Growth and Bubbles: The Interplay between Productive Investment and the Cost of Rearing Children

Xavier Raurich, Thomas Seegmuller

Growth and Bubbles: The Interplay between Productive Investment and the Cost of Rearing Children

Xavier Raurich
Thomas Seegmuller

WP 2017 - Nr 26

Growth and bubbles: the interplay between productive investment and the cost of rearing children

Xavier Raurich[*] and Thomas Seegmuller[†‡]

July 17, 2017

Abstract

As it is documented, investment of households in human capital is negatively related to the number of children individuals will have and requires some loans to be financed. We show that this negative relationship contributes to explain episodes of bubbles that are associated to higher growth rates. This conclusion is obtained in an overlapping generations model where agents choose to invest in a productive asset, that can be interpreted as human capital, and decide their number of children. A bubble allows to smooth consumption and expenses over the life-cycle, and can therefore be used to finance either productive investment or the cost of rearing children. The time cost of rearing children plays a key role in the analysis. If the time cost per child is sufficiently large, households have only a small number of children. The bubble then has a crowding-in effect because it is used to finance productive investment. On the contrary, if the time cost per child is low enough, households have a large number of children. Then, the bubble is mainly used to finance the total cost of rearing children and has a crowding-out effect on investment. Therefore, the new mechanism we highlight shows that a bubble enhances growth only if the economy is characterized by a high rearing time cost per child.

JEL classification: E44; G12; J11.

Keywords: Bubble; Sustained growth; Fertility.

[*]Departament de Teoria Econòmica and CREB, Universitat de Barcelona.

[†]Corresponding author. Aix-Marseille University, CNRS, EHESS, Centrale Marseille, AMSE. 5 Boulevard Maurice Bourdet CS 50498 F-13205 Marseille cedex 1, France. E-mail: thomas.seegmuller@univ-amu.fr.

[‡]This work has been carried out thanks to the financial support of the French National Research Agency (ANR-15-CE33-0001-01), the ECO2015-66701-R grant funded by the Government of Spain and the SGR2014-493 grant funded by the Generalitat of Catalonia. We would like to thank participants to the 21st T2M conference, Lisbon March 2017, the 2017 Asian Meeting of the Econometric Society, Hong Kong June 2017, the 16th Journées LAGV, Aix-en-Provence June 2017, The 17th SAET conference, Faro June 2017, for their helpful comments.

1 Introduction

As it is well documented in Caballero *et al.* (2006) or Martin and Ventura (2012), episodes of bubbles are associated with larger growth rates. These empirical facts contradict the results obtained in seminal contributions, where the existence of a bubble has a crowding-out effect on capital accumulation (see Tirole (1985) with exogenous growth or Grossman and Yanagawa (1993) with endogenous growth). Many recent papers have tried to conciliate the theory with the empirics, by providing some relevant explanations on the growth enhancing role of a bubble. Some of the underlying mechanisms are based on the existence of heterogenous productive investments and bubble shocks (Martin and Ventura (2012)), the existence of financial constraints (Fahri and Tirole (2012), Hirano and Yanagawa (2017), Kocherlakota (2009), Miao and Wang (2011)), the difference between liquid and illiquid assets (Raurich and Seegmuller (2015, 2017)), the existence of a bubble on a productive asset (Olivier (2000)), among others.

Some economists have also connected the asset prices to the population size (Abel (2001), Geanakoplos *et al.* (2004), Poterba (2005)). The main effect of population on asset valuation discussed in this literature is based on the number of savers with respect to dissavers. A relative larger number of savers raises the demand of asset, and therefore its price. This idea has in particular been formulated in Geanakoplos *et al.* (2004) considering an overlapping generations model with three-period lived households. They are savers at middle age, while they dissave when old and borrow when young. Therefore, the ratio of the number of middle age over young is a measure of the importance of savers. They show that when this ratio increases, it pushes up the price of the asset. Their theoretical findings are consistent with empirical evidence provided in their paper.

Two additional pieces of evidence allow us to argue that these two debates are not disconnected and there is a link between asset price, growth and demographics. First, there is a trade-off between investing in human capital and having children. Education can be seen as a proxy of human capital investment. As it is highlighted by Martinez *et al.* (2012),[1] women and men with a higher level of education have fewer children. For instance, in US during the period 2006-2010, the average number of children ever born or fathered for women aged 22-44 years is 2.5 when the woman has no high school diploma or equivalent, is 1.8 when she has a high school diploma or equivalent, is 1.5 when she was in some college, and is 1.1 when she has a Bachelor's degree or a higher diploma. We observe the same trend for men. Similar findings can be found in Jones and Tertilt (2008) or Preston and Hartnett (2011). This illustrates that agents investing more in human capital are also those having less children. More generally, it indicates that there exists a link between productive investment done at the beginning of the active life, such as investment in

[1] See especially their Figure 1.

human capital, and the number of children the household will have later.[2]

Second, productive investment requires some loans to finance it. This may be illustrated by the case of student loans in the US. As it is well-known, US student loan has drastically increased since the beginning of 2000's, to become larger than \$1 trillion in 2013 (Avery and Turner (2012), Dynarski (2014)). This amount is far from being negligible in comparison to other types of consumer loans, like auto loan or credit card. Since 2007, annual borrowing from US students is larger than \$100 billions (see College Board (2016), Figure 5). This illustrates that people engaging in an investment when young uses some assets to borrow and finance their investment.

In view of these evidences, we analyze the interplay between productive investment and having children, when these expenses may be financed by a loan. An example of this productive investment is human capital, representing training, education or acquisition of new skills (PhD or Executive MBA for instance). In addition, we consider that the support of the loan is a speculative asset without fundamental value, i.e. is a bubble when it is positively valued. The bubble is the financial instrument that is used to smooth consumption across generations and life-cycle periods.

To address the role played by a bubble on growth, through its effects on productive investment and fertility, we develop an overlapping generations model with three-period lived agents. A household invests when young and chooses the number of children when adult, facing a time cost of rearing children. She is working when young and adult, while she is engaged in home production when old. In addition, each household is a trader of the bubbly speculative asset when young and adult. Finally, firms produce the final good using an Ak technology to have sustained growth.

Our main concern is to investigate and understand why the existence of a bubble may promote growth. In such a case, we say that the bubble is productive. Comparing the bubbleless and bubbly balanced growth paths (BGP),[3] the bubble is productive when the time cost of rearing a child is high enough. Indeed, when the time cost per child is low, the number of children is large and the total cost of rearing children is also large. As a consequence, the bubble is used to transfer wealth from young to middle-age, implying less productive investment. The bubble has a crowding-out effect. Growth is lower at the bubbly than at the bubbleless BGP. On the contrary, when the time cost of rearing a child is high, the number of children is low and the total cost of rearing children is also small. As a consequence, the bubble is used to transfer resources from adult to young households. In this case, young agents are short-sellers of the bubble, that is used to increase their productive investment. The bubble has a crowding-in effect, which means that it is productive. Finally, we show that the bubble can be productive even if the young households are not short-sellers of the bubble. This situation happens when the time cost of rearing a

[2]This link may be more complex than a negative relationship. For instance, Hazan and Zoabi (2015) highlight a non-monotonic relationship between fertility and women education.

[3]As we will see in Section 5, the bubbly BGP is in fact asymptotic.

child takes intermediate values. In this case, the bubble raises capital because a larger amount of productive investment is required to finance savings at the middle-age.

Our model also allows us to contribute to the debate on the link between population size and the asset price. We associate the asset price to the bubble on the speculative asset and a larger ratio of adult over young households means that the main buyers of the speculative asset are relatively more. The comparison of the bubbly and bubbleless BGPs allows us to observe the same link than Geanakoplos *et al.* (2004) find between the asset price and the demography, i.e. the BGP associated with the larger value of the asset (the bubbly one) is also characterized by a larger ratio of adult over young households. However, our result should be interpreted in a different way: in our framework with endogenous fertility, the fall of the price of the speculative asset that follows a market crash of the bubble explains the smaller ratio of adult over young rather than the opposite.

This paper is organized as follows. In the next section, we present the model. In Section 3, we analyze the economy without bubble. In Section 4, we characterize an equilibrium with a bubble. In Section 5, we show the existence of an asymptotic bubbly BGP. We analyze whether a bubble may be productive in Section 6. In Section 7, we interpret our results according to the crowding-out versus crowding-in effects. Concluding remarks are provided in Section 8, while some technical details are relegated to an Appendix.

2 Model

Time is discrete ($t = 0, 1, ..., +\infty$) and there are two types of agents, households and firms.

2.1 Firms

Aggregate output is produced by a continuum of firms, of unit size, using labor, L_t, and capital, K_t, as inputs. This capital can be interpreted broadly to include human capital (training, master programs). In addition, production benefits from an externality that summarizes a learning by doing process and allows to have sustained growth. Following Frankel (1962) or Ljungqvist and Sargent (2004, chapter 14), this externality depends on the average capital-labor ratio.

Letting $k_t \equiv K_t/L_t$, $\bar{k}_t$ represents the average capital-labor ratio. Firms produce the final good with the following technology:

$$Y_t = F(K_t, \bar{k}_t L_t)$$

The technology $F(K_t, \bar{k}_t L_t)$ has the usual neoclassical properties, i.e. is a strictly increasing and concave production function satisfying the Inada conditions, and is homogeneous of degree one with respect to its two arguments.

Profit maximization under perfect competition implies that the wage w_t and the return of capital q_t are given by[4]

$$w_t = F_2(K_t, \bar{k}_t L_t)\bar{k}_t \quad (1)$$
$$q_t = F_1(K_t, \bar{k}_t L_t) \quad (2)$$

2.2 Households

We consider an overlapping generations economy populated by agents living for three periods. An agent is young at the first period of life, adult at the second period and old at the third period. As it is argued by Geanokoplos *et al.* (2004), such a demographic structure is a reasonable representation of the households' life-cycle.

Each household obtains utility from consumption at each period of time and from the number of children when she is an adult. Preferences of an individual born in period t are represented by the following utility function:

$$\ln c_{1,t} + \beta \ln c_{2,t+1} + \beta^2 \ln(c_{3,t+2} + \epsilon) + \mu \ln n_{t+1} \quad (3)$$

where $\beta \in (0,1)$ is the subjective discount factor, $\mu \in (0,1)$ measures the love for children and n_{t+1} is the number of children. At each age, the household can consume the market good produced by firms. Hence, $c_{j,t}$ amounts for consumption of this good when young ($j = 1$), adult ($j = 2$) and old ($j = 3$). We also assume that when old, the household has a consumption of a home-produced good $\epsilon > 0$, which is, to simplify, a perfect substitute of the market good.[5] As explained by Aguiar and Hurst (2005), Hurst (2008) and Schwerdt (2005), home production at the retirement age is quite a realistic feature since it seems to solve the consumption puzzle when households are retired.[6]

While each household uses her time endowment for home production when old, she supplies labor to firms when young and adult. When young, this labor supply is one unit. In contrast, labor supply is endogenous when adult. Indeed, each household has n_{t+1} children at the end of the first period and faces a rearing cost ψw_{t+1} per child in terms of the consumption good in middle-age, where $0 < \psi < 1$ measures the fraction of time an adult spends for each child. Having children takes time when adult. Such a specification is often used in the literature on fertility (de la Croix and Doepke (2003), Galor (2005)).

When young, the household invests an amount a_{t+1} in a productive asset. When it corresponds to an investment in human capital, it represents training, education or acquisition of new skills (PhD or Executive MBA for instance). It provides some adding returns $q_{t+1}a_{t+1}$ at the period where the agent is working for the market good sector, i.e. when she is adult, but no return when she

[4]We denote by $F_i(.,.)$ the derivative with respect to the ith argument of the function.

[5]For home-produced good, the reader can refer to Gronau (1986) for a survey and Benhabib *et al.* (1991) for a macro-model with home production.

[6]This puzzle is based on the observation that consumption expenditures are lower when individuals are retired, but the amount of consumption of goods is not lower.

is retired, i.e. when old.[7] This means that while the wage is the only income when young, the return of productive investment is a source of income complementary to the wage in the middle-age.[8]

Households may also invest in the speculative asset when young ($b_{1,t}$) and adult ($b_{2,t+1}$). This asset is supplied in one unit and $b_{i,t}$ represents the value of the share of this asset bought ($b_{i,t} > 0$) or sold ($b_{i,t} < 0$) by a young or an adult household. Since this asset has no fundamental value, it is a bubble if it has a strictly positive price. Given that the supply is in one unit, the price $\widehat{B}_t$ is given by the sum of the values of the shares hold by households living at the same time, $\widehat{B}_t = N_t b_{1,t} + N_{t-1} b_{2,t}$. When $\widehat{B}_t = 0$ and $b_{1,t} = b_{2,t} = 0$, the price of the speculative asset is zero and there is no bubble.

Accordingly, the budget constraints when young, middle age and old faced by a household born in period t are, respectively:

$$c_{1,t} + a_{t+1} + b_{1,t} = w_t \tag{4}$$

$$c_{2,t+1} + b_{2,t+1} = q_{t+1} a_{t+1} + R_{t+1} b_{1,t} + w_{t+1}(1 - \psi n_{t+1}) \tag{5}$$

$$c_{3,t+2} = R_{t+2} b_{2,t+1} \tag{6}$$

where $0 < n_{t+1} < 1/\psi$. Population size of the generation born at period t is N_t. Therefore, the evolution of the population size of the successive generations is given by $N_{t+1} = n_{t+1} N_t$.

2.3 Equilibrium

At the symmetric equilibrium, $k_t = \bar{k}_t$. Let us define $\alpha \equiv F_1(1,1)/F(1,1) \in (0,1)$ the capital share in total production and $A \equiv F(1,1) > 0$. Using (1) and (2), we deduce that:

$$w_t = (1 - \alpha) A k_t \tag{7}$$

$$q_t = \alpha A \tag{8}$$

Equilibrium in the labor market requires that:

$$L_t = N_t + N_{t-1}(1 - \psi n_t) = N_t(1 - \psi) + N_{t-1} \tag{9}$$

Equilibrium in the capital market is satisfied if $N_t a_{t+1} = K_{t+1}$. Using (9), we get:

$$a_{t+1} = \rho(n_{t+1}) k_{t+1} \tag{10}$$

with:

$$\rho(n_{t+1}) \equiv 1 + (1 - \psi) n_{t+1} \tag{11}$$

[7] This form of capital fully depreciates after one period of use.

[8] This assumption on the returns makes the investment, a_{t+1}, compatible with human capital.

3 The economy without bubble

We first analyze the model without bubble, i.e. $b_{1,t} = b_{2,t} = 0$. It corresponds to our benchmark case and it will allow us to compare the properties of equilibria with and without bubble.

3.1 Household's choices

Households smooth consumption between young and middle-ages using a_{t+1}. Maximizing utility (3) under the budget constraints (4)-(6) with $b_{1,t} = b_{2,t+1} = 0$, we obtain the level of productive investment and the number of children:

$$a_{t+1} = \frac{1}{1+\mu+\beta}\left[(\mu+\beta)w_t - \frac{w_{t+1}}{q_{t+1}}\right] \tag{12}$$

$$n_{t+1} = \frac{\mu}{1+\mu+\beta}\frac{w_t + w_{t+1}/q_{t+1}}{\psi w_{t+1}/q_{t+1}} \tag{13}$$

On the one hand, one can easily understand why productive investment, which corresponds to saving in the young age, increases with the wage earned when young and decreases with the present value of the wage received when adult. On the other hand, the number of children increases with the lifetime income, $w_t + w_{t+1}/q_{t+1}$, but decreases with the discounted value of the rearing cost of having children $\psi w_{t+1}/q_{t+1}$. As a result, the number of children decreases following an increase of the wage growth factor w_{t+1}/w_t.

3.2 Bubbleless BGP

Let us denote by $\gamma_{t+1} \equiv k_{t+1}/k_t$ the growth factor of the capital-labor ratio. Using (12) and (13), we obtain:

$$\gamma_{t+1} = \alpha A \frac{\psi[\mu+\beta(1-\alpha)] - \alpha\mu}{\psi(1+\alpha\beta)+\alpha\mu} \equiv \gamma_{wb} \tag{14}$$

where $\gamma_{wb} > 0$ if and only if $\psi > \alpha\mu/[\mu+\beta(1-\alpha)] \equiv \psi_{0wb}$. Note that the economy immediately jumps on this BGP. Using (13), (7) and (8), we also have:

$$n_{t+1} = \frac{\mu}{\psi(1+\mu+\beta)}\left(1+\frac{\alpha A}{\gamma_{t+1}}\right) \tag{15}$$

Since the wage increases with the capital-labor ratio and the number of children decreases with the wage growth factor, population growth reduces with the growth factor γ_{t+1}. As it is illustrated by Galor (2005), this negative relationship is in accordance with what we observe in Western Europe and in US, Australia, New Zealand and Canada since one century.

Substituting (14) in (15), we obtain the population growth factor at the BGP:

$$n_{wb} = \frac{\mu}{\psi[\mu+\beta(1-\alpha)] - \alpha\mu} \tag{16}$$

Note that $n_{wb} < 1/\psi$ if and only if $\psi > \psi_{nwb}$, which is larger than ψ_{0wb} and smaller than 1 if $\mu < \mu_{nwb}$, with:

$$\psi_{nwb} \equiv \frac{\alpha\mu}{\beta(1-\alpha)} \text{ and } \mu_{nwb} \equiv \frac{\beta(1-\alpha)}{\alpha} \tag{17}$$

We deduce the following proposition:

Proposition 1. *Without bubble ($b_{1,t} = b_{2,t} = 0$), the economy immediately jumps on the equilibrium $\gamma_{t+1} = \gamma_{wb}$ and $n_{t+1} = n_{wb}$, where $n_{wb} < 1/\psi$ if and only if $1 > \psi > \psi_{nwb}$ and $\mu < \mu_{nwb}$.*

Without bubble, there is no transitional dynamics. This is a standard property of models with endogenous growth and an Ak technology. We also note that there is sustained growth, $\gamma_{wb} > 1$, if the productivity A is high enough.

4 Equilibrium with a bubble

When there is a bubble ($b_{1,t} \neq 0$ and/or $b_{2,t} \neq 0$), a household maximizes her utility (3) under the budget constraints (4)-(6) taking into account that she can also smooth consumption using the speculative asset. By arbitrage, we get $R_{t+1} = q_{t+1}$. Then, the number of children and the consumptions over the life-cycle are given by:

$$n_{t+1} = \frac{\mu}{1+\beta+\beta^2+\mu} \frac{w_t + \frac{w_{t+1}}{R_{t+1}} + \frac{\epsilon}{R_{t+1}R_{t+2}}}{\psi\frac{w_{t+1}}{R_{t+1}}} \tag{18}$$

$$c_{1t} = \frac{1}{1+\beta+\beta^2+\mu}\left(w_t + \frac{w_{t+1}}{R_{t+1}} + \frac{\epsilon}{R_{t+1}R_{t+2}}\right) \tag{19}$$

$$c_{2t+1} = \frac{\beta R_{t+1}}{1+\beta+\beta^2+\mu}\left(w_t + \frac{w_{t+1}}{R_{t+1}} + \frac{\epsilon}{R_{t+1}R_{t+2}}\right) \tag{20}$$

$$c_{3t+2} = \frac{\beta^2 R_{t+1}R_{t+2}}{1+\beta+\beta^2+\mu}\left(w_t + \frac{w_{t+1}}{R_{t+1}}\right) - \frac{1+\beta+\mu}{1+\beta+\beta^2+\mu}\epsilon \tag{21}$$

Since when old, the market good produced by the firms is substitutable to a home-produced good, we should take care that c_{3t+2} is non-negative. However, as we will see later, the growing disposable income $w_t + w_{t+1}/R_{t+1}$ ensures a positive consumption $c_{3t+2} > 0$ at the long run equilibrium with sustained growth we are interested in.

Using (4), (6), (10), (19) and (21), we deduce that:

$$b_{1t} = \frac{\beta+\beta^2+\mu}{1+\beta+\beta^2+\mu}w_t - \frac{1}{1+\beta+\beta^2+\mu}\left(\frac{w_{t+1}}{R_{t+1}} + \frac{\epsilon}{R_{t+1}R_{t+2}}\right) - \rho(n_{t+1})k_{t+1} \tag{22}$$

$$b_{2t+1} = \frac{\beta^2 R_{t+1}}{1+\beta+\beta^2+\mu}\left(w_t + \frac{w_{t+1}}{R_{t+1}}\right) - \frac{\epsilon}{R_{t+2}}\frac{1+\beta+\mu}{1+\beta+\beta^2+\mu} \tag{23}$$

Because of the budget constraint (4), we observe that the share of the bubble bought by a young household decreases with productive investment and consumption at the first period of life. The first component means a negative effect of $\rho(n_{t+1})k_{t+1}$ on bubble holding when young, while the second one implies that b_{1t} decreases with the future wage because consumption when young linearly raises with the life-cycle income (see equation (19)). By inspection of equation (23), we deduce that since the share of the bubble hold in middle-age is the saving when adult, it increases with the disposable income $w_t + w_{t+1}/R_{t+1}$. Because the wage linearly increases with capital over labor, it explains that b_{2t+1} will increase with k_{t+1}, but at a lower rate. Finally, using equation (18), the number of children raises with the life-cycle income, but decreases with the discounted value of the rearing cost $\psi w_{t+1}/R_{t+1}$. Therefore, a higher wage when adult negatively affects the number of children.

These relations will allow us to deduce that b_{1t}/k_t, b_{2t}/k_t and n_{t+1} are decreasing with the growth factor γ_{t+1}. As we explained in the economy without bubble, the negative relationship between the number of children and the growth factor is observed in many developed countries since the beginning of the last century (Galor (2005)). To properly derive these negative relationships, let us note $\lambda_t \equiv \epsilon/k_t$. Then, using (7) and (8), we determine the shares of the bubble hold by the young and adult households detrended by the capital-labor ratio:

$$\frac{b_{1t}}{k_t} = (1-\alpha)A\frac{\beta+\beta^2+\mu}{1+\beta+\beta^2+\mu} \tag{24}$$

$$- \gamma_{t+1}\left[\rho(\gamma_{t+1},\lambda_{t+1}) + \frac{1}{1+\beta+\beta^2+\mu}\left(\frac{1-\alpha}{\alpha}+\frac{\lambda_{t+1}}{(\alpha A)^2}\right)\right] \equiv \widetilde{b}_1(\gamma_{t+1},\lambda_{t+1})$$

$$\frac{b_{2t}}{k_t} = \frac{\beta^2(1-\alpha)A}{1+\beta+\beta^2+\mu}\left(\frac{\alpha A}{\gamma_t}+1\right) - \frac{\lambda_t}{\alpha A}\frac{1+\beta+\mu}{1+\beta+\beta^2+\mu} \equiv \widetilde{b}_2(\gamma_t,\lambda_t) \tag{25}$$

with $\rho(\gamma_{t+1},\lambda_{t+1}) \equiv \rho(n(\gamma_{t+1},\lambda_{t+1}))$ and

$$n(\gamma_{t+1},\lambda_{t+1}) \equiv \frac{\mu/\psi}{1+\beta+\beta^2+\mu}\left(\frac{\alpha A}{\gamma_{t+1}}+1+\frac{\lambda_{t+1}}{\alpha(1-\alpha)A^2}\right) = n_{t+1} \tag{26}$$

Equilibrium on the speculative asset market means that:

$$N_{t+1}b_{1t+1} + N_t b_{2t+1} = R_{t+1}(N_t b_{1t} + N_{t-1}b_{2t}) \tag{27}$$

where we recall that the value of the bubble at time t is given by $\widehat{B}_t = N_t b_{1t} + N_{t-1}b_{2t}$. Let $B_t \equiv \widehat{B}_t/\left(k_t N_{t-1}\right)$. Using $R_{t+1} = q_{t+1}$, (8), (24)-(26), the evolution of the population size and the definition of λ_t, an intertemporal equilibrium is defined by:

$$B_t = n(\gamma_t,\lambda_t)\widetilde{b}_1(\gamma_{t+1},\lambda_{t+1}) + \widetilde{b}_2(\gamma_t,\lambda_t) \tag{28}$$

$$B_{t+1} = \frac{\alpha A}{n(\gamma_t,\lambda_t)\gamma_{t+1}}B_t \tag{29}$$

$$\lambda_{t+1} = \frac{1}{\gamma_{t+1}}\lambda_t \tag{30}$$

This system drives the dynamics of $(\gamma_t, B_t, \lambda_t) \in \mathbb{R}^3_{++}$ for all t and allows us to study the long run equilibrium. To study the transitional dynamics, it is important to note that B_t, which represents the price of the speculative asset detrended by the capital-labor ratio times the population size, is determined by expectations on the future and there is a bubble if and only if $B_t > 0$. Therefore, this variable is not predetermined. The growth factor γ_t is also a non-predetermined variable, because it is a function of the capital-labor ratio $k_t(= \gamma_t k_{t-1}) = K_t/L_t$ and $L_t = N_t(1-\psi) + N_{t-1}$ is not predetermined as it depends on the endogenous number of children $n_t = N_t/N_{t-1}$ chosen at period t. On the contrary, because k_t implies that γ_t is not predetermined, $\lambda_t = \epsilon/k_t$ is predetermined.

5 Asymptotic bubbly BGP

We focus on equilibria with sustained growth and a bubble growing at the same rate than capital, i.e. with a positive and constant value of B. Along such a dynamic path, $\lambda_t = \epsilon/k_t$ decreases and tends to 0 when time tends to $+\infty$. The dynamic system (28)-(30) admits no steady state, but may converge to a long run equilibrium with $\lambda^* = 0$, without attaining it. Such an equilibrium will correspond to an asymptotic BGP with a positive bubble and sustained growth. Using equations (28)-(30), it is a stationary solution $(\gamma^*, B^*, \lambda^*)$ satisfying $\lambda^* = 0$ and:

$$\gamma^* n(\gamma^*, 0) = \alpha A \tag{31}$$

$$B^* = n(\gamma^*, 0)\widetilde{b}_1(\gamma^*, 0) + \widetilde{b}_2(\gamma^*, 0) > 0 \tag{32}$$

Using equations (26) and (31), the growth factor γ^* is given by:

$$\gamma^* = \alpha A \frac{\psi(1+\beta+\beta^2+\mu) - \mu}{\mu} \tag{33}$$

The inequality $\psi > \mu/(1+\beta+\beta^2+\mu)$ ensures that $\gamma^* > 0$. Then, there is sustained growth ($\gamma^* > 1$) if and only if:

$$A > \frac{\mu}{\alpha[\psi(1+\beta+\beta^2+\mu) - \mu]} \equiv A_1 \tag{34}$$

Using (24), (25), (31) and (33), the population growth factor is given by:

$$n(\gamma^*, 0) = \frac{\alpha A}{\gamma^*} = \frac{\mu}{\psi(1+\beta+\beta^2+\mu) - \mu} \equiv n^* \tag{35}$$

and the shares of the bubble bought when young and adult by:

$$\widetilde{b}_1(\gamma^*, 0) = (1-\alpha)A - \frac{\psi}{\mu} A \left[1 + \alpha(\beta+\beta^2)\right] \equiv \widetilde{b}_1^* \tag{36}$$

$$\widetilde{b}_2(\gamma^*, 0) = \frac{\psi\beta^2(1-\alpha)A}{\psi(1+\beta+\beta^2+\mu) - \mu} \equiv \widetilde{b}_2^* \tag{37}$$

We deduce that the detrended value of the bubble is:

$$B^* = n^* \widetilde{b}_1^* + \widetilde{b}_2^* = \frac{A\left[1 + \alpha\beta + (2\alpha - 1)\beta^2\right]}{\psi(1 + \beta + \beta^2 + \mu) - \mu}(\psi_b - \psi) \tag{38}$$

with

$$\psi_b \equiv \mu \frac{1 - \alpha}{1 + \alpha\beta + (2\alpha - 1)\beta^2} \tag{39}$$

Then, we can show the following:

Proposition 2. *Let*

$$\underline{\psi} \equiv \frac{\mu}{1 + \beta + \beta^2} \tag{40}$$

Under $\alpha < \frac{\beta + 2\beta^2}{1 + 2\beta + 3\beta^2}$*, there is a unique asymptotic BGP with sustained growth (*$\gamma^* > 1$*,* $n^* < 1/\psi$*) and a positive bubble (*$B^* > 0$*) if* $A > A_1$ *and* $\underline{\psi} < \psi < \min\{\psi_b; 1\}$*. In addition, the equilibrium converges to this BGP.*

Proof. See Appendix A. □

This proposition establishes the existence of a unique asymptotic BGP with a positive bubble and sustained growth. Of course, there is sustained growth under a sufficiently high productivity A, but it also requires a high enough ψ (see equation (33)). Indeed, a high time cost per child ψ reduces the incentive to have children. This implies that only a small amount of adult time is devoted to the total time cost of rearing children ψn^* (see equation (35)) and a large part of household resources can be used to invest in the productive asset, which promotes growth.

As we have seen previously, the shares of the bubble hold by both young and middle-age households, and the number of children decrease with the growth factor. This implies that a positive bubble requires a not too large growth. As a result, the cost ψ should not be too high.

We now investigate more deeply the properties of this asymptotic bubbly BGP. We start by focusing on whether young and middle-age households are buyer ($\widetilde{b}_i^* > 0$) or rather short-sellers ($\widetilde{b}_i^* < 0$) of the speculative asset. By direct inspection of equation (37), there is no doubt that $\widetilde{b}_2^* > 0$ at the bubbly BGP. Adult households use the bubble to transfer purchasing power to their last period of life. In contrast, whether young agents are buyers or short-sellers of the bubble needs a deeper analysis (see equation (36)).

Corollary 1. *Let*

$$\widehat{\psi} \equiv \frac{(1 - \alpha)\mu}{1 + \alpha(\beta + \beta^2)} \tag{41}$$

Under $A > A_1$*, the asymptotic BGP with sustained growth and positive bubble is characterized by the following:*

1. *If* $\frac{\beta+\beta^2}{1+2\beta+2\beta^2} \leqslant \alpha < \frac{\beta+2\beta^2}{1+2\beta+3\beta^2}$*, young agents are short-sellers (*$\widetilde{b}_1^* < 0$*) for all* $\underline{\psi} < \psi < \min\{\psi_b; 1\}$*;*

2. *If* $\alpha < \frac{\beta+\beta^2}{1+2\beta+2\beta^2}$*, young agents are short-sellers (*$\widetilde{b}_1^* < 0$*) for* $\widehat{\psi} < \psi < \min\{\psi_b; 1\}$*, neither buy nor sell the bubble (*$\widetilde{b}_1^* = 0$*) for* $\psi = \widehat{\psi}$*, and buy the bubble (*$\widetilde{b}_1^* > 0$*) for* $\underline{\psi} < \psi < \widehat{\psi}$*.*

Proof. See Appendix B. □

A direct implication of this result is that the existence of a bubbly BGP does not always require $\widetilde{b}_1^* < 0$. An asymptotic bubbly BGP may exist if the young households buy the bubble and are not short-sellers of this asset to finance productive investment ($\widetilde{b}_1^* > 0$).

Corollary 1 shows that young households are short-sellers of the bubble if either the return of the productive investment αA or the time cost per child ψ are sufficiently large. In the first case, the high return of capital creates an incentive to finance productive investment by selling short the speculative asset. In the second case, the quite large time cost per child incites households to have only a few number of children. Therefore, with the log-linear utility, the total rearing cost, ψn^*, is relatively low (see equation (35)). This low total time cost of rearing children incites the households to borrow when young to foster productive investment and redistributes income from the middle to the young age. On the contrary, in the second configuration of the corollary, if ψ is low, the total time cost ψn^* is relatively high. Then, young households buy the bubble to finance this rearing cost of having children when adult.

6 Is a bubble productive?

We analyze now whether a bubble is productive. We say that it is productive when the growth factor at the asymptotic bubbly BGP is higher than at the bubbleless one. In this case, the positive valuation of the bubble raises the growth rate.

For such an analysis, the conditions for the existence of a bubbleless BGP, i.e. $1 > \psi > \psi_{nwb}$ and $\mu < \mu_{nwb}$, and those for the existence of an asymptotic bubbly BGP, i.e. $\alpha < \frac{\beta+2\beta^2}{1+2\beta+3\beta^2}$, $A > A_1$ and $\underline{\psi} < \psi < \min\{\psi_b; 1\}$, should all be satisfied and should not imply an empty admissible interval for some parameters. Then, we could compare the asymptotic bubbly and bubbleless BGPs. We show the following:

Proposition 3. *Let*

$$\psi_p \equiv \frac{\mu[(1-\alpha)(1+\mu+\beta) - \alpha\beta^2]}{(1+\alpha\beta)(1+\beta+\beta^2+\mu)} \tag{42}$$

If $A > A_1$, $\mu < \mu_{nwb}$ and $\alpha \leqslant \frac{\beta}{1+2\beta+\beta^2}$, we have:

1. *For $\max\{\underline{\psi}; \psi_{nwb}\} < \psi \leqslant \psi_p$, the growth factor at the asymptotic bubbly BGP is lower than at the bubbleless BGP ($\gamma^* \leqslant \gamma_{wb}$);*

2. *For $\psi_p < \psi < \min\{\psi_b; 1\}$, the growth factor at the asymptotic bubbly BGP is strictly larger than at the bubbleless BGP ($\gamma^* > \gamma_{wb}$).*

Proof. See Appendix C. □

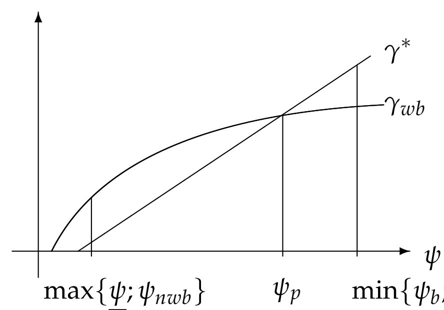

Figure 1 – Growth with versus without bubble

As it is illustrated on Figure 1, Proposition 3 shows that when the time cost per child ψ is relatively low, the economy without bubble is characterized by a higher growth rate than the economy with a bubble. On the contrary, when the time cost per child ψ is relatively high, the economy with bubble is characterized by a higher growth rate than the economy without bubble. In other words, if the time cost per child ψ is lower than the threshold ψ_p, the bubble is damaging for growth. If it is larger, the bubble or the positive valuation of the speculative asset is beneficial for growth. This last result is in accordance with the empirical facts underlying that episodes of bubble are associated with larger growth rates, as it is well documented in Caballero *et al.* (2006) or Martin and Ventura (2012).

While seminal papers like Tirole (1985) show that bubbles imply lower capital accumulation, various more recent contributions provide some mechanisms that reconcile the existence of rational bubbles with larger levels of capital (Fahri and Tirole (2012), Kocherlakota (2009), Martin and Ventura (2012), Miao and Wang (2011), Raurich and Seegmuller (2015)). In models with endogenous growth, Grossman and Yanagawa (1993) show that the existence of a bubble reduces the growth rate, but more recent results show that bubbles may be in accordance with higher growth rates. Olivier (2000) highlights that a bubble on equity raises the value of firms, which promotes firm creation and growth. On the contrary, bubbles on unproductive assets have the same effect than in Grossman and Yanagawa (1993). Hirano and Yanagawa (2017) discuss the existence of bubbles in a model with heterogenous investment projects and borrowing constraints. These authors are especially concerned with the interplay between growth and the existence of bubbles according to the degree of financial imperfections.

Our paper differs from these last two contributions. In contrast to Olivier (2000), growth is enhanced by the existence of a bubble even if the bubble is on an unproductive asset. Therefore, the mechanism that allows to have a productive bubble in our framework is different. We also depart from Hirano and Yanagawa (2017) since we do not consider heterogenous investment projects and do not discuss the results according to the level of financial frictions.

By inspection of Corollary 1 and Proposition 3, it is interesting to note that we can have ψ_p lower than $\widehat{\psi}$.[9] This means that for $\psi_p < \psi < \widehat{\psi}$, we can have a higher growth of the capital-labor ratio at the asymptotic bubbly BGP than at the bubbleless one ($\gamma^* > \gamma_{wb}$) even if young households are not short-sellers of the speculative asset. This is in contrast with Raurich and Seegmuller (2015, 2017) who study the existence of rational bubbles in overlapping generations economies with three period-lived households, vintage capital and exogenous growth. In both papers, capital is higher at the bubbly than at the bubbleless steady state only if either young or adult households are short-sellers of the bubble. Here, we obtain a different result, which means that the mechanism for the existence of a productive bubble is different than in these two papers.

Our model with endogenous fertility and rational bubble also allows us to refer to the debate on the link between population size and the value of asset (Abel (2001), Geanakoplos *et al.* (2004), Poterba (2005)), which explains that asset price is higher when the relative number of savers is more significant. In particular, Geanakoplos *et al.* (2004) consider an overlapping generations model with three-period lived agents and identify that the number of savers is relatively more important as the ratio of middle-age over young households is higher.

We contribute to this debate focusing on the price of the speculative asset. Obviously, the price of this asset is larger with the bubble than without. In the next corollary, we consider the range of parameter values for which the bubble is productive and we first make sure that, at the asymptotic bubbly BGP, middle-age households hold a larger amount of the bubble than young ones. Then, observing that the ratio of middle-age over young households is given by $MY_t \equiv N_{t-1}/N_t = 1/n_t$, we denote by MY^* and MY_{wb} the value of this ratio evaluated at the asymptotic bubbly and bubbleless BGPs, respectively, and we compare them:

Corollary 2. *If $A > A_1$, $\mu < \mu_{nwb}$, $\psi_p < \psi < \min\{\psi_b; 1\}$ and $\alpha \leqslant \frac{\beta}{1+2\beta+\beta^2}$, we have $\widetilde{b}_2^* > n^*\widetilde{b}_1^*$, $\gamma^* > \gamma_{wb}$ and $n^* < n_{wb}$, which means that $MY^* > MY_{wb}$.*

[9] We note that when Proposition 3 applies, it corresponds to case 2 of Corollary 1, because $\frac{\beta}{1+2\beta+\beta^2} < \frac{\beta+\beta^2}{1+2\beta+2\beta^2}$. Then, using (41) and (42), $\psi_p < \widehat{\psi}$ requires $\mu < \frac{1-\alpha(1-\alpha)+\alpha^2(\beta+\beta^2)}{\alpha(1-\alpha)}$.

Proof. See Appendix D. □

This corollary ensures first that middle-age households hold a larger amount of the bubble than young agents living at the same period ($\widetilde{b}_2^* > n^* \widetilde{b}_1^*$). Therefore, a higher MY is a relevant measure to argue that more traders buy the speculative asset. Then, comparing the asymptotic bubbly and bubbleless BGPs, we have not only that $\gamma^* > \gamma_{wb}$, but also $MY^* > MY_{wb}$ and the price of the speculative asset is of course larger at the asymptotic bubbly BGP, since it is zero at the bubbleless one. Hence, following a financial crash, which means that agents coordinate their expectations on the bubbleless BGP,[10] we may not only get a lower growth factor, as already discussed in Proposition 3, but the decrease of the price of the speculative asset (the crash of the bubble) occurs at the same time than a decrease in the ratio of middle-age over young households. This corroborates the theoretical and empirical findings of Geanakoplos *et al.* (2004) on the link between demography and asset prices. The basic mechanism they focus on is the following: the lower price of asset is explained by a relative lower number of savers, measured by a smaller ratio of adult over young households, which reduces the demand of asset. In our model, we associate the asset price to the bubble on the speculative asset and a larger ratio of adult over young households means that the main buyers of the speculative asset are relatively more. Even if we observe the same link than Geanakoplos *et al.* (2004) between the asset price and the demography, it should be interpreted in a different way: it is the market crash of the bubble that explains the smaller ratio of middle-age over young rather than the opposite.

7 Crowding-out versus crowding-in effect

To explain the economic mechanism that allows to have larger growth when there is a bubble, we next highlight three main ingredients that relate the explanation to the value of the time cost per child ψ.

1. The speculative asset allows to smooth consumption along the life-cycle. At middle-age, a household buys the bubble ($\widetilde{b}_2^* > 0$) to transfer purchasing power to the old age. Using $\widetilde{b}_1^*$, the household smooths consumptions and incomes between young and adult ages. As already discussed in the previous section, we learn from Corollary 1 and Proposition 3 that a growth enhancing bubble at the BGP is not equivalent to have $\widetilde{b}_1^* < 0$, i.e. young households are short-sellers of the bubble. Indeed, there is a value $\widehat{\psi}$ such that $\widetilde{b}_1^* > 0$ if $\psi < \widehat{\psi}$, $\widetilde{b}_1^* = 0$ if $\psi = \widehat{\psi}$, while $\widetilde{b}_1^* < 0$ if $\psi > \widehat{\psi}$.

[10] A financial crash can be associated to a market crash of the bubble, i.e. to the coordination of agents' expectations on the bubbleless BGP. One can refer to Weil (1987) who introduces a probability of bubble crash according to a sunspot process.

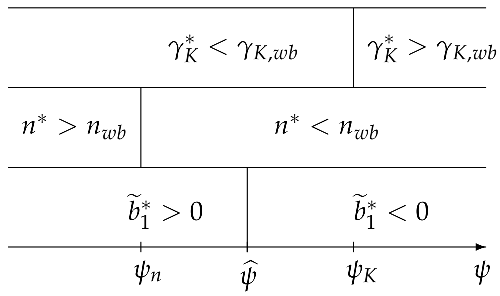

Figure 2 – The interplay between bubble, fertility and capital growth

2. In Corollary 2, we have shown that if $\gamma^* > \gamma_{wb}$, we get $n^* < n_{wb}$. Now, without comparing the growth factors, we deduce, using (16) and (35), that $n^* < n_{wb}$ if and only if $\psi > \psi_n$, with:

$$\psi_n \equiv \frac{\mu(1-\alpha)}{1+\beta(\alpha+\beta)} \tag{43}$$

3. Recall now that γ denotes the growth factor of the capital-labor ratio. It is often more usual to define the growth factor of capital per capita (or GDP per capita). This last one is defined by $K_t/N_t = k_t L_t/N_t = k_t(1-\psi+1/n_t)$. On a BGP, since n is constant, the growth factor of capital per capita is also equal to γ and the growth factor of capital is defined by:

$$\gamma_K \equiv \frac{K_{t+1}}{K_t} = \frac{k_{t+1}}{k_t}\frac{N_{t+1}}{N_t} = \gamma n$$

   Using (31), $\gamma_K = \alpha A \equiv \gamma_K^*$ at the asymptotic bubbly BGP and, using (14) and (16), $\gamma_K = \frac{\mu\alpha A}{\psi(1+\alpha\beta)+\alpha\mu} \equiv \gamma_{K,wb}$ at the bubbleless BGP. We deduce that $\gamma_K^* > \gamma_{K,wb}$ if and only if $\psi > \psi_K$, with:

$$\psi_K \equiv \frac{\mu(1-\alpha)}{1+\alpha\beta} \tag{44}$$

Using (41), (43) and (44), we have that $\psi_n < \widehat{\psi} < \psi_K$.[11] This allows us to draw Figure 2, which is useful to understand why the bubble raises or not

[11] One can further note that under Proposition 3, $\max\{\underline{\psi};\psi_{nwb}\} < \psi_n$ and $\psi_K < \min\{\psi_b;1\}$, which means that the critical values we focus on belong to the interval of ψ considered in Proposition 3.

growth of capital per capita. As follows from this figure, there are three main configurations according to the value of ψ: $\psi < \psi_n$, $\psi_n < \psi < \psi_K$ and $\psi > \psi_K$.

Before studying these different configurations, it is useful to note that since $\gamma = \gamma_K/n$, the threshold value ψ_p above which $\gamma^* > \gamma_{wb}$ belongs to (ψ_n, ψ_K). Of course, $\gamma^* > \gamma_{wb}$ for $\psi \geqslant \psi_K$ and $\gamma^* < \gamma_{wb}$ for $\psi \leqslant \psi_n$.

1. $\psi < \psi_n$: *Crowding-out effect*
   Since ψ is small, we have seen that the total time cost of rearing children ψn is large, whether or not the bubble exists. As a consequence, the labor supply and the income at middle-age are low. When the speculative asset is positively valued, young households buy the bubble to transfer income to the middle-age. Therefore, they invest less in capital when there is a bubble. Growth is lower when the bubble exists. This also reduces the cost of having children ψw_{t+1} in terms of consumption, which explains that population growth is larger at the asymptotic bubbly BGP.

2. $\psi_n < \psi < \psi_K$: *Indeterminate crowding effect*
   Both ψ and ψn^* take intermediate values. The main mechanism at stake is clearly different than in the previous configuration. To understand what happens, let us assume $\psi = \widehat{\psi}$. At the asymptotic bubbly BGP, young households neither buy, nor sell the bubble, i.e. $\widetilde{b}_1^* = 0$. Since $\widetilde{b}_2^* > 0$ to finance consumption when old, a middle-age household reduces the expenditures on children, implying a lower population growth at the asymptotic bubbly BGP. Then, productive investment a can be larger or lower at the asymptotic bubbly BGP than at the bubbleless one because of two opposite effects: on the one hand, fewer children expenses have to be covered; on the other hand, the purchase of the bubble to finance consumption when old has to be financed.

3. $\psi > \psi_K$: *Crowding-in effect*
   As we have seen, since ψ is large enough, ψn is relatively low, whether or not the bubble exists. Therefore, the main mechanism is exactly the opposite one than in the first configuration. In this case, the labor income when adult is large. When the bubble exists, there is a transfer of resources from the adult to the young age. The young household becomes a short-seller of the bubble and increases productive investment. Therefore, growth is larger at the asymptotic bubbly BGP. This larger growth also implies a larger cost of having children ψw_{t+1} in terms of consumption. Therefore, population growth is lower when there is a bubble.

8 Concluding remarks

We develop a model where a speculative asset (bubble) is used to finance two types of expenditures, a productive investment and costs of rearing children. If the time cost per child ψ is low, the household has a large number of children, meaning that the total time cost ψn is high. In this case, the bubble is mainly used to finance rearing children expenses instead of productive capital. Growth is lower when there is a bubble. If ψ is high, we have the opposite. The total time cost of having children ψn is low in this case and the bubble is used to finance productive investment, which enhances growth.

The comparison between equilibria with and without bubble can also be interpreted in terms of financial development. Such an interpretation of our results may contribute to the literature studying the link between financial development and growth.[12] In our framework, the bubbleless equilibrium describes an economy where the only asset is capital, whereas the bubbly one describes an economy where financial markets are more developed since there is an adding traded asset valued on such a market. Accordingly, we can say that the bubbleless economy is characterized by a less significant financial development than the bubbly economy. Under this interpretation, our model suggests that financial development is beneficial for growth only under some circumstances, i.e. if ψ is high enough.

Appendix

A Proof of Proposition 2

We have $n^* < 1/\psi$ if and only if $\psi > \underline{\psi}$, where this inequality ensures that $\psi > \mu/(1+\beta+\beta^2+\mu)$. Then, using (38), we immediately see that $B^* > 0$ if and only if $\psi < \psi_b$. We also note that $\underline{\psi} < 1$ because $\mu < 1$ and $\underline{\psi} < \psi_b$ is equivalent to $\alpha < \frac{\beta+2\beta^2}{1+2\beta+3\beta^2}$. Of course, there is sustained growth because $A > A_1$.

Let us focus now on the stability properties. Even if $(\gamma^*, B^*, 0)$ is an asymptotic BGP, it is important to note that from the mathematical point of view, $(\gamma_t, B_t, \lambda_t) = (\gamma^*, B^*, 0)$ is a steady state of the dynamic system (28)-(30). Therefore, to analyze the stability properties of the equilibrium $(\gamma^*, B^*, 0)$, we use standard mathematical tools.

Differentiating the dynamic system (28)-(30) in the neighborhood of the BGP $(\gamma^*, B^*, 0)$, we obtain a linear system of the following form:

$$\begin{aligned} dB_{t+1} &= a_{11} dB_t + a_{12} d\gamma_t + a_{13} d\lambda_t \\ d\gamma_{t+1} &= a_{21} dB_t + a_{22} d\gamma_t + a_{23} d\lambda_t \\ d\lambda_{t+1} &= a_{33} d\lambda_t \end{aligned}$$

Therefore, the characteristic polynomial can be written $P(\eta) \equiv (a_{33} - \eta)(\eta^2 - T\eta + D) = 0$, with $T = a_{11} + a_{22}$ and $D = a_{11}a_{22} - a_{12}a_{21}$. We immediately deduce that one eigenvalue is given by $\eta_1 = a_{33} = 1/\gamma^* \in (0, 1)$. To determine

[12]See Levine (2005) for a survey and Madsen and Ang (2016) for a recent contribution.

the other eigenvalues, we compute the terms a_{ij}, with $\{i,j\} = \{1,2\}$:

$$a_{11} = 1 - \frac{B^*}{\alpha A \partial \widetilde{b}_1^* / \partial \gamma_{t+1}} \tag{A.1}$$

$$a_{12} = -\frac{B^*}{n^*}\frac{\partial n_t}{\partial \gamma_t} + \frac{B^*}{\alpha A \partial \widetilde{b}_1^* / \partial \gamma_{t+1}}\left(\widetilde{b}_1^* \frac{\partial n_t}{\partial \gamma_t} + \frac{\partial \widetilde{b}_2^*}{\partial \gamma_t}\right) \tag{A.2}$$

$$a_{21} = \frac{\gamma^*}{\alpha A \partial \widetilde{b}_1^* / \partial \gamma_{t+1}} \tag{A.3}$$

$$a_{22} = -\frac{\gamma^*}{\alpha A \partial \widetilde{b}_1^* / \partial \gamma_{t+1}}\left(\widetilde{b}_1^* \frac{\partial n_t}{\partial \gamma_t} + \frac{\partial \widetilde{b}_2^*}{\partial \gamma_t}\right) \tag{A.4}$$

The minor D is given by:

$$D = \frac{\gamma^*}{\alpha A \partial \widetilde{b}_1^* / \partial \gamma_{t+1}}\left(\frac{B^*}{n^*}\frac{\partial n_t}{\partial \gamma_t} - \widetilde{b}_1^* \frac{\partial n_t}{\partial \gamma_t} - \frac{\partial \widetilde{b}_2^*}{\partial \gamma_t}\right)$$

Using $n^* = \alpha A / \gamma^*$ and $B^* = n^* \widetilde{b}_1^* + \widetilde{b}_2^*$, we obtain:

$$D = \frac{1}{(n^*)^2 \partial \widetilde{b}_1^* / \partial \gamma_{t+1}}\left(\widetilde{b}_2^* \frac{\partial n_t}{\partial \gamma_t} - \frac{\partial \widetilde{b}_2^*}{\partial \gamma_t} n^*\right)$$

From (25) and (26), we get:

$$\begin{aligned}
\frac{\partial n_t}{\partial \gamma_t} &= -\frac{\mu/\psi}{1+\beta+\beta^2+\mu}\frac{\alpha A}{\gamma_t^2} \\
\frac{\partial \widetilde{b}_2^*}{\partial \gamma_t} &= -\frac{\beta^2 \alpha A}{1+\beta+\beta^2+\mu}\frac{(1-\alpha)A}{\gamma_t^2}
\end{aligned}$$

Using (35) and (37), we deduce that $D = 0$. This means that one eigenvalue is zero, i.e. $\eta_2 = 0$. Using now (25), (26), (A.1) and (A.4), T rewrites:

$$T = 1 - \frac{B^*}{\alpha A \left(\frac{\alpha A}{\gamma^*} + 1\right) \frac{\partial \widetilde{b}_1^*}{\partial \gamma_{t+1}}}$$

We easily derive from (24) that:

$$\frac{\partial \widetilde{b}_1^*}{\partial \gamma_{t+1}} = -1 - \frac{(1-\psi)\mu/\psi}{1+\beta+\beta^2+\mu} - \frac{1}{1+\beta+\beta^2+\mu}\frac{1-\alpha}{\alpha} < 0$$

We deduce that the last eigenvalue is given by $\eta_3 = T > 1$.

B Proof of Corollary 1

Using (36), we see that $\widetilde{b}_1^* < 0$ if and only if $\psi > \widehat{\psi}$. We further have that $\widehat{\psi} < \psi_b$, $\widehat{\psi} < 1$ is ensured by $\mu < 1$ and $\widehat{\psi} > \underline{\psi}$ if and only if $\alpha < \frac{\beta+\beta^2}{1+2\beta+2\beta^2}$. If this last inequality is not satisfied, we have $\widehat{\psi} \leqslant \underline{\psi}$. Since $\frac{\beta+\beta^2}{1+2\beta+2\beta^2} < \frac{\beta+2\beta^2}{1+2\beta+3\beta^2}$, the corollary immediately follows.

C Proof of Proposition 3

We first take into account the conditions for the existence of a bubbleless BGP, i.e. $\mu < \mu_{nwb}$ and $\psi > \psi_{nwb}$. They are compatible with the conditions for the existence of an asymptotic bubbly BGP if $\psi_{nwb} < \psi_b$, which is satisfied if and only if $\alpha(1+2\beta+\beta^2) - \beta + 2\alpha\beta^2(\alpha-1) < 0$. This last inequality is ensured for $\alpha \leqslant \frac{\beta}{1+2\beta+\beta^2}$, which implies that $\alpha < \frac{\beta+2\beta^2}{1+2\beta+3\beta^2}$. Therefore, the bubbleless and asymptotic bubbly BGP coexist if $\max\{\underline{\psi};\psi_{nwb}\} < \psi < \min\{\psi_b;1\}$.

Using (14) and (33), we note that $\gamma^* > \gamma_{wb}$ if and only if $\psi > \psi_p$. We can further show that $\psi_p < 1$ and $\psi_p < \psi_b$ because $\alpha \leqslant \frac{\beta}{1+2\beta+\beta^2}$. Moreover, Using (17), (40) and (42), we can further show that under $\alpha \leqslant \frac{\beta}{1+2\beta+\beta^2}$, we have $\psi_p > \max\{\underline{\psi};\psi_{nwb}\}$. Indeed, $\psi_p > \underline{\psi}$ is equivalent to:

$$(1+\beta+\beta^2)[\beta - \alpha(1+2\beta+\beta^2)] > \mu[\alpha(1+2\beta+\beta^2) - (\beta+\beta^2)]$$

and $\psi_p > \psi_{nwb}$ to:

$$\mu[\beta - \alpha(1+2\beta)] > (1+\beta)[\alpha(1+2\beta+\beta^2) - \beta]$$

Both these inequalities are satisfied under $\alpha \leqslant \frac{\beta}{1+2\beta+\beta^2}$. This means that $\max\{\underline{\psi};\psi_{nwb}\} < \psi_p < \min\{\psi_b;1\}$ and the proposition follows.

D Proof of Corollary 2

Using (35)-(37), the inequality $\widetilde{b}_2^* > n^*\widetilde{b}_1^*$ is equivalent to $\psi > \psi_n$, with:

$$\psi_n \equiv \frac{\mu(1-\alpha)}{1+\alpha\beta+\beta^2}$$

Using now (42), $\psi_p > \psi_n$ is equivalent to:

$$\beta - \alpha(1+2\beta+\beta^2) + \mu(1-\alpha) > 0$$

which is always satisfied for $\alpha \leqslant \frac{\beta}{1+2\beta+\beta^2}$.

Using Proposition 3, we have $\gamma^* > \gamma_{wb}$. Using (16) and (35), we easily deduce that $n^* < n_{wb}$. This means that $MY^* > MY_{wb}$.

References

[1] Abel, A. B. (2001): ”Will bequests attenuate the predicted meltdown in stock prices when baby boomers retire”, *Review of Economics and Statistics* **83**, 589-595.

[2] Aguiar, M. and E. Hurst (2005): ”Consumption versus expenditure”, *Journal of Political Economy* **113**, 919-948.

[3] Avery, C. and S. Turner (2012): ”Student loans: do college students borrow too much - or not enough?”, *Journal of Economic Perspectives* **26**, 165-192.

[4] Benhabib, J., R. Rogerson and R. Wright (1991): "Homework in macroeconomics: household production and aggregate fluctuations", *Journal of Political Economy* **99**, 1166-1187.

[5] Caballero, R. J., E. Fahri and M. L. Hammour (2006): ”Speculative growth: hints from the U.S. economy”, *American Economic Review* **96**, 1159-1192.

[6] de la Croix, D. and M. Doepke (2003): ”Inequality and growth: why differential fertility matters”, *American Economic Review* **93**, 1091-1113.

[7] College Board (2016): *Trends in Student Aid 2016*, Trends in Higher Education Series.

[8] Dynarski, S. M. (2014): "An economist's perspective on student loans in the United States", Brookings Institution Working Paper.

[9] Fahri, E. and J. Tirole (2012): ”Bubbly liquidity”, *Review of Economic Studies* **79**, 678-706.

[10] Frankel, M. (1962): ”The production function in allocation and growth: a synthesis”, *American Economic Review* **52**, 996-1022.

[11] Galor, O. (2005): "From stagnation to growth: unified growth theory", in *Handbook of Economic Growth*, Aghion, P. and S. N. Durlauf (eds.), North-Holland, Amsterdam, 171-293.

[12] Geanakoplos, J., M. Magill and M. Quinzii (2004): ”Demography and the long-run predictability of the stock market”, *Brookings Papers on Economic Activity* **1**, 241-325.

[13] Gronau, R. (1986): "Home production - a survey", in *Handbook of Labor Economics*, Ashenfelter, O. and R. Layard (eds.), North-Holland, Amsterdam, 274-304.

[14] Grossman, G. M. and N. Yanagawa (1993): ”Asset bubbles and endogenous growth”, *Journal of Monetary Economics* **31**, 3-19.

[15] Hirano, T. and N. Yanagawa (2017): ”Asset bubbles, endogenous growth, and financial frictions”, *The Review of Economic Studies* **84**, 406-443.

[16] Hurst, E. (2008): ”The retirement of a consumption puzzle”, NBER Working Paper 13789.

[17] Jones, L. E. and M. Tertilt (2008): "An economic history of fertility in the US: 1826-1960", in *Frontiers of Family Economics*, Rupert, P. (ed.), Emerald Group Publishing, 165-209.

[18] Kocherlakota, N. (2009): ”Bursting bubbles: consequences and cures”, mimeo, Federal Reserve Bank of Minneapolis.

[19] Levine, R. (2005): "Finance and growth: theory and evidence", in *Handbook of Economic Growth*, Aghion, P. and S. N. Durlauf (eds.), North-Holland, Amsterdam, 865-934.

[20] Ljungqvist, L. and T. J. Sargent (2004). *Recursive Macroeconomic Theory*. Cambridge: MIT Press.

[21] Madsen, J. B. and J. B. Ang (2016): ”Finance-led growth in the OECD since the nineteenth century: how does financial development transmit to growth?”, *The Review of Economics and Statistics* **98**, 552-572.

[22] Martin, A. and J. Ventura (2012): ”Economic growth with bubbles”, *American Economic Review* **102**, 3033-58.

[23] Martinez, G., K. Daniels and A. Chandra (2012): ”Fertility of men and women aged 15-44 years in the United States: national survey of family growth, 2006-2010”, *National Health Statistics Report* **51**.

[24] Miao, J. and P. Wang (2011): ”Bubbles and credit constraints”, Working paper, University of Boston.

[25] Olivier, J. (2000): ”Growth-enhancing bubbles”, *International Economic Review* **41**, 133-151.

[26] Poterba, J. (2005): ” The impact of population aging on financial markets”, in *Global Demographic Change: Economic Impact and Policy Challenges*, G. H. Sellor Jr. (ed.), Federal Reserve Bank of Kansas City, Kansas City, 163-216.

[27] Preston, S. H. and C. S. Hartnett (2011): ” The future of american fertility”, in *Demography and the Economy*, J. Shoven. (ed.), The University of Chicago Press, Chicago, 11-36.

[28] Raurich, X. and T. Seegmuller (2015): ”On the interplay between speculative bubbles and productive investment”, AMSE Working Paper 2015-42.

[29] Raurich, X. and T. Seegmuller (2017): ”Income distribution by age group and productive bubbles”, mimeo.

[30] Schwerdt, G. (2005): “Why does consumption fall at retirement? Evidence from Germany”, *Economics Letters* **89**, 300-305.

[31] Tirole, J. (1985): "Asset bubbles and overlapping generations", *Econometrica* **53**, 1071-1100.

[32] Weil, P. (1987): "Confidence and the real value of money in overlapping generations models", *Quarterly Journal of Economics* **102**, 1-22.