Formation of coalition structures as a non-cooperative game

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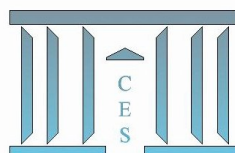Dmitry Levando. Formation of coalition structures as a non-cooperative game. 2017. halshs-01491935v2

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**Formation of coalition structures as
a non-cooperative game**

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Formation of coalition structures as a non-cooperative game

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Abstract

Traditionally social sciences are interested in structuring people in multiple groups based on their individual preferences. This paper suggests an approach to this problem in the framework of a non-cooperative game theory.

Definition of a suggested finite game includes a family of nested simultaneous non-cooperative finite games with intra- and inter-coalition

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externalities. In this family, games differ by the size of maximum coalition, partitions and by coalition structure formation rules.

A result of every game consists of partition of players into coalitions and a payoff profile for every player. Every game in the family has an equilibrium in mixed strategies with possibly more than one coalition. The results of the game differ from those conventionally discussed in cooperative game theory, e.g. the Shapley value, strong Nash, coalition-proof equilibrium, core, kernel, nucleolus.

We discuss the following applications of the new game: cooperation as an allocation in one coalition, Bayesian games, stochastic games and construction of a non-cooperative criterion of coalition structure stability for studying focal points.

Keywords: Noncooperative Games, Nash equilibrium, Shapley value, strong equilibrium, core.
JEL : C71, C72, C73

1   Introduction

There is a conventional dichotomy in game theory applications, the cooperative game theory (CGT) versus the non-cooperative game theory (NGT). CGT deals with coalitions as elementary items, NGT deals with strategic individual behavior.

However, the picture is not so smooth. CGT traditionally disregards issues of strategic interactions between individual players, with players in the same coalition and players in other coalitions. CGT substitutes an individual gain by a value of a coalition, a total and not personified gain of everybody. As a result, value of a coalition does not depend on an allocation of all other players, and construction of an individual gain requires additional arrangements. From another side NGT with an individual impact on an equilibrium overlooks structural aspects of player's partition into groups or into coalition

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structures.[1]

The two theories even pursue different goals, their outcomes vary in many ways, and are hard to compile. CGT concentrates on socially desirable allocations of players, while efficiency is defined in aggregate terms, rather than individual, as in the standard definition of Pareto-efficiency. At the same time NGT aspires towards equilibrium conditions, which later could be Pareto-improved.

Thus it is not surprising that there are questions beyond the range of concern for any of the two theories. The present study concentrates on games, where players can be allocated to different coalitions in an equilibrium, still preserving traditional for NGT individual strategies and individual payoffs. This results in two types of externalities from every player, intra-coalition and inter-coalition ones.

Consider two similar examples: a voluntarily division of participants into paintball teams or a voluntarily division of a class into studying groups. Before they are engaged into a team/a group every player/student makes a decision from self-interest considerations, concerning with whom he/she is, and how all others are allocated. A decision includes a choice of a preferable coalition structure, and what to do after it is formed.

In these examples the participants know about commonly accepted mechanisms of coalition formation before any activity starts. These mechanisms resolve possible conflicts between individual choices. Members of the team/the group may perform different actions inside a team/a group[2] formed.[3]

Intra- and inter-coalition externalities for the paintball game are clear. Let us discuss two type of externalities for studying group game. There are maybe intra-coalition externalities in studying groups in a form of mutual

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[1]Coalition structure terminology was used by Aumann and Dreze (1976).

[2]Indefinite article is used as a formed team/group may not coincide with preferable. See an example in Section 2.

[3]This is different from an approach within cooperative game theory, where players from a coalition perform the same action.

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assistance. Inter-coalition externalities for studying groups could be a mutual noise, or a slow WiFi connection from an external service.

The common features of the examples include multi-coalition frameworks, formed from self-concerned (self-interested) behavior of everybody, and two types of externalities. Thus both examples occur in an area between the existing cooperative and non-cooperative game theories.

In this paper 'coalition structure', or 'partition' [4] for short, is a disjoint union of non-overlapping subsets from a set of players. A group, or a coalition, is an element of a coalition structure or of a partition.[5]

The examples above may seem to be related to cooperation, as suggested by Nash (1953). Now his suggestion is known as the Nash Program (Serrano, 2004), and cooperation is understood as an activity inside a group with positive externalities between players. However both examples above are more complicated than the existing theories suggest. For example, due to a multi-coalition framework and two types of externalities.

The best analogy for the difference of the Nash Program from the research of this paper is the difference between a partial and a general strategic equilibrium analysis[6] in economics. The former isolates a market and ignores cross-market interactions, the latter explicitly studies cross market interactions from individual strategic actions of self-interested traders.

The novelty of the paper consists in: a construction of a *family* of nested non-cooperative games such that every game has an equilibrium in mixed strategies. Based on the game, the paper presents a non-cooperative criterion for coalition structure stability, that leads to a non-cooperative reconsideration of focal points of Schelling. These results reveal new properties of well-known games, and ask new research questions.[7] Using a game similar

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[4]Existing literature uses both terms.

[5]the same.

[6]meaning, general equilibrium framework with strategic market games of Shapley and Shubik.

[7]Recently Dasgupta and Maskin (2016) wrote that 'game theory is the study of strategic interaction: how a person should and will behave if her actions affect others and their

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to Prisoner's Dilemma the paper shows that an efficient and not equilibrium outcome does not imply allocation of players in one coalition. Further reading to the literature on this subject is available in the Discussion section of this paper.

The paper addresses only construction and equilibrium issues of a non-cooperative game, but does not study efficiency of an equilibrium concept.

The paper has the following structure: Section 2 presents an example based on Prisoner's Dilemma and demonstrates that there is actually no explicit cooperation in it. Then Section 3 follows with the main model. Section 4 contains a discussion of the result and a comparison with existing literature. Section 5 contains a formal definition of a sufficient criterion for cooperation. All further sections deal with reconsidering well-known games, Bayesian games (Section 6), stochastic games (Section 7), and demonstrating their new properties, including a construction of non-cooperative criterion of coalition structure stability and reconsidering focal points of Shelling (Section 8). The final section of this paper is a Conclusion.

2 Modified Prisoner's Dilemma (PD)

The example demonstrates that there is no explicit cooperation in the standard PD game, if to think about a cooperation as an allocation of players in the same coalition. Formal analysis of cooperation is in Section 5.[8]

Consider a game of two players $i = 1, 2$, where each can choose to be alone and has two activity levels: $L(ow)_{i,alone}$ or $H(high)_{i,alone}$, $i = 1, 2$. The players can make only one coalition structure $\{\{1\}, \{2\}\}$. Payoffs for this

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actions affect her'. The novelty of this paper is that players can be allocated in different coalitions still making externalities for all others.

[8]This section contains an example for two players, what may make to think that the players care about coalitions, not about coalition structures. The suggested mechanism is constructed to deal with coalition structures. A game of two "introvert" players further in this paper demonstrates importance of coalition structures. An example, where coalition structure matter explicitly is in Section 6, Corporate lunch game.

case are in Table 1. Every cell contains a payoff profile for both players and an allocation of the players over coalitions. The unique equilibrium $(-2; -2)^*$, marked with one star *, is inefficient. But the desirable and non-equilibrium payoff outcome $(0; 0)$ is efficient. Hence the game reproduces payoff properties of the Prisoner's Dilemma with an explicit allocation of players over coalitions structures.

Table 1: Payoff for the standard Prisoners Dilemman

	$L_{2,alone}$	$H_{2,alone}$
$L_{1,alone}$	(0;0) $\{\{1\}, \{2\}\}$	(-5;3) $\{\{1\}, \{2\}\}$
$H_{1,alone}$	(3;-5) $\{\{1\}, \{2\}\}$	$(-2; -2)^*$ $\{\{1\}, \{2\}\}$

Let the game be more complicated, and the players can choose to be either alone or together, and as above to choose between two activity levels for every case. Then set of strategies $S_i(K = 2)$ for player i is:

$$S_i(K = 2) = \Big\{(L_{1,alone}, H_{1,alone}), (L_{1,together}, H_{1,together})\Big\}.$$

Parameter $K = 2$ indicates that maximum available coalition size for the players is 2. For simplicity we assume the simplest rule for coalition structure formation: if at least one of the players chooses to be alone, then a coalition of two cannot be formed. Payoff matrix for this game is in Table 2.

Every cell in Table 2 contains a payoff profile and a coalition structure. The upper-left part of Table 2 corresponds to strategies $L_{i,alone}$ and $H_{i,alone}$, $i = 1, 2$, and coincides with the previous game in Table 1. The previous equilibrium is marked by one star *, the newly discovered equilibria are marked by two stars **. It is clear that all new equilibria are still inefficient, but belong to different coalition structures. An increase in a number of possible coalition structures enriches a set of equilibria of the game and a set of

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Table 2: Payoff for an extension of Prisoner's Dilemma.

	$L_{2,alone}$	$H_{2,alone}$	$L_{2,together}$	$H_{2,together}$
$L_{1,alone}$	(0;0) $\{\{1\},\{2\}\}$	(-5;3) $\{\{1\},\{2\}\}$	(0;0) $\{\{1\},\{2\}\}$	(-5;3) $\{\{1\},\{2\}\}$
$H_{1,alone}$	(3;-5) $\{\{1\},\{2\}\}$	$(-2;-2)^{*,**}$ $\{\{1\},\{2\}\}$	(3;-5) $\{\{1\},\{2\}\}$	$(-2;-2)^{**}$ $\{\{1\},\{2\}\}$
$L_{1,together}$	(0;0) $\{\{1\},\{2\}\}$	(-5;3) $\{\{1\},\{2\}\}$	(0;0) $\{1,2\}$	(-5;3) $\{1,2\}$
$H_{1,together}$	(3;-5) $\{\{1\},\{2\}\}$	$(-2;-2)^{**}$ $\{\{1\},\{2\}\}$	(3;-5) $\{1,2\}$	$(-2;-2)^{**}$ $\{1,2\}$

equilibrium coalition structures, but not efficiency.

In the coalition structure $\{\{1\},\{2\}\}$ two players have externalities within coalitions, which will be further referred to as inter-coalition externalities. But in the coalition structure $\{1,2\}$ the same players experience between externalities, will be further referred to as intra-coalition externalities.

Formation of $\{1,2\}$ requires that two players choose to be together, whatever they plan to do inside the coalition structure. It is impossible to implement $\{1,2\}$ only from a choice of one player, given coalition structure formation rule. Inside this coalition structure both players can deviate simultaneously from the efficient outcome to the equilibrium.

The feature of the example is that there is no instruction for players on how to deviate: unilaterally, independently, or within only one coalition, as in the cooperative game theory. *The sets of individual strategies anticipate all combinations of possible deviations for both players.* A formal model in Section 2 generalizes this idea.

A resulting coalition structure is formed as *a result of choices from all players* for a given coalition structure formation rule. Exogenous rules resolve conflicts between choices of players becoming components of a game.

Table 2 contains four efficient outcomes: (0;0) located in different coali-

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Table 3: Payoff for two extrovert players who prefer to be together. Uniqueness of an equilibrium is fixed.

	$L_{2,alone}$	$H_{2,alone}$	$L_{2,together}$	$H_{2,together}$
$L_{1,alone}$	(0;0) $\{\{1\},\{2\}\}$	(-5;3) $\{\{1\},\{2\}\}$	(0;0) $\{\{1\},\{2\}\}$	(-5;3) $\{\{1\},\{2\}\}$
$H_{1,alone}$	(3;-5) $\{\{1\},\{2\}\}$	$(-2;-2)^{*}$ $\{\{1\},\{2\}\}$	(3;-5) $\{\{1\},\{2\}\}$	$(-2;-2)$ $\{\{1\},\{2\}\}$
$L_{1,together}$	(0;0) $\{\{1\},\{2\}\}$	(-5;3) $\{\{1\},\{2\}\}$	$(0+\epsilon;0+\epsilon)$ $\{1,2\}$	$(-5+\epsilon;3+\epsilon)$ $\{1,2\}$
$H_{1,together}$	(3;-5) $\{\{1\},\{2\}\}$	(-2;-2) $\{\{1\},\{2\}\}$	$(3+\epsilon;-5+\epsilon)$ $\{1,2\}$	$(-2+\epsilon;-2+\epsilon)^{**}$ $\{1,2\}$

tion structures: this means that by observing only the payoff profiles $(0;0)$ we can't conclude whether players are in one coalition or not. This differs from a common understanding that the Prisoner's Dilemma can serve as an example of unsuccessful cooperation. If to consider a cooperation as an allocation of players in one coalition, we observe, that information about payoff profile is not enough to assert cooperation existence.

An allocation of players in one coalition can be fixed by addressing to the game of "extrovert" players. Two players have preferences over coalition structures and prefer to be together. Let $\epsilon > 0$ be an additional individual gain (a corporate gain) by contrast from being separate, but only if the grand coalition $\{1,2\}$ is formed. Table 3 contains payoffs for this game. Now the equilibrium marked with ** is unique and appears when both players choose to be together. However it is non-efficient again.

If both players are "introverts", then each prefers to be alone. If a desirable coalition structure, $\{\{1\},\{2\}\}$, is formed, then every player obtains an additional markup $\delta > 0$. Payoff matrix for this game is in Table 4. The additional option of being together does not change the equilibrium in com-

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parison to the aforementioned game, where strategies are $(L_{i,alone}, H_{i,alone})$, $i = 1, 2$.

Table 4: Payoff for two "introvert" players, who prefer to be alone. Uniqueness of an equilibrium is fixed.

	$L_{2,alone}$	$H_{2,alone}$	$L_{2,together}$	$H_{2,together}$
$L_{1,alone}$	$\dfrac{(0+\delta;0+\delta)}{\{\{1\},\{2\}\}}$	$\dfrac{(-5+\delta;3+\delta)}{\{\{1\},\{2\}\}}$	$\dfrac{(0;0)}{\{\{1\},\{2\}\}}$	$\dfrac{(-5;3)}{\{\{1\},\{2\}\}}$
$H_{1,alone}$	$\dfrac{(3+\delta;-5+\delta)}{\{\{1\},\{2\}\}}$	$\dfrac{(-2+\delta;-2+\delta)^{*,**}}{\{\{1\},\{2\}\}}$	$\dfrac{(3;-5)}{\{\{1\},\{2\}\}}$	$\dfrac{(-2;-2)}{\{\{1\},\{2\}\}}$
$L_{1,together}$	$\dfrac{(0;0)}{\{\{1\},\{2\}\}}$	$\dfrac{(-5;3)}{\{\{1\},\{2\}\}}$	$\dfrac{(0;0)}{\{1,2\}}$	$\dfrac{(-5;3)}{\{1,2\}}$
$H_{1,together}$	$\dfrac{(3;-5)}{\{\{1\},\{2\}\}}$	$\dfrac{(-2;-2)}{\{\{1\},\{2\}\}}$	$\dfrac{(3;-5)}{\{1,2\}}$	$\dfrac{(-2;-2)}{\{1,2\}}$

Consider a case, where the players are different: player $i = 1$ is "extrovert", but player $i = 2$ is "introvert". Table 5 is the payoff matrix for this game. Following the rule of unanimous agreement to form a coalition the grand coalition, $\{1, 2\}$, can not be formed, and equilibrium strategy profile for $i = 2$ does not change in comparison to the first example. For player $i = 1$ an equilibrium strategy profile is a mixed strategy with equal weights over two pure strategies $H_{1,alone}$ and $H_{1,together}$. Final coalition structure $\{\{1\}, \{2\}\}$ is not affected by this randomization and another coalition structure is unfeasible. Players have inter-coalition externalities and inefficient outcome. Hence the mixed strategies induce inter-coalition externalities and two coalitions in an equilibrium. Games with such properties are not described in existing literature.

We can summarize the section: expansion of traditional non-cooperative game theory for coalition structures formation requires additional arrangements. This includes:

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Table 5: Payoff for a game when player 1 is "extrovert" and player 2 is "introvert".

	$L_{2,alone}$	$H_{2,alone}$	$L_{2,together}$	$H_{2,together}$
$L_{1,alone}$	$(0; 0 + \delta)$ $\{\{1\}, \{2\}\}$	$(-5; 3 + \delta)$ $\{\{1\}, \{2\}\}$	$(0; 0 + \delta)$ $\{\{1\}, \{2\}\}$	$(-5; 3 + \delta)$ $\{\{1\}, \{2\}\}$
$H_{1,alone}$	$(3; -5 + \delta)$ $\{\{1\}, \{2\}\}$	$(-2; -2 + \delta)^{*,**}$ $\{\{1\}, \{2\}\}$	$(3; -5 + \delta)$ $\{\{1\}, \{2\}\}$	$(-2; -2 + \delta)$ $\{\{1\}, \{2\}\}$
$L_{1,together}$	$(0; 0 - 5 + \delta)$ $\{\{1\}, \{2\}\}$	$(-5; 3 + \delta)$ $\{\{1\}, \{2\}\}$	$(0 + \epsilon; 0)$ $\{1, 2\}$	$(-5 + \epsilon;)$ $\{1, 2\}$
$H_{1,together}$	$(3; -5 + \delta)$ $\{\{1\}, \{2\}\}$	$(-2; -2 + \delta)^{**}$ $\{\{1\}, \{2\}\}$	$(3 + \epsilon; -5)$ $\{1, 2\}$	$(-2 + \epsilon; -2)$ $\{1, 2\}$

1. to be aware of a maximum available coalition size;

2. to enumerate possible coalition structures for it;

3. individual strategy sets should be related to possible coalition structures;

4. to embed some mechanism to resolve conflicts between players.

The next section presents a formal model for a non-cooperative game with coalition structure formation.

3   Formal setup of the model

This section presents a construction of the main model. Nash (1950, 1951) suggested a non-cooperative game, that consists of a set of players N, with a general element i, sets of individual finite strategies $S_i, i \in N$, and individual payoffs, $U_i(s)$, defined for every strategy profile $s \in S = \times_{i \in N} S_i$, as a mapping from a set of all strategies S into a bounded set $U_i \colon S \mapsto \mathbb{R}$,

$U_i(s) < \infty$. Thus every strategy profile $s \in S$ is assigned a vector of payoffs $\left(U_i(s) \right)_{i \in N} \in \mathbb{R}^N_+$, where $u_i(s)$ is an individual payoff of i for a strategy profile s. However, the non-cooperative game introduced by Nash says nothing about coalition structure formation.

The game suggested in this paper follows similar logic of Nash, but with a preliminary modification for a set of strategy profiles. Every player has a set of strategies for every relevant coalition structure. A set of all strategy profiles S, constructed as a Cartesian (direct) product of all individual strategies, is portioned into non-overlapping domains, as in Section 2. Every constructed domain is a set of strategy profiles. Every strategy profile inside a domain is assigned the same coalition structure and a profile of individual payoffs. This approach reproduces the method of state-contingent payoffs for Arrow-Debreu securities.

In this section we introduce components for a game, how to assemble them into a game and now to construct an equilibrium concept for the game. The common feature of the components is their nested structure, that enables us to construct a nested family of games.

3.1   Players and partitions

Consider a set of agents N, with a general element i, and denote a size of N as $\#N$, a finite integer, $2 \leq \#N < \infty$. If there is no ambiguity we will use N also for a size of the set of players.

Introduce a parameter K, taking values from 1 to N. It has two equivalent interpretations: maximum coalition size and maximum number of deviators. From one side, in any available coalition structure there is no coalition with maximum size greater than K. From another, at most K agents are required to dissolve any coalition in any such coalition structure.

Let $n(k)$ be a subset of players with a size no more than K, $n(k) \subset N$, $\#n(k) \leq K$. *A subset $n(k)$ is not a coalition, but a subset of players from N with an upper restriction on a number of elements.* After a coalition structure

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is formed the players from $n(k)$ may appear in different final coalitions.

Every fixed K induces a family of coalition structures (or a family of partitions) $\mathcal{P}(K)$:

$$\mathcal{P}(K) \;=\; \Big\{ P \colon P \;=\; \{g_j \colon g_j \;\subset\; N, \quad \#g \;\leq\; K, \quad \sqcup_j g_j \;=\; N\} \Big\} \quad (3.1)$$

An element $P = \{g_j\}$ of $\mathcal{P}(K)$ is a partition of N into coalitions, where j is an index of a coalition in P. Every coalition has an upper boundary on its size, $\#g_j \leq K$. The notation $\sqcup_j g_j = N$ means that a player may participate only in one coalition: $\forall j_1 \neq j_2, \Rightarrow g_{j_1} \cap g_{j_2} = \emptyset$. Size of $\mathcal{P}$ is described by a partition function $p(N, K)$ (reference).

If we increase K by one, then we need to add new coalition structures from $\mathcal{P}(K+1) \backslash \mathcal{P}(K)$, and finally we obtain the nested families of partitions:

$$\mathcal{P}(K = 1) \subset \dots \subset \mathcal{P}(K) \subset \dots \subset \mathcal{P}(K = N).$$

The bigger K is , the more coalition structures (or partitions) are included into consideration. The grand coalition, i.e. a coalition of size N, belongs to the family $\mathcal{P}(K = N)$, only.

3.2 Strategies

Take a fixed K and consider a relevant family of coalition structures $\mathcal{P}(K)$. For every partition P from $\mathcal{P}(K)$ agent i has a finite strategy set $S_i(P)$ and a set of strategies of i for $\mathcal{P}(K)$ is $S_i(K) = \cup_{P \in \mathcal{P}(K)} S_i(P)$.[9]

Definition 1 (individual strategy set)**.** *A set of strategies of agent i for a family of coalition structures $\mathcal{P}(K)$ is*

$$S_i(K) = \Big\{ s_i \colon s_i \in \cup_{P \in \mathcal{P}(K)} S_i(P) \Big\}$$

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[9]Finite strategies are used as in Nash (1950).

with a general element $s_i(K)$ or s_i for simplicity.

Agent i chooses s_i from $S_i(K)$. All agents make their choices simultaneously. *A choice is a choice of a **desirable** partition and an action for this partition.*[10] If we increase K by one, then we need to construct additional strategies only for the newly available coalition structures from $\mathcal{P}(K+1) \setminus \mathcal{P}(K)$. This makes strategy sets for different K's to be nested:

$$S_i(K = 1) \subset \ldots \subset S_i(K) \subset \ldots \subset S_i(K = N).$$

We construct sets of strategies of all players from individual strategies in the usual way, $S(K) = \times_{i \in N} S_i(K)$, and it inherits the nesting property:

$$S(K = 1) \subset \ldots \subset S(K = N).$$

A choice of all players, a strategy profile, is $s(K) = \Big(s_1(K), \ldots, s_N(K) \Big) \in S(K)$, or if there is no ambiguity $s \equiv s(K)$.

One may ask a question, is it possible to make a choice in two stages: first, all players choose a partition, second, everybody chooses a coalition. Such reformulation of the game leads to the generalization of strategic equilibrium, introduced by Mertens (1995), also in Hillas, Kohlberg (2002), that goes beyond the scope of this paper.

3.3   Coalition structure mechanism

For every K we have a family of coalition structures and a set of strategy sets. However, individual choices of agents may be inconsistent. To overcome this deficiency we define a coalition structure formation mechanism

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[10]A desirable partition may not realize due to conflicts in individual choices. It was demonstrated with examples in the previous section. A sequential game for dynamic formation coalition structures is in progress with Marc Kelbert and Olga Pushkareva. Both versions of the game need a mechanism for conflict resolution.

$\mathcal{R}(K)$, (a mechanism or a rule for short) to resolve possible conflicts.[11] The rule portions $S(K)$ into non-overlapping coalition structure specific domains $S(K) = \sqcup_{P \in \mathcal{P}(K)} S(P)$, $S(P) = \times_{i \in N} S_i(P)$, where disjoint union $\sqcup$ means that domains for different coalition structures do not overlap. An example was in Section 2.

A unique coalition structure P from $\mathcal{P}(K)$ is assigned for every strategy profile s from $S(K)$. A set of such strategy profiles makes a coalition structure strategy domain $S(P) = \{s = (s_1, \ldots, s_i, \ldots, s_N)\}$, which may not be convex, (see examples in Section 2), $s_i \in S_i(K)$. It is clear that $S(K) = \sqcup_{P \in \mathcal{P}(K)} S(P)$. For some players a realized coalition structure P may not coincide with a chosen one.

Definition 2 (a coalition structure formation mechanism). *For every K **a coalition structure formation mechanism** $\mathcal{R}(K)$ is a set of measurable mappings such that:*

1. *A domain of $\mathcal{R}(K)$ is a set of all strategy profiles of $S(K)$.*

2. *A range of $\mathcal{R}(K)$ is a finite number of subsets $S(P) \subset S(K)$, $P \in \mathcal{P}(K)$. Every $S(P)$ is a strategy set for only one coalition structure P.*

3. *$\mathcal{R}(K)$ divides $S(K)$ in such a way that the union of all $S(P)$ constructs the original set $S(K) = \sqcup_{P \in \mathcal{P}(K)} S(P)$.*

Formally the same:

$$\mathcal{R}(K) \colon S(K) = \times_{i \in N} S_i(K) \mapsto \Big(S(P), P \in \mathcal{P}(K) \Big) \colon \sqcup_{P \in \mathcal{P}(K)} S(P)$$

There are two ways to construct $S(K)$: in terms of initial individual strategies, $S(K) = \times_{i \in N} S_i(K)$, or in terms of realized partition strategies, $S(K) = \sqcup_{P \in \mathcal{P}(K)} S(P)$.

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[11]They can be social norms or social institutions, which people take for granted.

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Holt and Roth (2004) suggested to include a mechanism to eliminate a distinction between non-cooperative and cooperative games: 'One trend in modern game theory, often referred as *Nash program*, is to erase this distinction by including any relevant enforcement mechanism in the model, so that all games can be modeled as non-cooperative'.

Informally, every coalition structure becomes itself a non-cooperative game: for every coalition structure P there is a set of players, a non-trivial set of strategies and partition-specific payoffs, described further.

If K increases we need to add a mechanism for strategy sets from $S(K + 1) \setminus S(K)$ only. This supports consistency of coalition structure formation mechanisms for different K, and the family of mechanisms is nested:

$$\mathcal{R}(K = 1) \subset \ldots \subset \mathcal{R}(K) \subset \ldots \subset \mathcal{R}(K = N).$$

It is important to note that a role of a mechanism is to identify domains for coalition structures, but not to deal with efficiency of results (see inefficient equilibria in Section 2).[12] Studying efficiency requires studying properties of all possible measurable transformations $\mathcal{R}(K)$, that goes far beyond this paper. Discussion of some properties of $\mathcal{R}(K)$ is in the next section.

3.4 Payoffs

For every K we have a family of coalition structures, a set of strategy sets and coalition structure formation rules. And now within every given coalition structure for every strategy profile we define a payoffs profile. Payoffs are considered as von Neumann-Morgenstern utilities.

Definition 3 (coalition structure specific individual payoff). *For every coalition structure P from $\mathcal{P}(K)$ player i, $i \in N$, has a payoff function*

$$U_i(P) \colon S(P) \to \mathbb{R}_+,$$

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[12]Efficiency in non-cooperative and cooperative game theories is understood differently.

such that $U_i(P)$ is bounded, $U_i < \infty$.

Definition of payoffs for all feasible coalition structures generalizes the definition above.

Definition 4 (family of coalition structure specific individual payoffs). *For given maximum coalition size K, family of coalition structures $\mathcal{P}(K)$, set of strategy profiles $S(K)$, payoffs of player i from N, make a family:*

$$\mathcal{U}_i(K) = \Big\{U_i\Big(P\Big)\colon P \in \mathcal{P}(K), S(K) = \sqcup_{P \in \mathcal{P}(K)} S(P)\Big\},$$

where $U_i\Big(P\Big)$ is a set of coalition structure specific payoffs for i.

Every coalition structure has a strategy domain $S(P)$, and a set of payoffs, $\Big(U_i\Big(S(P)\Big)\Big)_{i \in N}$, defined on this domain. Thus, every coalition structure is a non-cooperative game itself.

An increment in K increases a number of possible partitions and a set of feasible strategies for every player. Thus we need to add only payoffs for newly added coalition structures from $\mathcal{P}(K+1) \setminus \mathcal{P}(K)$. Not surprisingly we obtain a nested family of payoff functions:

$$\mathcal{U}_i(K = 1) \subset \ldots \subset \mathcal{U}_i(K) \subset \ldots \subset \mathcal{U}_i(K = N).$$

It is clear that in every coalition structure players can have both intra and inter coalition externalities.

3.5   Game

For every K we have a family of coalition structures, strategy sets, coalition structure formation rules and families of payoffs and now, we define a game.

Definition 5 (a simultaneous coalition structure formation game). *A non-cooperative game for coalition structure formation with maximum coalition*

size K is

$$\Gamma(K) = \left\langle N, \left\{ K, \mathcal{P}(K), \mathcal{R}(K) \right\}, \left(S_i(K), \mathcal{U}_i(K) \right)_{i \in N} \right\rangle,$$

where $\left\{ K, \mathcal{P}(K), \mathcal{R}(K) \right\}$ *- coalition structure formation mechanism (a social norm, a social institute),* $\left(S_i(K), \mathcal{U}_i(K) \right)_{i \in N}$ *- properties of players in* N, *(individual strategies and payoffs), such that:*

$$\times_{i \in N} S_i(K) \stackrel{\mathcal{R}(K)}{\rightarrow} \left\{ S(P) \colon P \in \mathcal{P}(K) \right\} \rightarrow \left\{ \left(\mathcal{U}_i(K) \right)_{i \in N} \right\}_{P \in \mathcal{P}(K)}.$$

If we omit the mechanism part of the game, i.e. take $K = 1$, and forget about the coalition structures, then we obtain the traditional non-cooperative game of Nash. It is clear that for different K's the constructed games are nested.

Definition 6 (family of games). *A family of games is* ***nested*** *if :*

$$\mathcal{G} = \Gamma(K = 1) \subset \ldots \subset \Gamma(K) \subset \ldots \subset \Gamma(K = N).$$

Nested games are important to study changes in an equilibrium with an increase in a number of deviators or maximum coalition's size K.

3.6 Equilibrium

Let $S_i(K)$ be a finite set of strategies of player i. Let $\Delta_i(K)$ be a set of all mixed strategies (probability measures or probability distributions) of player i, i.e. a space of probability measures over $S_i(K)$,

$$\Delta_i(K) = \left\{ \sigma_i(K) \colon \sum_{P \in \mathcal{P}(K)} \sum_{s_i \in S_i(K)} \sigma_i(s_i) = 1 \right\}.$$

Let $\sigma_i(K)$ be a mixed strategy of i, $\sigma_i(K) \in \Delta_i(K)$.

Sets of mixed strategies for players different from i are defined in the standard way:

$$\Delta_{-i}(K) = \left\{\sigma_{-i} = \prod_{j \neq i} \sigma_j(K) \colon \forall j \neq i\right\}.$$

Set of all strategies $S(K)$ has two representations, $S(K) = \times_{i \in N} S_i(K)$ and $S(K) = \sqcup_{P \in \mathcal{P}(K)} S(P)$. Thus there are two parallel and equivalent ways to construct expected utility.

Expected utility of i in terms of individual strategies $(S_i(K))_{i \in N}$ is

$$EU_i^{\Gamma(K)}\Big(\sigma_i(K), \sigma_{-i}(K)\Big) = \sum_{P \in \mathcal{P}(K)} \sum_{s_i \in S_i(P)} U_i(s_i, s_{-i}) \sigma_i(K) \sigma_{-i}(K),$$

and is defined in terms of realized coalition structures from $\mathcal{P}(K)$.

Definition 7 (an equilibrium in a game $\Gamma(K)$). *There is a non-cooperative game for coalition structure formation*

$$\Gamma(K) = \left\langle N, \left\{K, \mathcal{P}(K), \mathcal{R}(K)\right\}, \Big(S_i(K), \mathcal{U}_i(K)\Big)_{i \in N}\right\rangle.$$

A mixed strategies profile $\sigma^*(K) = \left(\sigma_i^*(K)\right)_{i \in N}$ is an equilibrium for a game $\Gamma(K)$ if for every subset $n(k)$ from N, with a size $1 \leq k \leq K$, and for every player $i \in n(k)$ a deviation from an equilibrium, $\sigma_i(K) \neq \sigma_i^*(K)$, does not generate an individual gain:

$$EU_i^{\Gamma(K)}\Big(\sigma_i^*(K), \sigma_{-i}^*(K)\Big) \geq EU_i^{\Gamma(K)}\Big(\sigma_i(K), \sigma_{-i}^*(K)\Big).$$

The difference with Nash equilibrium is that an equilibrium must be satisfied not for every player i from N, but for every player i from every subset $n(k)$ from N, where a size of $n(k)$ does not exceed K. Informally this means that any $k \leq K$ deviators do not have restriction to move either separately or together. Players from $n(k)$ can deviate in any combinations, what is anticipated in their strategy sets in construction of the game,

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Possible deviations from an equilibrium are constructed differently from those used in the strong Nash equilibrium, in the core or in the coalition-proof equilibrium. This property of the simultaneous deviation already appeared in the Prisoner's Dilemma example, where if players have an option to be together, then they must be aware of a deviation of both in an equilibrium.

Theorem 1. *A game $\Gamma(K)$ has an equilibrium in mixed strategies.*

A set of finite strategies has a simplex as a set of mixed strategies. Simplex is a compact set. Set of mixed strategies of all players is a combination of finite number of simplex, what is again a compact set. Continuous utility function function reaches a maximum value on this simplex. Thus there is always a Nash equilibrium for finite strategies in $\Gamma(K)$. Nash theorem with finite strategies is the partial case for $K = 1$. The result of the paper is not in a trivial mathematical theorem, but in introduction of a new game to solve earlier untreatable problems with any number of deviators structured by groups. This new game follows the same rules, as the game suggested by Nash.

The equilibrium result is different from results of the cooperative game theory, where games with empty cores may exist. The suggested non-cooperative equilibrium does not demand super-additivity, transferable / non-transferable utilities, axioms on payoffs or weights. Existence of an equilibrium does not imply that it is efficient.

The advantage of the theorem is that it can be generalized for any relevant K. New insight of the result is not in complexity of mathematical objects, but in constructing a more general structure of a game, which allow to ask new questions and to reconsider old games.

Theorem 2. *The family of games $\mathcal{G} = \{\Gamma(K), K = 1, 2, \ldots, N\}$ has equilibrium mixed strategies profiles for every K:*

$$\sigma^*(\mathcal{G}) = \Big(\sigma^*(K = 1), \ldots, \sigma^*(K)_{i \in N}, \ldots, \sigma^*(K = N)\Big),$$

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where $\sigma^*(K) = \left(\sigma_i^*(K)\right)_{i \in N}$ is an equilibrium for a game $\Gamma(K)$.

The proof is simple. A players i's finite set of pure strategies has simplex as a set of mixed strategies. Then a set of all strategies of a game $\Gamma(K)$ is a convex subset of $\mathcal{R}^{p(N,K)}$, where $p(N, K)$ is partition function. Then by Brouwer's fixed point theorem a continuous mapping from a convex compact set in a finite dimensional Euclidean space into a convex subset of the same finite dimensional Euclidean space has a fixed point.

Thus every game for a coalition structure formation has an equilibrium for any number of deviators. It is clear that further development of the model requires setting the standard problem of equilibrium refinement.

An equilibrium in the game can also be characterized by equilibrium partitions.

Definition 8 (equilibrium coalition structures or partitions). *A set of partitions $\{P^*(K)\}$, $\{P^*(K)\} \subset \mathcal{P}(K)$, of a game $\Gamma(K)$, is a set of equilibrium partitions, if it is induced by an equilibrium strategy profile $\sigma^*(K) = \left(\sigma_i^*(K)\right)_{i \in N}$.*

Robustness of an equilibrium to an increase in K is addressed in Section 8.

4 Discussion

At the moment there are two competing game theories: the non-cooperative and the cooperative. In early 60-s Aumann (1960) wrote that 'the non-cooperative game differs from the cooperative game chiefly in that the use of correlated strategy vectors that are not also mixed strategies vectors are forbidden'. The current paper demands another description of the difference: the non-cooperative game theory is based on mapping of a strategy profile of all players (a subset from $\mathbb{R}^N$) into a payoff profile for all players (a bounded

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subset from $\mathbb{R}^N$). The cooperative game theory is based on mapping[13] of subsets of integer numbers (a subset from $\mathbb{N}$) into a subset of real numbers (a subset from $\mathbb{R}$).

The model of this paper explicitly assumes that every self-interested agent produces a relevant impact for all other players, independently from their coalition location. The resulting impact can be either positive or negative. This idea requires modification of the above to be explained missed in existing non-cooperative and existing cooperative game theories.

A possible irrelevance of cross-coalition externalities leaves a space for a co-existence of the non-cooperative and the cooperative game theories.

Inedequacy of cooperative game theory to study coalitions and coalition structures was noted earlier by many authors. Maskin (2011) wrote that 'features of cooperative theory are problematic because most applications of game theory to economics involve settings in which externalities are important, Pareto inefficiency arises, and the grand coalition does not form'.[14]. Myerson (p.370, 1991) noted that 'we need some model of cooperative behavior that does not abandon the individual decision-theoretic foundations of game theory'. Therefore, there is a demand for a non-cooperative game to study non-cooperative formation of coalition structures.

4.1 Referring to existing literature

There are volumes of literature on coalitions, and this list of highly respected list authors is far from complete: Aumann, Hart, Holt, Kurz, Maschler, Maskin, Moulen, Myerson, Peleg, Roth, Serrano, Shapley, Schmeidler, Weber, Winter, Wooders and many others. All the authors address highly significant specific issues. However addressing a particular issue is not the same as a solution for a general problem, formation of coalition structures

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[13]a characteristic function

[14]or that 'characteristic function ... rules out externalities, situations when a coalition payoff depends on what other coalitions are doing', Maskin, (2016)

from self-interest fundamentals. The solution for a fundamental problem is not abundance in literature and is not abundant in current literature. For the purpose of this paper one needs to seek answers for certain questions:

Problem Identification: What are the specific properties of a general problem?

Solution Existence: Do existing solutions satisfy these properties?

Choice of a tool: Do existing tools challenge them enough?

4.1.1 Identification of a problem

There are different views on complexities for non-cooperative formation of coalition structures. There are two opinions above, two recent ones are below.

Serrano (2014) wrote: 'the axiomatic route find difficulties identifying solutions', and that for studying coalition formation 'it maybe worth to use strategic-form games, as proposed in the Nash Program'. Ray and Vohra (2015) wrote on complexity and contradictions in existing approaches. They offer a systematic view on the field, based on 'collection of coalitions', or a modified cooperative game theory:

'Yet as one surveys the landscape of this area of research, the first feature that attracts attention is "the fragmented nature of the literature" ... The literature on coalition formation embodies two classical approaches that essentially form two parts of this chapter: (i) The blocking approach, in which we require the immunity of a coalition arrangement to blocking, perhaps subcoalitions or by other groups which intersect the coalition in question... (ii) Noncooperative bargaining, in which individuals make proposals to form a coalition, which can be accepted or rejected...

After all, the basic methodologies differ apparently at "an irreconcilable level" over cooperative and noncooperative game approaches... '

Every presented view describes only a special part of the general problem and suggests a partial solution for it. Existing diversity of views, 'irrecon-

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cilable' approaches and 'the fragmented nature', enable us to conclude: the long lasting problem of structuring people in multiple groups based on their individual preferences and individual actions is still not well-identified: a 'difficult to identify problem' cannot be easily solved in a consistent way

This paper considers the general problem as: how to construct coalition structures with multiple coalitions, intra/inter coalition externalities only from non-cooperative actions of players.

4.1.2 Existence of a solution

'A problem cannot be solved if it's bounds are unknown' (A. Tchekmarev[15]). The current paper dares to suggest a general identification for the problem and a consistent, natural solution for it. This game generalizes the concept of non-cooperative games introduced by Nash to study formation of coalition structure with many possible coalitions.

4.1.3 A tool

How to deal with the problem, which has not been solved and has 'the fragmented nature'? The answer comes from Albert Einstein: 'The significant problems we have cannot be solved at the same level of thinking with which we created them'. The current paper dares to suggest a new tool, presented in Section 3.

4.2 Comparison with existing approaches

4.2.1 A threat

A threat as a tool for coalition formation analysis was suggested by Nash (1953), and earlier by von Neumann and Morgenstern. It starts from a strategy profile, a threat, from *a subset* of players, possibly allocated in

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[15]Personal communication on engineering design.

different coalitions. Let this profile be a threat to someone beyond this subset. The threatening players may produce externalities for each other (and negative externalities are not excluded!). Then how can we describe the externalities for members of the subset if the threat is an elementary tool? At the same time there maybe some other player(s) beyond the subset, who may benefit from the threat. But they may have no motivation to join those who are threatening. For example, some additional adverse externalities may appear. Such issues should be included into consideration of coalition structure formation.

There is a parallel argument against using the threat as an elementary tool. Assume there are several agents, who individually cannot make a threat to some others, and these small agents are allocated in different coalitions. A credible threat may come only from many small players. Does this mean that the small agents should join together? They may have their own contradictions to join in one coalition. Maybe a formal example will be more illustrative here, but the volume of the paper does is excessive.

4.2.2   Usage of the non-cooperative approach

The justification for usage of non-cooperative games as a tool for coalition structure formation, comes from Maskin (2011) and the recent remark from Serrano (2014): 'it maybe worth to use strategic-form games, as proposed in the Nash Program'.

There is a difference in the research agenda of this paper from the Nash Program (Serrano 2004). Nash program aims to study a non-cooperative formation of *one* coalition, but this paper aims to study non-cooperative formation of coalition structures, possibly with more than one coalition. The constructed finite non-cooperative game allows to study what can be a cooperative behavior, when the individuals 'rationally expand their individual interests' (Olson, 1971), see the next section.

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4.2.3 Novelties of the paper

Nash (1950, 1951) suggested to construct a non-cooperative game as a mapping from a set of strategies into a set of payoffs,

$$(U_i)_{i \in N} \colon \times_{i \in N} S_i \to \mathbb{R}^N, \quad (U_i)_{i \in N} < \infty.$$

This paper has two additions in comparison to his paper: construction of a non-cooperative game with an embedded coalition structure formation mechanism and a parametrization of games by a number of deviators K, $K = 1, \ldots, \#N$:

$$\left\{ \left(\mathcal{U}_i(S(P)) \right)_{i \in N} \right\}_{P \in \mathcal{P}(K)} \colon \times_{i \in N} S_i(K) \overset{\mathcal{R}(K)}{\to} \{ S(P) \colon P \in \mathcal{P}(K) \} \to (\mathbb{R}^N)^{p(N,K)},$$

where M is a number of coalition structures from N players with the restriction K on coalition sizes. The game suggested by Nash becomes a partial case for these games, where coalition structures do not matter and only one player deviates, $K = 1$.

Every result of the suggested game consists of two parts: 1/ a payoff profile for all players; 2/ an allocation of all players over coalitions. Equilibrium in mixed strategies always exists and may not be efficient like in traditional non-cooperative games.

We can take $\mathcal{R}(K)$, $K = 1, \ldots, N$, as a trustworthy social institute. Application of $\mathcal{R}(K)$ evokes two issues: how to construct an equilibrium (modelled in this paper), and how to achieve social efficiency equilibrium (beyond the scope of this paper). Studying efficiency requires an explicit description for all possible coalition structure formation rules. Then one can define social desirability in terms of final individual payoffs and allocations of players. This is the big research program left for the future.

Further development of this research leads to a non-cooperative welfare theory with reconsideration of the first and second welfare theorems and

Arrow's impossibility theorem. These projects can be applied in sociology, management, social and political theories, where disagreement on rules of coalition structure formation can be interpreted as a social or a political conflict. All such issues are left for the future. The author believes that this new research direction will enrich non-cooperative game theory and provide non-cooperative fundamentals for social and mechanism design.

4.2.4 Differences from cooperative game theory concepts

There are volumes of great literature on the cooperative game theory, with the famous equilibrium concepts: the strong Nash equilibrium (sometimes it is considered as the non-cooperative equilibrium concept), the Shapley value, the coalition-proof equilibrium, the nucleolus, the kernel, the bargaining set and some other.

They share a list of common differences with the equilibrium concept introduced here: every player does not make an individual choice and does not obtain an individual payoff. In equilibrium there are no intra- and inter-coalition externalities for every player. A coalition value, (in terminology of cooperative game theory, a sum of individual payoffs of all coalition members), does not depend on a whole coalition structure.

There are other more specific differences with every existing cooperative game theory equilibrium concept. Differences from the core approach of Aumann (1960) are clear: no restriction that only one group deviates, no restriction on the direction of a deviation (inside or outside), and a construction of individual payoffs from a strategy profile of all players. There is no need to assume transferable/non-transferable utilities for players.

The approach does not need to use the blocking coalition approach, which does not study simultaneous deviation of more than one coalition, and ignores externalities between deviators in the coalition and the original set of players. Construction of a sequential game with coalition structure formation will

follow in the next paper.[16]

The role of a central planner in the current paper is different from the one introduced by Nash, who 'argued that cooperative actions are the result of some process of bargaining' Myerson (p.370, 1991). The central planner offers a predefined family of coalition structure formation mechanisms, a family of eligible partitions and a family of rules to construct these partitions from individual strategies of players.

Based on the properties of the game we can propose a criterion of stability of coalition structures and study self-enforcing properties of an equilibrium. These properties cannot be designed within existing game theories.

5   Formal definition of cooperation

In the examples from Section 2 we have seen that efficiency does not imply cooperation. This section formalizes cooperation as an allocation of players in one coalition independently from Pareto efficiency. A suggested cooperation criterion is sufficient and can be relaxed in many ways.

Definition 9 (complete cooperation in a coalition). *There is a non-cooperative game $\Gamma(K)$ with an equilibrium $\sigma^*(K) = (\sigma_i^*)_{i \in N}$. A set of players g, "cooperate completely in the coalition g" if for every player $i \in g$ there are:*

ex ante: *for every i in g, a coalition g always belongs to an individually chosen coalition structure, P_i , i.e. if s_i is chosen by i, and $s_i \in S_i(P_i)$, then $g \in P_i$.* [17]

ex post: *every realized equilibrium coalition structure contains g, i.e. $g \in \forall P^*$, where P^* is a formed equilibrium partition of $\Gamma(K)$,*

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[16]M.Kelbert, O.Pushkareva, D. Levando Formation of coalition structures as an extended form game, mimeo.

[17]However coalition structures chosen by different players maybe different.

Cooperation is defined for a game $\Gamma(K)$ with a fixed K. If K increases the cooperation may evaporate.

After the game is over the coalition g always belongs to every final equilibrium coalition structure, *disregarding allocation of players in other coalitions*. A final partition may differ from an individual choice, but will contain the desirable coalition.

6   Application: Bayesian games

Equilibrium mixed strategies may exist inside one coalition, inducing intra-coalition externalities. To show this, we reconsider the standard Battle of the Sexes (BoS) game.

There are two players, Ann and Bob. Each has two options: to be together or to be alone. In every option each can choose where to go: Boxing or Opera. Every player has four strategies, in total 16 outcomes. Every outcome consists of a payoff profile and a partition (or a coalition structure). Assume both players have preferences over coalition structures: they prefer to be together, as in the game above with two extrovert players.

The rules of coalition structure formation mechanism are:

1. If they both choose to be together, i.e. both choose the coalition structure $P_{together} = \{Ann, Bob\}$ then:

   (a) if they choose the same action (i.e. both chooses Boxing or both chooses Opera), then they they both made the same choice;

   (b) otherwise they do not go anywhere, but enjoy just being together;

2. if at least one of them chooses to remain alone, i.e. chooses a partition $P_{alone} = \{\{Ann\}, \{Bob\}\}$, then each goes alone to where she/he chooses, maybe to a different Opera or to a different Boxing match.

<div align="center">28</div>

Table 6: Payoff for the Battle of the Sexes game. B is for Boxing, O is for Opera. If the players choose to be together, and it is realized, then each obtains an additional fixed payoff $\epsilon > 0$ in any outcome.

	$B_{Bob,alone}$	$O_{Bob,alone}$	$B_{Bob,together}$	$O_{Bob,together}$
$B_{Ann,alone}$	$(2;1)^*$ $\{\{1\},\{2\}\}$	$(0;0)$ $\{\{1\},\{2\}\}$	$(2;1)$ $\{\{1\},\{2\}\}$	$(0;0)$ $\{\{1\},\{2\}\}$
$O_{Ann,alone}$	$(0;0)$ $\{\{1\},\{2\}\}$	$(1;2)^*$ $\{\{1\},\{2\}\}$	$(0;0)$ $\{\{1\},\{2\}\}$	$(1;2)$ $\{\{1\},\{2\}\}$
$B_{Ann,together}$	$(2;1)$ $\{\{1\},\{2\}\}$	$(0;0)$ $\{\{1\},\{2\}\}$	$(2+\epsilon;1+\epsilon)^{**}$ $\{1,2\}$	$(\epsilon;\epsilon)$ $\{1,2\}$
$O_{Ann,together}$	$(0;0)$ $\{\{1\},\{2\}\}$	$(1;2)$ $\{\{1\},\{2\}\}$	$(\epsilon;\epsilon)$ $\{1,2\}$	$(1+\epsilon;2+\epsilon)^{**}$ $\{1,2\}$

Formally the coalition structure formation rules are:

$$\mathcal{R}(K=1)\colon S(K=1) \mapsto S(P = \{\{Ann\},\{Bob\}\}),$$
$$\forall s \in S_i(K=1) = \{O_{Ann,alone}, B_{Ann,alone}\} \times \{O_{Bob,alone}, B_{Bob,alone}\}$$

and

$$\mathcal{R}(K=2)\colon S(K=2) \mapsto \begin{cases} S(P = \{Ann, Bob\}) = \Big\{ s\colon \\ s \in \{O_{Ann,together}, B_{Ann,together}\} \times \{O_{Bob,together}, B_{Bob,together}\} \\ S\Big(P = \{\{Ann\}, \{Bob\}\}\Big) = \\ S(K=2) \setminus S(P = \{Ann, Bob\}\}) \quad \text{otherwise} \end{cases}$$

Table 6 corresponds to the game with $K = 2$, where the game for $K = 1$ is a nested component. If Ann and Bob play the game with $K = 1$, a single coalition structure $\{\{Ann\}, \{Bob\}\}$ game, then the payoffs for this game are in the two-by-two top-left corner of Table 6, and this is the standard game.

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If Ann and Bob are together, then each obtains an additional payoff ϵ, and the corresponding cells make the two-by-two bottom-right corner of Table 6.

BoS game with one deviator ($K = 1$) has two equilibria in pure strategies and one in mixed. For $K = 2$ only mixed strategy equilibrium survives, as two player can deviate simultaneously. Mixed equilibrium for $K = 2$ differs from one for $K = 1$: another domain of pure strategies makes another coalition structure and another payoff profile. Mixed strategies equilibrium for Ann is: $\sigma^*(B_{Ann,together}) = (1 + \epsilon)/(3 + 2\epsilon)$, $\sigma^*(O_{Ann,together}) = (2 + \epsilon)/(3 + 2\epsilon)$.

Equilibrium mixed strategies may appear for every $K = 1, 2$. But for $K = 2$ equilibrium coalition structure is the grand coalition, where individual payoffs fluctuate. More of that, for $K = 2$ a value of the grand coalition holds constant for equilibrium mixed strategies. Such games are not described in available literature.

7 Application: Stochastic games

Shapley (1953) defined stochastic games as 'the play proceeds by steps from position to position, according to transition probabilities controlled jointly by the two player'. This section demonstrates how to construct this type of games.

The example below differs from the one above in two respects. A set of equilibrium mixed strategies induces more than one equilibrium coalition structure. Another difference is that an individual payoff of a coalition depends on a coalition structure to which it belongs. This game is the closest to studying group/paintball team examples in the introduction of the paper.

7.1   Corporate lunch game

There is a set of four identical players $N = \{A, B, C, D\}$. A coalition structure is an allocation of players over no more than four separate tables. [18] A strategy of i is a coalition structure, or an allocation of all players across tables during a lunch. Different from previous examples there are no strategies inside coalitions, a pure strategy is a coalition structure.

A rule of coalition structure formation is that any coalition in any coalition structure can be formed only from a unanimous agreement from members of this coalition.

A player has preferences over coalition structures: she/he prefers to eat with another one, but also prefers that all other players eat individually. Possible motivation could be a dissipation of information or gossips. From the other side, if one eats alone he/she is hurt by formed coalition of others.

If individual choices of coalition structures are inconsistent, then we apply the rule of coalition structure formation. The first column of Table 7 contains formed coalition structures.

Payoffs in Table 7 are organized to satisfy the individual preferences described above. Take column 4 and compare payoff for the coalition $\{A, B\}$ in different coalition structures: line 1, $\{\{A, B\}, \{C\}, \{D\}\}$ and line 8, $\{\{A, B\}, \{C, D\}\}$. Player C prefers to eat with D, but also prefers A and B do not eat together. This is described as that payoffs of the coalition $\{C, D\}$ depend on the whole coalition structure. *This result is impossible in existing cooperative game theory: a coalition value depends on allocation of all players.*

Let maximum number of players in a coalition be 2, $K = 2$. In equilibrium every player chooses those coalition structures, where she/he is with someone, while others are separate. Equilibrium mixed strategies are indicated only for player A.

Several different coalition structures can form in an equilibrium. Each of them can be considered as a state of a stochastic game.

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[18]Possibly empty tables do not matter, and tables can not be moved.

Table 7: Office lunch game: strategies and payoff profiles. Full set of equilibrium mixed strategies are indicated only for player A.

num	Coalition structure	Payoff profile U_A, U_B, U_C, U_D	Coalition values as in cooperative game theory
1^*	$\{A, B\}, \{C\}, \{D\}$: $\sigma_A^* = \sigma_B^* = 1/3$	(10,10,3,3)	$20_{A,B}, 3_C, 3_D$
2^*	$\{A, C\}, \{B\}, \{D\}$: $\sigma_A^* = \sigma_C^* = 1/3$	(10,3,10,3)	$20_{A,C}, 3_B, 3_D$
3^*	$\{A, D\}, \{C\}, \{B\}$: $\sigma_A^* = \sigma_D^* = 1/3$	(10,3,3,10)	$20_{A,C}, 3_C, 3_B$
4	$\{A\}, \{B\}, \{C, D\}$	(3,3,10,10)	$3_A, 3_B, 20_{C,D}$
5	$\{A\}, \{D\}, \{B, C\}$	(3,10,10, 3)	$3_A, 3_D, 20_{C,B}$
6	$\{A\}, \{C\}, \{B, D\}$	(3,10,3,10)	$3_A, 3_C, 20_{B,D}$
7	$\{A\}, \{B\}, \{C\}, \{D\}$	(3,3,3,3)	$3_A, 3_B, 3_C, 3_D$
8	$\{A, B\}, \{C, D\}$	(3,3,3,3)	$6_{A,B}, 6_{C,D}$
9	$\{A, C\}, \{B, D\}$	(3,3,3,3)	$6_{A,C}, 6_{B,D}$
10	$\{A, D\}, \{B, C\}$	(3,3,3,3)	$6_{A,D}, 6_{B,C}$
11	all other with $K = 3, 4$	(0,0,0,0)	$= 0$

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An increase in K, maximum coalition size, decreases payoffs for all players. An increment of K from 2 to 3 and then to 4 does not change an equilibrium in mixed strategies. Thus the equilibrium for $K = 2$ is robust to greater Ks. This issue is addressed Section 8.

7.2 A formal definition of a stochastic game of coalition structure formation

Let $\Gamma(K)$ be a non-cooperative game as defined above.

Definition 10. *A game $\Gamma(K)$ is a stochastic game if a set of equilibrium partitions is bigger than one, $\#(\{P^*\}(K)) \geq 2$, where a state is an equilibrium partition P^*, $P^* \in \{P^*\}(K)$.*

Studying properties of stochastic games with non-cooperative coalition structure formation is left for the future.

8 Application: non-cooperative criterion for stability

The Nash equilibrium, as defined in Nash (1950, 1951), does not specify how a player may deviate within a group. This leads to cases, when desirable outcomes cannot be supported by Nash equilibrium, while an outcome intuitively requires studying possible deviations of more than one player. Sometimes such points are called 'focal points' (Schelling).

Aumann (1990) demonstrates absence of a self-enforcement property of Nash equilibrium for a focal point in 'a stag and hare game'. The key problem is that standard Nash equilibrium does not have a tool to study deviations of more than one player. An example below explains how to change the basic game to make joint hunting for a stag become an equilibrium focal

point robust to a deviation of two players. Then the paper presents a non-cooperative criterion to measure stability of a coalition structure.

There are two hunters $i = 1, 2$. If they can hunt (only) separately, then $K = 1$, and the only available partition is $P_{alone} = \{\{1\}, \{2\}\}$. An individual strategy set of i is

$$S_i(K = 1) = \left\{ (P_{alone}, hare), (P_{alone}, stag) \right\}$$

with a general element s_i. Every s_i consists of two terms: who is the hunting partner (oneself) and what is the animal to hunt. For example, a strategy $s_i = (P_{separ}, hare)$ is interpreted as a strategy for i to hunt alone for a hare. A set of corresponding strategies for the game with $K = 1$ is

$$S(K = 1) = S_1(K = 1) \times S_1(K = 1).$$

For a game with $K = 2$ every hunter can additionally choose to hunt in a company of two and as before to choose a target for hunting: a hare or a stag. A set of strategies of i is

$$S_i(K = 2) = \left\{ (P_{alone}, hare), (P_{alone}, stag), (P_{together}, hare), (P_{together}, stag) \right\}.$$

A set of strategies of the game is a direct (Cartesian) product,

$$S(K = 2) = S_1(K = 2) \times S_2(K = 2).$$

We do not rewrite the rules for coalition structure formation, as they are the same as in the Battle of the Sexes game above. The difference is in renaming strategies. Every player knows, which game is played, either with $K = 1$ or with $K = 2$, and this is the common knowledge.

Payoffs for the games $\Gamma(K = 1)$ and $\Gamma(K = 2)$ are in Table 8. Some payoff profiles have a special interpretation: $(8; 8)$ - every hunter obtains a hare, $(4; 4)$ - two hunters obtain a hare for two, $(100; 100)$ - both hunters

obtain one stag. The point $(100; 100)$ is a desirable outcome and a focal point, which cannot be supported as the Nash equilibrium with one deviator Aumann (1990).

For a game with $K = 1$ maximum achievable payoff is $(8, 8)$, when each hunts individually for a hare. An equilibrium strategy profile is

$$s^*(K = 1) = \Big((P_{alone}, hare), (P_{alone}, hare)\Big)$$

with the payoff profile $(8; 8)$. In the game $\Gamma(K = 1)$ the players cannot reach the efficient outcome $(100, 100)$. It is available only if both hunters can decide to hunt together. This focal point (in terminology of Schelling) can be reached only within the game $\Gamma(K = 2)$, but not for the game $\Gamma(K = 1)$. This is the explanation for the problem posed by Aumann: there can be an attractive outcome of a game, but it cannot be described in terms of a Nash equilibrium of a traditional non-cooperative game. Deviation of two players demands additional tools for the model.

An equilibrium strategy profile is one when hunters can operate together:

$$s^*(K = 2) = \Big((P_{together}, hare), (P_{together}, hare)\Big)$$

and reach the payoff profile $(100; 100)$. The focal point $(100; 100)$ is an equilibrium for the game $\Gamma(K = 2)$, but it is unfeasible for the game $\Gamma(K = 1)$.

Table 8: Expanded stag and hare game

	P_{alone}, hare	P_{alone}, stag	$P_{together}$,hare	$P_{together}$, stag
P_{alone},hare	$(8; 8)^*$; $\{\{1\}, \{2\}\}$	$(8;0)$; $\{\{1\}, \{2\}\}$	$(8;8)$;$\{\{1\}, \{2\}\}$	$(8;0)$; $\{\{1\}, \{2\}\}$
P_{alone}, stag	$(0;8)$; $\{\{1\}, \{2\}\}$	$(0;0)$; $\{\{1\}, \{2\}\}$	$(0;8)$; $\{\{1\}, \{2\}\}$	$(0;0)$; $\{\{1\}, \{2\}\}$
$P_{together}$, hare	$(8;8)$; $\{\{1\}, \{2\}\}$	$(8;0)$; $\{\{1\}, \{2\}\}$	$(4;4)$; $\{1, 2\}$	$(8;0)$; $\{1, 2\}$
$P_{together}$, stag	$(0;8)$; $\{\{1\}, \{2\}\}$	$(0;0)$;$\{\{1\}, \{2\}\}$	$(0;8)$;$\{1, 2\}$	$(100; 100)^{**}$; $\{1, 2\}$

If it is uncertain, which game is played, either $\Gamma(K = 1)$ or $\Gamma(K = 2)$, then players will randomize between two strategies: $(P_{alone}, hare)$ and $(P_{separ}, stag)$, and the game become a stochastic game.

8.1 Criterion of coalition structure (a partition) stability

There is a nested family of games

$$\mathcal{G} = \{\Gamma(K = 1), \ldots, \Gamma(K), \ldots \Gamma(K = N)\}:$$

such that

$$\Gamma(K = 1) \subset \ldots \subset \Gamma(K) \subset \ldots \subset \Gamma(K = N).$$

A list of corresponding mixed strategies equilibria is

$$\Big(\sigma^*(1), \ldots, \sigma^*(K), \ldots, \sigma^*(K = N)\Big),$$

where $\sigma^*(K) = (\sigma_i^*(K))_{i \in N}$. Resulting equilibrium coalition structures (partitions) are :

$$\Big(\{P^*\}(1), \ldots, \{P^*\}(K), \ldots, \{P^*\}(K = N)\Big),$$

where for every $K = 1, \#N$ there is $\{P^*\}(K) \subset \mathcal{P}(K)$.

The family of games has an equilibrium expected payoff profiles:

$$\Big((EU_i^{\Gamma(1)})_{i \in N}^*, \ldots, (EU_i^{\Gamma(K)})_{i \in N}^*, \ldots, (EU_i^{\Gamma(K=N)})_{i \in N}^*\Big),$$

where $(EU_i^{\Gamma(K)})_{i \in N}^* \equiv (EU_i^{\Gamma(K)}(\sigma^*))_{i \in N}$.

Take a game $\Gamma(K_0)$ from $\mathcal{G}$ with maximum coalition size K_0. It has $\sigma^*(K_0)$ as an equilibrium mixed strategy profile. The question is: what is the condition when an equilibrium strategy profile does not change with an increase in maximum coalition size K?

The suggested criterion is based on the idea that a set of mixed strategies should not change with an increase in K, $\sigma^*(K_0) = \sigma^*(K_0+1) = \ldots = \sigma^*(N)$, with the restriction $Dom(\sigma^*(K_0)) = Dom(\sigma^*(K_0+1)) = \ldots = Dom(\sigma^*(N))$, where Dom is a domain of equilibrium mixed strategies. The same in other words, if a game $\Gamma(K_0)$ is substituted sequentially by games $\Gamma(K+1)$ upto $\Gamma(N)$, then an equilibrium strategy profile will not change in domain and in range. The next criterion is a sufficient criterion for coalition structure stability.

Definition 11. *Coalition structure (partition) stability criterion* *for a game $\Gamma(K_0)$ is a maximum coalition size K^*, when an equilibrium still holds true, i.e. for all $i \in N$ there is a number K^* such that*

1.

$$K^* = \max_{\substack{K=K_0,\ldots,N \\ \Gamma(K_0)\ldots,\Gamma(K=N)}} \left\{ EU_i^{\Gamma(K_0)}\Big(\sigma_i^*(K_0), \sigma_{-i}^*(K_0)\Big) \geq EU_i^{\Gamma(K)}\Big(\sigma_i^*(K), \sigma_{-i}^*(K)\Big) \right\},$$

2. $Dom\ \sigma^*(K^*) = Dom\ \sigma^*(K_0)$

where $\sigma^*(K_0)$ is an equilibrium in <u>the</u> game $\Gamma(K_0)$, while $\sigma^*(K)$ is an equilibrium in <u>a</u> game $\Gamma(K)$, $K = K_0, \ldots, N$, and Dom is a domain of equilibrium mixed strategies set.

The definition is operational, it can be constructed directly from a definition of a family of games $\mathcal{G}$. It guarantees stability of both payoffs and partitions. In Prisoner's Dilemma example (where both players are "non-extroverts", "non-introverts") an increment in K increases a number of equilibrium coalition structures without a change in payoffs. This increases a number of equilibria, without rejecting one for $K = 1$. The criterion suggests $K^* = 2$ for this game. The lunch game (Section 7) example has robust equilibrium for $K^* = 2$. Some applications may require weaker forms of the criterion.

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In the extended version of stag and hare game an increase in K changed the equilibrium. The same took place in a variation of Battle of the Sexes game. However this did not happen in the Corporate Lunch game.

The proposed criterion may serve as a measure of trust to an equilibrium or as a test for self-enforcement of an equilibrium. This criterion can be applied to study opportunistic behavior in coalition partitions. If players in a coalition g of a game $\Gamma(K_1)$ have perfect cooperation (Section 5), this does not mean that in a wider game $\Gamma(K_2)$, $K_1 < K_2$, they will still cooperate.

Studying stability is tightly connected to studying focal points. The suggestion of the criterion is only the first step to the big field of research.

9   Conclusion

Traditionally social sciences are interested in structuring people in multiple groups based on their individual preferences. The paper presents a family of non-cooperative finite games for coalition structure formation motivated by self-interest benefits. Every game in a family has an equilibrium in mixed strategies. Examples demonstrate how the approach can be applied to study some popular games. The paper introduces a non-cooperative criterion to measure stability of coalition structures based on maximum number of deviators.

The novelty of the paper is not in mathematical, but in the structural result. The suggested family of games can help to study new issues, earlier beyond the scope of research, for example, focal points. Development of the model for repeated and network games are in other forthcoming papers from the author and his colleagues.

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