x_i)) \in P_s \times I \times \mathcal{A}$ meet the conditions of the definition of P_s. For every $n > 1$, we let $x_j^n \in \mathbb{R}_+^{L \times S'_j}$ be such that $x_{js}^n := (1 - \frac{1}{n})x_{js}$ and $x_{js'}^n := x_{js'}$ for $s' \neq s$. It satisfies $p_s \cdot (x_{js}^n - x_{js}) < 0$ (since $p_s \cdot x_{js} \geqslant p_s \cdot \widetilde{e} > 0$).

Let $E := (E_{s'}^{l'}) \in \mathbb{R}_+^{L \times S'_j}$ be defined by $E_s^l = 1$ and $E_{s'}^{l'} = 0$, for every $(s', l') \neq (s, l)$. Along Assumption *A2*, there exists $n \in \mathbb{N}$, such that $y := (x_j^n + (1 - \frac{1}{n})E)$ satisfies $u_j(y_0, y_s) > u_j(x_{j0}, x_{js})$, implying $p_s \cdot x_{js} \leqslant p_s \cdot y_s = p_s \cdot (x_{js}^n + (1 - \frac{1}{n})e) < p_s \cdot x_{js} + (1 - \frac{1}{n})p_s \cdot e$. Hence, $p_s^l = p_s \cdot e > 0$. The mapping $\varphi_{(s,l)} : P_s \to \mathbb{R}_{++}$, defined by $\varphi_{(s,l)}(p) := p_s \cdot e$ is continuous and attains its minimum for some element $\underline{p}$ on the compact set P_s, say $\delta_{(s,l)} > 0$. Then, Assertion (ii) holds for $\delta := \min \delta_{(s,l)}$, for $(s,l) \in \underline{\mathbf{S}} \times L$. $\square$

Lemmata 1 proves the first part of Lemma 1. Since δ was chosen independent of $V \in \mathcal{V}_1$ and of $(e_i) \in E_1$, the second part of Lemma 1 also holds. $\square$

Lemma 1 also holds for the economy of sub-Section 5.2 above. To see this, for every $s \in \underline{\mathbf{S}}$, and every $n \in \mathbb{N}$, we replace in the definition of the above sets P_s, the price set, $\mathcal{P}$, of Section 2 by those of Section 5, namely, $\mathcal{P}^n := \{p \in \mathbb{R}_{++}^{L \times \underline{\mathbf{S}}'} : p \cdot e_1^n = 1\}$. We let the reader check that Lemma 1 holds by the very same arguments as above (with a bound, δ, which does not depend on $n \in \mathbb{N}$). $\square$

Lemma 2 *For each $(i,n) \in I \times \mathbb{N}$, we may choose the above $\mathcal{P}_i^n$, Ω_i^n, Φ_i^n, such that:*

(i) *$\Omega_i^n \subset \Omega_i^{n+1}$ and $\mathcal{P}_i^{n+1}$ is finer than $\mathcal{P}_i^n$;*

(ii) *$\cup_{n \in \mathbb{N}} \Omega_i^n$ is everywhere dense in Ω_i;*

(iii) for every $\omega \in \Omega_i$*,* $\omega = \lim_{n\to\infty} \Phi_i^n(\omega)$*, and* $\Phi_i^n(\omega)$ *converges uniformly to* ω*;*

(iv) there exist $N \in \mathbb{N}$*, such that* (Ω_i^n) *is arbitrage-free for every* $n \geqslant N$*.*

For simplicity, we henceforth assume that $N = 1$*.*

Proof Let $i \in I$, $n \in \mathbb{N}$ and $K^n := \{1, ..., 2^{n-1}\}^L$ be given (letting $\mathbb{N}$ start from $n = 1$).

From the definition, $\Omega_i := \cup_{s \in S_i} \{s\} \times P_s^i \subset S \times P$. For each pair $(s, k := (k^l)) \in S_i \times K^n$, we define the (possibly empty) subset, $\Omega_{(i,n)}^{(s,k)} := \{s\} \times (P_s^i \cap \times_{l \in L}]\frac{k^l - 1}{2^{n-1}}, \frac{k^l}{2^{n-1}}])$, of Ω_i. To simplify notations, we let $K_{(i,n)} := \# \{(s,k) \in S_i \times K^n : \pi_i(\Omega_{(i,n)}^{(s,k)}) > 0\}$ and identify the latter set, $\{(s,k) \in S_i \times K^n : \pi_i(\Omega_{(i,n)}^{(s,k)}) > 0\}$, to the subset, $\{1, ..., K_{(i,n)}\}$, of $\mathbb{N}$. Then, the partitions, $\mathcal{P}_i^n := \{\Omega_{(i,n)}^k\}_{1 \leqslant k \leqslant K_{(i,n)}}$, of Ω_i are ever finer as $n \in \mathbb{N}$ increases.

For every integer, $k \leqslant K_{(i,n)}$, we choose one element, $\omega_{(i,n)}^k \in \Omega_{(i,n)}^k$, and just one. We may always construct the sets, $\Omega_i^n := \{\omega_{(i,n)}^k\}_{1 \leqslant k \leqslant K_{(i,n)}}$, such that $\Omega_i^n \subset \Omega_i^{n+1}$, for every $n \in \mathbb{N}$. And we define the mapping, Φ_i^n, as in sub-Section 4.1. Then, Assertions (i)-(ii)-(iii) of Lemma 2 hold. □

Assertion (iv): for each $(i,n) \in I \times \mathbb{N}$, we let $Z_i^n := \{z \in \mathbb{R}^J : V(\omega) \cdot z = 0, \forall \omega \in \Omega_i^n\}$ include $Z_i := \{z \in \mathbb{R}^J : V(\omega) \cdot z = 0, \forall \omega \in \Omega_i\}$. Since the sequence $\{(Z_i^n)\}$ is non-increasing in $\mathbb{R}^{J \times I}$, it is stationary. We let the reader check, from Assertion (ii) and the continuity of the scalar product that its limit is (Z_i). Hence, there exists $N \in \mathbb{N}$, such that $(Z_i^n) = (Z_i)$ for every $n \geqslant N$. For simplicity, we assume costlessly that $N = 1$. Then, for every pair $(i,n) \in I \times \mathbb{N}$, we let $Z_i^{n\perp} = Z_i^\perp$ be the orthogonal of $Z_i^n = Z_i$ and $Z := \{(z_i) \in \times_{i \in I} Z_i^\perp : \|(z_i)\| = 1, \ (\sum_{i \in I} \ z_i) \in \sum_{i \in I} Z_i\}$ be a compact set.

Assume, by contraposition, that Assertion (iv) fails. Then, from De Boisdeffre's (2016) Claim 2, for every $n \in \mathbb{N}$, there exist $n' \geqslant n$ and portfolios, $(z_i^{n'}) \in Z$, such that: $V(\omega_i) \cdot z_i^{n'} \geqslant 0$, for every $(i, \omega_i) \in I \times \Omega_i^{n'}$. The sequence, $\{(z_i^{n'})\}$, may be assumed to converge, say to $(z_i) \in Z$. From the continuity of the scalar product, Assertion (ii)

and above, the relations $V(\omega_i)\cdot z_i \geqslant 0$ hold, for every $(i,\omega_i)\in I\times\Omega_i$. The latter imply $(z_i)\in \times_{i\in I} Z_i\cap Z=\varnothing$, from above, and from De Boisdeffre's (2016) Claim 2 jointly with the fact that (Ω_i) is arbitrage-free. This contradiction completes the proof. $\square$

Lemma 3 *For each $i\in I$, we let $Z_i := \{z\in\mathbb{R}^J : V(\omega)\cdot z = 0,\ \forall\omega\in\Omega_i\}$, $Z_i^{\perp}$ be its orthogonal complement and $Z := \sum_{i\in I} Z_i$. Given $\{\mathcal{C}^n\}$, we let $z_i^n = z_i^{on} + z_i^{\perp n}$ be the decomposition of z_i^n on $Z_i\times Z_i^{\perp}$, for each $(i,n)\in I\times\mathbb{N}$. The following Assertions hold:*

(i) the price sequence $\{(p^n,q^n)\}$ may be assumed to converge to $(p^*,q^*)\in\overline{\mathcal{P}}\times Q$, such that $\{(s,p_s^*)\}_{s\in\underline{\mathbf{S}}}\subset\overline{\Delta}\subset(\cap_{i\in I}\Omega_i)$;

(ii) the sequences $\{(x_{is}^n)_{s\in\underline{\mathbf{S}}'}\}$ and $\{(z_i^{\perp n})_{i\in I}\}$ may be assumed to converge, say to $(x_{is}^*)_{s\in\underline{\mathbf{S}}'}$ and $(z_i^{\perp *})\in\mathbb{R}^{J\times I}$, such that $\sum_{i\in I}(x_{is}^* - e_{is})_{s\in\underline{\mathbf{S}}'} = 0$ and $\sum_{i\in I} z_i^{\perp *}\in Z$;

(iii) there exists $(z_i^*)\in\mathbb{R}^{J\times I}$, such that $\sum_{i\in I} z_i^* = 0$ and $(z_i^* - z_i^{\perp *})\in Z_i$ for every $i\in I$.

Proof Assertion (i) is obvious from the definitions, Lemma 1, the relations $p^n\in\Lambda$ (in sub-Section 4.2) for every $n\in\mathbb{N}$, Assumption *A3* and compactness arguments. $\square$

Assertion (ii) The non-negativity and market clearance conditions over auxiliary equilibrium allocations imply that $\{(x_{is}^n)_{s\in\underline{\mathbf{S}}'}\}$ is bounded, hence, may be assumed to converge. The market clearance conditions of equilibrium, $\sum_{i\in I}(x_{is}^n - e_{is})_{s\in\underline{\mathbf{S}}'} = 0$, which hold for each $n\in\mathbb{N}$, yield the limit: $\sum_{i\in I}(x_{is}^* - e_{is})_{s\in\underline{\mathbf{S}}'} = 0$.

By contraposition, assume that there exists an extracted sequence, $\{(z_i^{\perp\varphi(n)})\}$, such that $\lim_{n\to\infty} k_{\varphi(n)} := \|(z_i^{\perp\varphi(n)})\| = \infty$. To simplify, we assume w.l.o.g. that $\varphi(n) = n$ for every $n\in\mathbb{N}$, and we let $\alpha := \sup\|e'\| > 0$, for $e' := (e_i')\in E_1$. From the definition, for every $n\in\mathbb{N}$, the matrix V^n of sub-Section 5.2 is identical to V in all rows except those of states $s\in\underline{\mathbf{S}}$, that is, $V^n(\omega_i^n) = V(\omega_i^n)$ for every $(i,\omega_i^n)\in I\times\Omega_i^n$. Then, for every $n\in\mathbb{N}$, the definition of (Ω_i), the budget constraints and market clearing conditions of the equilibrium, $\mathcal{C}^n$, yield, in both sub-Sections 4.2 and 5.2:

$(\sum_{i\in I}\ z_i^{\perp n}) \in Z$ and $V(\omega_i^n)\cdot z_i^{\perp n} \geqslant -\alpha,\ \forall (i,n,\omega_i^n) \in I\times\mathbb{N}\times\Omega_i^n$.

For every $(i,n) \in I\times\mathbb{N}$, let $z_i'^n := \frac{z_i^{\perp n}}{k_n}$. The bounded sequence $\{(z_i'^n)\}$ admits a cluster point, (z_i), such that $\|(z_i)\| = 1$. The above relations and Lemma 2 yield:

$(\sum_{i\in I}\ z_i'^n) \in Z$ and $V(\omega_i^n)\cdot z_i'^n \geqslant -\alpha/k_n,\ \forall (i,n,\omega_i^n) \in I\times\mathbb{N}\times\Omega_i^n$, and

$(\sum_{i\in I}\ z_i) \in Z$ and $V(\omega_i)\cdot z_i \geqslant 0,\ \forall (i,\omega_i) \in I\times\Omega_i$, when passing to the limit.

The structure $(\Omega_i) \in \mathcal{AS}$ is arbitrage-free, along Definition 2, above. The latter relations, imply $z_i \in Z_i \cap Z_i^\perp = \{0\}$, from De Boisdeffre's (2016) Claim 2 and above, for each $i \in I$. This contradicts the fact that $\|(z_i)\| = 1$. It follows that the sequence, $\{(z_i^{\perp n})\}$, is bounded and may be assumed to converge, say to $(z_i^{\perp *})$, and the above relations, $(\sum_{i\in I}\ z_i^{\perp n}) \in Z$, for all $n \in \mathbb{N}$, pass to the limit, that is, $(\sum_{i\in I}\ z_i^{\perp *}) \in Z$. □

Assertion (iii) is obvious from the definitions and Assertion (ii). □

Lemma 4 *Let* $B_i(\omega,z) = \{x \in \mathbb{R}_+^L : p\cdot(x-e_{is}) \leqslant V(\omega)\cdot z\}$*, be given sets, for every* $z \in \mathbb{R}^J$ *and all* $\omega := (s,p) \in \Omega_i$*. Along Lemma 3, the following Assertions hold for all* $i \in I$:
(i) *the correspondence* $\omega \in \Omega_i \mapsto \arg\max\ u_i(x_{i0}^*, x)$*, for* $x \in B_i(\omega, z_i^*)$*, is a continuous map, whose embedding,* $x_i^* : \omega \in \Omega_i' \mapsto x_{i\omega}^*$*, is a consumption, that is,* $x_i^* \in X_{\Omega_i}$*;*
(ii) $U_i^{\pi_i}(x_i^*) = \lim_{n\to\infty} u_i^n(x_i^n)$.

Proof Assertion (i) Let $i \in I$ be given. We denote simply $\mathcal{C}^n := \{p^n, q^n, (x_i^n), (z_i^n)\}$, for each $n \in \mathbb{N}$, the equilibrium chosen in either sub-Sections 4.2 or 5.2.

To simplify notations, we henceforth let $\varpi := (\omega, z)$ for every $(\omega,z) \in \Omega_i\times\mathbb{R}^J$, we let $\varpi_i^* := (\omega, z_i^{\perp *})$, for every $(i,\omega) \in I\times\Omega_i$ and $\varpi_i^n := (\Phi_i^n(\omega), z_i^{\perp n})$, for every $(i,\omega,n) \in I\times\Omega_i\times\mathbb{N}$.

We recall that in sub-Section 5.2, the relation $V^n(\omega) = V(\omega)$ holds, and we notice that $B_i(\omega,z) = B_i(\omega, z^\perp)$, for every $(n,\omega,z) \in \mathbb{N}\times\Omega_i^n\times\mathbb{R}^J$, where $z^\perp$ is the orthogonal projection of z on $Z_i^\perp := \{z \in \mathbb{R}^J : V(\omega)\cdot z = 0,\ \forall \omega \in \Omega_i\}^\perp$, whereas $B_i(\omega, z_i^*) = B_i(\varpi_i^*)$.

For every $(\omega, n) \in \Omega_i \times \mathbb{N}$, the fact that $\mathcal{C}^n$ is an equilibrium of $\mathcal{E}^n$ (or $\mathcal{E}^n_{(\pi_i)}$) and Assumption *A2* imply: $\{x^n_{i\Phi^n_i(\omega)}\} = \arg\max_{x \in B_i(\varpi^n_i)} u_i(x^n_{i0}, x)$.

Let R be the subset of $\Omega_i \times \mathbb{R}^J$ upon which the correspondence $\varpi \mapsto B_i(\varpi)$ has non-empty values. These values are convex compact from the definition of Ω_i. As standard from Berge Theorem (see, e.g., Debreu, 1959, p. 19) the correspondence (a mapping from Assumption *A2*), $(x_0, \varpi) \in \mathbb{R}^L_+ \times R \mapsto \arg\max_{x \in B_i(\varpi)} \; u_i(x_0, x)$, is continuous, since u_i is continuous, and, from the definition of Ω_i, B_i is also continuous.

From Lemmas 2 and 3, the relations $(x^*_{i0}, \varpi^*_i) = \lim_{n\to\infty}(x^n_{i0}, \varpi^n_i)$ hold for every $(i, \omega) \in I \times \Omega_i$. Hence, from Berge's theorem, the relations, $\{x^n_{i\Phi^n_i(\omega)}\} = \arg\max_{x \in B_i(\varpi^n_i)} u_i(x^n_{i0}, x)$, for $n \in \mathbb{N}$, pass to the limit and yield a continuous map, $\omega \in \Omega_i \mapsto x^*_{i\omega} := \arg\max_{x \in B_i(\varpi^*_i)} u_i(x^*_{i0}, x)$, whose embedding, $x^*_i : \omega \in \{0\} \cup \Omega_i \mapsto x^*_{i\omega}$, is a consumption of the economy $\mathcal{E}_{(\pi_i)}$. □

Assertion (ii) Let $i \in I$ be given and $x^*_i \in X_{\Omega_i}$ be defined from above. By the same token (with same notations as above), we let $\varphi_i : (x_0, \varpi) \in \mathbb{R}^L_+ \times R \mapsto \arg\max_{x \in B_i(\varpi)} \; u_i(x_0, x)$ be defined continuous on its domain. The continuity of u_i implies that of $U_i : (x_0, \varpi) \in \mathbb{R}^L_+ \times R \mapsto u_i(x_0, \varphi_i(x_0, \varpi))$. Moreover, the relations $(x^*_{i0}, \varpi^*_i) = \lim_{n\to\infty}(x^n_{i0}, \varpi^n_i)$, $u_i(x^*_{i0}, x^*_{i\omega}) = U_i(x^*_{i0}, \varpi^*_i)$ and $u_i(x^n_{i0}, x^n_{i\Phi^n_i(\omega)}) = U_i(x^n_{i0}, \varpi^n_i)$ hold, for every $(\omega, n) \in \Omega_i \times \mathbb{N}$. Then, Lemma 2 and the uniform continuity of u_i and U_i on compact sets, yield:

(I) $\qquad \forall \varepsilon > 0,\ \exists N \in \mathbb{N} : \forall n > N,\ \forall \omega \in \Omega_i,\ |\ u_i(x^*_{i0}, x^*_{i\omega}) - u_i(x^n_{i0}, x^n_{i\Phi^n_i(\omega)})\ |\ < \varepsilon.$

Moreover, we recall the following definitions, for every $n \in \mathbb{N}$:

(II) $\qquad U^{\pi_i}_i(x^*_i) := \int_{\omega \in \Omega_i} u_i(x^*_{i0}, x^*_{i\omega}) d\pi_i(\omega);$

(III) $\qquad u^n_i(x^n) := \frac{1}{n\#\underline{\mathbf{S}}} \sum_{s \in \underline{\mathbf{S}}} u_i(x^n_0, x^n_s) + (1 - \frac{1}{n}) \sum_{\omega \in \Omega^n_i} u_i(x^n_0, x^n_\omega)\pi^n_i(\omega).$

Then, Assertion (ii) results immediately from relations (I)-(II)-(III) above. □

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