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Abstract The distortion operator proposed by Wang (2000) has been developed in the actuarial literature and that are now part of the risk measurement tools inventory available for practitioners in finance and insurance. In this article, we propose an alternative class of distortion operators with explicit analytical inverse mapping. The distortion operators are based on tangent function allowing to transform a symmetrical unimodal distribution to an asymmetrical multimodal distribution.

Keywords Distortion operator · Multimodal distribution · Asymmetry · Invertibility
JEL classification C20 · G32

1 Introduction

To integrate financial and actuarial insurance pricing theories, Wang (2000) proposes a form of insurance risk pricing based on the standard Gaussian cumulative distribution function (cdf) distortion operator with one parameter. He points out that this operator is either concave (when the parameter is

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positive) or convex (when the parameter is negative). Hamada and Sherris (2003) suggests that this operator can shift the quantile of a distribution to the left, thereby assigning higher probabilities to low outcomes. A number of papers propose applications of distortion operators. For example, Härlimann (2004) obtains an optimal economic capital formula, under suitable assumptions on insurance market prices, the collection of possible losses, and the distortion function. Lin and Cox (2005) successfully applies Wang transform to price mortality risk bonds. Hamada et al. (2006) shows a formal treatment of risk measures based on distortion functions in discrete-time setting. They conclude that the risk neutral computational approach is well adapted to portfolio optimisation with such measures that do not lie within the expected utility framework. De Jong and Marshall (2007) provides a method for analysing and projecting mortality based on the Wang Transform. Denuit et al. (2007) designs the survivor bonds with the help of Wang transform (Wang (2000)) which could be issued directly by insurers.

However, Godin et al. (2012) argues that it is a well-known fact that the returns of most financial assets have semi-heavy tails. Consequently he recognizes that a downside of the normal distortion of Wang (2000) is its underlying symmetrical that poses some constraints in applications. More precisely, Guégan and Hassani (2015) criticize that Wang (2000) applies the same perspective of preference to quantify the risk associated to gain and risk. Thus, the risk manager evaluates the risk associated to the upside and downside risks with the same function that implies a symmetrical consideration for the two effects due to the distortion.

Accordingly, a number of papers make proposals on how to avoid the problem of symmetry in the previous distortion operators. For example, van der Hoek and Sherris (2001) proposes to use two different distortion functions $g(x)$ and $h(x)$ (for instance when $g(x)$ is concave, a convex $h(x) = 1 - g(1 - x)$ would be one possible choice), to allow a different treatment of the upside and downside of the risk. In Wang (2004) a Student-t distribution based distortion operator is introduced allowing for skewness. Sereda et al. (2010) proposes to use two different functions issued from the same polynomial with different coefficients. Godin (2012) introduces a distortion operator based on the Normal Inverse Gaussian (NIG) distribution, which can asymmetrically distort the underlying distribution. Guégan and Hassani (2015) suggests the use of an inverse S-shaped polynomial function of degree 3 as a distortion operator to create an asymmetrical distribution. Additionally this operator can have a concave part and a convex part by varying its parameters.

Wirch and Hardy(1999) concludes that there is an interest in investigating measures for capital adequacy that utilise the shape of the loss distribution, particularly in the right tail. Specifically, multimodality is one of the important characteristics of probability distribution. A number of papers related to the income distribution find evidence of multimodality (for a review, see, e.g., Bianchi (1997), Jones (1997), Quah (1996a, 1996b, 1997) and Zhu (2005)), for instance the results of Zhu (2005) indicate that the US personal income distribution has been multimodal in 1962, 1972, 1982, 1992 and 2000. Also, he suggests that changes in the shape of the income distribution over the entire income range provide rich information and shed light on issues of income inequality, poverty traps and convergence. Moreover, he argues that theoretical models should be capable of explaining and generating multimodal income distributions.

Importantly, motivated by the crisis, the multimodal characteristic of distributions of some economic variables, for example some stock market indexes like S&P 500 index (the Standard & Poor's 500), SHCOMP index (Shanghai Stock Exchange Composite Index) and FTSE Index (the Financial Time Stock Exchange 100 Index), can include useful information of systemic risk. Accordingly it is necessary to find a model such that it is flexible enough to accommodate various shapes of continuous distributions with leptokurtic, skewed and multimodal characteristics.

Additionally, the financial industry has extensively used quantile-based downside risk measures based on the Value-at-Risk (VaR). In actuarial terms, VaR is a quantile reserve, often using the p^{th} percentile of the loss distribution. We should emphasize that when we compute the VaR, the explicit analytical form of the inverse mapping of cdf is crucial. Consequently, we propose an alternative class of distortion operators allowing to build an asymmetrical multimodal distribution, with explicit analytical inverse mapping.

We proceed as follows. Section 2 describes the definition and some basic properties of distortion operators. Section 3 presents our model. Section 4 assess the properties of three distortion operators by simulation. Section 5 concludes.

2 Distortion operator

For a continuous distribution with cdf F (f is its probability density function (pdf)), we recall the definitions of distortion operator and multimodal distribution.

Definition 1 (Distortion operator) From Wang (2000), a mapping g: $[0,1] \to [0,1]$ is a distortion operator if: g is continuous and increasing; $g(0) = 0$ and $g(1) = 1$.

Definition 2 (Multimodal distribution) We call F a multimodal distribution if its pdf f has multiple local maxima.

Then we give a definition

Definition 3 (Changing point of concave-convex property) A point $x_0 \in (0,1)$ is called a changing point of concave-convex property for a distortion operator g if $g''(x_0) = 0$, $g''(x) < 0$ when $x \in [0, x_0)$ and $g''(x) > 0$ when $x \in (x_0, 1]$.

In his article Wang (2000) specifies that the distortion operator g can be applied to any distributions. A distortion operator g can always transform a cdf F to another cdf $g \circ F$ (denotes the composition function of g and F). Denoting Φ the cdf of the standard Gaussian distribution with mean 0 and variance 1 (N(0,1)), Wang (2000) proposes a distortion operator g_W as follows:

$$g_W(x) = \Phi[\Phi^{-1}(x) + a], \quad x \in [0,1] \tag{1}$$

By illustrating the impact of g_W on the logistic distribution, Guégan and Hassani (2015) observes the shift of the mode of the initial distribution only. Further more the close form of g_W is not straightforward.

To create multi-modal distributions $g \circ F$, assuming f is differentiable and g is twice differentiable, we can derive its associated pdf $(g \circ F(x))' = (g(F(x)))' = g'(F(x))f(x)$ and

$$(g \circ F(x))'' = g''(F(x))f^2(x) + g'(F(x))f'(x). \tag{2}$$

By definition, $g'(F(x))$ is always positive. Thus to add hump in $(g \circ F(x))'$ for a given F, we need to manipulate the sign of $g''(F(x))$. Consequently concave-convex property of g need to be considered.

Guégan and Hassani (2015) provides an inverse S-shaped polynomial function of degree 3 given by the following equation and characterized by a location parameter δ and a shape parameter β

$$g_p(x) = a[\frac{x^3}{6} - \frac{\delta}{2}x^2 + (\frac{\delta^2}{2} + \beta)x], \quad x \in [0,1] \tag{3}$$

where $a = (\frac{1}{6} - \frac{\delta}{2} + \frac{\delta^2}{2} + \beta)^{-1}$, $\delta \in [0,1]$ and $\beta \in R$. They remark that g_p's curve exhibits a concave part and a convex part. We can derive that $g'' = a(x - \delta)$. Thus when $F(x) = \delta$, $g''(F(x))$ equals to 0. However, the sign of $g''(F(x))$ depends on the sign of a, for $F(x) \in [0, \delta)$ and $F(x) \in (\delta, 1]$. Consequently to locate the changing point of concave-convex property of g by δ is confusing.

3 A new class of distortion operators

The objective is to construct a distortion operator g with property below

Property 1 *It is twice differentiable. It contains a changing point of concave-convex property x_0, with a location parameter to locate x_0 and with a shape parameter to control the concave-convex level of g (characterised by $|g''|$: the absolute value of g''). Furthermore the inverse mapping of g has a straightforward closed-form.*

In order to construct a g satisfying Property 1, first we construct a distortion operator g_1 by tangent function (since $tan(x)$, $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$ is a smooth function (it has derivatives of all orders) containing x_0, and its inverse mapping is $arctan(x)$)

Definition 4 (Distortion operator g_1) For $x \in [0,1]$ with a shape parameter $0 < a < \pi$

$$g_1(x) = \frac{1}{2tan\frac{a}{2}}(tan(ax - \frac{a}{2}) + tan\frac{a}{2}) \tag{4}$$

The first and second derivatives of g_1 are

$$g_1'(x) = \frac{a}{2tan\frac{a}{2}} \frac{1}{cos^2(ax - \frac{a}{2})} \tag{5}$$

$$g''(x) = \frac{a^2}{tan\frac{a}{2}} tan(ax - \frac{a}{2}) \tag{6}$$

The inverse mapping of $g_1(x)$ is

$$g_1^{-1}(x) = \frac{1}{2} + \frac{1}{a} arctan(2xtan\frac{a}{2} - tan\frac{a}{2}) \tag{7}$$

One can verify that $g_1(x)$ is a distortion operator and it has $x_0 = \frac{1}{2}$. Thus the concave-convex property of g_1 is symmetrical.

Comparing with Property 1 for g_1, a location parameter to locate x_0 is needed, which allows g_1 to have an asymmetrical concave-convex property.

Definition 5 (Distortion operator g_2) For $x \in [0,1]$ with shape parameter $0 < a < \pi$ and location parameter $b \in (\frac{1}{2}, \infty)$

$$\begin{cases} g_2(x) = \dfrac{1}{2btan\frac{a}{2}} tan(abx - \dfrac{a}{2}) + \dfrac{1}{2b}, & 0 \le x \le \dfrac{1}{2b} \\ g_2(x) = \dfrac{2b-1}{2btan\frac{a}{2}} tan(\dfrac{ab}{2b-1}x + \dfrac{a}{2-4b}) + \dfrac{1}{2b}, & \dfrac{1}{2b} < x \le 1 \end{cases} \tag{8}$$

One can verify that g_2 satisfies all conditions in Property 1, whose first and second derivatives are

$$\begin{cases} g_2'(x) = \dfrac{a}{atan\frac{a}{2}} \dfrac{1}{cos^2(abx - \frac{a}{2})}, & 0 \leq x \leq \dfrac{1}{2b} \\ g_2'(x) = \dfrac{a}{atan\frac{a}{2}} \dfrac{1}{cos^2(\frac{ab}{2b-1}x + \frac{a}{2-4b})}, & \dfrac{1}{2b} < x \leq 1 \end{cases} \tag{9}$$

$$\begin{cases} g_2'' = \dfrac{a^2b}{tan\frac{a}{2}} tan(abx - \dfrac{a}{2}), & 0 \leq x \leq \dfrac{1}{2b} \\ g_2'' = \dfrac{a^2b}{(2b-1)tan\frac{a}{2}} tan(\dfrac{ab}{2b-1}x + \dfrac{a}{2-4b}), & \dfrac{1}{2b} < x \leq 1 \end{cases} \tag{10}$$

The inverse mapping of g_2 is

$$\begin{cases} g_2^{-1} = \dfrac{1}{ab} arctan(2bxtan\dfrac{a}{2} - tan\dfrac{a}{2}) + \dfrac{1}{2b}, & 0 \leq x \leq \dfrac{1}{2b} \\ g_2^{-1} = \dfrac{2b-1}{ab} arctan(\dfrac{2btan\frac{a}{2}}{2b-1}x - \dfrac{tan\frac{a}{2}}{2b-1}) + \dfrac{1}{2b}, & \dfrac{1}{2b} < x \leq 1 \end{cases} \tag{11}$$

4 Simulation

In this section, first we show the evolution of g_p, g_1 and g_2 w.r.t (with respect to) different values of their parameters by simulation (g_p is the benchmark), which allows to compare these distortion operators and remark the difference of two properties of them: concave-convex property level and location of x_0. Second g_2 is applied to transform a unimodal F to a multimodal $g_2 \circ F$. We plot the pdf associated to $g_2 \circ F$ to check the influence of g_2.

4.1 Evolution of distortion operators

Firstly we check the concave-convex level of g_p, g_1 and g_2 affected by the shape parameter. Thus we always fix the location parameter. In Fig. 1, the value of δ is given by 0.5, then we plot the function g_p for different values of β ($\beta = 0.00003, 0.003, 0.03, 0.3$). The purpose of Fig. 1 is to show how β can effect the concave-convex level of g_p. We observe in this picture that the curve has symmetrical concave and convex parts, and low values of β are corresponding to high concave-convex level.

For g_1, to understand the influence of the parameter a on the shape of curve, we plot g_1 for $a = 0.1\pi, 0.8\pi, 0.9\pi, 0.95\pi$ in Fig. 2. The curves show that the function g_1 is always symmetrical. We observe that if a tends to 0 then g_1 tends to the identity mapping and when a tends to π the curve exhibits higher concave-convex level.

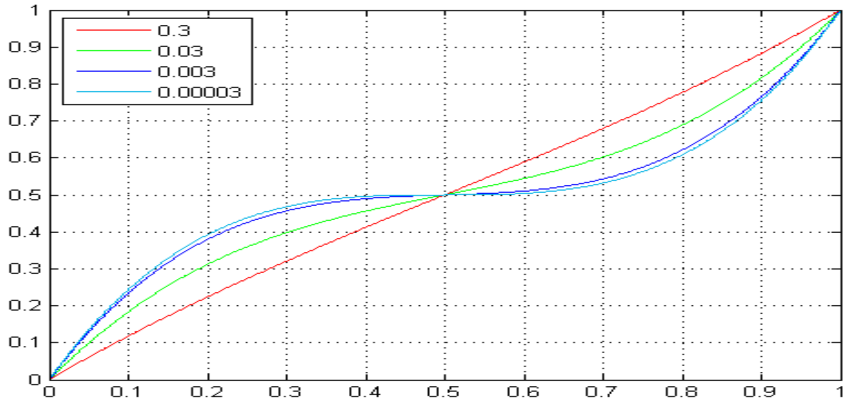

Fig. 1 Curves of the distortion function g_p introduced in equation (3) for several value of β ($\beta = 0.00003, 0.003, 0.03, 0.3$ and fixed value of $\delta = \frac{1}{2}$.

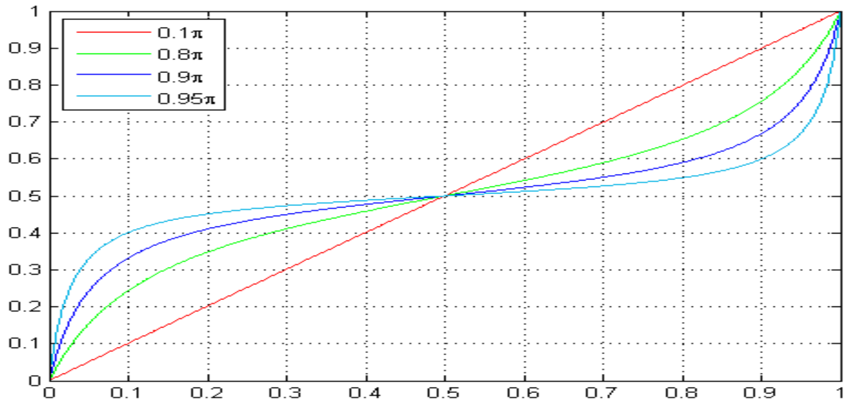

Fig. 2 we plot g_1 in equation (4) for $a = 0.1\pi, 0.8\pi, 0.9\pi, 0.95\pi$.

Fig. 3 illustrates the effect of the shape parameter a on g_2 using the same values of a as in Fig. 2 and fixed $b = 1$. We observe that the curve is symmetrical and in this case the shape parameter has the same effects as in Fig. 2.

Secondly we check the effects of location parameters of g_p and g_2 on the location of x_0. The location of x_0 characterises the asymmetrical concave-convex property of distortion operators.

For g_p, to illustrate the role of δ on the location of x_0, we use two graphs in Fig. 4. The left graph corresponds to the curve of g_p for $\delta = 0.45$ and $\beta = 0.00003, 0.003, 0.03, 0.3$. The right graph provides the curve of g_p for

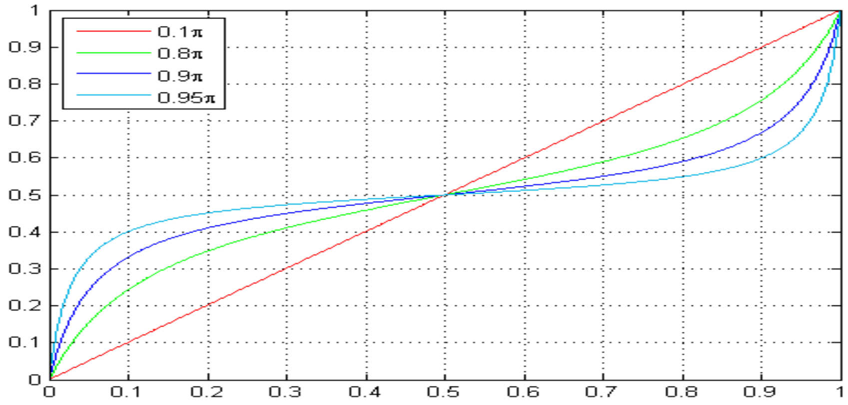

Fig. 3 we plot g_2 in equation (8) for $a = 0.1\pi, 0.8\pi, 0.9\pi, 0.95\pi$ and fixed $b = 1$.

$\delta = 0.3$ and $\beta = 0.00003, 0.003, 0.03, 0.3$. We observe that through δ one can indeed manipulate the location of x_0, but the relationship of them are confusing.

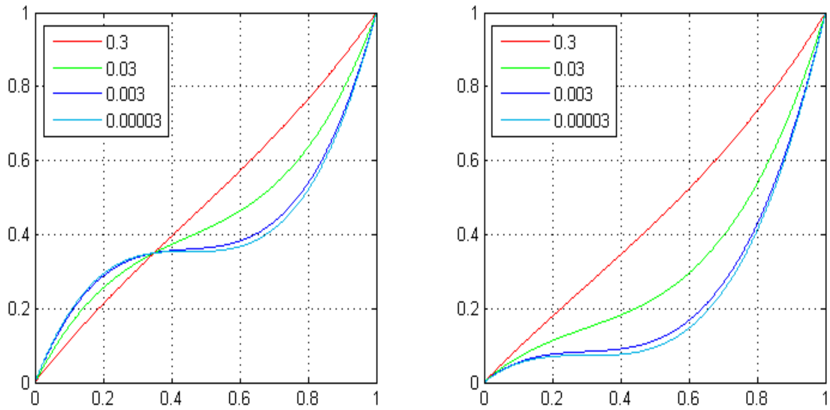

Fig. 4 The left graph corresponds to the curve of g_p for $\delta = 0.45$ and $\beta = 0.00003, 0.003, 0.03, 0.3$. The right graph provides the curve of g_p for $\delta = 0.3$ and $\beta = 0.00003, 0.003, 0.03, 0.3$.

For g_2, to understand the influence of the parameter b on the location of x_0, we provide two graphs in Fig. 5. The left graph corresponds to the curve of g_2 for $b = 2.5$ and $a = 0.1\pi, 0.8\pi, 0.9\pi, 0.95\pi$. The right graph provides the curve of g_2 for $= \frac{5}{8}$ and $a = 0.1\pi, 0.8\pi, 0.9\pi, 0.95\pi$. We observe the location of x_0 is exactly $\frac{1}{2b}$.

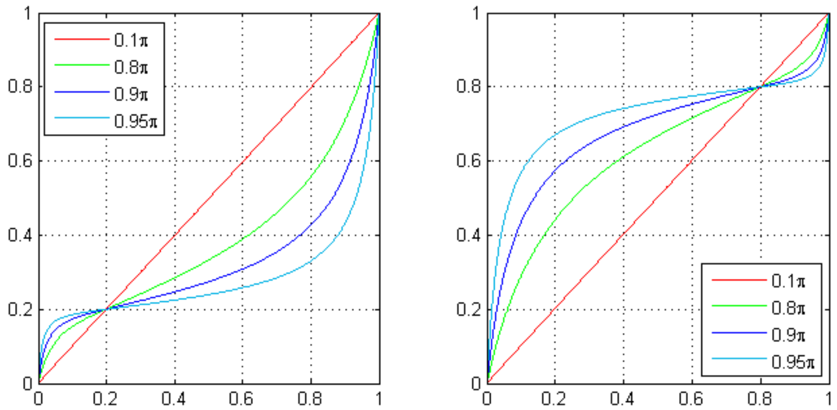

Fig. 5 The left graph corresponds to the curve of g_2 for $b = 2.5$ and $a = 0.1\pi, 0.8\pi, 0.9\pi, 0.95\pi$. The right graph provides the curve of g_2 for $= \frac{5}{8}$ and $a = 0.1\pi, 0.8\pi, 0.9\pi, 0.95\pi$.

From the argumentation and simulation above, we can summarize that g_2 meets all the conditions in Property 1.

4.2 Transform a unimodal distribution to a multimodal distribution

To explain how to use g_2 to transform a given unimodal F to a new asymmetrical multimodal $g_2 \circ F$, we begin with the N(0,1) with cdf F_G and pdf f_G.

By plotting the pdf of N(0,1) and the distorted density $g_2'(F_G(x))f_G(x)$ with fixed $b = 1$ and $a = 0.5\pi, 0.8\pi, 0.9\pi, 0.99\pi$, Fig. 6 shows the effect of shape parameter a of g_2 on N(0,1). For $b = 1$, $g_2'(F_G(x))f_G(x)$ are always symmetrical. As a increases, $g_2'(F_G(x))f_G(x)$ associates a small probability in the centre of the distribution and puts bigger weight in the tails. We observe two humps in the distorted density when a is large enough.

To investigate how the location parameter b of g_2 introduces asymmetrical property into $g_2'(F_G(x))f_G(x)$, we provide two graphs in Fig. 7 using the same values of the shape parameters than those used in Fig. 6, but b is 2.5 for the left graph and b is $\frac{5}{8}$ for the right one. From Fig. 7 we can remark that the values of a control the information under the humps and the values of b control the locations of the humps. When $b \neq 1$, $g_2'(F_G(x))f_G(x)$ is asymmetrical. More precisely, when the original distribution is symmetrical, the relatively

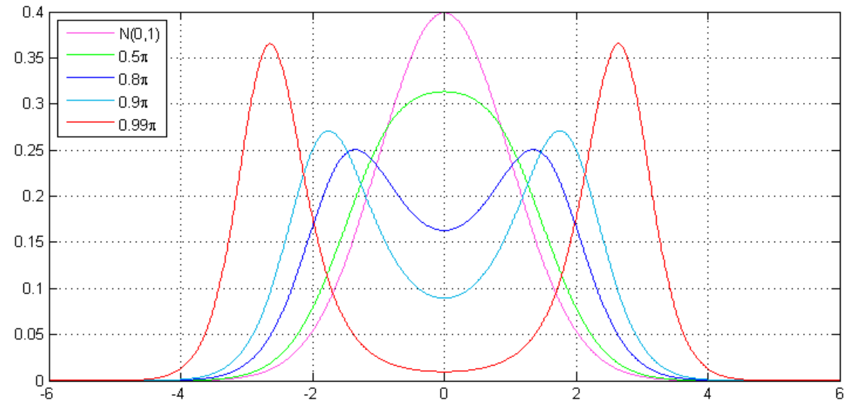

Fig. 6 We plot the pdf of N(0,1) and distorted density $g_2'(F_G(x))f_G(x)$ with fixed $b = 1$ and $a = 0.5\pi, 0.8\pi, 0.9\pi, 0.99\pi$.

high hump of the distorted density is in the right tail for the high value of b in g_2; the relatively high hump of the distorted density is in the left tail for the low value of b in g_2.

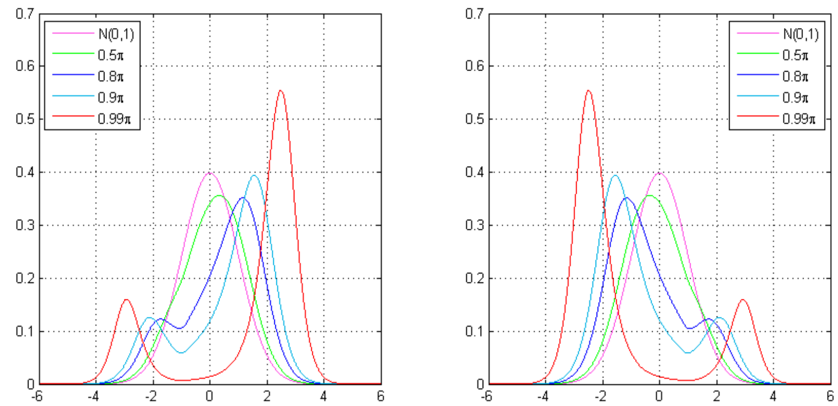

Fig. 7 The left graph corresponds to the curves of $g_2'(F_G(x))f_G(x)$ for $b = 2,5$ and $a = 0.5\pi, 0.8\pi, 0.9\pi, 0.99\pi$. The right graph provides the curves of $g_2'(F_G(x))f_G(x)$ for $b = \frac{5}{8}$ and $a = 0.5\pi, 0.8\pi, 0.9\pi, 0.99\pi$.

Besides assessing the location parameter b with F coming from the symmetrical, thin-tail distribution from the elliptical distribution family, it is also necessary to check the influence of b with F coming from the asymmetrical, fat-tail distribution. Let F_E (f_E the associated density) be GEV(0.2,0.05,-0.01) (Generalized extreme value distribution with shape parameter $k = -0.4$, scale parameter $\sigma = 0.05$ and location parameter $\mu = -0.01$.). Using the same value of the shape parameter $a = 0.95\pi$ but different location parameters $b = 0.625, 0.54$, we plot the distorted density $g_2'(F_E(x))f_E(x)$ in Fig. 8. Unexpectedly, instead of controlling the locations of the humps, we observe b controls the information under the humps. Especially lower value of b is associated to higher left hump and lower right hump.

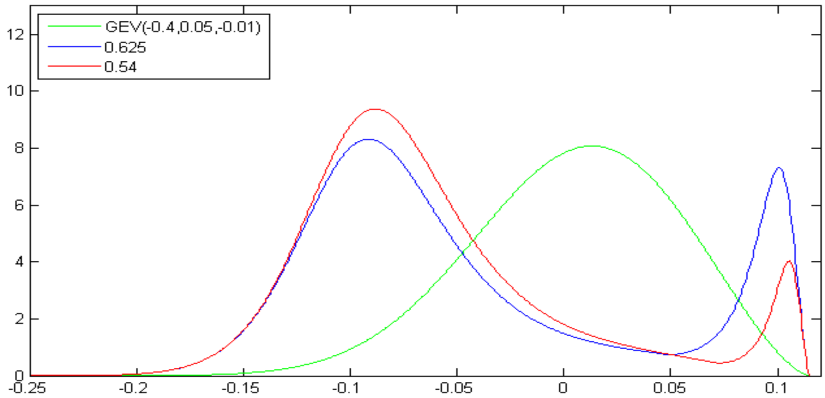

Fig. 8 Using the same value of the shape parameter $a = 0.95\pi$ but different location parameters $b = 0.625, 0.54$, we plot the distorted density $g_2'(F_E(x))f_E(x)$.

To check if parameter a can affect the locations of the humps or not, we plot the pdf of GEV(0.2,0.05,-0.01) and $g_2'(F_E(x))f_E(x)$ with the same $b = 1$ but different $a = 0.95\pi, 0.999\pi$ in Fig. 9. The graph suggests that the parameter a indeed effects the locations of the humps. We observe that the left hump shifts to the left when a increases. It is important to point out that in this simulation, the concave-convex property of g_2 is symmetrical since $b = 1$.

Motivated by Fig. 9, we plot the pdf of GEV(0.2,0.05,-0.01) and $g_1'(F_E(x))f_E(x)$ in Fig. 10 using the same values of the shape parameters than those used in Fig. 9. Comparing Fig. 9 and Fig. 10, we observe the same result. Consequently, we can remark that when the original distribution is asymmetrical, it is enough to use g_1 if the main purpose is to create an asymmetrical distorted density with two humps, which possesses the flexibility of shifting the positions of the humps.

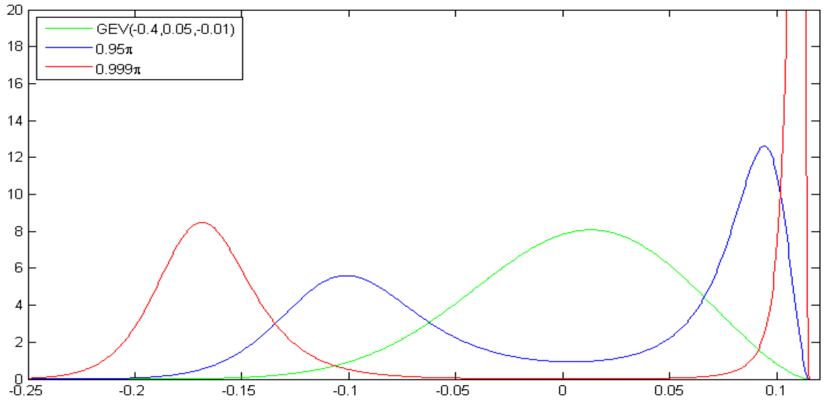

Fig. 9 We plot $g_2'(F_E(x))f_E(x)$ with the same $b = 1$ but different $a = 0.95\pi, 0.999\pi$.

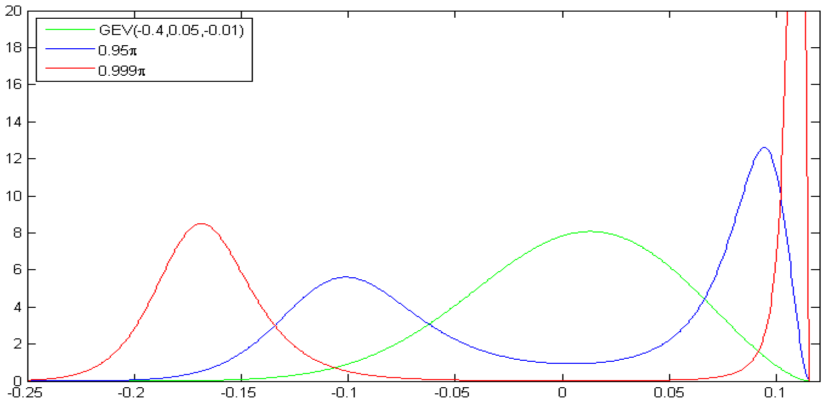

Fig. 10 We plot $g_1'(F_E(x))f_E(x)$ using the same values of the shape parameters than those used in Fig. 9., i.e. $a = 0.95\pi, 0.999\pi$.

5 Conclusion

In this article, we propose an alternative class of distortion operators with explicit analytical inverse mapping. The distortion operators are based on tangent function allowing to transform a symmetrical unimodal distribution to an asymmetrical multimodal distribution.

More precisely, when the original distribution is symmetrical, the first distortion operator with just shape parameter can only generate symmetrical distorted density. Its shape parameter controls the information under the humps. Consequently, to introduce asymmetry into the distorted density in this case, it is necessary to use the second distortion operator with both shape parameter and location parameter. Especially, the values of shape parameter control the information under the humps and the values of location parameter control the locations of the humps.

However, when the original distribution is asymmetrical, unexpectedly we observe that instead of controlling the locations of the humps, the location parameter of the second distortion operator controls the information under the humps. Further more, in this case its shape parameter indeed effect the locations of the humps. Additionally we remark that it is enough to use the first distortion operator which is more concise, if the main purpose is to create an asymmetrical distorted density with more than one humps from an asymmetrical original density, and at the same time possess the flexibility of shifting the positions of the humps.

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