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Secular Satiation

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ABSTRACT

Satiation of need is generally ignored by growth theory. I study a model where consumers may be satiated in any given good but new goods may be introduced. A social planner will never elect a trajectory with long-run satiation. Instead, he will introduce enough new goods to avoid such a situation. In contrast, the decentralized equilibrium may involve long run satiation. This, despite that the social costs of innovation are second order compared to their social benefits.

Multiple equilibria may arise: depending on expectations, the economy may then converge to a satiated steady state or a non satiated one. In the latter equilibrium, capital and the number of varieties are larger than in the former, while consumption of each good is lower. This multiplicity comes from the following strategic complementary: when people expect more varieties to be introduced in the future, this raises their marginal utility of future consumption, inducing them to save more. In turn, higher savings reduces interest rates, which boosts the rate of innovation.

When TFP grows exogenously and labor supply is endogenized, the satiated equilibrium generically survives. For some parametrer values, its growth rate is positive while labor supply declines over time to zero. Its growth rate is then lower than that of the non satiated equilibrium. Hence, the economy may either coordinate on a high leisure, low growth, satiated "leisure society" or a low leisure, high growth, non satiated "consumption society".

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1 Introduction

Limits to growth are generally considered to lie on the supply side: limited resources, running out of good ideas, etc. Yet as societies become more affluent, the question arises as to whether consumers will be able to absorb an ever increasing output level. Indeed we observe that over time growth becomes increasingly qualitative as opposed to quantitative. Productivity improvements are matched by greater product quality and variety as well as a secular decline in working time.

This paper analyzes growth dynamics when needs may be subject to satiation. I assume that utility derived from any individual good reaches a maximum at a finite consumption level. An immediate implication is that there may be two kinds of long-term steady states: a non satiated one where the capital stock is determined by the standard Keynes-Ramsey condition, and a satiated one where the marginal utility of consumption is zero, and there may be excess capital (in that its net marginal product is negative) in the long run.

I then allow for new products to be introduced. While each product is subject to satiation, there is no satiation in the taste for variety: utility can potentially be raised without bounds if enough new goods are introduced. I show that a social planner will never elect a trajectory which converges to a satiated steady state. In such a steady state, the marginal utility of consumption is zero, and so is the opportunity cost of introducing an additional variety, which in turn has a first order positive effect on utility. I then consider a decentralized economy where, in a standard fashion, new varieties are introduced by profit-seeking innovators, who derive subsequent monopoly rents. It is shown that the economy may converge to satiated steady states. At the margin of those equilibria, the market does not deliver the required innovation level to lift the economy out of its satiation trap, despite that the social opportunity cost of innovation is zero. The associated market failure goes beyond the usual appropriability and business stealing issues that the literature has identified since Dixit and Stiglitz (1977). Relative prices are pinned down by the production side, and equal to marginal rates of transformation. But marginal rates of substitution are not defined since the marginal utility of consuming any variety is zero. The price system does not convey the information that the marginal utility of consumption is zero. A reduction in the consumption of existing goods entails second-order welfare losses, while affecting these resources to innovation generates first-order welfare gains. By pricing innovation using prevailing prices, however, markets evaluate costs and benefits as being of the same order of magnitude; they do not internalize the facts that new products would eliminate satiation and restore a strictly positive marginal utility of consumption, while absent innovation opportunity costs remain second order.

The model also highlights an important strategic complementary, which generates multiple equilibrium trajectories. Unlike vertical innovation, horizontal innovation raises the marginal utility of aggregate consumption. Consequently, expecting more varieties to be introduced in the future raises the incentives to save. Higher savings in turn reduces interest rates, which favors innovation. I show that equilibrium trajectories that converge to a non satiated equilibrium where the modified golden rule holds coexist with trajectories that converge to satiated equilibria. In the latter, the capital stock and the number of varieties are lower than in the former, while interest rates are higher and wages are lower.

The model can be extended in several ways. I discuss the consequences of consumer heterogeneity in rates of time preference, characterize balanced growth paths when TFP follows a deterministic trend, and discuss the implications of endogenous labor supply. In particular, if TFP grows, endogenous labor supply does not rule out balanced growth paths such that consumers are asymptotically satiated and the economy grows at a positive rate. It may be that labor supply does not fall to zero because wages grow faster than the decline in the marginal utility of consumption. It may also be that while labor supply falls to zero, output nevertheless grows due to technical progress and capital accumulation. Under endogenous labor supply, a "consumer society" balanced growth trajectory such that working time remains constant and consumers are not satiated may coexist with a "leisure society" trajectory such that working time trends downwards, converging to zero, while consumers are asymptotically satiated. This feature is reminiscent of debates about the relative merits of the "American model" vs. the "European model", and suggests that people may have coordinated on different equilibrium paths on different sides of the Atlantic. It is shown that the leisure society is associated with lower growth in output, physical capital and the number of varieties than the consumer society, while being more intensive in innovation relative to physical capital.

Satiation in the consumption of many category of goods is consistent with the study of Engel curves. For example, Moneta and Chai (2014) estimate cross-sectional Engel curves for various range of products in various years and find that they reach an interior maximum in many instances. Of course, a mode may indicate that the good becomes inferior beyond some income level. However, this is implausible for some goods (such as travel services, which has a mode in 1968 and 1988), while for some other goods the Engel curve flattens out rather than having a strict mode, which is more consistent with satiation (this is the case for leisure goods or household goods in 1968)[1]. Interestingly, these curves shift over time, in such a way that a category of goods which exhibits satiation may no longer do so, or vice-versa. This may be due to factors that are unrelated to the theme of this paper, such as shifts in relative prices. For example, energy has a mode in the 1968 Engel curve, but the post oil shock 1978 one is strictly monotonic. But it is plausible that the mode disappears because innovation lifts the economy out of satiation, as is probably the case for "leisure services": The range of goods included in this category, as argued by Gallouj and Weinstein (1997), has presumably widened over time, explaining why the mode that appears in its 1968 Engel curve has disappeared. Of course, this evidence does not tell us how this wave of innovation relates to low real interest rates nor whether another trajectory with less innovation, higher interest rates and converging to satiation might have existed. Nevertheless it is consistent with the mechanisms highlighted in this paper.

The present paper is related to several strands of literature. Non homothetic preferences have been introduced into growth models in order to explain structural change (Echevarria (1997), Laitner (2000), Ngai and Pissarides (2007), Foellmi and Zweimüller (2008)) and to study the interplay between income distribution and growth through demand channels (Matsuyama (2002), Foellmi and Zweimüller (2006), Saint-Paul (2006)). Yet to my knowledge the role of satiation as a limit to growth and its relationship to innovation has been under-researched by the mainstream of economics[2]. Rather, research has

[1]If utility is smooth, one cannot be satiated in one good unless one is satiated in all other goods. This is what occurs in this paper's model. However, if utility has a kink at the satiation point, this no longer holds: one may be at the satiation point for some goods but not others. This rationalization is consistentn with the structuralist literature a la Pasinetti.

[2]As an example, Gordon (2012) lists six "headwind" factors that will slow down US economic growth, none of them coming from consumer demand.

focused on supply-side limits to growth such as complementarities and bottlenecks (Baumol et al., 1985) and whether innovation may endogenously lift those limits (Acemoglu (2002)).[3]

Alternative approaches have paid more attention to the possibility of saturation, perhaps, since at least Veblen (1973), as a by-product of a more general critique of "consumer society". Hence, saturation of demand for existing goods, and the implied need to reallocate R and D effort towards new goods, plays an important role in Pasinetti's (1981) theory of structural change[4]. An important theme of this literature is how capitalism may manipulate preferences and institutions so as to avoid satiation and maintain a large consumer basis (Galbraith (1958), Scitovsky (1976), Schor (1998), Lee (2000))[5]. In contrast, this paper pursues a standard neo-classical approach: preferences are exogenous and differ from usual ones only in that they include a bliss point[6].

The paper is organized as follows. Section 2 lays out the basic model without innovation and analyzes whether the economy's growth trajectory converges to a satiated steady state. Section 3 extends the model by allowing for new goods to be introduced. It derives the paper's two key results: first, that the decentralized equilibrium may fail to lift the economy out of satiation through innovation even though a central planner would always choose to do so; second, that the strategic complementarity between innovation and savings may lead to multiple equilibrium trajectories associated with multiple steady states. Section 4 studies how the economy reacts to an increase in TFP and to a fall in the cost of R and D. In equilibria such that the economy is not satiated in the long run, these shocks

[3] Aoki and Yoshikawa (2002) study a growth model with "saturation of demand" in each good and horizontal innovation. However, what they refer to is not satiation, but an S-shaped demand pattern following the introduction of each good. In their model, which is in the fashion of Young's (1991) models of product life cycle, demand for each good is bounded in equilibrium because there is no vertical innovation, and the S-shape comes from the assumption that the utility derived by consumers from any individual good, which is logarithmic and therefore has no bliss point, is itself multiplied by a logistic factor. Furthermore, horizontal innovation is assumed to be entirely exogenous. Accordingly, none of the results and welfare analysis studied here hold.

[4] See Andersen (1998) for an evolutionary learning model, based on Pasinetti's ideas.

[5] See Benhabib and Bisin (2002) for a formalization of those approaches based on endogenous preferences.

[6] At some level of abstraction, however, one can always interpret the introduction of new goods as implying a change in the consumers' preferences. For horizontal innovation, the formalism of new growth theory does not allow to distinguish between the preferences for new products being genuine versus artificially created.

unambiguously increase output, capital and product variety. Things are more mixed, however, for equilibria with satiation: In trajectories such that innovation fails to pick up, an increase in TFP is met by a fall in investment; in a trajectory which converges to a satiated steady state with innovation, a reduction in the cost of R & D eventually reduces the number of varieties. In Section 5, I consider what happens if agents differ in their rate of time preference. If the most patient type is patient enough, the long-term equilibrium exhibits segmentation: agents below a critical discount rate are all satiated, while those above it end up with negligible wealth and consumption. While no satiation equilibria disappear, the strategic complementarity that we have highlighted remains in another form: a higher interest rate reduces the net present value of innovation, but, as it boosts incentives to save, raises the long-run proportion of satiated consumers and therefore the market size for innovation. As a result, multiple equilibrium interest rates may coexist, if the latter effect dominates over some range. Section 6 shows that if there is a trend of TFP growth, the steady states that have been constructed can be generalized as balanced growth paths. Section 7 discusses how the features of those trajectories change when one allows for endogenous labor supply. I highlight the possibility of a "leisure society" trajectory with positive growth in output, capital, and product variety, negative growth in labor supply, and asymptotic satiation. Section 8 concludes.

2 The model without innovation

How does the possibility of satiation affect the analysis of the standard neoclassical growth model? My starting point is the Ramsey model. A representative consumer, endowed with one unit of labor, maximizes

$$\max \int_0^{+\infty} u(C_t) e^{-\rho t} dt, \tag{1}$$

where C_t is consumption at date t. Output Y_t is produced using labor and capital, and the representative consumer is endowed with one unit of labor:

$$Y_t = AK_t^{\alpha} L_t^{1-\alpha} = AK_t^{\alpha}. \tag{2}$$

Each unit of output can be converted into one unit of consumption or one unit of capital, which depreciates at rate δ. Hence the consumer maximizes (1) subject to the law of motion of capital given by

$$\dot{K}_t = AK_t^\alpha - C_t - \delta K_t. \tag{3}$$

Let us assume that needs may be satiated, by picking a quadratic specification for $u()$:

$$u(C) = C - \beta \frac{C^2}{2}.$$

Note that labor supply is assumed exogenous. Consequently, the economy may not escape satiation through working time reduction. This may occur if there exists a minimum level of working hour for which there is no disutility, or if, as suggested by Schor (1995), institutional barriers prevent working time from being reduced. In Section 7 I relax the assumption of exogenous labor supply, and I show that as long as there is productivity growth, it does not rule out asymptotic satiation.

Clearly the solution to this problem is

$$C_t = \max(\frac{1-\lambda_t}{\beta}, 0),$$

where λ_t, the marginal value of capital, follows the law of motion

$$\dot{\lambda}_t = \lambda_t(\rho - r(K_t)), \tag{4}$$

where $r(K) = \alpha AK^{\alpha-1} - \delta$ denotes the net marginal product of capital.

Two types of steady state are possible. In a non satiated steady state, $\lambda > 0$, $C < 1/\beta$ and the capital stock satisfies the standard modified golden rule: $K = r^{-1}(\rho) = K_{MGR}$. Aggregate consumption then is equal to $C_{MGR} = AK_{MGR}^\alpha - \delta K_{MGR}$. In a satiated steady state, $\lambda = 0$, $C = 1/\beta$ and the equilibrium capital stock must solve $AK^\alpha - \delta K = 1/\beta$. Whenever a solution exists, I will denote by K_{S1} and K_{S2} the smaller and larger solutions. Clearly $K_{MGR} < K_{S2}$. The solution is then described by the following proposition:

Proposition 1 – Let $K^* = \arg\max AK_t^\alpha - \delta K_t = r^{-1}(0)$ *be the Golden Rule capital stock and* $C^* = AK^{*\alpha} - \delta K^*$ *the corresponding consumption level. Let* K_0 *be the initial capital stock.*

A. If $1/\beta > C^*$ *then the economy converges to a unique, non satiated steady state* $K = K_{MGR}$.

B. If $1/\beta < C^*$ *and* $K_{S1} > K_{MGR}$ *then*

(i) If $K_{S1} < K_0 < K_{S2}$ *the economy converges to a satiated steady state such that* $K = K_{S2}$. *Throughout the convergence path, the consumer is satiated:* $C_t = 1/\beta$.

(ii) If $K_0 < K_{S1}$ *the economy converges to a non satiated steady state such that* $K = K_{MGR}$. *The consumer is never satiated.*

C. If $1/\beta < C^*$ *and* $K_{MGR} > K_{S1}$ *then*

(i) If $K_0 > K_{S1}$ *the economy converges to a satiated steady state such that* $K = K_{S2}$. *Throughout the convergence path, the consumer is satiated:* $C_t = 1/\beta$.

(ii) If $K_0 < K_{S1}$ *the economy converges to a satiated steady state such that* $K = K_{S1}$.*Throughout the convergence path,* $C_t < 1/\beta$.

The Proof is in the Appendix. Figure 1 illustrates Proposition 1. The convergence path is always unique. In case A the satiation level is too high to be feasible in the long run: the only steady state is a non satiation one. In cases B and C satiation is sustainable or not depending on whether the initial capital stock is high enough. In case B people are impatient enough so that they would never elect to remain satiated in the long run unless they can do so throughout the entire accumulation path. Consequently there are two long-run stable steady states. If initial capital is large, setting consumption at $C = 1/\beta$ throughout does not deplete the capital stock. The economy ends at $K = K_{S2}$, an apparently dynamically inefficient steady state (but because of satiation there is no point in picking a path with higher consumption). Otherwise, the economy converges to the modified golden rule steady state. In case C people are patient enough so that the marginal utility of consumption falls to zero asymptotically. For high enough capital they can afford to be satiated throughout. Otherwise they gradually accumulate capital so as to reach satiation asymptotically.

3 Innovation

I now introduce the possibility of product innovation. I assume that there is a bliss point for each good, however one may in principle escape from satiation by consuming a greater variety of goods. I first compute the first-best allocation and show that the economy cannot settle in a satiated steady state. If that were to happen, it would be optimal to devote some resources to innovation and broaden the range of goods being consumed. Starting from a satiated situation, it is always optimal to deviate by introducing more goods and reducing the consumption of each good. The losses from doing so are second order while the gains are first order.

Next, I consider the case of market-determined innovation. I show that the economy may converge to a satiated steady state without any innovation despite that in such a situation the marginal social cost of innovation is zero. Depending on initial condition, such an equilibrium may be at the margin of innovation (type I), i.e. innovation takes place along the convergence paths but eventually stops despite satiation, or it may be such that innovation never occurs (type II).

The reason for this discrepancy between optimal and equilibrium outcomes is that the set of relative prices that drive production and innovation no longer reflect the consumers' marginal willingnesses to pay. Producers and innovators fail to internalize the fact that the marginal productivities of the activities they undertake are multiplied by that of consumption, which is zero. That is, while relative prices are equal to marginal rates of transformation, marginal rates of substitution are not defined. Economic choices are driven by equilibrium prices, that are themselves determined from the production side; because both marginal costs and marginal benefits, in terms of utility, are equal to zero, economic decisions by producers and innovators do not internalize the effect of those decisions on welfare. In particular, they do not internalize the fact that more innovation has a first-order effect on welfare by just lifting consumers out of the satiation trap.

Finally, I show that the economy may converge to a (type III) more conventional steady state with positive innovation, no satiation, and a capital stock equal to its modified golden rule level, and discuss how the same economy with the same initial conditions may, for some parameter range, follow either a trajectory converging to such a steady state or, alternatively, a trajectory with long-run satiation.

3.1 The first best

In this subsection, I introduce the relevant assumptions and notations and I compute the first best allocation.

At any date t there are N_t goods and the consumer's total utility is

$$U(\{c_{it}\}) = \int_0^{N_t} u(c_{it})di,$$

where c_{it} denotes consumption of good i and again $u(c) \equiv c - \beta c^2/2$.

The individual goods are produced using a single good, "output", which is again produced according to (2) and can also be transformed into capital or new varieties. Therefore, the capital accumulation equation is:

$$\dot{K}_t = AK_t^\alpha - C_t - \delta K_t - R_t, \tag{5}$$

where R_t denotes the resources devoted to innovation ("R and D") and C_t is equal to the amount of output devoted to the production of varieties.

The production function for any variety i is

$$c_{it} = z_{it},$$

where z_{it} is the amount of output used in the production of good i. Therefore, by symmetry, it is optimal to consume the same amount of each good, defined as $c_t = C_t/N_t$. We can then rewrite utility as

$$U(C_t, N_t) = C_t - \beta \frac{C_t^2}{2N_t}. \tag{6}$$

Clearly, $U_2 > 0$, reflecting the consumers' taste for diversity.

The production process for new goods is as follows: One unit of output in the R and D sector produces γ new goods per unit of time. Consequently, the law of motion for N_t is given by

$$\dot{N}_t = \gamma R_t. \tag{7}$$

The social planner maximizes

$$\max \int_0^{+\infty} U(C_t, N_t)e^{-\rho t}dt, \tag{8}$$

subject to the law of motions (5) and (7) and the constraint $R_t \geq 0$.

Proposition 2 characterizes the optimal solution in a steady state:

Proposition 2 – Let c^* be the unique solution in the $(0, 1/\beta)$ range to the following equation

$$\frac{\beta\gamma}{2}c^2 = \rho(1 - \beta c). \tag{9}$$

Then in any steady state $c \leq c^*$ and $N \geq N^* = C_{MGR}/c^*$. Furthermore $K = K_{MGR}$ and $C = C_{MGR}$.

Proof – See Appendix

Intuitively, c^* is the minimum consumption level per variety beyond which, in a steady state, it is socially profitable to innovate. Equation (9) equates the marginal benefit per unit of time of introducing a new variety, $\frac{\beta\gamma}{2}c^2$, to its annuity cost in terms of utility, equal to the product of the discount rate ρ and the marginal utility of consumption $1 - \beta c$. Clearly, the solution is always strictly smaller than $1/\beta$. In the long run, the representative consumer will never elect a consumption level per item c greater than c^*, because in such a situation it is preferable to sacrifice some consumption units to raise N, which eventually brings down c below c^*. Consequently, a satiated steady state does not exist.

3.2 Equilibrium satiation

I now compute the equilibrium level of innovation and I show that if innovation is determined in a decentralized fashion, a satiated steady state is no longer ruled out. This, despite that, as illustrated by (9), the social marginal cost of innovating in such a situation is zero.

Following the literature, I assume that innovators get a permanent patent on the variety they have invented. However, competitive imitators can produce the same good without authorization, at a marginal cost in terms of output equal to $\mu > 1$. If μ is not too large, the patent holder will charge at a markup equal to μ. Using output as the numéraire, μ is also equal to the price of the good. Consequently, the profits generated by any variety at date t are given by

$$\pi_t = (\mu - 1)c_t.$$

The market value of a patent at t, V_t, follows the usual law of motion

$$r_t V_t = \dot{V}_t + \pi_t, \tag{10}$$

where r_t is the real interest rate which, in equilibrium, equates the net marginal product of capital:

$$r_t = r(K_t).$$

At date t, the cost of producing a new variety is $1/\gamma$. Therefore the economy can be in one of two regimes:

A. The no innovation regime: $V_t < 1/\gamma$ and $R_t = 0$.

B. The innovation regime: $V_t = 1/\gamma$ and $R_t > 0$.

The consumer maximizes (8) with respect to $\{C_t\}$, or equivalently $\{c_t\}$, subject to his intertemporal budget constraint:

$$\int_0^{+\infty} \mu c_t N_t e^{-\int_0^t r_u du} dt \leq \int_0^{+\infty} w_t e^{-\int_0^t r_u du} dt + N_0 V_0 + K_0, \tag{11}$$

where w_t is the wage, also equal to labor income.[7]

From this problem, denoting by λ the Lagrange multiplier of (11), we get the optimality conditions for consumption:

$$c_t = \max\left(\frac{1 - \lambda\mu e^{\rho t - \int_0^t r_u du}}{\beta}, 0\right). \tag{12}$$

In particular, as long as $c > 0$ it satisfies the following Euler equation:

$$\dot{c}_t = \left(\frac{1}{\beta} - c_t\right)(r_t - \rho). \tag{13}$$

In partial equilibrium, if $r > \rho$, the consumer converges to satiation regardless of his income. Satiation only depends on people's degree of patience: if they are patient

[7] As will be clear below, because of satiation, the economy may converge to a steady state such that $r \leq 0$. In this case the RHS of (11) is not defined. But N_t is bounded and the intratemporal budget constraint $\dot{W}_t = r_t W_t + w_t - N_t c_t$ is stable and the consumer can clearly pick $c = 1/\beta$ throughout. Therefore, this special situation does not invalidate (13).

enough relative to the market interest rates, they will eventually have accumulated enough resources to reach satiation.

The following Proposition establishes the existence of long-run steady states that are satiated[8]. Furthermore, they are stable in the sense that there exist trajectories, starting from the initial values of the capital stock K_0 and of the number of varieties N_0, that converge to these steady states, for sets of initial conditions of positive measure.

Proposition 3 – Let $K_S = r^{-1}\left(\frac{\gamma(\mu-1)}{\beta}\right)$. *Let* $N_S = \beta(AK_S^\alpha - \delta K_S)$. *Assume* $N_0 < \min(\beta C_{MGR}, N_S)$. *Let* $\hat{K} < K_S$ *such that* $N_0 = \beta(A\hat{K}^\alpha - \delta\hat{K})$.

A. If $\hat{K} < K_0 < K_S$ *there exists an equilibrium trajectory such that (i)* $c_t = 1/\beta$ *throughout, (ii) there exists a critical date* T *such that* $\dot{K} > 0, \dot{R} = 0$ *for* $t < T$ *and* $\dot{K} = 0, \dot{R} > 0$ *for* $t > T$, *(iii) the economy converges to a satiated (type I) steady state such that* $N = N_S$ *and* $K = K_S$

B. If $K_0 < \hat{K}$ *there exists an equilibrium trajectory such that (i)* $c_t < 1/\beta$ *throughout, (ii)* $\dot{R} = 0$ *throughout (iii) the economy converges to a satiated (type II) steady state such that* $c = 1/\beta$, $K = \hat{K}$, *and* $N = N_0$.

Proof – See Appendix.

In the trajectories described in Proposition 3, agents are patient enough to accumulate enough capital so as to be satiated in the long run. However, there is no point in accumulating wealth to consume beyond the satiation level. In case B, no innovation takes place and people target an equilibrium level of the capital stock so as to be asymptotically satiated. At this level, interest rates are too high for innovation to be profitable. In case A, initial capital is high enough for people to choose a path such that they are satiated throughout while accumulating capital at a positive rate. At some point the interest rate falls to a level where innovation becomes profitable. It is then optimal to devote

[8] In Propositions 3 and 4 I only focus on the regimes that are most relevant. For example, if $\beta C_{MGR} < N_0 < N_S$, implying in particular $K_S > \hat{K} > K_{MGR}$, and $\rho > \frac{\gamma(\mu-1)}{\beta}$ for $K_0 < \hat{K}$ the economy follows a standard Ramsey path with no innovation and convergence of the capital stock to K_{MGR}.

Also, I do not analyze paths with falling capital stocks, which may arise if the initial capital stock is large, because one would then have to either allow for instantaneous conversion of a mass of capital into new goods, or for a corner regime where no gross investment takes place, and the marginal product of capital is lower than the interest rate which would be pinned down by the innovation equilibrium condition $V = 1/\gamma$.

all investment to innovation (any further accumulation of physical capital would make innovation strictly superior to capital accumulation) while raising aggregate consumption so as to maintain consumption of each variety equal to its satiation level. As this process continues, less and less resources remain available for innovation, which eventually dies out as the economy approaches steady state.[9]

Figure 2 illustrates Proposition 3 in the (K, c) plane[10]. The II schedule is the steady free-entry condition in the R and D sector, i.e. $r(K) = \gamma(\mu - 1)c$. In steady state, the economy cannot be above this locus, otherwise one would have $V > 1/\gamma$. The horizontal line SS is the satiation level $c = 1/\beta$. The economy's trajectory cannot be above this schedule. The $KK(N)$ schedule is the $\dot{K} = 0$ locus associated with a stationary mass of varieties equal to N. Whenever the economy is innovating, this schedule shifts to the right. In case A, the trajectory is below $KK(N_0)$. At date T the economy hits point E which is the intersection of II and SS. Thereafter both K and c remain constant while N goes up. The $\dot{K} = 0$ schedule gradually shifts right and asymptotically converges to $KK(N_S)$ which intersects II and SS at point E. In case B, the equilibrium trajectory is an asymptotically satiated saddle path without innovation. Throughout this trajectory $V_t < 1/\gamma$ and innovation does not take place. Dynamics are given by the Euler equation (13) and the $KK(N_0)$ schedule.

3.3 Equilibrium non-satiation

It is also possible to construct equilibrium trajectories which converge to a modified golden rule steady state at the margin of the innovation regime. These trajectories are characterized in Proposition 4:

Proposition 4 – Assume $\rho < \frac{\gamma(\mu-1)}{\beta}$. *Let* $N_N = C_{MGR}\frac{\gamma(\mu-1)}{\rho}$. *If* $N_0 \leq N_N$ *and* $K_0 \leq K_{MGR}$ *then there exists an equilibrium trajectory such that.*

(i) Until some date T, $\dot{K} > 0$, $\dot{N} = 0$, *and* $\dot{c} = 0$

(ii) For any $t \geq T$, $K_t = K_{MGR}$, $c_t = \frac{\rho}{\gamma(\mu-1)} = c_N < \frac{1}{\beta}$, $\dot{N}_t > 0$

[9] For simplicity, nonnegativity constraints on investment have been ignored. For $K_0 > K_S$, we conjecture that in equilibrium all capital in excess of K_S would be immediately converted into new varieties.

[10] It has been assumed that $K_S < K_{MGR}$, i.e. $\rho < \frac{\gamma(\mu-1)}{\beta}$

(iii) The economy converges to a (type III) steady state such that $K = K_{MGR} > K_S$, $c = c_N < \frac{1}{\beta}$, $C = C_{MGR} > C_S$, $N = N_N > N_S$.

Proof – See Appendix

Figure 3 illustrates the trajectory elicited in Proposition 4. This trajectory converges in finite time to the modified Golden rule capital stock, with a consumption level which is lower. The surplus of savings at $K = K_{MGR}$ after date T is devoted to innovation. As innovation proceeds, aggregate consumption gradually goes up as people consume a greater variety of goods, each in the same quantity. This process continues until, asymptotically, no resources are left for further innovations.

As long as $\rho < \frac{\gamma(\mu-1)}{\beta}$ and $N_0 < N_S$, these trajectories coexist with those converging to satiated steady states and characterized in Proposition 3[11]. Therefore, there exist multiple self-fulfilling trajectories that converge to different steady states. A trajectory that converges to a type I satiated steady state is associated with lower innovation and lower capital accumulation than a trajectory converging to a non satiated steady state. In a satiated steady state, the market size for innovation is higher than in a non satiated one, but that is compensated by higher interest rates since such a steady state is less capitalistic.

This strategic complementarity is based on the following mechanism: Suppose people expect more goods to be introduced in the future. This raises the marginal utility of future aggregate consumption C and therefore the incentives to save[12]. These incremental savings in turn reduce real interest rates, which makes innovation more profitable[13]. Conversely, expecting a lower level of innovation or no innovation at all would lead to lower savings and higher interest rates, which would deter innovation.

The possibility of satiation is important for this strategic complementarity to lead to

[11] If $\rho < \frac{\gamma(\mu-1)}{\beta}$ then $K_S < K_{MGR}$ and $N_S < \beta C_{MGR}$, implying that if $N_0 < N_S$ then $N_0 < \beta C_{MGR}$, so that Proposition 3 applies.

[12] In that respect, horizontal innovation is very different from vertical innovation. The latter raises the future physical amount of goods available to the economy, and interest rates must be higher for consumers to absorb those goods. In contrast, horizontal innovation raises the marginal utility of future consumption, which makes it more valuable to save at any given interest rate.

[13] This accumulation process in anticipation of a future wave of innovation is reminiscent of Greenwood and Yorukoglu's (1994) analysis of the IT revolution. The convergence path highlighted here has two phases: a capital accumulation phase and an innovation phase. If the contribution of innovation is mismeasured, the second phase may be wrongly interpreted as a 'slump'.

multiple steady states. Otherwise, there would be a unique long-run equilibrium capital stock given by the modified golden rule level K_{MGR}. It is easy to check that this is consistent with a unique value of c and a unique value of N , which may or may not be in the innovation regime depending on initial conditions.

As in Dixit and Stiglitz (1977), the equilibrium number of varieties in Proposition 4 may be either higher or lower than the social optimum. An appropriability effect which tends to make innovation too low has to be balanced against a business stealing effect which tends to make it too high.

4 Some comparative statics

Whenever the economy converges to a satiated equilibrium, the conventional results regarding the effects of greater productivity are partially overturned.

In the standard Ramsey model, a permanent increase in TFP raises capital accumulation. In a type II steady state, however, a strong income effect dominates: In the long run, absent innovation, one can finance the same satiated level of consumption with a lower capital stock.

In standard models of horizontal innovation, the number of varieties goes up when the cost of R and D falls. In a type I steady state, however, the number of varieties goes down, because the economy accumulates less physical capital.

4.1 Effect of productivity shocks

I fist study the effect of productivity shocks. I show that they have different qualitative effects on the equilibrium depending on whether or not the economy lies at the margin of the innovation regime in the long-run. This is summarized in Proposition 5:

Proposition 5 – Consider a small increase in A. Then

(i) If the economy converges to a type I steady state, in the long run K, Y, C and N go up. c is unchanged at $c = 1/\beta$.

(ii) If the economy converges to a type II steady state, in the long run K and Y fall. C, N, and $c = 1/\beta$ are unchanged.

(iii) If the economy converges to a type III steady state, in the long run K, Y, C and N goes up, c is unchanged at $c = \frac{\rho}{\gamma(\mu-1)}$. Furthermore, if the economy is initially in steady state and A goes up, then c falls upon impact.

Proof – This follows immediately from Propositions 3 and 4.

The effect of TFP growth is quite different depending on the type of steady state. If there is no innovation and people are eventually satiated (type II), the capital stock falls so as to leave aggregate consumption C unchanged in the long run. The extra output allowed by productivity growth is essentially consumed, allowing to consume more over the convergence trajectory. Accumulating more capital would have zero value since people are satiated in the long run and there is no innovation. In a type III steady state, however, higher productivity makes it worthwhile to accumulate more capital, because this eventually allows to invent more varieties. Indeed, starting from steady state, consumers start saving more at the time of the shock, which further speeds up capital accumulation. The economy transitions from the capital accumulation regime to the innovation regime with a higher capital stock, which allows to devote more output to innovation[14]. The economy ends up with both more capital and more varieties, while consumption of each variety remains unchanged, since there is only one market size which delivers zero net profits from innovating at the equilibrium interest rate $r = \rho$.

To summarize: a positive productivity shock generates an investment boom if innovation is expected to occur, and an investment slump if that is not the case.

4.2 Effects of shocks to the cost of R and D

I now study an increase in the productivity of R and D, i.e. an increase in γ. A shock to γ also has very different effects on capital and innovation depending on whether or not the economy converges to a steady state with satiation:

Proposition 6 – Consider a small increase in γ. Then

[14] Thus, in Regime III, a "vertical" innovation, i.e. an increase in TFP, triggers a phase of capital accumulation boosted by higher savings, followed by a halt to capital accumulation and a phase of vertical innovation. This bears some similarities to Schumpeterian long waves such as those analyzed by Aghion et al. (2014).

(i) If the economy converges to a type I steady state, in the long run K, Y, C and N fall. c is unchanged at $c = 1/\beta$.

(ii) If the economy converges to a type II steady state, this steady state is unchanged.

(iii) If the economy converges to a type III steady state, in the long run K is unchanged as is Y. C is unchanged, c falls and N goes up.

Proof – Immediate from Propositions 3 and 4.

To understand these paradoxical results, consider first the most standard case, i.e. a type III steady state (figure 4a). As the cost of innovation falls, people anticipate that more varieties will be introduced in the future, which raises their marginal utility of aggregate consumption in the future compared to the present. For this reason they save more, reducing their consumption of each individual good c. As capital is accumulated more quickly, the critical date when innovation kicks in is reached earlier. Furthermore, agents coordinate their consumption plans so that r and K are still equal to their modified Golden Rule levels, otherwise the subsequent trajectory would not converge to a steady state. Thus the equilibrium value of c must be lower: as the cost of innovation is lower and the interest rate is the same, market size must fall for the free entry condition in innovation to be satisfied. This in turn frees more resources available for innovation: in the long run, more varieties have been introduced.

Consider now (Figure 4b) an economy that converges to a type I steady state. A higher γ shifts the II locus down and to the left, meaning that since innovation is less costly, the economy will innovate at higher interest rates, i.e. lower capital stocks. Hence the trajectory hits point E earlier than if γ had remained smaller; people benefit from an increase in the number of varieties earlier. However that is a mixed blessing because the economy stops raising its capital stock after this date, investing all savings in innovation. In the long run, as the capital stock is lower, there are also fewer resources left to innovate, and this also reduces the equilibrium number of varieties – As the economy follows a self-fulfilling trajectory with satiation in all varieties, the mechanism allowing for variety to go up thanks to a reduction in c does not operate.

5 Heterogeneous agents

I now discuss how the analysis changes if agents are heterogeneous. I assume that people differ in their rate of time preference ρ. I assume that it is distributed with c.d.f. F and density $F' = f$. To simplify matters I make two key assumptions:

ASSUMPTION 1 – F has full support over $(0, +\infty)$ and no mass point.

ASSUMPTION 2 – f is hump-shaped, implying F is $S-$shaped.

Let us now consider steady states. For any equilibrium interest rate $r > 0$, there exists a positive measure of agents such that $\rho < r$. According to (12), they are necessarily satiated. By the same token, agents such that $\rho > r$ have a zero consumption level. While in the Ramsey model with heterogeneous agents the most patient dynasty own all the wealth in the long run, while other dynasties have zero consumption, here consumers for an entire range of values of ρ have positive consumption in the long run, and they are all satiated.

Given that a fraction $F(r)$ of consumers consume $1/\beta$ of each good and the others zero, the profit level for any variety is equal to

$$\pi = \frac{\mu - 1}{\beta} F(r).$$

Let us focus on a steady state where there is innovation at the margin. The free entry condition $V = 1/\gamma$ reads

$$\frac{\mu - 1}{\beta} F(r) = \frac{r}{\gamma}. \tag{14}$$

This condition determines the equilibrium interest rate. Interestingly, both sides are increasing functions of r, so that the equilibrium interest rate may not be unique. This is due to the following: While a higher interest rate reduces the net present value of innovating through the usual discount effect, it also induces a greater fraction of the population to save in order to eventually be satiated, which in turn raises the market size for innovations and hence the flow of profits it generates[15].

Figure 5 depicts the determination of r. Under Assumption A2 there are at most three equilibrium values for r : $0, r_1$ and r_2. It is easy to see that $r = 0$ cannot be an

[15] In some sense, type I and III equilibria in the preceding section are extreme versions of the ones highlighted here.

equilibrium[16]. Intuitively, r_1 can be ruled out on economic stability grounds. Consider a small perturbation, say an increase in r. Then in the neighborhood of equilibrium r_1 the LHS of (14) goes up by more than its RHS, meaning that the market size effect on innovation dominates the effect of the cost of capital. We expect innovators to innovate more, which raises the demand for loanable funds and should push r further up. That is, let $V(r) \equiv \frac{\mu-1}{\beta}\frac{F(r)}{r}$. If we define a stable equilibrium as an equilibrium r which satisfies $V'(r) < 0$ in addition to $V(r) = 1/\gamma$, then r_1 may be ruled out and, since F is $S-$shaped, the only stable equilibrium is such that $r = r_2$. From there it is easy to compute the equilibrium capital stocks and number of varieties[17]:

$$K_2 = r^{-1}(r_2), \tag{15}$$

$$N_2 = \frac{\beta}{F(r_2)}\left(AK_2^\alpha - \delta K_2\right). \tag{16}$$

As in the analysis of Section 4.2, an increase in γ raises the equilibrium r. Again from (15)-(16) fewer varieties exist in equilibrium and the capital stock is lower. On the other hand, the proportion of agents that are satiated, $F(r)$, goes up.

6 Balanced growth paths

This section extends the results of Section 3 by allowing for growth in the TFP parameter A. Clearly, since the critical capital stock $\hat{K}$ goes to zero as A goes up, a type II balanced growth path (BGP) is precluded: as consumption of each good is bounded, capital accumulation would push interest rates down to zero; however innovation would pick up before such a point is reached. On the other hand, it is easy to construct BGPs of types I and III. These are characterized by the following propositions.

Proposition 7 – Assume $A_t = A_0 \exp(g_A t)$. *Assume* $g_A < \frac{1-\alpha}{\alpha}\left(\frac{\gamma(\mu-1)}{\beta} + \delta(1-\alpha)\right)$. *Then there exists an equilibrium trajectory such that*

[16] At $r = 0$ (14) does not in fact apply. The RHS of any consumer's budget constraint becomes infinite. All consumers therefore choose $c = 1/\beta$. Then $\pi = \frac{\mu-1}{\beta} > 0$, implying $V = +\infty$, which contradicts the requirement that $V = 1/\gamma$.

[17] An equilibrium with no innovation can also be computed. Let N_0 be the initial number of varieties. Then the equilibrium capital stock $K(N_0)$ is the unique solution to $0 = AK^\alpha - N_0F(r(K))/\beta - \delta K$. For no innovation to indeed take place in equilibrium, $K(N_0)$ must be such that $r(K) > (\mu-1)F(r(K))/\beta$.

(i) $K_t = kA_t^{\frac{1}{1-\alpha}}$, *where* $k = \left(\frac{\alpha}{\delta + \frac{\gamma(\mu-1)}{\beta}}\right)^{\frac{1}{1-\alpha}}$,

(ii) $c_t = 1/\beta$

(iii) $r_t = \frac{\gamma(\mu-1)}{\beta}$

(iv) Y, K *and* N *grow at common rate* $g = \frac{g_A}{1-\alpha} = g_{CS}$,

(v) $N_t = nA_t^{\frac{1}{1-\alpha}}$, *where* $n = k\frac{(\frac{\gamma(\mu-1)}{\beta}+\delta)/\alpha - g - \delta}{1/\beta + g/\gamma} > 0$.

Proof–See Appendix

Proposition 8 – Assume $A_t = A_0 \exp(g_A t)$. *Assume* $g_A < (1-\alpha)\rho$ *and* $\rho < \frac{\gamma(\mu-1)}{\beta}$. *Then there exists an equilibrium trajectory such that*

(i) $K_t = k' A_t^{\frac{1}{1-\alpha}}$, *where* $k' = \left(\frac{\alpha}{\delta+\rho}\right)^{\frac{1}{1-\alpha}}$,

(ii) $c_t = \frac{\rho}{\gamma(\mu-1)} < \frac{1}{\beta}$

(iii) $r_t = \rho$

(iv) Y, K *and* N *grow at common rate* g_{CS},

(v) $N_t = n' A_t^{\frac{1}{1-\alpha}}$, *where* $n' = k' \frac{(\rho+\delta)/\alpha - g - \delta}{\frac{\rho}{\gamma(\mu-1)} + g/\gamma} > 0$.

Proof – See Appendix.

The BGPs constructed in Propositions 7 and 8 generalize the type I and type III steady states, respectively. Again, as the conditions for Proposition 7 and Proposition 8 to hold are not mutually exclusive, the two trajectories may coexist in the same economy[18]. Furthermore, it is easy to check that the non satiated BGPs have more innovation and capital than the satiated ones.[19]

[18]The condition $g_A < (1-\alpha)\rho$ is the standard one that is needed for a Ramsey MGR BGP to deliver summable utility. However if $\frac{1-\alpha}{\alpha}\left(\frac{\gamma(\mu-1)}{\beta} + \delta(1-\alpha)\right) < (1-\alpha)\rho$ and $\rho > \frac{\gamma(\mu-1)}{\beta}$, then a BGP cannot be constructed for $\frac{1-\alpha}{\alpha}\left(\frac{\gamma(\mu-1)}{\beta} + \delta(1-\alpha)\right) < g_A < (1-\alpha)\rho$.

[19]Since $\rho < \frac{\gamma(\mu-1)}{\beta}$, $k' > k$. Furthermore, in both BGPs, $N = \frac{Y-(g+\delta)K}{g/\gamma + r/(\gamma(\mu-1))}$. Since r is lower in the non satiated BGP, the numerator is lower. Since K is higher in the non satiated BGP but, since $r > g$, lower than the Golden Rule level, the numerator is higher in the non satiated BGP. Therefore, N is higher in the non satiated BGP than in the satiated one.

7 Long-term growth, near satiation and the leisure society

In this section I make labor supply endogenous, while still assuming the existence of a productivity growth trend. I assume all units of labor have a disutility. Absent growth, then, satiation would not arise because if it did, labor supply would be equal to zero, and the economy could not produce enough output for people to be satiated. Under growth, however, the story is different.

I assume that the consumer's flow of utility is now given, instead of (6), by

$$U(C_t, N_t, L_t) = C_t - \beta \frac{C_t^2}{2N_t} - v(L_t),$$

where L_t denotes labor supply and $v(L_t)$, the marginal disutility of labor, is given by

$$\begin{aligned} v(L_t) &= \tau \frac{L_t^2}{2} \text{ if } L_t \leq 1 \\ &= +\infty \text{ if } L_t > 1. \end{aligned}$$

At any date t, the real consumption wage us w_t/μ and the marginal utility of consumption is $1 - \beta c_t$. Therefore, labor supply is given by

$$\begin{aligned} L_t &= 1 \text{ if } w_t(1 - \beta c_t)/\mu > \tau, \\ &= \frac{w_t(1 - \beta c_t)}{\tau \mu} \text{if } w_t(1 - \beta c_t)/\mu \leq \tau \end{aligned}$$

Since, along a non satiated BGP, w grows over time while c is constant and lower than $1/\beta$, the equilibria described in Proposition 8 still exist, provided the initial level of wages satisfies $w_0(1 - \beta c)/\mu > \tau$. In those equilibria, labor supply is at its maximum level. In contrast, equilibria with satiation no longer exist. Interestingly, however, we can construct trajectories that converge to satiation while output does not go to zero, either because labor supply is at its maximum, or because TFP growth more than offsets the effect of shrinking labor supply in the long run. These trajectories are characterized in the following two propositions, again proved in the Appendix:

Proposition 9 – Assume that

$$g_A > (1 - \alpha) \left(\frac{\gamma(\mu - 1)}{\beta} - \rho \right). \tag{17}$$

Then there exists $\eta, \varepsilon > 0$ and a decreasing function $A()$, such that for all $k_0 \in (k, k+\eta)$ and $n_0 \in (n-\varepsilon, n+\varepsilon)$, and for $A_0 > A(k_0)$ and $A_t = A_0 \exp(g_A t)$, there exists an equilibrium trajectory (K_t, c_t, N_t, L_t) such that

(i) $K_0 = k_0 A_0^{\frac{1}{1-\alpha}}$

(ii) $N_0 = n_0 A_0^{\frac{1}{1-\alpha}}$

(iii) $\dot{c}_t > 0$ *and* $\lim_{t\to\infty} c_t = 1/\beta$

(iv) $L_t = 1$, $\forall t$

(v) $\dot{N}_t > 0$ *and* $\lim_{t\to\infty} \frac{N_t}{A_0^{\frac{1}{1-\alpha}}} = n$

(vi) $\lim_{t\to\infty} \frac{K_t}{A_0^{\frac{1}{1-\alpha}}} = k$

(vii) The economy is in the innovation regime throughout. In particular, $r_t = \gamma(\mu - 1)c_t = \alpha A_t K_t^{\alpha-1} - \delta$.

Proposition 9 has constructed a trajectory which is near the BGP of Proposition 7 and therefore near satiation. Furthermore this trajectory converges to that BGP and consumers are asymptotically satiated. Nevertheless, it is rational for them to supply one unit of labor forever, because wage growth overtakes the decline in the marginal utility of consumption.

If (17) is violated, but growth is not too low, we can construct a trajectory such that labor supply shrinks over time, and converges to zero, while consumers are asymptotically satiated. Growth is strictly positive because technical progress and capital accumulation more than offset the attrition in labor supply.

Proposition 10 (Leisure society) – Assume that $\rho < \frac{\gamma(\mu-1)}{\beta}$, *that* $A_t = A_0 \exp(g_A t)$, *and that*

$$\frac{1}{2}\left(\frac{\gamma(\mu-1)}{\beta} - \rho\right) < \frac{g_A}{1-\alpha} < \frac{\gamma(\mu-1)}{\beta} - \rho. \tag{18}$$

Let

$$g = \frac{2}{1-\alpha} g_A - \left(\frac{\gamma(\mu-1)}{\beta} - \rho\right) = g_{LS} > 0$$

and

$$z = \frac{\frac{1}{\alpha}\left(\frac{\gamma(\mu-1)}{\beta} + \delta\right) - g - \delta}{1/\beta + g/\gamma} > 0.$$

Then there exists $\eta, \zeta, \varepsilon > 0$ *such that for any initial conditions* (K_0, N_0) *such that* $K_0 < \min(\eta A_0^{\frac{2}{1-\alpha}}, \zeta A_0^{\frac{1}{1-\alpha}})$ *and* $N_0 \in (K_0 z(1-\varepsilon), K_0 z(1+\varepsilon))$, *there exists an equilibrium trajectory* (K_t, c_t, N_t, L_t) *from those initial conditions such that*

(i) $\dot{c}_t > 0$ *and* $\lim_{t\to\infty} c_t = 1/\beta$

(ii) $L_t < 1$, *and* $\lim_{t\to\infty} \frac{\dot{L}_t}{L_t} = \frac{1}{1-\alpha} g_A - (\frac{\gamma(\mu-1)}{\beta} - \rho) = g_L < 0$. *Therefore,* $\lim_{t\to\infty} L_t = 0$

(iii) $\lim_{t\to\infty} \frac{\dot{K}_t}{K_t} = \lim_{t\to\infty} \frac{\dot{Y}_t}{Y_t} = \lim_{t\to\infty} \frac{\dot{N}_t}{N_t} = g.$

(iv) $\lim_{t\to\infty} \frac{N_t}{K_t} = z$

(v) The economy is in the innovation regime throughout. In particular, $r_t = \gamma(\mu - 1)c_t = \alpha A_t K_t^{\alpha-1} - \delta.$

Proposition 10 constructs a leisure society where, despite innovation, capital accumulation proceeds at a faster pace than the introduction of new goods, eventually leading to satiation. Over time, labor supply falls to zero, but at a lower rate than the contribution of TFP growth and capital accumulation to GDP, thus making room for long-run satiation.

As long as $\rho < \frac{\gamma(\mu-1)}{\beta} < 3\rho$, there exists an interval of values of g_A for which a leisure society trajectory characterized by Proposition 10 coexists with a consumer society characterized by Proposition 8. Since for the leisure society to exist it must be that $g_A < (1-\alpha)\left(\frac{\gamma(\mu-1)}{\beta} - \rho\right)$, one clearly has $g_{LS} < g_{CS}$: the leisure society grows less fast than the consumer society. By comparing the expression for z to its counterpart for the consumer society, which may be obtained from (v) in Proposition 2, we find that the leisure society has a greater range of goods, *relative to* K, than the consumer society. This is due to several factors: first, the market size for new goods is larger in the leisure society, because people are satiated in those goods. Second, as the economy grows less fast, it invests less in physical capital, which leaves more resources for innovation. Nevertheless, over time, the consumer society will widen its gap relative to the leisure society both in terms of capital and number of varieties.

The above analysis also suggests that, contrary to the claims by Schor (1995), the persistence of high labor supply despite satiation may well be an equilibrium phenomenon rather than the outcome of institutional constraints. Not only the economy may never approach satiation because it will endogenously introduce new goods, but in the configuration of Proposition 9 near satiation does not preclude labor supply from being in its maximum. It is also interesting to note that differences between European-style "leisure societies" and American-style "consumer societies" may be explained by multiple equilibria instead of different exogenous institutions.

8 Conclusion

The possibility of satiation of needs substantially changes our analysis of growth models. The market may fail to introduce new goods so as to lift the economy out of a satiation trap. As expectations of introducing new goods generate a strategic complementarity between savings and innovation, there are multiple equilibria. When one allows for endogenous labor supply, satiated ("Leisure Society") equilibria may involve a lower growth rate, and an ever decreasing labor supply, compared with non-satiated ("Consumer Society") equilibria.

There is no overproduction in the economy analyzed here, since it is always at a full employment equilibrium. Nevertheless, I have shown that because of satiation, an increase in total factor productivity may lead to a reduction in the long-term capital stock, thus triggering an investment slump. Furthermore, for this slump to occur it must be that the economy remains in the (type II) no innovation regime. This is reminiscent of both Schumpeter's (1939) analysis of long waves and recent New Keynesian analyses of contractionary technical progress[20]. Indeed, a natural direction for further research would be to analyze the short and medium term implications of satiation, in models where sticky prices may generate overproduction and underemployment.

[20] Gali (1999), Basu et al. (2006)..

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9 Appendix

9.1 Proof of Proposition 1

A. Remember that $C^* = \max_K AK^\alpha - \delta K$. Therefore if $1/\beta > C^*$ the equation $AK^\alpha - \delta K = 1/\beta$ has no solution and the only steady states are non satiated. Consequently, $K = K_{MGR}$ in steady state.

B. (i). Assume $C_t = 1/\beta$ throughout. Then $\dot{K}_t = AK^\alpha - \delta K - 1/\beta$. Over $(K_{S1}, K_{S2}]$, K clearly converges to K_{S2}. Furthermore, this trajectory is optimal as it delivers the maximum possible utility level.

(ii) By (4), the consumer is either always satiated or never satiated. Since K would become equal to zero in finite time if he were always satiated, the consumer is never satiated. Assume the economy converges to a satiated steady state, implying $\lim_{t\to\infty} K = K_{S,i}$. Since $K_{S2} > K_{S1} > K_{MGR}$, for some T and $t > T$, $K > K_{MGR}$. Then $\dot{C} < 0$ for $t > T$, which contradicts the fact that $\lim_{t\to\infty} C = 1/\beta$. Therefore the economy converges to a non satiated steady state, implying $\lim K = K_{MGR}$.

C. (i) Same proof as B (i)

(ii) Again the consumer is never satiated. Assume the economy converges to a non satiated steady state. Then $\lim_{t\to\infty} K = K_{MGR}$. Since $K_{MGR} > K_{S1}$, for some T and $t > T$, $K_{S2} > K > K_{S1}$. Let $t_0 > T$. Consider the following deviation $\tilde{C}_t$ from the actual trajectory: $\tilde{C}_t = C_t$, $t < t_0$, $\tilde{C}_t = 1/\beta$, $t \geq t_0$. Then the corresponding capital stock $\tilde{K}_t$ is such that $\tilde{K}_t = K_t$, $t < t_0$. Therefore $K_{S2} > K_{t_0} > K_{S1}$. Clearly, then $\lim \tilde{K} = K_{S2}$. Hence the deviation is feasible. Furthermore, as $C_t < 1/\beta$ throughout, $\tilde{C}_t > C_t$ for $t \geq t_0$. It follows that the deviation delivers a strictly higher utility level than the initial trajectory, which therefore cannot be optimal. Consequently, the economy necessarily converges to a satiated steady state.

QED

9.2 Proof of Proposition 2

The Hamiltonian is given by

$$H = (C_t - \beta \frac{C_t^2}{2N_t})e^{-\rho t} + \lambda_t e^{-\rho t}(AK_t^\alpha - C_t - \delta K_t - R_t) + \mu_t e^{-\rho t}\gamma R_t$$

The representative consumer maximizes H subject to $R_t \geq 0$ and $C_t \geq 0$. Let $c_t = C_t/N_t$, $r(K) = \alpha A K^{\alpha-1} - \delta$. The first order conditions are

$$\begin{aligned} c_t &= \max(0, \frac{1-\lambda_t}{\beta}) \\ -\dot{\lambda}_t + \rho\lambda_t &= \lambda_t r(K_t); \\ \lambda_t &\geq \gamma\mu_t \text{ and } (\lambda_t - \gamma\mu_t) R_t = 0; \\ -\dot{\mu}_t + \rho\mu_t &= \frac{\beta c_t^2}{2}. \end{aligned}$$

Consider any steady state. Then either $\lambda > \gamma\mu$ or $\lambda = \gamma\mu$.

Assume $\lambda > \gamma\mu$. Since $\mu \geq 0$, $\lambda > 0$. Therefore, the steady state is non satiated. Then clearly $\rho = r(K)$, implying $K = K_{MGR}$. Since $\dot{K} = \dot{N} = 0$, it must be that $C = C_{MGR}$. In particular, $c = C/N > 0$, so that $\lambda = 1 - \beta c$. Since $\rho\mu = \frac{\beta c^2}{2}$, $\lambda = 1 - \beta c$ and $\lambda > \gamma\mu$, it must be that $\rho(1-\beta c) > \frac{\gamma\beta c^2}{2}$, or equivalently $c < c^*$. Therefore $N = C/c = C_{MGR}/c > C_{MGR}/c^* = N^*$.

Assume $\lambda = \gamma\mu$. Since $\rho\mu = \beta c^2/2$, one cannot have $c = 0$. Otherwise one would have $\lambda = \mu = 0$, implying $c = 1/\beta$, a contradiction. Therefore $c > 0$, implying $\mu > 0$ and therefore $\lambda > 0$ and satiation cannot hold. Again $\rho = r(K)$, $K = K_N$ and $C = C_N$. Since $\lambda = 1 - \beta c = \gamma\mu = \gamma\beta c^2/(2\rho)$, $c = c^*$ and $N = C/c = C_N/c^* = N^*$.

QED

9.3 Proof of Proposition 3

A. We first construct a trajectory such that $R = 0$, $N = N_0$, and $c = 1/\beta$. Along this trajectory, the law of motion for K is

$$\dot{K}_t = AK_t^\alpha - \frac{N_0}{\beta} - \delta K_t.$$

Since $K_0 > \hat{K}$, one has $\dot{K} > 0$ throughout and the trajectory converges to $\tilde{K}$, which is the larger solution to $N_0 = \beta(AK^\alpha - \delta K)$. In particular, $r(\tilde{K}) < 0$, implying $\tilde{K} > K_S$. Let T be the date at which, along this trajectory, $K_t = K_S$.

Our equilibrium trajectory coincides with that trajectory for $t \leq T$. Thereafter, we assume that $c_t = 1/\beta$, $K_t = K_S$, and therefore

$$\dot{N}_t = \gamma \left(AK_S^\alpha - \frac{N_t}{\beta} - \delta K_S \right).$$

Clearly, N converges monotonically to $N_S > N_0$.

We now show that this trajectory satisfies all the equilibrium conditions.

First, the Euler condition (13) is always satisfied since $\dot{c} = 0$ and $c = 1/\beta$. The law of motion (5) is also satisfied. For $t \geq T$, we have that $r_t = r(K_S) = \frac{\gamma(\mu-1)}{\beta}$ and $\pi_t = (\mu - 1)/\beta$. Therefore, from (10), $V_t = 1/\gamma$. The economy is in the innovation regime, consistent with our assumption that $\dot{N} > 0$. For $t < T$, integrating (10) yields

$$V_t = \int_t^T \frac{\mu - 1}{\beta} e^{-\int_t^s r_u du} ds + \frac{1}{\gamma} e^{-\int_t^T r_u du}.$$

Since $K_t < K_S$, $r_s > \frac{\gamma(\mu-1)}{\beta}$. Straightforward algebra then implies that $V_t < 1/\gamma$. Consequently, the economy is in the no innovation regime for $t < T$, consistent with $\dot{N} = 0$.

Thus, the constructed trajectory satisfies all the equilibrium conditions.

B. Assume now that $K_0 < \hat{K}$ and consider the following system

$$\dot{K}_t = AK_t^\alpha - N_0 c_t - \delta K_t. \tag{19}$$

$$\dot{c}_t = \left(\frac{1}{\beta} - c_t \right) (r(K_t) - \rho). \tag{20}$$

This is a standard system and there exists a unique saddle-path converging to its steady state such that $c = 1/\beta$ and $K = \hat{K}$. Note that since $N_0 < \beta C_{MGR}$, $\hat{K} < K_{MGR}$. Therefore, along this saddle-path trajectory, $\dot{K} > 0$ and $\dot{c} > 0$, since $r(K_t) > r(\hat{K}) > r(K_{MGR}) = \rho$. We can express this trajectory as $c = c_{SP}(K)$, where $c_{SP}' > 0$.

Assume $c_{SP}(K) < 0$ for $0 < K < K_c$. Then for any $K < K_C$ we replace $c_{SP}(K)$ by a zero value. It is then straightforward to check that the trajectory defined by $c_t = c_{SP}(K)$ and $\dot{K}_t = AK_t^\alpha - N_0 c_{SP}(K) - \delta K_t$ satisfies the consumer's optimality conditions throughout.

Assume that the trajectory followed by the economy is such that (c, K) are along this saddle-path and $N = N_0$. To prove that this is an equilibrium trajectory, we only need to show that $V \leq 1/\gamma$ throughout. From (10) we have that

$$V_t = \int_t^{+\infty} (\mu - 1)c_t e^{-\int_t^s r_u du} ds.$$

Since $N_0 < N_S$, $\hat{K} < K_S$. Therefore $r_t = r(K_t) > r(\hat{K}) > \frac{\gamma(\mu-1)}{\beta}$ and $c_t < 1/\beta$. Clearly, then, $V_t < 1/\gamma$.

QED.

9.4 Proof of Proposition 4

First, we can construct a unique saddle path (K_t, c_t) such that (i) (K_t, c_t) satisfies the system (19-20), (ii) $K = K_0$ initially and (iii) $c = \frac{\rho}{\gamma(\mu-1)}$ at $K = K_{MGR}$. Given our assumption that $K_0 < K_{MGR}$ and $N_0 < N_N$, along this trajectory $\dot{c} > 0$ and $\dot{K} > 0$. If the mathematical solution is such that $c < 0$, we replace it by $c = 0$ as in the proof of Proposition 3, B. Furthermore, at $c = \frac{\rho}{\gamma(\mu-1)}$, $\dot{K} = (N_N - N_0)\frac{\rho}{\gamma(\mu-1)} > 0$, implying that the point (K_{MGR}, c) is reached at a finite date T.

For $t \leq T$ we assume the economy follows this trajectory.

For $t \geq T$ we assume that c and K remain constant at $c = \frac{\rho}{\gamma(\mu-1)}$, $K = K_{MGR}$, while

$$\dot{N} = C_{MGR} - N\frac{\rho}{\gamma(\mu - 1)}.$$

Since $r = \rho$ and c is constant, the consumer's optimality conditions are clearly satisfied for $t \geq T$. Since c is C1 as a function t, they are also satisfied locally around T.

Since the point $c = \frac{\rho}{\gamma(\mu-1)}$, $K = K_{MGR}$ lies on II, i.e. $r(K_{MGR}) = \rho = \gamma(\mu - 1)c_N$, we clearly have, for $t \geq T$, $V = \frac{(\mu-1)c_N}{r} = \frac{1}{\gamma}$, consistent with the free entry condition for innovation.

Last, we need to prove that $V < 1/\gamma$ for $t < T$. As in the proof of Proposition 3, A, we have that for $t < T$

$$V_t = \int_t^T (\mu - 1)c_t e^{-\int_t^s r_u du} ds + \frac{1}{\gamma} e^{-\int_t^T r_u du}.$$

Since $K_t < K_T = K_{MGR}$, $r_t > \rho$. Furthermore, $c_t < \frac{\rho}{\gamma(\mu-1)}$. Therefore $V_t < \int_t^T \frac{\rho}{\gamma} e^{-\rho(s-t)} ds + \frac{1}{\gamma} e^{-\rho(T-t)} = 1/\gamma$.

Finally, the property that $N_N > N_S$ is straightforward, since $N_N = C_{MGR} \frac{\gamma(\mu-1)}{\rho} > \beta C_{MGR} > \beta C_S = N_S$.

QED

9.5 Proof of Proposition 7

Consider a trajectory such that (i), (ii) and (v) hold. It follows from (i) that the net marginal product of capital satisfies (iii). It is also true from (ii) that the consumption Euler equation (13) holds. Property (iv) holds by construction and from the production function $Y_t = A_t K_t^\alpha$. In particular Y is proportional to K, $Y = \frac{r+\delta}{\alpha} K = yA$. From (iii) and (ii) we get that the equilibrium condition for innovation $V = \pi/r = 1/\gamma$ holds. Given the equilibrium values of r and g, the property $n > 0$ follows from the assumption that $g_A < \frac{1-\alpha}{\alpha} \left(\frac{\gamma(\mu-1)}{\beta} + \delta(1-\alpha) \right)$. To complete the proof, just substitute the values of $N = nA$, $K = kA, Y = yA$ into the capital accumulation equation (5), which can be rewritten as

$$\dot{K}_t = A_t K_t^\alpha - C_t - \delta K_t - \dot{N}_t/\gamma, \tag{21}$$

and check that it is satisfied. QED.

9.6 Proof of Proposition 8

The proof is similar to that of Proposition 8, with the following differences. The Euler equation now holds because $r = \rho$. The equilibrium consumption level is the one which guarantees equilibrium innovation at $r = \rho$. The assumption $\rho < \frac{\gamma(\mu-1)}{\beta}$ guarantees that this level is below satiation. In addition, the consumer's transversality condition has to hold (it always holds under satiation). For this to be the case we need that $r > g$, which is implied by the assumption that $g_A < (1-\alpha)\rho$. This in turn guarantees that $(\rho + \delta)/\alpha - g - \delta > 0$.

QED.

9.7 Proof of Proposition 9

We construct a trajectory based on a path $\{c_t\}$. From the Euler equation, provided its initial value is between $\rho/(\gamma(\mu-1))$ and $1/\beta$, $c_t \longrightarrow 1/\beta$. Furthermore, let us impose that the economy is in the innovation regime throughout. Then $r_t = \gamma(\mu-1)c_t$, implying that

$$K_t = \left(\frac{\alpha A_t}{\gamma(\mu-1)c_t+\delta}\right)^{\frac{1}{1-\alpha}}. \tag{22}$$

Since K is a state variable, this pins down the initial c, c_0. Furthermore, for $k_0 = K_0/A_0^{\frac{1}{1-\alpha}}$ above and close enough to k, c_0 will be below and close to $1/\beta$, and therefore above $\rho/(\gamma(\mu-1))$, implying indeed that $c_t \longrightarrow 1/\beta$.

To have $L_t = 1$, throughout, we need that $w_t(1-\beta c_t) > \mu\tau$. Note that

$$\begin{aligned} w_t &= (1-\alpha)A_tK_t^{\alpha} \\ &= \left(\frac{\alpha}{\gamma(\mu-1)c_t+\delta}\right)^{\frac{\alpha}{1-\alpha}}(1-\alpha)A_t^{\frac{1}{1-\alpha}}. \end{aligned}$$

A sufficient condition for the quantity $A_t^{\frac{1}{1-\alpha}}(\gamma(\mu-1)c_t+\delta)^{-\frac{\alpha}{1-\alpha}}(1-\beta c_t)$ to grow over time is

$$\frac{1}{1-\alpha}g_A - \frac{\alpha}{1-\alpha}\left(\frac{1}{\beta}-c_t\right)(\gamma(\mu-1)-\rho/c_t) - (\gamma(\mu-1)c_t-\rho) > 0.$$

Clearly, since we have assumed that $g_A > (1-\alpha)\left[\frac{\gamma(\mu-1)}{\beta}-\rho\right]$, this holds if c close enough to $1/\beta$, i.e. again $K/A^{\frac{1}{1-\alpha}}$ close enough to k. Along our trajectory c goes up and $K/A^{\frac{1}{1-\alpha}}$ falls. Therefore, we can always pick initial conditions so that these two quantities are arbitrarily close to $1/\beta$ and k, respectively. In such a case, the inequality $w_t(1-\beta c_t) > \mu\tau$ will hold for all t, provided it holds initially, that is

$$\left(\frac{\alpha}{\gamma(\mu-1)c_0+\delta}\right)^{\frac{\alpha}{1-\alpha}}(1-\alpha)A_0^{\frac{1}{1-\alpha}}(1-\beta c_0) > 0.$$

Clearly, given the initial value of k_0 and hence of c_0, the preceding inequality holds for A_0 large enough.

To summarize, there exists η such that we can choose $k_0 \in (k, k+\eta)$ and $A_0 > A(k_0)$ in such a way that for $K_0 = k_0A_0^{\frac{1}{1-\alpha}}$, the trajectory for K_t defined by (22), with $\{c_t\}$

the unique solution to (13) such that (22) holds at $t = 0$, is such that the representative consumer would pick $L = 1$ throughout. Furthermore, by construction the economy is in the innovation regime throughout, and since $c_t \longrightarrow 1/\beta$, $k_t = K_t/A_t^{\frac{1}{1-\alpha}} \longrightarrow k$.

To complete our construction, we have to check the feasibility of the implied trajectory for N_t. Let $g = g_A/(1-\alpha)$ and $n_t = N_t/A_t^{\frac{1}{1-\alpha}}$. Using (21) we get that

$$\dot{n}_t/\gamma = k_t^\alpha - \delta k_t - \dot{k}_t - g k_t - c_t n_t - g n_t/\gamma.$$

Note that $n = \frac{k^\alpha - (g+\delta)k}{1/\beta + g/\gamma}$Clearly, then, as $c_t \to 1/\beta$ and $k_t \to k, n_t \to n$. For the trajectory for N to be feasible, we need that $\dot{N}_t \geq 0$ throughout. Note that we can always choose η small enough such that the quantity $k_t^\alpha - (g+\delta)k_t$ is arbitrarily close to $(1/\beta + g/\gamma)n$ and c_t is arbitrarily close to $1/\beta$. Then

$$\frac{\dot{n}_t}{\gamma n_t} = -\frac{\dot{k}_t}{n_t} + \left(\frac{k_t^\alpha - (g+\delta)k_t}{n_t} - \frac{(1/\beta + g/\gamma)n}{n_t}\right) + \left(\frac{(1/\beta + g/\gamma)n}{n_t} - (c_t + \frac{g}{\gamma})\right).$$

If n_t is sufficiently close to n, the two terms in brackets are arbitrarily small. Since the first term is > 0, this can be made larger than any negative number. In particular, larger than $-g/\gamma$. Thus we can pick k_0 and n_0 simultaneously close enough to k and n to make sure that $\dot{N}/N = \dot{n}/n + g > 0$.

QED

9.8 Proof of Proposition 10

We construct our trajectory in a similar fashion as for Proposition 9. Given a path for c_t, we make sure that we are in the innovation regime and that there is an interior solution for labor supply. That is

$$\begin{aligned} r_t &= \alpha A_t K_t^{\alpha-1} L_t^{1-\alpha} - \delta \\ &= \gamma(\mu - 1)c_t \end{aligned}$$

and

$$\begin{aligned} L_t &= \frac{w_t}{\tau\mu}(1 - \beta c_t) \\ &= \left[\frac{(1-\alpha)A_t}{\tau\mu}\right]^{\frac{1}{1+\alpha}} K_t^{\frac{\alpha}{1+\alpha}} (1 - \beta c_t)^{\frac{1}{1+\alpha}}, \end{aligned}$$

where the marginal productivity condition $w_t = (1-\alpha)A_t K_t^{\alpha} L_t^{-\alpha}$ has been used to derive this last expression. In turn, we can solve for K and L as a function of c :

$$K_t = \varphi A_t^{\frac{2}{1-\alpha}}(1-\beta c_t)(\gamma(\mu-1)c_t+\delta)^{-\frac{1+\alpha}{1-\alpha}}, \tag{23}$$

$$L_t = \psi A_t^{\frac{1}{1-\alpha}}(1-\beta c_t)(\gamma(\mu-1)c_t+\delta)^{-\frac{\alpha}{1-\alpha}}. \tag{24}$$

The composite parameters φ and ψ are defined as follows:

$$\begin{aligned} \varphi &= \alpha^{\frac{1+\alpha}{1-\alpha}}\left(\frac{1-\alpha}{\tau\mu}\right), \\ \psi &= \left(\frac{1-\alpha}{\tau\mu}\right)^{\frac{1}{1+\alpha}} \varphi^{\frac{\alpha}{1+\alpha}}. \end{aligned}$$

We note that (23) defines K as a decreasing function of c which maps $(0, 1/\beta)$ onto $(0, +\infty)$. Therefore, there exists a unique c_0 associated with the initial capital stock K_0. Furthermore, if $\kappa_0 = K_0/A_0^{\frac{2}{1-\alpha}}$ is not too large, then $c_0 > \rho/(\gamma(\mu-1))$. Also, from (23), for κ_0 c_0 is arbitrarily close to $1/\beta$.

Since (13) here is equivalent to

$$\dot{c}_t = \left(\frac{1}{\beta} - c_t\right)(\gamma(\mu-1)c_t - \rho), \tag{25}$$

we clearly have that $\dot{c} > 0$ throughout and $c_t \to 1/\beta$. Then, using (25) and (23, we have that

$$\frac{\dot{K}_t}{K_t} = \frac{2}{1-\alpha}g_A - (\gamma(\mu-1)c_t - \rho) - \frac{1+\alpha}{1-\alpha}\frac{(\gamma(\mu-1)c_t - \rho)(1/\beta - c_t)}{\gamma(\mu-1)c_t + \delta}. \tag{26}$$

Therefore

$$\lim_{t\to\infty}\frac{\dot{K}_t}{K_t} = \frac{2}{1-\alpha}g_A - (\frac{\gamma(\mu-1)}{\beta} - \rho) = g > 0,$$

where the inequality comes from the assumption that $g_A > \frac{1-\alpha}{2}(\gamma(\mu-1)/\beta - \rho)$. A corollary is that for $0 < g' < g$, if c_t is close enough to $1/\beta$, the RHS of (26) is $> g'$. Therefore, we can always pick κ_0 small enough so that $\frac{\dot{K}_t}{K_t} > g' > 0$ throughout our entire constructed trajectory.

It is also possible to compute $Y_t = A_t K_t^\alpha L_t^{1-\alpha}$, and we get

$$Y_t = \chi A_t^{\frac{2}{1-\alpha}} (1 - \beta c_t)(\gamma(\mu - 1)c_t + \delta)^{-\frac{2\alpha}{1-\alpha}},$$

where $\chi = \varphi^\alpha \psi^{1-\alpha}$. Again $\lim_{t\to\infty} \frac{\dot{Y}_t}{Y_t} = g$.

Turning now to labor supply, we have to make sure that L_t as defined by (24) remains < 1 throughout. We have that

$$\frac{\dot{L}_t}{L_t} = \frac{1}{1-\alpha} g_A - (\gamma(\mu - 1)c_t - \rho) - \frac{\alpha}{1-\alpha} \frac{(\gamma(\mu - 1)c_t - \rho)(1/\beta - c_t)}{\gamma(\mu - 1)c_t + \delta}. \quad (27)$$

Consequently,

$$\lim_{t\to\infty} \frac{\dot{L}_t}{L_t} = \frac{1}{1-\alpha} g_A - (\frac{\gamma(\mu - 1)}{\beta} - \rho) = g_L < 0,$$

where the inequality comes from the assumption that $g_A < (1 - \alpha)(\gamma(\mu - 1)/\beta - \rho)$.

Clearly, then, if c_t is close enough to $1/\beta$, the RHS of (27) is negative. Since c_t grows over time, all we need for this to be the case for all t is that c_0 is close enough to $1/\beta$, or equivalently, again, that κ_0 is not too large. Then, for L_t to be lower than 1, we just need it to be lower than 1 initially. Substituting (23), into (24), we get that

$$L_t = \frac{\psi}{\varphi} A_t^{-\frac{1}{1-\alpha}} K_t (\gamma(\mu - 1)c_t + \delta)^{\frac{1}{1-\alpha}}.$$

Thus we have $L_0 < 1$ provided $K_0 A_0^{-\frac{1}{1-\alpha}}$ is small enough.

To conclude the proof, we need to check that the implied trajectory for N_t is feasible. Let $z_t = N_t/K_t$. The law of motion (21) can be rewritten as

$$\frac{\dot{z}_t}{\gamma} = \frac{\chi}{\varphi}(\gamma(\mu - 1)c_t + \delta) - z_t(c_t + \frac{1}{\gamma}\frac{\dot{K}_t}{K_t}) - \delta - \frac{\dot{K}_t}{K_t}.$$

Clearly, then $z_t \to \frac{\frac{\chi}{\varphi}(\frac{\gamma(\mu-1)}{\beta} + \delta) - g - \delta}{1/\beta + g/\gamma} = z$. Since $\chi/\varphi = 1/\alpha$, this expression is positive iff

$$g_A < \frac{1-\alpha}{2}\left[\frac{1+\alpha}{\alpha}\frac{\gamma(\mu - 1)}{\beta} + \frac{1-\alpha}{\alpha}\delta - \rho\right].$$

The expression on the RHS is always larger than $(1 - \alpha)\left(\frac{\gamma(\mu-1)}{\beta} - \rho\right)$, which is $> g_A$ by (18).

Recall that by picking κ_0 small enough, we can have c_t arbitrarily close to $1/\beta$ and $\frac{\dot{K}_t}{K_t}$ arbitrarily close to g throughout the entire trajectory. If in addition to that, we pick z_0 close enough to z, $\dot{z}_t/z_t$ will be arbitrarily close to zero. Consequently, $N_t = z_t K_t$ will grow at a strictly positive rate throughout. This proves that the trajectory for N is feasible.

QED

Figure 1-A

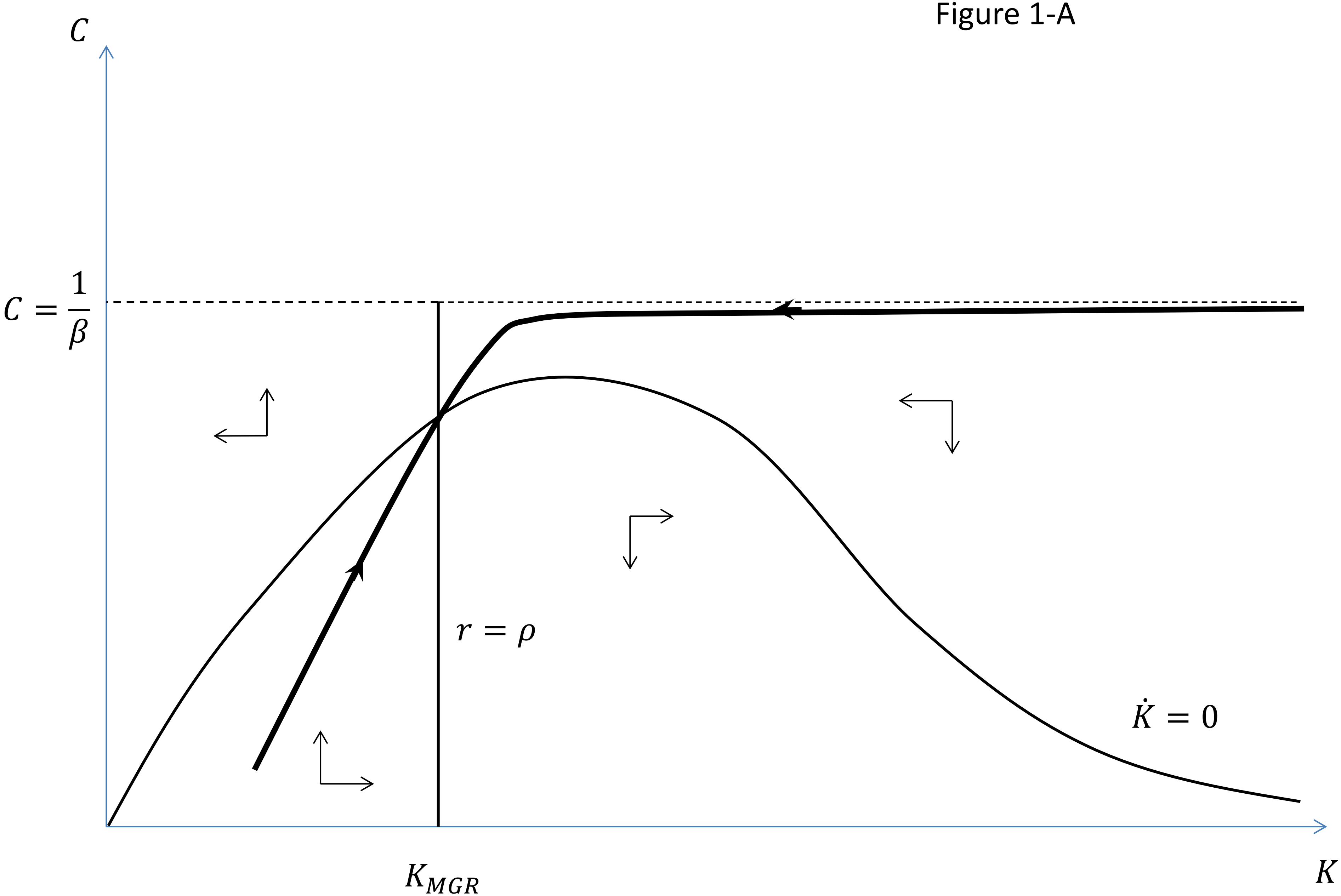

Figure 1-B

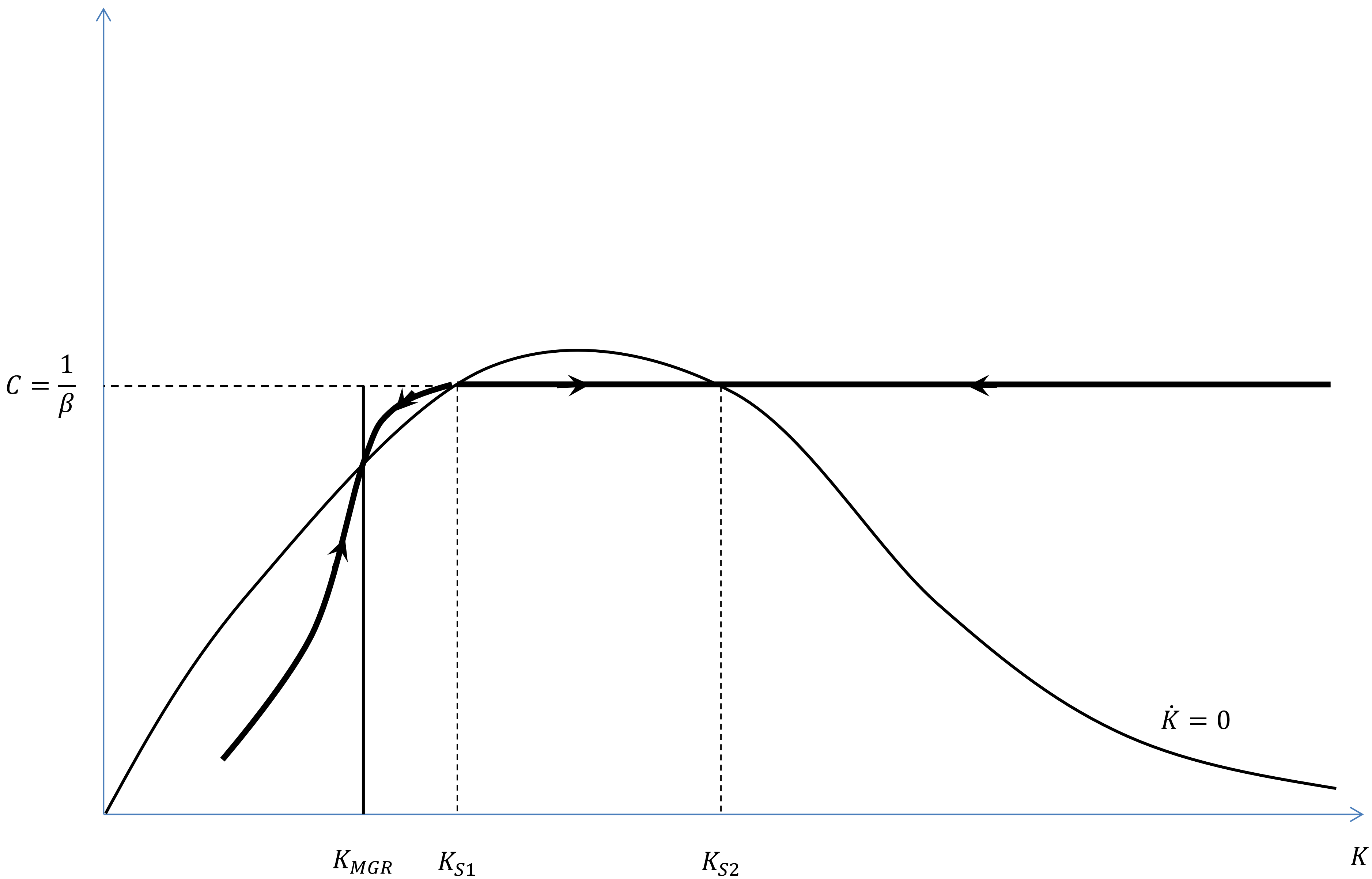

Figure 1-C

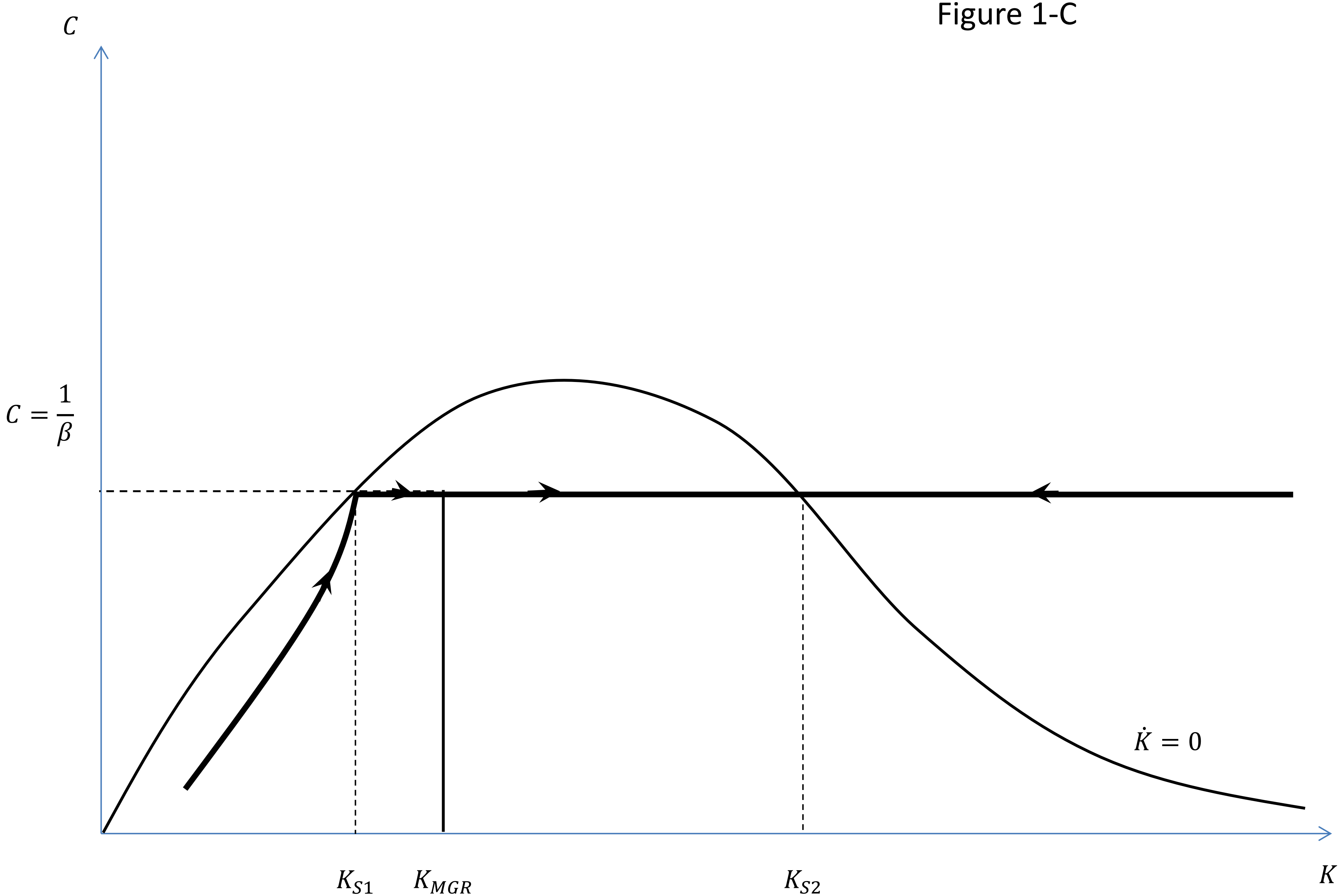

Figure 2 – long run satiation

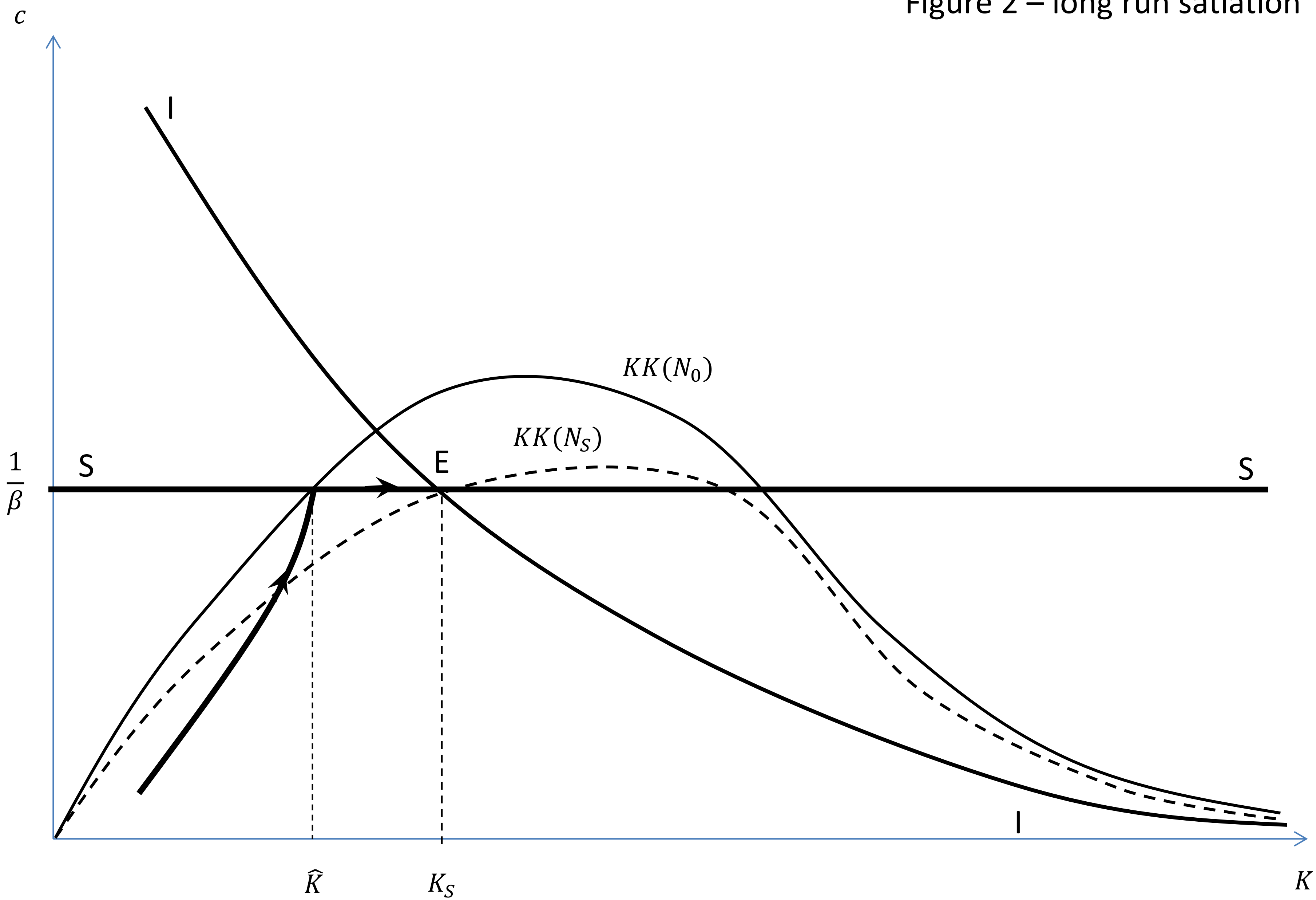

Figure 3 – No satiation, innovation

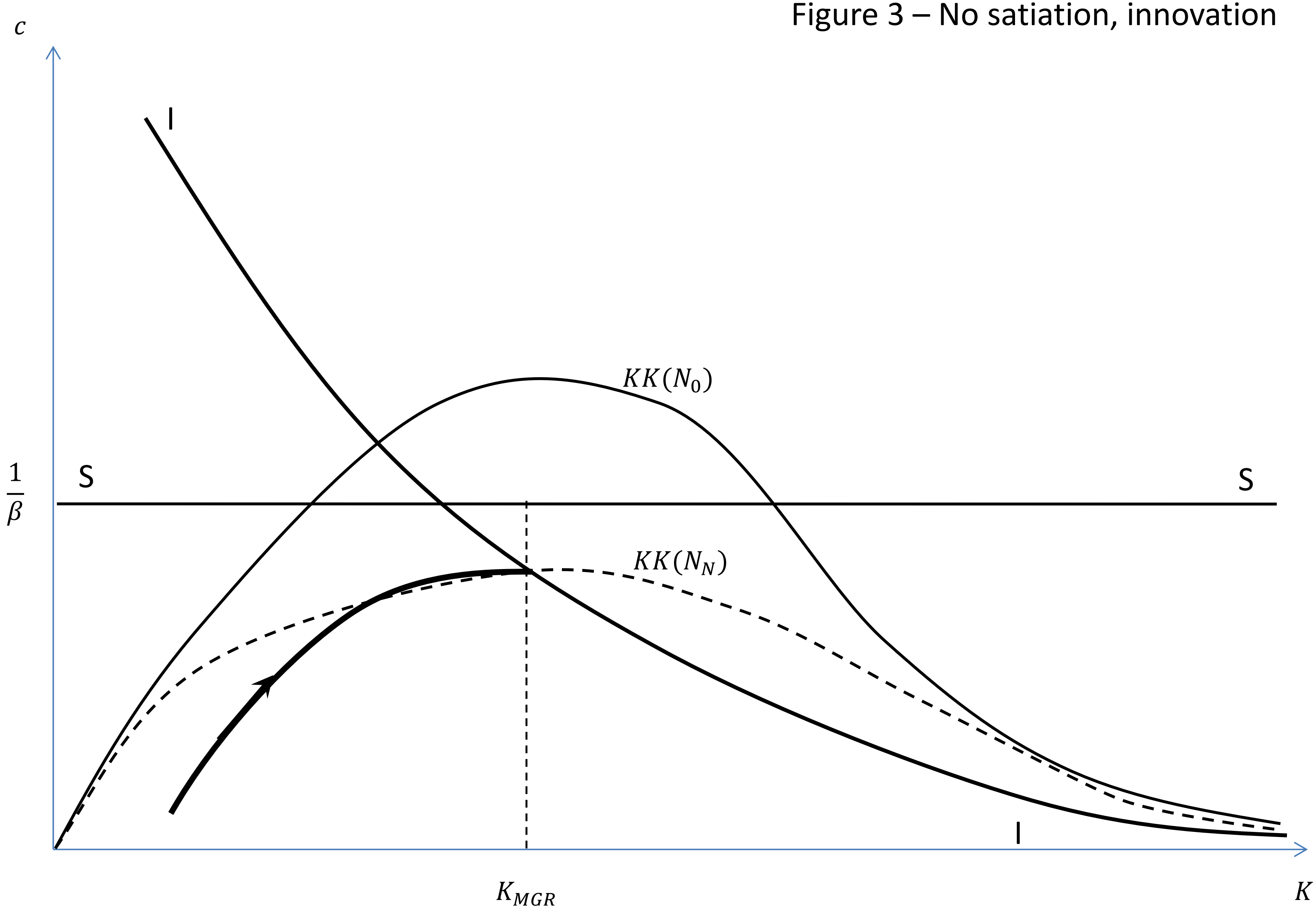

Figure 4a

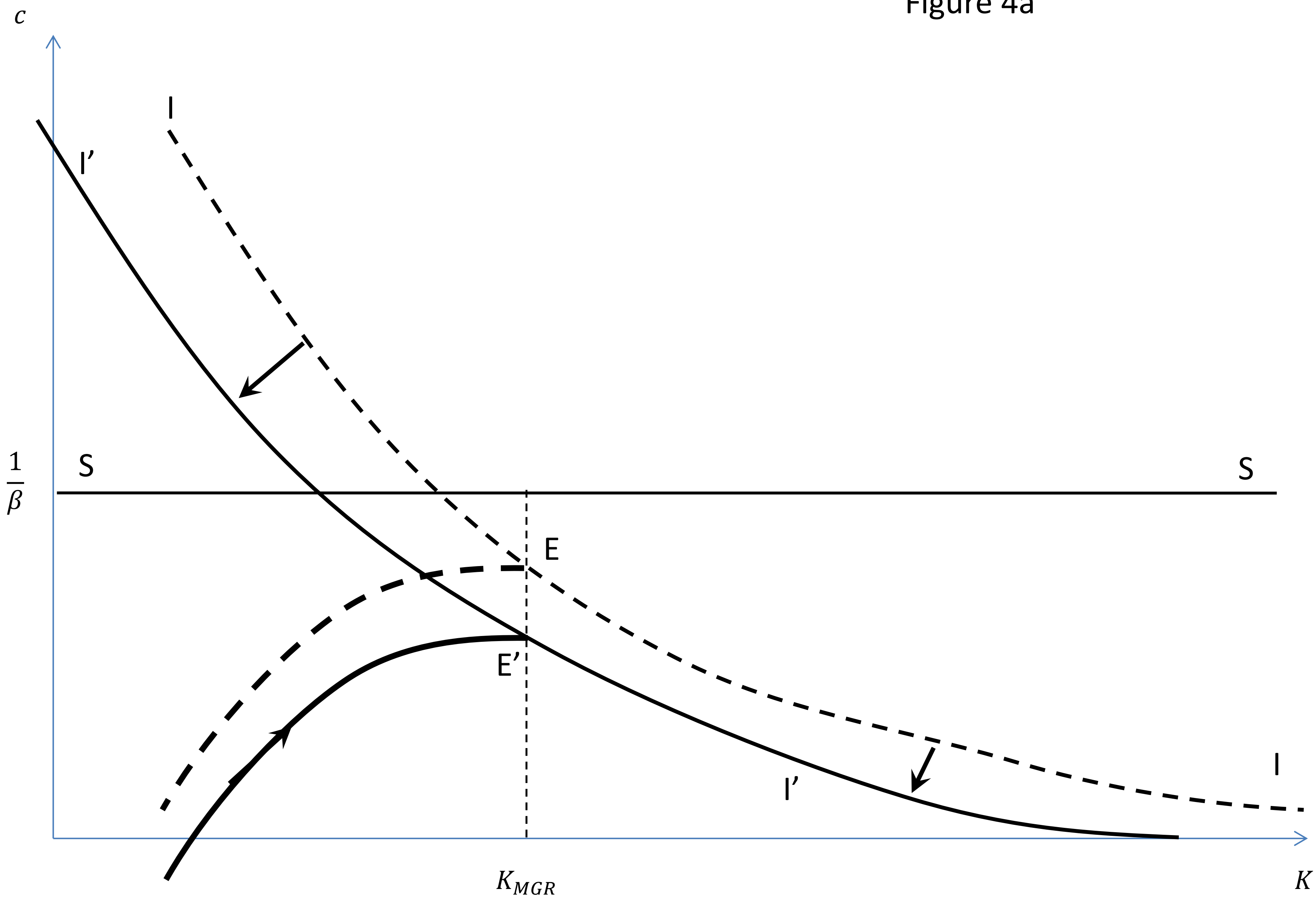

Figure 4b

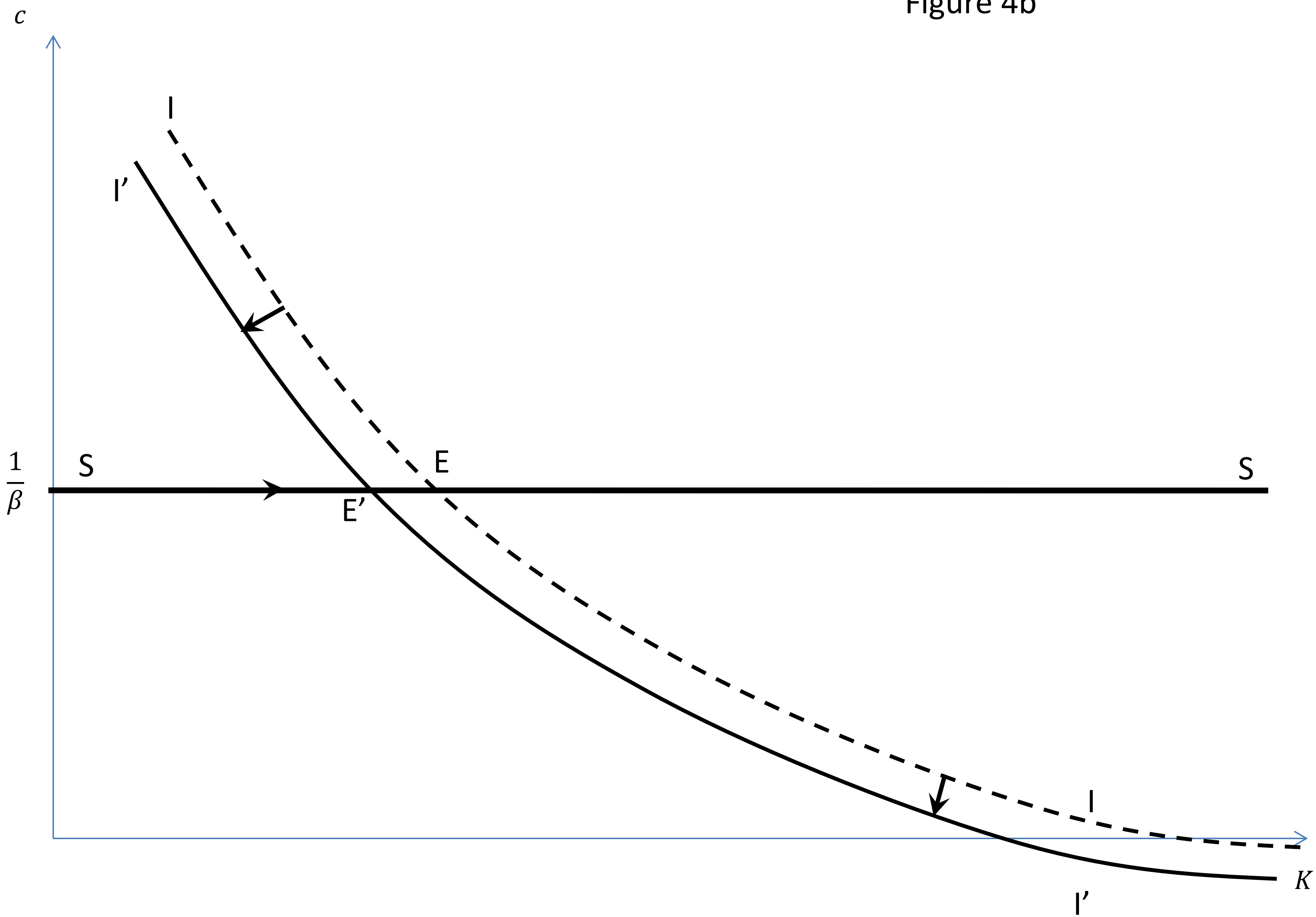

Figure 5 – Interest rate determination under heterogeneous agents