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Simon Schopohl

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Communication Games with Optional Verification*

Simon Schopohl†

EDEEM, Université Paris 1, Universität Bielefeld and Université catholique de Louvain

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Abstract

We consider a Sender-Receiver game in which the Sender can choose between sending a cheap-talk message, which is costless, but also not verified and a costly verified message. While the Sender knows the true state of the world, the Receiver does not have this information, but has to choose an action depending on the message he receives. The action then yields to some utility for Sender and Receiver. We only make a few assumptions about the utility functions of both players, so situations may arise where the Sender's preferences are such that she sends a message trying to convince the Receiver about a certain state of the world, which is not the true one. In a finite setting we state conditions for full revelation, i.e. when the Receiver always learns the truth. Furthermore we describe the player's behavior if only partial revelation is possible. For a continuous setting we show that additional conditions have to hold and that these do not hold for "smooth" preferences and utility, e.g. in the classic example of quadratic loss utilities.

Keywords : cheap-talk, communication, costly disclosure, full revelation, increasing differences, Sender-Receiver game, verifiable information.

JEL Classification : C72, D82.

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†mail: SimonSchopohl@gmail.com

1 Introduction

In the early 1980s the first papers on communication games got published and nearly from the beginning there have been two strains dealing with a different type of communication. Crawford and Sobel (1982) introduced the cheap-talk models, in which the content of the message can be whatever the Sender wants it to be. The Sender does not have to tell the truth and so the Receiver either believes the message or not. Given this model the different types of the Sender (corresponding to our different states of the world) may either send the same messages or different types send different messages. While in the first case the Receiver cannot trust the messages and there is a so called babbling equilibrium, in the second case the Receiver can get information from the message.

While in this setting the Sender has no possibility to verify that she tells the truth, she can only tell the truth in the models of Grossman (1981) and Milgrom (1981). In the models of verifiable messages, each message consists of a set of states, which has to include the true state. In equilibrium all different types of the Sender are separated, caused by the unraveling argument. In our model we give the Sender the choice between sending a cheap-talk message or a verified message. The limitation we do is to only allow for the entire truth as the verified message, i.e. the Sender cannot choose a set of states, but can either tell the complete truth or send a cheap-talk message.

In this paper we do not model communication in a very own way, but combine the most basic settings of cheap-talk and verifiable messages. There is one player who has detailed information about the state of the world. She is called the Sender, as she sends a message about the state to the second player, the Receiver. In our model the Sender can choose between cheap-talk messages corresponding to the different states of the world, or a costly verified message. The Receiver reads this message and chooses an action, which yields to some payoff for both players. While the verified message reveals the true state of the world, the cheap-talk messages do not. The content of the cheap-talk messages can be the truth, can contain the truth, but also can be a lie. The Receiver's behavior after receiving a cheap-talk message may vary depending on the preferences of both players. In a situation where there is no benefit for the Sender to lie, the Receiver can trust the cheap-talk messages, while he should not do so if the Sender's preference differs too much from his own.

There are many economic and non-economic examples for Sender-Receiver-games: The most classic is the idea of an employer (the Receiver) and an agent (the Sender) going back to Spence (1973). The Sender has different interests than the Receiver, but still wants to be employed or get a contract. Her effort cannot be observed, but she reports it to the Receiver. Our model adds the possibility of costly reporting a certified effort, e.g. giving a proof of skill enhancement or other training. The Receiver has to choose an action according to the message he gets. While the verifiable message gives him guaranty, the receiving of a cheap-talk message also reveals further information, as the Sender has chosen not to invest into the verifiable message.

Another example is the Lemon Market by Akerlof (1970). There is a seller (the Sender), who knows the quality of the good she is selling and a buyer (the Receiver), who can decide to buy the product or not. The cheap-talk message corresponds to the idea that the Sender simply tells the quality of the good. We often see these messages in internet auctions or in online car advertisements. On the other hand the Sender could also invest some money and certify the quality of his product. In the real world there are several ways to do so, for example by independent consultants and experts. Of course the Sender will never verify if the quality is low, but for high quality it might be reasonable to assume that the Sender is willing to invest some money into the verification to get his good sold.

Related Literature

Since the introduction of cheap-talk models, extensions in many different directions have been made. Farrell and Gibbons (1989) introduce an additional Receiver. They observe how the existence of a second Receiver changes the report of the Sender. McGee and Yang (2013) and Ambrus and Lu (2014) do a similar step with multiple Senders. While McGee and Yang, 2013 focus on two Senders with complementary information, Ambrus and Lu (2014) model several Senders who observe a noisy state. Noise is also added to the signaling game by Haan, Offerman, and Sloof (2011). The authors derive equilibria depending on the level of noise and confirm their results by an experiment. A different extension of cheap-talk is done by Blume and Arnold (2004). They model learning in cheap-talk games and derive a learning rule depending on common interest. The idea of Baliga and Morris (2002) is closer to our paper. The authors give conditions for full communication, but also state conditions under which cheap-talk does not change the equilibrium.

An overview over cheap-talk models and models with verifiable information can be found in Sobel (2009). The author describes several models and gives some economic examples. Most of these examples can be extended to fit our setting by adding a reasonable verifiable message.

At the same time other models focus more on disclosure of information and costly communication. Hagenbach, Koessler, and Perez-Richet (2014) analyse a game with a set of players, where each player can tell the truth about his type or can masquerade as some other type. As in the literature of verifiable messages, the player who deviates from telling the truth is punished by the other players. If a player masquerades, the other players assume a worst case type and punish him by choosing the action this type of player dislikes. The authors state conditions for full revelation depending on the possible masquerades of each player.

Bull and Watson (2007) and Mookherjee and Tsumagari (2014) deal with communication and mechanism design. While Bull and Watson (2007) focus on costless disclosure, Mookherjee and Tsumagari (2014) add communication costs to prevent full revelation of information. Communication costs are also introduced by Hedlund (2015). The author derives two types of equilibria: For high costs there exists a pooling equilibrium, while for not too high costs a separating equilibrium exists.

Verrecchia (2001) provides an overview over different models of disclosure, which is extended by Dye (2001).

It remains to mention that there are several works in which the authors have created their own way of modeling messages, which are most often in between the models of Crawford and Sobel (1982) and Grossman (1981) and Milgrom (1981). Kartik (2009) introduces a model, where the Sender sends a message about her type, but has the incentive to make the Receiver believe that her type is higher than it actually is. If the Sender lies in the message, she has to pay some costs for lying, which depend on the distance between the true type and the type stated in the message. Dewatripont and Tirole (2005) analyse the communication, where the Sender has to invest into effort to make a message understandable, while the Receiver's investment into effort is to understand the message. They motivate this model by the idea that very confusing written messages as well as reading messages without paying a lot of attention yield to misunderstandings. The probability of understanding the message is influenced by the effort of both players.

Austen-Smith and Banks (2000) introduce the possibility for the Sender to send a costly message with the same content as a costless message. By this way of burning money the Sender has an additional possibility of signaling. The authors show that conditions exist under which both message types are used. Verrecchia (1983) deals with the idea that the disclosure of information works as a signal itself.

The most closely related to this paper is the work by Eső and Galambos (2013). They start with a similar idea, but make some different assumptions in their model. While they focus on optimal actions for Sender and Receiver which are closely related to the paper of Crawford and Sobel (1982), we start with fewer assumptions and allow for a wider class of utility functions. Under their assumptions, Eső and Galambos (2013) state conditions under which the equilibria are split into different intervals depending on the Sender's type.

The paper is structured as follows. We start by defining a discrete model in Section 2 and focus on a finite set of states and actions. We state different conditions for full revelation. Full revelation where the Sender just sends cheap-talk messages is only possible when the preferences of both players are similar. Even if the preferences differ, there still can be full revelation: The Receiver threatens the Sender by answering cheap-talk with an action the Sender dislikes. By this the Sender has an incentive to pay for the costly verifiable message. As some actions are incredible threats, we define a set of possible threat points. We provide conditions under which the Receiver can use an action from this set to enforce full revelation. Furthermore we look at the cases of partial revelation and state all the different maximization problems. We get similar conditions as we got for the fully revealing equilibria, but for partial revelation these conditions just have to hold for some states. While the solving by hand might be complex, we give some detailed ideas for algorithms. We simplify all these conditions for utility functions that have increasing or decreasing differences.

In Section 3 we modify our setting to a continuous model. We provide necessary conditions for the existence of fully revealing equilibria and also give some examples. We illustrate and prove that in a fully revealing equilibrium the Sender will never use both types of messages, if the utility functions of both players are continuous. A very detailed analysis is done for the quadratic loss function. Since full revelation is impossible in a continuous state space, we give conditions for full revelation in a discrete version. Extension possibilities are mentioned several times within the paper, but discussed in detail in Section 4. Section 5 concludes. All proofs are relegated to the Appendix.

2 Discrete Model

We start with a model with only a finite set of states of the world and a finite set of actions the Receiver can choose from. Let $\Omega = \{\omega_1, \dots, \omega_L\}$ denote the set of the L different states of the world, where each state ω_i has the probability $\mathbb{P}[\omega_i]$.

The timing is as follows: The Sender learns the true state of the world and then she sends a message to the Receiver. We assume that the set of possible cheap-talk messages M corresponds to the states of the world Ω and that verifiable message v is unique in each state of the world. The Sender chooses a message from $M \cup \{v\}$, so either sends a cheap-talk message or the verifiable message v. There is no possibility for partial disclosure. While sending any cheap-talk message is free, the Sender has to pay costs $c > 0$ if she sends the verifiable message. An economic explanation for these costs can be either a payment for a certificate or the investment into effort. For simplicity we assume that the costs are state independent, but state dependent costs are discussed in a later part as an extension.

The Receiver reads the message and chooses an action from $A = \{a_1, \dots a_N\}$. By $\Delta(A)$ we denote the set of mixed strategies. Both players have preferences about the actions, resulting in different von Neumann-Morgenstern utility functions for both players, depending on the action and state of the world.

For the Sender it is given by $\tilde{u}_S : A \times M \cup \{v\} \times \Omega \to \mathbb{R}$ with $\tilde{u}_S(a, m, \omega) = u_S(a, \omega) \, \forall m \in M$, $\tilde{u}_S(a, v, \omega) = u_S(a, \omega) - c$ and $u_S : A \times \Omega \to \mathbb{R}$. So we can split the utility function of the Sender up into two parts: First a utility function depending on action and state of the world. Additionally we have to subtract the costs for the message if there are any.

For the Receiver the utility function is not depending on the type of the message, but just on the action and state: $u_R : A \times \Omega \to \mathbb{R}$. The utility functions show that there is neither a punishment for lying nor a direct reward for telling the truth. We assume that these utility functions are common knowledge. Let $a^*_R(\omega_i)$ denote the action the Receiver prefers in the state ω_i. We will make some more assumptions about this function later. We denote the Sender's behavior by the function $f : \Omega \to M \cup \{v\}$. This function f maps each state of the world to the message she sends. We assume that the Sender does not mix different messages.

The Receiver chooses the action, depending on the message he received: $g : M \cup \Omega \to \Delta(A)$. In equilibria we have to define the behavior of the Sender for every state, so $f(\omega)$ and the Receiver's action after each message, i.e. $g(m) \, \forall m \in M$ and $g(v)$.

The equilibrium concept we use is the Perfect Bayesian Equilibrium.

Definition 1. *A Perfect Bayesian Equilibrium in a dynamic game of incomplete information is a strategy profile (f^*, g^*) and a belief system μ^* for the Receiver such that*

- *The strategy profile (f^*, g^*) is sequentially rational.*
- *The belief system μ^* is consistent whenever possible, given (f^*, g^*).*

In other words each equilibrium consists of optimal strategies for Sender and Receiver, which are sequentially rational. Furthermore the Receiver has a belief system over the true state of the world depending on the message he receives. This belief system is updated by Bayes rule whenever possible. For Perfect Bayesian Equilibria the actions off the equilibrium path have to be the best actions for the Receiver for at least one belief system. We can neglect this if we limit our attention to actions that are undominated for the Receiver.

We are specially interested in equilibria with full revelation:

Definition 2. *A Perfect Bayesian Equilibrium is fully revealing, if the Receiver knows the true state of the world, after reading the Sender's message.*

If there is full revelation, the Sender either sends different cheap-talk messages in each state, or just verifiable messages, or different cheap-talk messages in some states and verifiable messages in the other states.

For the entire paper we make two assumptions:

Assumption 1. *Let us assume that for each action $a_j \in A$ there exists at least one belief system μ such that a_j is the Receiver's best strategy under the belief system μ.*
By $\hat{\Delta}(A) \subseteq \Delta(A)$ we denote the set of mixed strategies that satisfy this assumption, i.e.

$$\forall \hat{a} \in \hat{\Delta}(A) : \exists \mu : \hat{a} \in \arg\max_{a} \sum_{\omega \in \Omega} \mu(\omega) \cdot u_R(a, \omega)$$

Assumption 1 requires that each action is optimal for the Receiver at least under one belief system, which means that there are no dominated actions. Our results depend on the idea that the Receiver uses an action as a threat and so enforces the Sender to send verified messages. The threat is only credible, if this action is an element of $\hat{\Delta}(A)$.

We can think about different equilibrium refinements as introduced in several papers. The most common ones are the Divinity Criterion by Banks and Sobel (1987) and the Intuitive Criterion by Cho and Kreps (1987). Using any of them adds more conditions for the threat points, so the set $\hat{\Delta}(A)$ gets smaller and the Receiver has less possibilities to make a threat.

Assumption 2. *Let us assume that in every state of the world the Receiver has strict preferences.*

Under Assumption 2, $a^*_R(\omega_i)$ is a single action, which will help for the upcoming results. This is to avoid the situation that the Receiver is indifferent between two actions.

2.1 Full revelation

In this first part we focus our attention on fully revealing equilibria. We will state conditions for full revelation, where the Sender just uses the cheap-talk messages, conditions where she uses only verified messages and conditions where she uses both types of message, depending on the state. Even if conditions for one of these fully revealing equilibria are satisfied, there might be other equilibria at the same time. By examples we show that the existence of these different types of full revelation are independent of each other.

Assumption 3. *Let us assume that for all states $\omega_i \neq \omega_j$ the Receiver prefers different actions, i.e. $a^*_R(\omega_i) \neq a^*_R(\omega_j)$.*
In other words the function $a^*_R : \Omega \to A$ has to be injective.

This assumption assures that there can be a fully revealing equilibrium, even if the Sender uses cheap-talk messages in several states. For the case that the Sender just uses the cheap-talk messages and we still have full revelation, the Sender is not allowed to have any incentive to deviate to another cheap-talk message. It is not necessary that the preferences in all states are the same for Sender and Receiver. Crucial is that the action the Receiver chooses when he knows the true state $a^*_R(\omega_i)$ generates a higher utility for the Sender than the Receiver's most preferred action of any other state $a^*_R(\omega_j)$. There is also the possibility that there exists an action the Sender prefers, but which is never included by the Receiver as long as he knows the true state of the world.

Theorem 1 (Full Revelation just by Cheap-Talk Messages)**.**
Let Assumption 3 hold. There is a fully revealing equilibrium with only costless messages sent if and only if:

$$\forall \omega_i \in \Omega : u_S(a^*_R(\omega_i), \omega_i) > u_S(a^*_R(\omega_j), \omega_i) \; \forall \omega_j \neq \omega_i \tag{1}$$

Remark. *If Assumption 3 does not hold, i.e. if there exist two states ω_i, ω_j such that $a^*_R(\omega_i) = a^*_R(\omega_j)$, there is no fully revealing equilibrium. Still the Receiver can get the highest possible utility in every state, while the Sender just sends cheap-talk messages.*

Theorem 2 (Full Revelation just by Verifiable Messages)**.**
There is a fully revealing equilibrium with only verifiable messages sent if and only if:

$$\begin{aligned} \exists \hat{a} \in \hat{\Delta}(A) : &1) \; \forall \omega_i : \hat{a} \neq a^*_R(\omega_i) \\ &2) \; \forall \omega_i : u_S(a^*_R(\omega_i), \omega_i) - c > u_S(\hat{a}, \omega_i) \end{aligned}$$

The idea behind Theorem 2 is that the Sender replies to cheap-talk messages with an action $\hat{a}$ the Sender really dislikes. With this threat point $\hat{a}$ the Receiver forces the Sender to use the verified message. The same idea can be found in several existing papers dealing with verifiable messages, e.g. in Hagenbach, Koessler, and Perez-Richet (2014), which we already mentioned in the introduction.

We can combine both theorems and get conditions for full revelation, where the Sender uses both types of messages.

Theorem 3 (Full Revelation by Cheap-Talk and Verifiable Messages)**.**
Let Assumption 3 hold. There is a fully revealing equilibrium with both message types used if and only if there exists $\hat{\Omega} \subsetneq \Omega$ with $\hat{\Omega} \neq \emptyset$ such that

$$\begin{aligned} \exists \hat{a} \in \hat{\Delta}(A) : &1) \; \forall \omega_i \notin \hat{\Omega} : u_S(a^*_R(\omega_i), \omega_i) - c > u_S(a^*_R(\omega_j), \omega_i) \; \forall \omega_j \in \hat{\Omega} \\ &2) \; \forall \omega_i \in \hat{\Omega} : u_S(a^*_R(\omega_i), \omega_i) > u_S(a^*_R(\omega_j), \omega_i) \; \forall \omega_j \in \hat{\Omega}, \omega_j \neq \omega_i \end{aligned}$$

Theorem 3 allows that the Receiver trusts the Sender in some states ($\hat{\Omega}$), but in the other states he enforces the use of verifiable messages as in Theorem 2. To have both message types used $\hat{\Omega}$ has to be a real subset of Ω and non-empty, otherwise just one message type is used. The two conditions in this theorem are quite similar to those of the previous theorems. Instead of a single threat point $\hat{a}$, every $a^*_R(\omega_j)$ with $\omega_j \in \hat{\Omega}$ has to work as a threat. In addition the Sender is not allowed to have an incentive to deviate to another cheap-talk message if the true state is in $\hat{\Omega}$.

There might be several possibilities for the set of states $\hat{\Omega}$, where the Receiver trusts the cheap-talk messages. Those possibilities do not have to be subsets of each other, but also can be disjoint. For the case that there are several subsets we can say that for smaller sets Condition 1) has to hold for more states, but Condition 2) for less states.

Remarks.

- *For the result of Theorem 3 we need Assumption 3 just for the states in $\hat{\Omega}$. So even if there exist two states $\omega_i, \omega_j \in \Omega/\hat{\Omega}$ such that $a^*_R(\omega_i) = a^*_R(\omega_j)$, Theorem 3 still holds.*
- *If Assumption 3 does not hold and there exist two states $\omega_i, \omega_j \in \hat{\Omega}$ such that $a^*_R(\omega_i) = a^*_R(\omega_j)$, Theorem 3 does not hold, but under the conditions of that theorem, the Receiver still gets the highest possible utility in every state.*

Theorems 1, 2 and 3 give conditions for different types of fully revealing equilibria. It can happen that there is no fully revealing equilibrium just by cheap-talk or just by verifiable messages, but one by a combination of both message types:

Example 1. *Assume that there are two states (ω_1, ω_2) and two actions (a_1, a_2). The Receiver prefers a_1 in ω_1 and a_2 in ω_2, while the Sender always prefers a_1. Obviously there is no fully revealing equilibrium just with cheap-talk, because the Sender always wants the action a_1 and so she would lie. Furthermore there is no equilibrium just with verifiable messages, because there is no threat available:*

For the mixed strategy that plays a_1 with probability p and a_2 with probability $(1-p)$, we use the notation $pa_1 + (1-p)a_2$. Denote $\hat{a} = pa_1 + (1-p)a_2$. For $p = 0$, the Sender will not use the verifiable message in ω_2, because she gets the same action by sending cheap-talk, but verifiable messages are costly. Also for $p > 0$ the Sender will not use the verifiable message in ω_2, because she prefers a_1 over a_2 and so she also prefers $\hat{a}$ over a_2.

Still there is full revelation possible if c is low enough. Let us assume that costs c are small, i.e. $c < u_S(a_1, \omega_1) - u_S(a_2, \omega_1)$. If the Receiver answers every cheap-talk message by a_2, the Sender will use the verifiable message in ω_1, yielding to action a_1. The utility the Sender gets is $u_S(a_1, \omega_1) - c > u_S(a_2, \omega_1)$, while her utility would be $u_S(a_2, \omega_1)$ if she sends the cheap-talk message. In the second state ω_2, the Sender will use the cheap-talk message. Both message types will result in action a_2, so the Sender prefers the costless message.

Even though we stated conditions for full revelation, it might happen that there is no revelation at all. The easiest example can be done just by two states and two actions:

Example 2. *Assume that the Receiver prefers a_1 in ω_1 and a_2 in ω_2 and the Sender's preferences are switched, i.e. she prefers a_2 in ω_1 and a_1 in ω_2. Clearly there is no full revelation just by cheap-talk, because the Sender will always lie. Furthermore there can be no revelation just by verifiable messages. Assume that the threat point is $\hat{a} = pa_1 + (1-p)a_2$, with the notation used as in the previous example.*

For $p = 0$, the Sender will not use the verifiable message in ω_1, because she prefers a_2 over a_1. The same argument holds even for $p > 0$: Using the cheap-talk message resulting in $\hat{a}$ gives the Sender at least a little chance of a_2. Therefore $u_S(\hat{a}, \omega_1) > u_S(a_1, \omega_1)$ and this implies $u_S(\hat{a}, \omega_1) > u_S(a_1, \omega_1) - c$.

So the only possibility is to have a fully revealing equilibrium with both message types used. Doing the same steps again for Theorem 3 proves that there is no full revelation. So in this example where the preferences of Sender and Receiver differ a lot, the Receiver has no possibility to enforce the full revelation.

2.1.1 Increasing and Decreasing Differences

The previous results have to be checked for every state, which might be not easy to do. If the utility function of the Sender satisfies either the increasing or decreasing differences property, we can state weaker conditions. The idea is that we just need to check the previous conditions, for one state and then can easily get full revelation for all states, if some additional properties are satisfied.

Definition 3. *We say $u_S(a, \omega)$ has increasing (decreasing) differences in (a, ω), if*
$\forall a' \geq a, \forall \omega' \geq \omega : u_S(a', \omega') - u_S(a, \omega') \geq (\leq)\ u_S(a', \omega) - u_S(a, \omega)$.

Proposition 1 (Full revelation under increasing differences)**.**
Let $\Omega = \{\omega_1, \ldots, \omega_L\}$ and sort A such that $A = \{a_R^*(\omega_1), \ldots, a_R^*(\omega_L)\}$. We can ignore all actions, which are never the best reply for the Receiver in a single state.
There is a fully revealing equilibrium with $\hat{a} = a_R^*(\omega_j)$ if:

$$\begin{aligned}
&1)\ u_S \textit{ has increasing differences}\\
&2.1)\ u_S(a_R^*(\omega_{j+1}), \omega_{j+1}) - c > u_S(a_R^*(\omega_j), \omega_{j+1})\\
&2.2)\ u_S(a_R^*(\omega_{j-1}), \omega_{j-1}) - c > u_S(a_R^*(\omega_j), \omega_{j-1})\\
&3.1)\ \forall \omega_i > \omega_j : u_S(a_R^*(\omega_i), \omega_i) \geq u_S(a_R^*(\omega_{i-1}), \omega_i)\\
&3.2)\ \forall \omega_i < \omega_j : u_S(a_R^*(\omega_i), \omega_i) \geq u_S(a_R^*(\omega_{i+1}), \omega_i)
\end{aligned}$$

The fully revealing equilibrium is such that the Sender sends cheap-talk in ω_j and verifiable messages in all other states.

Corollary 1.
We can replace Condition 3.1) by

$$3.1') \ \forall \omega_i > \omega_j : u_S(a_R^*(\omega_i), \omega_i) \geq u_S(a_R^*(\omega_{j+1}), \omega_i)$$

and Condition 3.2) by

$$3.2') \ \forall \omega_i < \omega_j : u_S(a_R^*(\omega_i), \omega_i) \geq u_S(a_R^*(\omega_{j-1}), \omega_i)$$

An interpretation of these properties can be done easily, if we look at the following Corollary.

Corollary 2.
Let $\Omega = \{\omega_1, \dots, \omega_L\}$ and sort A such that $A = \{a_R^*(\omega_1), \dots, a_R^*(\omega_L)\}$. We can ignore all actions, which are never the best reply for the Receiver in a single state.
There is a fully revealing equilibrium with $\hat{a} = a_R^*(\omega_1)$ if:

1) u_S *has increasing differences*
2) $u_S(a_R^*(\omega_2), \omega_2) - c > u_S(a_R^*(\omega_1), \omega_2)$
3) $\forall \omega_i \in \{\omega_2, \dots, \omega_L\} : u_S(a_R^*(\omega_i), \omega_i) \geq u_S(a_R^*(\omega_{i-1}), \omega_i)$

The fully revealing equilibrium is such that the Sender sends cheap-talk in ω_1 and verifiable messages in all other states.

The threat point here is $a_R^*(\omega_1)$. Condition 2) ensures that the Sender prefers the verifiable message in the state after, which is ω_2. Increasing Differences mean that the gains from a higher action increase, if the state gets higher. With Condition 3) combined, we get that the Sender also prefers the verifiable message in all states higher than ω_2. We can get a similar result with $a_R^*(\omega_L)$, where we have to replace ω_2 in Condition 2) by ω_{L-1} and adjust Condition 3) as well. An application can be found in Section 3.1.1.

Similar changes for decreasing differences can be made easily:

Proposition 2 (Full revelation under decreasing differences)**.**
Let $\Omega = \{\omega_1, \dots, \omega_L\}$ and sort A such that $A = \{a_R^*(\omega_1), \dots, a_R^*(\omega_L)\}$. We can ignore all actions, which are never the best reply for the Receiver in a single state.

There is a fully revealing equilibrium with $\hat{a} = a_R^*(\omega_j)$ if:

$$\begin{aligned}
&1)\ u_S \textit{ has decreasing differences}\\
&2.1)\ u_S(a_R^*(\omega_L), \omega_L) - c > u_S(a_R^*(\omega_j), \omega_L)\\
&2.2)\ u_S(a_R^*(\omega_1), \omega_1) - c > u_S(a_R^*(\omega_j), \omega_1)\\
&3.1)\ \forall \omega_i > \omega_j : u_S(a_R^*(\omega_i), \omega_i) \geq u_S(a_R^*(\omega_{i+1}), \omega_i)\\
&3.2)\ \forall \omega_i < \omega_j : u_S(a_R^*(\omega_i), \omega_i) \geq u_S(a_R^*(\omega_{i-1}), \omega_i)
\end{aligned}$$

The fully revealing equilibrium is such that the Sender sends cheap-talk in ω_j and verifiable messages in all other states.

Changing the conditions as in Corollary 1 and Corollary 2 is possible.

2.2 Maximization without full revelation

Whenever there is no full revelation, the Receiver cannot get the highest possible utility in all states. Depending on the preferences, utility and costs, there might be partial revelation or no revelation at all. We start our analysis by an example with just three states and give conditions for partial revelation and no revelation. In a numerical example we will show that each of this possibilities can be the best strategy for the Receiver.

In the second part we generalize: In a setting with more than three states, partial revelation can be of one of three different types: Verifiable messages in some states, revealing the true state. Unique cheap-talk messages can have the same effect, but cheap-talk messages can also partial reveal information to the Receiver, such that it splits the state space into disjoint subsets. In that case the Receiver might just know whether he is in the first or second state, or in the third or fourth state.

We give conditions for all the different possibilities of partial revelation and also for combinations of those. Furthermore we again use utility functions with increasing or decreasing differences to simplify these conditions.

2.2.1 Three state examples

Assume $|\Omega| = 3$ and assume that the Receiver prefers different actions in different states.
In general the Receiver can maximize his utility by three different possibilities:

1. Use the same action in every state.

2. Reply with one action to one cheap-talk message and with another to the remaining messages.

3. Use the same action as a reply to any cheap-talk message, enforcing the Sender to send the verifiable message in exactly one state, i.e. revelation of one state.

First possibility

$$\begin{aligned}\max_{\hat{a}} &\sum_{i=1}^{3} \mathbb{P}[\omega_i] u_R(\hat{a}, \omega_i)\\ \text{s.t. } &\forall \omega_i : u_S(\hat{a}, \omega_i) > u_S(a_R^*(\omega_i), \omega_i) - c\end{aligned}$$

Second possibility (Revelation in ω_1 (wlog) by cheap-talk)

$$\begin{aligned}\max_{\hat{a}} \ &\mathbb{P}[\omega_1] u_R(a_R^*(\omega_1), \omega_1) + \sum_{i=2}^{3} \mathbb{P}[\omega_i] u_R(\hat{a}, \omega_i)\\ \text{s.t. } &u_S(a_R^*(\omega_1), \omega_1) > u_S(\hat{a}, \omega_1)\\ &u_S(\hat{a}, \omega_i) > u_S(a_R^*(\omega_1), \omega_i) \ \forall \omega_i, i \in \{2, 3\}\end{aligned}$$

Third possibility (Revelation in ω_1 (wlog) by a verifiable message)

$$\begin{aligned}\max_{\hat{a}} \ &\mathbb{P}[\omega_1] u_R(a_R^*(\omega_1), \omega_1) + \sum_{i=2}^{3} \mathbb{P}[\omega_i] u_R(\hat{a}, \omega_i)\\ \text{s.t. } &u_S(a_R^*(\omega_1), \omega_1) - c > u_S(\hat{a}, \omega_1)\\ &u_S(\hat{a}, \omega_i) > u_S(a_R^*(\omega_i), \omega_i) - c \ \forall \omega_i, i \in \{2, 3\}\end{aligned}$$

These are the different maximization problems the Receiver has to solve to find the best strategy. With different example we will show that either of the strategies can be the best choice. For that we have to keep in mind that the Receiver cannot commit to any strategies, but plays his best possibility given his beliefs. Especially in the case where he knows the true state, the Receiver will always play the action that yields the highest utility for him.

Example 3. *Assume $c > 2$, $|\Omega| = |A| = 3$,*

$$\begin{array}{lll} u_R(a_3, \omega_1) = 4 & u_R(a_1, \omega_1) = 2 & u_R(a_2, \omega_1) = 1 \\ u_R(a_1, \omega_2) = 4 & u_R(a_2, \omega_2) = 2 & u_R(a_3, \omega_2) = 1 \\ u_R(a_2, \omega_3) = 4 & u_R(a_1, \omega_3) = 2 & u_R(a_3, \omega_3) = 1 \end{array}$$

and $u_S(\cdot, \omega_1) = u_R(\cdot, \omega_1)$, but $u_S(\cdot, \omega_2) = u_R(\cdot, \omega_3)$ and $u_S(\cdot, \omega_3) = u_R(\cdot, \omega_2)$. So Sender and

Receiver have the same preferences in ω_1, but switched between ω_2 and ω_3.

<u>No full revelation</u>

Clearly there is no full revelation just by cheap-talk messages. The proof why there is no full revelation just by verifiable messages and also not by both message types used, follows the same idea: Assume that $\hat{a} = \pi_1 a_1 + \pi_2 a_2 + (1 - \pi_1 - \pi_2) a_3$, therefore

$$
\begin{aligned}
u_S(a_3, \omega_1) - c &> u_S(\hat{a}, \omega_1) \\
u_S(a_1, \omega_2) - c &> u_S(\hat{a}, \omega_2) \\
u_S(a_2, \omega_3) - c &> u_S(\hat{a}, \omega_3)
\end{aligned}
$$

have to hold. Rewriting this yields to

$$
\begin{aligned}
4 - c &> \pi_1 \cdot 1 + \pi_2 \cdot 2 + (1 - \pi_1 - \pi_2) \cdot 4 \\
2 - c &> \pi_1 \cdot 2 + \pi_2 \cdot 4 + (1 - \pi_1 - \pi_2) \cdot 1 \\
2 - c &> \pi_1 \cdot 4 + \pi_2 \cdot 2 + (1 - \pi_1 - \pi_2) \cdot 1
\end{aligned}
$$

and finally to

$$
\begin{aligned}
c &< 3\pi_1 + 2\pi_2 \\
1 - c &> \pi_1 + 3\pi_2 \\
1 - c &> 3\pi_1 + \pi_2
\end{aligned}
$$

This is impossible for $c \geq 1$. For the full revelation by both message types, $\hat{a}$ can also be equal to a_1, a_2 or a_3, but all these possibilities still contradict at least one condition.

<u>Maximization</u>

1.) No revelation:

The Receiver solves $\max \frac{1}{3} \cdot (2\pi_1 + 1\pi_1 + 4(1 - \pi_1 - \pi_2)) + \frac{1}{3} \cdot (4\pi_1 + 2\pi_1 + 1(1 - \pi_1 - \pi_2)) + \frac{1}{3} \cdot (2\pi_1 + 4\pi_1 + 1(1 - \pi_1 - \pi_2))$, which yields to $\hat{a} = a_1$ and expected utility for the Receiver given by $\mathbb{E}[u_R] = \frac{1}{3}(2 + 4 + 2) = \frac{8}{3}$. The conditions for the Sender not to deviate are:

$$
\begin{aligned}
u_S(\hat{a}, \omega_1) > u_S(a_3, \omega_1) &\quad \Leftrightarrow \quad 2 > 4 - c \\
u_S(\hat{a}, \omega_2) > u_S(a_1, \omega_1) &\quad \Leftrightarrow \quad 2 > 4 - c \\
u_S(\hat{a}, \omega_3) > u_S(a_2, \omega_1) &\quad \Leftrightarrow \quad 4 > 2 - c.
\end{aligned}
$$

All these conditions hold for $c > 2$.

2.) Partial revelation of ω_1 by cheap-talk:

The Receiver has to answer with a_3 to one cheap-talk message and with $\hat{a}$ to the others. The maximization problem yields that $\hat{a} = \pi_1 a_1 + (1 - \pi_1)a_2$, with $\pi_1 \in [0, 1]$. The conditions for the Sender's utility are

$$\begin{aligned} u_S(a_3, \omega_1) > u_S(\hat{a}, \omega_1) &\Leftrightarrow 4 > u_S(\hat{a}, \omega_1) \\ u_S(\hat{a}, \omega_2) > u_S(a_3, \omega_1) &\Leftrightarrow u_S(\hat{a}, \omega_2) > 1 \\ u_S(\hat{a}, \omega_3) > u_S(a_3, \omega_1) &\Leftrightarrow u_S(\hat{a}, \omega_3) > 1, \end{aligned}$$

which are clearly satisfied. Here the Receiver's expected utility is $\mathbb{E}[u_R] = \frac{1}{3}(4 + 6) = \frac{10}{3}$.

3.) Partial revelation of ω_1 by a verifiable message:
For example we can take $\hat{a} = a_2$. The conditions for the Sender's utility are

$$\begin{aligned} u_S(a_3, \omega_1) - c > u_S(a_2, \omega_1) &\Leftrightarrow 4 - c > 1 \\ u_S(a_2, \omega_2) > u_S(a_1, \omega_1) - c &\Leftrightarrow 4 > 2 - c \\ u_S(a_2, \omega_3) > u_S(a_2, \omega_1) - c &\Leftrightarrow 2 > 2 - c, \end{aligned}$$

which all hold for $c < 3$. The Receiver's expected utility is $\mathbb{E}[u_R] = \frac{1}{3}(4 + 6) = \frac{10}{3}$.

In this example it is possible to get partial revelation in ω_1 either by cheap-talk or by verifiable message if $c \in (2, 3)$. For $c > 3$ partial revelation is just possible by cheap-talk.

Example 4. *Assume $c > 2$, $|\Omega| = |A| = 3$,*

$$\begin{aligned} &u_R(a_3, \omega_1) = 4 \quad u_R(a_1, \omega_1) = 2 \quad u_R(a_2, \omega_1) = 1 \\ &u_R(a_1, \omega_2) = 4 \quad u_R(a_2, \omega_2) = 2 \quad u_R(a_3, \omega_2) = 1 \\ &u_R(a_2, \omega_3) = 4 \quad u_R(a_1, \omega_3) = 2 \quad u_R(a_3, \omega_3) = 1 \end{aligned}$$

and

$$\begin{aligned} &u_S(a_3, \omega_1) = 4 \quad u_R(a_1, \omega_1) = 2 \quad u_R(a_2, \omega_1) = 1 \\ &u_S(a_3, \omega_2) = 4 \quad u_R(a_2, \omega_2) = 2 \quad u_R(a_1, \omega_2) = 1 \\ &u_S(a_3, \omega_3) = 4 \quad u_R(a_1, \omega_3) = 2 \quad u_R(a_2, \omega_3) = 1. \end{aligned}$$

No full revelation

Obviously there is no fully revealing equilibrium just by cheap-talk message, because Sender and Receiver prefer different actions in two states. To see that full revelation just by verifiable messages is impossible, we take a closer look at ω_2: To have an incentive to send the verifiable information the Sender has to prefer $u_s(a_1, \omega_2) - c$ over $u_S(\hat{a}, \omega_2)$. Since the left part equals $1 - c$ and the right part something larger than 1, this is impossible. The same arguments contradict the full revelation by both message types for mixed strategies.

For $\hat{a} = a_1$, the Sender does not use the verifiable message in ω_3 and for $\hat{a} = a_2$ she uses

cheap-talk in ω_2. So there is no full revelation possible.
<u>Maximization</u>

1.) No revelation:
The maximization here is the same as in the previous example. It is possible for $c > 2$ and the expected utility is $\mathbb{E}[u_R] = \frac{8}{3}$.

2.) Partial revelation by cheap-talk
Getting partial revelation in ω_1 is impossible, because the Sender will use the same cheap-talk message in both other states. To get partial revelation in ω_2, $u_S(a_1, \omega_2) > u_S(\hat{a}, \omega_2)$ has to hold, which is impossible for any $\hat{a} \neq a_1$. Same arguments work with a_2 in ω_3. This means in this example it is not possible to achieve partial revelation by different answers to cheap-talk.

3.) Partial revelation of ω_1 by a verifiable message:
For example we can take $\hat{a} = a_2$. The conditions for the Sender's utility are

$$\begin{aligned}
u_R(a_3, \omega_1) - c > u_S(a_2, \omega_1) &\Leftrightarrow 4 - c > 1 \\
u_R(a_2, \omega_2) > u_S(a_1, \omega_1) - c &\Leftrightarrow 2 > 1 - c \\
u_R(a_2, \omega_3) > u_S(a_2, \omega_1) - c &\Leftrightarrow 1 > 1 - c,
\end{aligned}$$

which again all hold for $c < 3$. The Receiver's expected utility is $\mathbb{E}[u_R] = \frac{1}{3}(4 + 6) = \frac{10}{3}$.

In this example partial revelation is only possible by verifiable information and just if $c \in (2, 3)$ holds. For $c > 3$ partial revelation is impossible and the Receiver maximizes his utility as he would do without communication.

2.2.2 General results

Again we would like to underline that the Receiver cannot commit to any actions, but maximizes his utility. Then it should be obvious that the Receiver always prefers partial revelation over no revelation at all. If there is a partial revelation of one state, the Receiver will maximize his expected utility in the remaining states. It might happen that several actions (pure or mixed) solve this maximization problem.

Definition 4. *For $\Omega' \subseteq \Omega$ we define $\dot{A}(\Omega') = \arg\max_a \sum_{\omega \in \Omega'} \mu[\omega] u_R(a, \omega)$.*

$\dot{A}(\Omega')$ is the set of actions which maximize the Receiver's utility on a given state space Ω' according to the Receiver's belief system μ.

In a general model with more than three states, there can be different types of partial revelation: Partial revelation can be either achieved by verifiable messages, which then fully reveal a subset of states or by cheap-talk messages. Partial revelation by cheap-talk creates a partition of state subsets, each element of the partition can contain a single state or several states. Elements with just a single state have the same effect as verifiable messages: The Receiver knows whether specific state is the true state. For simplicity we split partial revelation by cheap-talk up into two cases.

1 Partial revelation by verifiable messages $\Rightarrow$ Full Revelation of a subset of states

2 A Partial revelation by cheap-talk $\Rightarrow$ Full Revelation of a subset of states
  B Partial revelation by cheap-talk $\Rightarrow$ Dividing the state space into disjoint subsets.

The case 2A contains just the special cases, in which the partition consists of some subsets with just one element and a subset containing the remaining states. Note that also in case 2 there can be a full revelation of subsets of states.

In a world with four states $\{\omega_1, \dots \omega_4\}$ partial revelation by type 2B for example means that the Receiver just knows whether the true state is in $\{\omega_1, \omega_2\}$ or in $\{\omega_3, \omega_4\}$. Of course there can also be a combination of type 1 with 2A or with 2B.

Even with just 4 states most often it is impossible to see, which partial revelation is possible without calculating all possibilities. We state conditions for each of the different types and their combinations. With these conditions it is easy to write an algorithm and let a computer check all the possibilities.

Partial Revelation just by one message type

Proposition 3 (Partial Revelation just by Verifiable Messages)**.**
There is a partial revealing equilibrium, where the Sender uses verifiable messages to reveal the true state just in the states in $\Omega^{vI} \subsetneq \Omega$ if

$$\begin{aligned} \exists \hat{a} \in \dot{A}(\Omega \setminus \Omega^{vI}) : & \; 1) \; \forall \omega \in \Omega^{vI} : u_S(a_R^*(\omega), \omega) - c > u_S(\hat{a}, \omega) \\ & \; 2) \; \forall \omega \in \Omega \setminus \Omega^{vI} : u_S(a_R^*(\omega), \omega) - c < u_S(\hat{a}, \omega). \end{aligned}$$

With this theorem we can define the family of subsets in which partial revelation by verifiable information is possible.

Definition 5.

$$\boldsymbol{\Omega}^{vI}(\Omega) = \Big\{ \Omega^{vI} \subsetneq \Omega \;\Big|\; \exists \hat{a} \in \dot{A}(\Omega \setminus \Omega^{vI}) \textit{ such that } \\ \forall \omega \in \Omega^{vI} : u_S(a_R^*(\omega), \omega) - c > u_S(\hat{a}, \omega) \textit{ and } \\ \forall \omega \in \Omega \setminus \Omega^{vI} : u_S(a_R^*(\omega), \omega) - c < u_S(\hat{a}, \omega) \quad \Big\}$$

This implies that partial revelation by verifiable information is impossible if $\mathbf{\Omega}^{vI}(\Omega) = \{\emptyset\}$. We can also define the set of all tuples of actions and subsets of states $(\hat{a}, \Omega^{vI})$, where the action maximizes the Receiver's utility on $\Omega \setminus \Omega^{vI}$, but for the states in Ω^{vI} this action works as a threat to enforce the Sender to use the verifiable message.

Definition 6.

$$\begin{aligned}\mathbf{\Omega}_A^{vI}(\Omega) = \Big\{ \big(\hat{a}, \Omega^{vI}\big) \Big| \; & 1)\ \hat{a} \in \dot{A}(\Omega \setminus \Omega^{vI}) \\ & 2)\ \forall \omega \in \Omega^{vI} : u_S(a_R^*(\omega), \omega) - c > u_S(\hat{a}, \omega) \\ & 3)\ \forall \omega \in \Omega \setminus \Omega^{vI} : u_S(a_R^*(\omega), \omega) - c < u_S(\hat{a}, \omega) \quad \Big\}\end{aligned}$$

This definition will help to combine different types of partial revelation. We can make similar statements for partial revelation of type 2A:

Proposition 4 (Partial Revelation just by Cheap-Talk)**.**
There is a partial revealing equilibrium, where the Sender uses verifiable messages to reveal the true state just in all states in $\Omega^{ct} \subsetneq \Omega$ if

$$\begin{aligned}\exists \hat{a} \in \dot{A}(\Omega \setminus \Omega^{ct}) : & 1)\ \forall \omega \in \Omega^{ct} : u_S(a_R^*(\omega), \omega) > u_S(\hat{a}, \omega) \\ & 2)\ \forall \omega \in \Omega \setminus \Omega^{ct} : u_S(a_R^*(\omega), \omega) < u_S(\hat{a}, \omega).\end{aligned}$$

Definition 7.

$$\begin{aligned}\mathbf{\Omega}^{ct}(\Omega) = \Big\{ \Omega^{ct} \subsetneq \Omega \Big| \; & \exists \hat{a} \in \dot{A}(\Omega \setminus \Omega^{ct}) \textit{ such that} \\ & \forall \omega \in \Omega^{ct} : u_S(a_R^*(\omega), \omega) > u_S(\hat{a}, \omega) \textit{ and} \\ & \forall \omega \in \Omega \setminus \Omega^{ct} : u_S(a_R^*(\omega), \omega) < u_S(\hat{a}, \omega) \quad \Big\}\end{aligned}$$

Definition 8.

$$\begin{aligned}\mathbf{\Omega}_A^{ct}(\Omega) = \Big\{ \big(\hat{a}, \Omega^{ct}\big) \Big| \; & 1)\ \hat{a} \in \dot{A}(\Omega \setminus \Omega^{ct}) \\ & 2)\ \forall \omega \in \Omega^{ct} : u_S(a_R^*(\omega), \omega) > u_S(\hat{a}, \omega) \\ & 3)\ \forall \omega \in \Omega \setminus \Omega^{ct} : u_S(a_R^*(\omega), \omega) < u_S(\hat{a}, \omega) \quad \Big\}\end{aligned}$$

For partial revelation of type 2B the conditions look a little bit different.

Proposition 5 (Partial Revelation just by Cheap-Talk)**.**
There is a partial revealing equilibrium, where the state space Ω is split up into disjoint subsets if there exists a series of sets $(\Omega_j^{div})_{j=1,\dots,J}$ such that

1. $\bigcup_{j=1}^{J} \Omega_j^{div} = \Omega$ *and* $\forall k \neq l : \Omega_k^{div} \cap \Omega_l^{div} = \emptyset$.

2. $\forall \Omega_j^{div}\ \exists \hat{a}_j \in \dot{A}(\Omega_j^{div})$ *such that* $u_S(\hat{a}_k, \omega) > u_S(\hat{a}_l, \omega)\ \forall \omega \in \Omega_k^{div}$ *with* $k \neq l$.

The first condition says that the subsets have to be disjoint and add up to the complete state space. The second condition ensures that the Sender has no incentive to lie if the Receiver chooses the actions that maximize his expected utility for each subset. As before, we can write this as a set, this time consisting of series of tuples of actions and subsets of the state space:

Definition 9.

$$\begin{aligned}\mathbf{\Omega}_A^{div}(\Omega) = \Big\{ \left(\hat{a}_j, \Omega_j^{div}\right)_j \Big| \;& 1)\ \forall \Omega_k^{div} : a_k \in \dot{A}(\Omega_k^{div}) \\ & 2)\ \bigcup_j \Omega_j^{div} = \Omega \text{ and } \forall k \neq l : \Omega_k^{div} \cap \Omega_l^{div} = \emptyset \\ & 3)\ \forall \omega \in \Omega_k^{div} : \forall \hat{a}_k \neq \hat{a}_l : u_S(\hat{a}_k, \omega) > u_S(\hat{a}_l, \omega) \quad \Big\}\end{aligned}$$

This set contains all the different possibilities of series of tuples that split the state space into subsets.

Partial revelation by a combination of verifiable messages and cheap-talk

For the combination of two types of partial revelation it is not sufficient to combine $\mathbf{\Omega}^{vI}$ and $\mathbf{\Omega}^{ct}$, because we need to use the same action $\hat{a}$ for the states, that are not revealed.

Theorem 4 (Partial Revelation by type 1 and 2A)**.**
All combinations of revelation by verifiable message and cheap-talk (type 2A) are given by:

$$\begin{aligned}\mathbf{\Omega}^{vI+ct}(\Omega) = \Big\{ \left(\Omega^{vI}, \Omega^{ct}\right) \Big| \;& 1)\ \Omega^{vI} \cap \Omega^{ct} = \emptyset \\ & 2)\ \exists \hat{a} \in \dot{A}\left(\Omega \setminus (\Omega^{vI} \cup \Omega^{ct})\right) \text{ such that} \\ & \quad (\hat{a}, \Omega^{vI}) \in \mathbf{\Omega}_A^{vI}(\Omega \setminus \Omega^{ct}) \text{ and} \\ & \quad (\hat{a}, \Omega^{ct}) \in \mathbf{\Omega}_A^{ct}(\Omega \setminus \Omega^{vI}) \quad \Big\}\end{aligned}$$

This means that all states in Ω^{vI} are revealed by verifiable messages and those in Ω^{ct} by cheap-talk. Therefore it is necessary that $\hat{a}$ maximizes the Receiver's utility for the remaining states $\Omega \setminus (\Omega^{vI} \cup \Omega^{ct})$. By the definition of $\mathbf{\Omega}^{vI}$ and $\mathbf{\Omega}^{ct}$ it is ensured that $\Omega^{vI} \cup \Omega^{ct} \subsetneq \Omega$ holds, because otherwise there would be full revelation.

Similar to the combination of type 1 and type 2A, it is also possible to combine type 1 and type 2B. This means that there are some states revealed by verifiable information (type 1) and the remaining states are divided into subsets of the state space (type 2B).

Theorem 5 (Partial Revelation by type 1 and 2B).
All combinations of revelation by verifiable message and cheap-talk (type 2B) are given by:

$$\mathbf{\Omega}^{vI+div}(\Omega) = \left\{ \left(\left(\hat{a}_j, \Omega_j^{div} \right)_j, \Omega^{vI} \right) \;\middle|\; \begin{array}{l} 1)\ \left(\bigcup_j \Omega_j^{div} \right) \cup \Omega^{vI} = \Omega \\ 2)\ \forall \Omega_k^{div} :\ \Omega_k^{div} \cap \Omega^{vI} = \emptyset \\ 3)\ (\hat{a}_j, \Omega_j^{div})_j \in \mathbf{\Omega}_A^{div}(\Omega \setminus \Omega^{vI}) \\ 4)\ \forall \hat{a}_k : (\hat{a}_k, \Omega^{vI}) \in \mathbf{\Omega}_A^{vI} \left(\Omega \setminus \left(\bigcup_{l \neq k} \Omega_l^{div} \right) \right) \end{array} \right\}$$

Condition 1) and 2) ensure that the subsets of states are disjoint, but united are equal to the entire state space. Condition 3) makes sure that the states are split up, if there is no revelation by a verifiable message. The Receiver plays different actions on different subsets of states, with Condition 4) the Sender will send the verifiable message in all states in Ω^{vI} and will not deviate to an action $\hat{a}_k$ from the series $(\hat{a}_j)$.

2.2.3 Increasing and Decreasing Differences

For increasing (or decreasing) differences, we can state the existence of partial revealing equilibria with verifiable messages in a way similar to Proposition 1. The most important change is that the answer to cheap-talk is no longer a_R^*, but $\hat{a}$ such that this action maximizes the Receiver's utility on the non-revealed states.

Proposition 6 (Partial Revelation by verifiable messages under increasing differences).
Let $\Omega = \{\omega_1, \ldots, \omega_L\}$ and sort A such that $A = \{a_R^*(\omega_1), \ldots, a_R^*(\omega_L)\}$. We can ignore all actions, which are never the best reply for the Receiver in a single state.
There is a partial revealing equilibrium that reveals the states just in $[\underline{\omega}, \overline{\omega}]$ by verifiable messages if

$$\exists \hat{a} \in \dot{A}(\Omega \setminus [\underline{\omega}, \overline{\omega}]) \text{ with } a_R^*(\underline{\omega}) > \hat{a} \text{ such that}$$

$$1)\ u_S \text{ has increasing differences on } \Omega' = [\underline{\omega}, \overline{\omega}] \text{ and } A' = [a_R^*(\underline{\omega}), a_R^*(\overline{\omega})]$$

$$2)\ u_S(a_R^*(\underline{\omega}), \underline{\omega}) - c > u_S(\hat{a}, \underline{\omega})$$

$$3)\ \forall \omega_i \in [\underline{\omega}, \overline{\omega}] : u_S(a_R^*(\omega_i), \omega_i) \geq u_S(a_R^*(\omega_{i-1}), \omega_i)$$

$$4)\ \forall \omega_j \in \Omega \setminus [\underline{\omega}, \overline{\omega}] : u_S(\hat{a}, \omega_j) > u_S(a_R^*(\omega_j), \omega_j) - c$$

Proposition 7 (Partial Revelation by verifiable messages under increasing differences)**.**
Let $\Omega = \{\omega_1, \ldots, \omega_L\}$ *and sort* A *such that* $A = \{a^*_R(\omega_1), \ldots, a^*_R(\omega_L)\}$. *We can ignore all actions, which are never the best reply for the Receiver in a single state.*
There is a partial revealing equilibrium that reveals the states just in $[\underline{\omega}, \overline{\omega}]$ *by verifiable messages if*

$$
\begin{aligned}
&\exists \hat{a} \in \dot{A}(\Omega \setminus [\underline{\omega}, \overline{\omega}]) \text{ with } a^*_R(\overline{\omega}) < \hat{a} \text{ such that} \\
&1)\ u_S \text{ has increasing differences on } \Omega' = [\underline{\omega}, \overline{\omega}] \text{ and } A' = [a^*_R(\underline{\omega}), a^*_R(\overline{\omega})] \\
&2)\ u_S(a^*_R(\overline{\omega}), \overline{\omega}) - c > u_S(\hat{a}, \overline{\omega}) \\
&3)\ \forall \omega_i \in [\underline{\omega}, \overline{\omega}] : u_S(a^*_R(\omega_i), \omega_i) \geq u_S(a^*_R(\omega_{i+1}), \omega_i) \\
&4)\ \forall \omega_j \in \Omega \setminus [\underline{\omega}, \overline{\omega}] : u_S(\hat{a}, \omega_j) > u_S(a^*_R(\omega_j), \omega_j) - c
\end{aligned}
$$

We can do a similar change to Condition 3) as before and also get the same results for decreasing differences by the same changes as done between Propositions 1 and 2.
In addition it is possible to rewrite these conditions that they hold for more than just a single interval $[\underline{\omega}, \overline{\omega}]$, but for a disjoint series of intervals $\left([\underline{\omega_k}, \overline{\omega_k}]\right)_k$.

3 Continuous model

In many settings it is not enough to focus on a finite action or state space, but assume that both of them are continuous. For example at wage negotiations or any discussions concerning prices, we have to deal with a continuous interval. In this section we do not limit our attention any more to the discrete setting, but switch to a continuous model. So in general we can assume that $A = \Omega = [0, 1]$. We state different conditions under which there is no possibility for a fully revealing equilibrium. Afterwards we use the example of the quadratic loss function to visualize our results. Theorem 1 and Theorem 2, which give the conditions for fully revealing equilibria with only a single message type, still hold. The conditions in these theorems still have to hold for every state, which is more strict in the continuous model. The following Theorems 6 to 8 give us necessary conditions for the existence of different fully revealing equilibria, where the continuity of u_S and a^*_R are the most important factors. Combined with the results from the discrete model we also get the sufficient conditions.

Theorem 6 (Full Revelation under continuous u_S and a^*_R)**.**
Assume that $a^*_R(\omega) : \Omega \to A$ *is continuous and that* $u_S(a, \omega) : A \times \Omega \to \mathbb{R}$ *is continuous in both arguments. Then full revelation can only be achieved either by cheap-talk messages in every state or by verifiable messages only.*

Theorem 7 (Full Revelation under continuous $u_S(a,\omega)$).
Assume that $u_S(a,\omega)$ is continuous. There can be a fully revealing equilibrium with both message types used if there exists $[\underline{\omega},\overline{\omega}] \subseteq [0,1]$ such that for all $\omega \in [0,1]$

1) $\lim\limits_{\omega \nearrow \underline{\omega}} a_R^*(\omega) \neq a_R^*(\underline{\omega})$

2) $\lim\limits_{\omega \searrow \overline{\omega}} a_R^*(\omega) \neq a_R^*(\overline{\omega})$

3) If $\underline{\omega} \neq \overline{\omega} : \forall \omega_i \in [\underline{\omega},\overline{\omega}] : u_S(a_R^*(\omega_i),\omega_i) > u_S(a_R^*(\omega_j),\omega_i)\ \forall \omega_j \in [\underline{\omega},\overline{\omega}], \omega_j \neq \omega_i$

holds.

Remarks.

- *If $\underline{\omega} = 0$, then the first condition is always satisfied.*
- *If $\overline{\omega} = 1$, then the second condition is always satisfied.*
- *There may exist more than one interval satisfying the conditions of Theorem 7.*

Theorem 7 states that if u_S is continuous in both arguments, the function a_R^* has to be discontinuous. The interval $[\underline{\omega},\overline{\omega}]$ gives the interval of states in which the Receiver believes the Sender's cheap-talk message. For that a_R^* has to be neither right-continuous nor left-continuous at a single $\hat{\omega}$ or not right-continuous at $\underline{\omega}$ and not left-continuous at $\overline{\omega} > \underline{\omega}$. In the second case the Sender also is not allowed to have any incentive to deviate to a different cheap-talk message for states in $[\underline{\omega},\overline{\omega}]$.

Corollary 3.
There is a fully revealing equilibrium with both message types used if there exists $[\underline{\omega},\overline{\omega}] \subseteq [0,1]$ such that:

- $[\underline{\omega},\overline{\omega}]$ *satisfies the conditions of Theorem 7.*
- *Theorem 3 is satisfied with $\hat{\Omega} = [\underline{\omega},\overline{\omega}]$.*

Figure 1 shows three different discontinuous functions $a_R^*(\omega)$. For the blue graph there can be a fully revealing equilibrium with both message types, where the threat point is at $a_R^*(\frac{1}{2})$. The red graph shows a situation where the possible threat point is at the border of the interval, here at $a_R^*(1)$. So Condition 2) of Theorem 7 is always satisfied. As the function is discontinuous for $\omega = 1$, Condition 1) also holds. An example where Theorem 7 implies that there can be no full revelation is given by the green graph. The function is continuous coming from below and so does not satisfy Condition 1).

If a_R^* is continuous the previous Theorem does not hold, but we need that u_S is discontinuous to achieve full revelation under the usage of both message types.

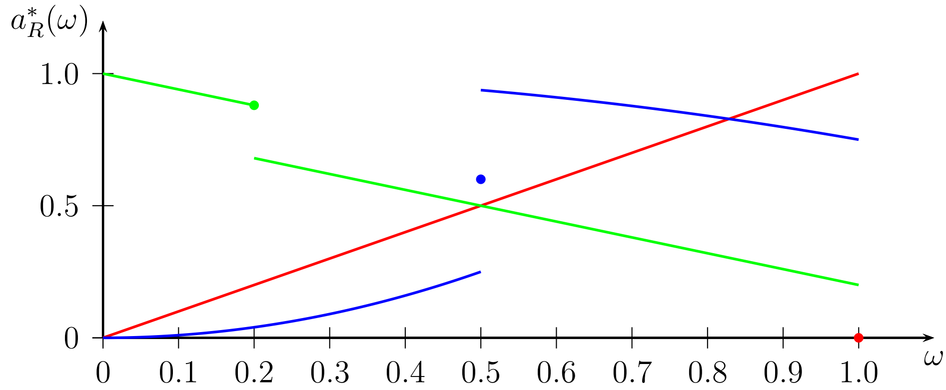

Figure 1: Different examples for $a_R^*(\omega)$.

Theorem 8 (Full Revelation under continuous $a_R^*(\omega)$)**.**
Assume that $a_R^*(\omega)$ is continuous. Only if $u_S(a, \omega)$ is not continuous in at least one argument, there can only be a fully revealing equilibrium with both message types used.

Remark. *Theorems 7 and 8 just state necessary, but not sufficient conditions for the different types of fully revealing equilibria.*

3.1 Quadratic loss function

For this second part we like to focus on the quadratic loss utility for the Receiver and a biased quadratic loss utility for the Sender. We show how our general results from the continuous model work and what the intuition behind the missing of the fully revealing equilibria is. The utility functions are $u_R = -(a-\omega)^2$ and $u_S = -(a-\omega-b(\omega))^2$, where $b(\omega) \in \mathbb{R}$ is the state dependent bias function of the Sender. We assume that this bias function is continuous. For positive values of b, the Sender wants to have a higher action than state, while for negative values she wants to have a lower action than state. This is similar to the example Crawford and Sobel (1982) use, but we allow that the bias function is state-dependent.

Clearly we have the problem that $a_R^*(\omega)$ and $u_S(a, \omega)$ are continuous and therefore all fully revealing equilibria just include the use of one message type. As for $A = \Omega = [0, 1]$ the function $a_R^*(\omega)$ is bijective and so every action is the best reply for one state, we can focus on pure strategies. It will happen that we misuse notation a little and denote actions by ω as well. Then we simply mean the action $a = \omega$.

As an immediate conclusion from Theorem 6 we see that there can be no fully revealing equilibrium with both message types used. As long as the bias function $b(\omega)$ is not constant equal to 0, the Sender will not always tell the truth by cheap-talk. In addition it is also impossible to have a fully revealing equilibrium where the Sender just uses the verifiable messages, because every possible threat point $\hat{a}$ is the Receiver's best reply to one state. This means that in that state the Receiver will never use the verifiable message, but prefers to save the costs and goes for cheap-talk.

Corollary 4.
For $A = \Omega = [0, 1]$ and quadratic loss utility functions for the players, there are no fully revealing equilibria.

This follows immediately from the continuity of u_S and a^*_R and Theorem 6. We can see it in more detail with the help of the following Lemma:

Lemma 1.
There is a fully revealing equilibrium if

$$\exists \hat{\omega} : 1)\ \forall \omega > \hat{\omega} : b(\omega) > \frac{\hat{\omega} - \omega}{2} - \frac{c}{2(\hat{\omega} - \omega)}$$
$$2)\ \forall \omega < \hat{\omega} : b(\omega) < \frac{\hat{\omega} - \omega}{2} - \frac{c}{2(\hat{\omega} - \omega)}$$

Lemma 1 states the condition for a fully revealing equilibrium, where the Sender uses a cheap-talk message in $\hat{\omega}$ and the verified messages in all other states. We can state the same result for a set of states with cheap-talk messages, but use this case to illustrate the problem of the continuous model.

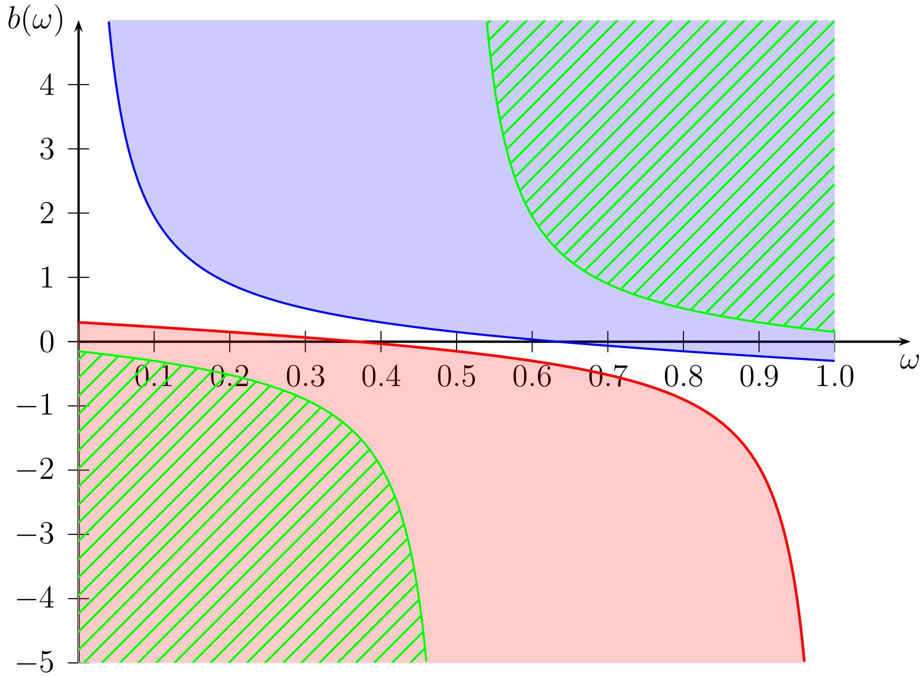

Figure 2: Regions of fully revealing equilibria, for $\hat{\omega} = 0, 0.5$ and 1 with $c = 0.4$

Figure 2 shows Lemma 1 for three different values of $\hat{\omega}$. For $\hat{\omega} = 0$, the function $b(\omega)$ has to above the blue curve (in the blue area). If the Receiver answers every cheap-talk message by the action $\hat{\omega} = 0$, the Sender should not prefer this action over the one responding to the true state. This can be achieved by a positive bias function, or for some values also by a slightly negative one. For $\hat{\omega} = 1$, the function $b(\omega)$ has to be below the red curve (in the red area). For $\hat{\omega} = 0.5$, the function $b(\omega)$ has to be below the green curve for $\omega < 0.5$ and above for $\omega > 0.5$ (in the green shaded area).

This figure already reveals a problem with this setting: No matter the value of $\hat{\omega}$, it is necessary that the bias function $b(\omega)$ gets either really high or low values. The problem here is that the bias function has values in the real numbers, but Conditions 1) or 2) require $|b(\omega)| = \infty$, for some ω. This means if the Receiver answers every cheap-talk message with $\hat{\omega}$ there is always a neighborhood around $\hat{\omega}$ where the Sender prefers sending the costless cheap-talk message over sending the costly verifiable message. The Sender's utility loss by the quadratic loss function (difference between action and state) is less than the utility loss resulting from the costs c.

3.1.1 Discretization

One way to achieve full revelation, while keeping the quadratic loss functions, is to discretize the type space.

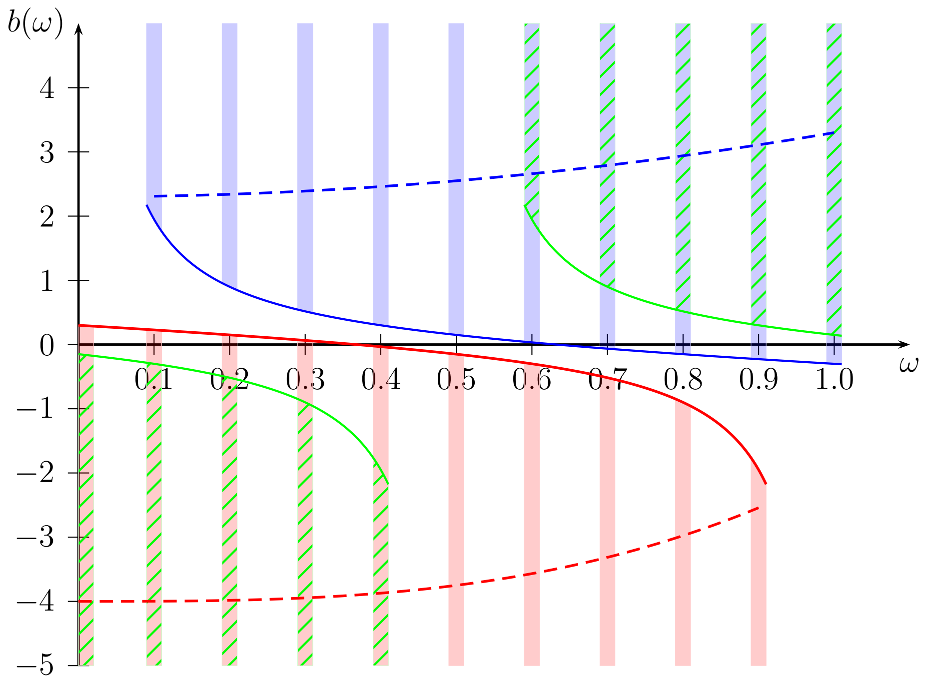

Figure 3: Regions of fully revealing equilibria, for $\hat{\omega} = 0$, 0.5 and 1 with $c = 0.4$

Figure 3 shows a discretization for example for $\Omega = \{0, 0.1, 0.2, \dots, 1\}$. There can be a fully revealing equilibria, even with costly verification, quadratic loss functions and a continuous action space. Again for $\hat{\omega} = 0$, the function $b(\omega)$ has to above the blue curve (in the blue area) for $\omega \in \{0.1, 0.2, \dots, 1\}$. As we do not need this condition for values close to $\hat{\omega}$, but just starting with 0.1 the area is cut off at 0.1. This avoids that the bias function needs too high values. Analogue for $\hat{\omega} = 1$, the area is cut off at $\omega = 0.9$ and the function $b(\omega)$ has to be below the red curve (in the red area) for $\omega \in \{0, 0.1, \dots, 0.9\}$. There are two cuts for $\hat{\omega} = 0.5$. One at the state lower than 0.5, which is 0.4 and the other one at the next higher state, 0.6. The function $b(\omega)$ has to be below the green curve for $\omega \in \{0, 0.1, \dots, 0.4\}$ and above for $\omega \in \{0.6, 0.7, \dots, 1\}$(in the green shaded area).

Lemma 2.
Assume that the Sender's utility is modeled by a quadratic loss function $u_S(a, \omega) = -(a - \omega - b(\omega))^2$. *If* $b(\omega)$ *is non-decreasing,* u_S *satisfies increasing differences.*

The application of Proposition 1 and the following corollaries can be seen in Figure 3 as well. For $\hat{a} = 0$ and $b(\omega)$ increasing we have as first condition that $b(\omega)$ should be above the blue curve. An example is given by the dashed blue curve. The maximal cost c have to be lower than the utility difference is 0.1, which is illustrated as the difference between the blue curve and the dashed blue curve. Similar for $\hat{a} = 1$ and the red dashed curve, here the critical condition is that $b(\omega)$ stays below the red curve even for $\omega = 0.9$.

4 Extensions

In this section we want to give some ideas of extension possibilities to fit our model into different situations. We do not go much into detail, but just state our ideas and possible implications.

State dependent costs

One simple extension of our model is to allow that the costs for the verifiable message c are state dependent, i.e. $c(\omega) : \Omega \to \mathbb{R}$. As long as the costs are strictly positive for all states, there are no mayor changes. Of course all conditions of the previous results are slightly different for each state, but the ideas stay the same. A more dramatic change would happen if we allow that the costs c are equal to zero for some states. Then there is no possibility to guarantee that there exists a fully revealing equilibrium, where the Sender uses just cheap-talk. The simple reason is that, in the states where the verifiable message is for free, she is indifferent as both messages yield to the same utility. So Theorem 1 does not hold any longer.

The most dramatic changes happen to our results in the continuous model: Even with continuous utility u_S and continuous a_R^*, there can be a fully revealing equilibrium if $c(\omega) = 0$ holds for some ω. This might be an interesting point for future research.

Sender mixing

As long as there is full revelation, the Sender will never mix messages, so the only part where the mixing plays a role is for the cases of partial revelation. Only under special circumstances the Sender has an incentive to mix, for example if she is indifferent between several actions resulting from different messages. If the Sender does not mix, the Receiver will use the probabilities of each state to maximize his utility, as stated in section 2.2. Knowing that the Sender might mix his actions, the Receiver will use Bayes' rule to update his beliefs and maximize according to them. This means we have to replace the probabilities $\mathbb{P}$ in the maximization problems with the Receiver's beliefs μ. We state three examples for this:

Example 5.
Assume that $\Omega = \{\omega_1, \dots, \omega_l\}$ and for simplicity $\mathbb{P}[\omega_i] = \frac{1}{L} \forall \omega_i \in \Omega$. Furthermore we assume that the Sender sends m_1 in ω_1 and ω_2, but nowhere else. After reading the message m_1 the Receiver updates his beliefs to $\frac{1}{2}\omega_1, \frac{1}{2}\omega_2$.

Example 6.
Assume that $\Omega = \{\omega_1, \dots, \omega_l\}$, $\mathbb{P}[\omega_i] = \frac{1}{L} \forall \omega_i \in \Omega$ and that the Sender mixes such that:

$$\begin{array}{lclcl} \omega_1 \rightarrow & \frac{1}{3}m_1 & + & \frac{2}{3}m_2 \\ \omega_2 \rightarrow & \frac{1}{2}m_1 & + & \frac{1}{2}m_2 \end{array}$$

In all other states she does not send m_1 or m_2. Then if the Receiver gets the message m_1, he beliefs that the true state is: ω_1 with probability $\frac{1/3}{1/3+1/2} = \frac{2}{5}$ and ω_2 with probability $\frac{1/2}{1/3+1/2} = \frac{3}{5}$. Analogue for message m_2.

Example 7.
Assume that $\Omega = \{\omega_1, \dots, \omega_l\}$, $\mathbb{P}[\omega_i] = \frac{1}{L} \forall \omega_i \in \Omega$ and that the Sender mixes as follows:

$$\begin{array}{lccccc} \omega_1 \rightarrow & \frac{1}{3}m_1 & + & \frac{2}{3}m_2 & & \\ \omega_2 \rightarrow & & & \frac{1}{2}m_2 & + & \frac{1}{2}m_3 \\ \omega_3 \rightarrow & & & & & m_3 \end{array}$$

Then the Receivers beliefs are:

$$\begin{array}{llll} m_1 \rightarrow & \omega_1 & & \\ m_2 \rightarrow & \frac{2}{5}\omega_1 & \frac{3}{5}\omega_2 & \\ m_3 \rightarrow & & \frac{1}{3}\omega_2 & \frac{2}{3}\omega_3 \end{array}$$

States as Sender's types

Many economic examples do not consider different states of the world, but different Sender types. We can easily see our states as types. For the typical notation we replace Ω by Θ with elements θ_1 to θ_L instead of $\omega_1, \ldots, \omega_L$.

A standard example is to see the Sender as an agent looking for a job, with skills θ, while the Receiver is the employer who wants to hire a well skilled worker. He can either choose to hire the Sender or not. Of course his action depends on his belief of the Sender's skills. The Sender either just mentions her skill sets, which is the cheap-talk message, or she can verify it by some certificates. If we assume that the Sender's utility just depends on his employment and not on his skills, the Receivers threat point here is not to hire the Sender after a cheap-talk message, but just if the verifiable message shows that the Sender has the necessary skill level.

Receiver's utility

To avoid multiple equilibria and the waste of money for verifiable messages when they are not necessary, we can modify the utility function of the Receiver: $\hat{u}_R(a, \omega, \tilde{u}_S) = u_R(a, \omega) + \epsilon_R \cdot \tilde{u}_S$, with u_R and $\tilde{u}_S$ as before and $\epsilon_R > 0$, but small. We assume ϵ to be such small that

$$|\epsilon_R| \cdot \max_{a_x} \hat{u}_S(a_x, m, \omega) < |u_R(a_i, \omega) - u_R(a_j, \omega)| \; \forall \omega \; \forall a_i, a_j, i \neq j \; \forall m \in M \cup \{v\}. \tag{2}$$

Equation (2) implies that the Receiver aims to maximize his utility directly by u_R. He will not choose an action that gives him not the highest u_R, but a high utility through $\tilde{u}_S$. With this assumption we assure that the solutions of our maximization problems stay the same, but if there are multiple solutions, the Receiver will choose the solution giving the highest utility to the Sender. Especially for the case, where the preferences of the Sender and Receiver are the same, this is helpful. In this setting there can be several fully revealing equilibria, but just the one where the Receiver beliefs every cheap-talk message is Pareto-efficient.

Sender's utility

In the previous sections we ignored the possibility that the Sender might be indifferent between sending a cheap-talk message or the verifiable message. We can change the utility function to $\hat{u}_S(a, \omega, m, u_R) = \tilde{u}_S(a, \omega, m) + \epsilon_S \cdot u_R$ with $\tilde{u}_S$ and u_R as before. Under some assumptions it is reasonable that the Sender prefers cheap-talk, under other assumptions she should prefer the verifiable messages. The best reason for cheap-talk is the one stated in the previous extension: The Sender wants to maximize his utility, but at the same time she prefers a higher utility for the Receiver over a lower one, in that case ϵ_S should be positive. On the other hand one can argue that sending a verifiable message gives her certainty, what she might prefer. Then ϵ_S should be negative. In both cases $|\epsilon_S|$ should be small enough not to interfere with the Sender's maximization problem, as we stated in equation (2) for the Receiver.

5 Conclusion

In this paper we have combined the cheap-talk model of Crawford and Sobel (1982) with the models dealing with verifiable messages. In our Sender-Receiver game the informed Sender can choose between verifiable and non-verifiable messages. While the Receiver only learns the true state for sure after reading a verifiable message, a cheap-talk message will not reveal the true state to him, but just let him update his belief system. We stated conditions for a discrete setting under which the Sender reveals the true state to the Receiver. The main idea behind is known from other models as well: The Receiver punishes the Sender for not using the verifiable message by answering every cheap-talk message with an action the Sender dislikes. As we limit our attention to non-dominated action, there always exists a belief system which makes this action best reply and so it makes the threat credible.

If such action does not exist, full revelation can only be achieved by common interests. In that case the Sender has no reason to lie and the Receiver can trust every cheap-talk message. Otherwise there can be only partial revelation or no revelation at all. In the first case we differ between different ways of revelation, for each of them we state conditions. We have not only analyzed different examples for partial revelation and have shown that there exist several ways for the Receiver to maximize his utility, but also stated general results. For the case that the utility functions have increasing or decreasing differences, all conditions simplify and make them easier to check.

In a continuous model the enforcement of full revelation is more difficult. Under continuous utility functions for the Sender and the Receiver there is no full revelation possible. We have illustrated that at the standard example of the quadratic loss function and also have shown a way to counteract it: By discretization of the state space. All in all we stated results that allow to check whether there are fully revealing equilibria or not. Therefore we differ between three different types of fully revealing equilibria: The one where both message types are used and those where the Sender always sticks to one kind of message.

For future research it might be interesting to make more than one verifiable message available. Letting the Sender send intervals of states, in which the true state has to be included, might change our results. By that we could combine the cheap-talk literature and the verifiable information literature even further. We also like to characterize the group of utility functions further, which allow for full revelation, either by using specific properties as single-crossing or by discretization of specific utility functions. There are several ways in which we can push these ideas, but with this model we created a suitable foundation.

Appendix

Proof. Theorem 1
Only if: Clearly there is a fully revealing equilibrium just with cheap-talk messages if Condition (1) holds: The Receiver will trust every cheap-talk message and the Sender has no incentive to deviate.
If: Proof by contradiction. Let us assume that $\exists \omega_k$ such that $u_S(a_R^*(\omega_k), \omega_k) \not> u_S(a_R^*(\omega_j), \omega_k) \, \forall \omega_j \neq \omega_k$. This implies that there exists ω_j such that $u_S(a_R^*(\omega_k), \omega_k) < u_S(a_R^*(\omega_j), \omega_k)$ holds. Then the Receiver has an incentive to lie in ω_k and send the cheap-talk message ω_j, so there is no full revelation. □

Proof. Theorem 2
Only if: Follows directly.
If: Proof by contradiction.
Step 1) Let us assume that Condition 2) does not hold. Then there exists a ω_j such that $u_S(a_R^*(\omega_j), \omega_j) - c < u_S(\hat{a}, \omega_j)$ holds. This implies that the Sender prefers sending a cheap-talk message and getting action $\hat{a}$ over sending the verifiable message and action $a_R^*(\omega_j)$. So she will deviate in ω_j and there will be no full revelation.
Step 2) Let us assume that Condition 1) does not hold, then Condition 2) does not hold and we can follow Step 1). □

Proof. Theorem 3
Only if: The equilibrium is as follows: For $\omega \in \hat{\Omega}$ the Receiver trusts the cheap-talk and in all other states the Sender uses the verifiable message.
If: Proof by contradiction.
Step 1) Let us assume that Condition 1) does not hold. This implies that there exist $\omega_i \notin \hat{\Omega}$ and $\omega_j \in \hat{\Omega}$ such that $u_S(a_R^*(\omega_j), \omega_i) > u_S(a_R^*(\omega_i), \omega_i) - c$ holds. So the Sender prefers cheap-talk (and action $a_R^*(\omega_j)$) over the verifiable message (and action $a_R^*(\omega_i)$) and there will be no full revelation, because $a_R^*(\omega_i) \neq a_R^*(\omega_j)$.
Step 2) We assume that Condition 2) does not hold and follow the same steps as in the proof of Theorem 1. □

Proof. Proposition 1
We split the proof in two parts. First we show that for states higher than ω_j, the Sender prefers the verifiable message, then we do the same for lower states. Assume $\omega_i > \omega_j$ holds. For $\omega_i = \omega_{j+1}$ we have condition 2.1), which states that the Sender prefers the costly verifiable message (yielding $a_R^*(\omega_{j+1})$), over the cheap-talk message (yielding $\hat{a}$).

It remains to show that for all $\omega_i > \omega_{j+1} : c < u_S(a_R^*(\omega_i), \omega_i) - u_S(a_R^*(\omega_j, \omega_i)$ hold.

$$\begin{aligned} u_S(a_R^*(\omega_i), \omega_i) - u_S(a_R^*(\omega_1), \omega_i) &\overset{3.1)}{\geq} u_S(a_R^*(\omega_{i-1}), \omega_i) - u_S(a_R^*(\omega_j), \omega_i) \\ &\overset{1.)}{\geq} u_S(a_R^*(\omega_{i-1}), \omega_{i-1}) - u_S(a_R^*(\omega_j), \omega_{i-1}) \end{aligned}$$

We repeat these two steps until

$$\begin{aligned} &\geq u_S(a_R^*(\omega_{j+1}), \omega_{j+1}) - u_S(a_R^*(\omega_j), \omega_{j+1}) \\ &\overset{2.1)}{>} c \end{aligned}$$

Analogue steps yield the proof for $\omega_i < \omega_j$. □

Proof. Proposition 2

The proof follows the same ideas as the proof of Proposition 1. The only difference is that we go step by step from the boundary states and actions to the threat point, while in the previous proof we moved from the threat point towards the boundaries.

For example for $\omega_i > \omega_j$:

$$\begin{aligned} u_S(a_R^*(\omega_i), \omega_i) - u_S(a_R^*(\omega_j), \omega_i) &\overset{3.1)}{\geq} u_S(a_R^*(\omega_{i+1}), \omega_i) - u_S(a_R^*(\omega_j), \omega_i) \\ &\overset{1.)}{\geq} u_S(a_R^*(\omega_{i+1}), \omega_{i+1}) - u_S(a_R^*(\omega_j), \omega_{i+1}) \end{aligned}$$

We repeat these two steps until

$$\begin{aligned} &\geq u_S(a_R^*(\omega_L), \omega_L) - u_S(a_R^*(\omega_j), \omega_L) \\ &\overset{2.1)}{>} c \end{aligned}$$

□

Proof. Proposition 3

If Condition 1) holds, the Sender has an incentive to use the verifiable message in all states in Ω^{vI}. She also sends the cheap-talk message in all other states, because of Condition 2). For the Receiver $\hat{a}$ is by definition an action that maximizes his expected utility on $\Omega \setminus \Omega^{vI}$. As the states in Ω^{vI} are revealed, he will play the action he likes the most there. So both players have no incentive to deviate and we have a partial revealing equilibrium. □

Proof. Proposition 4

Analogue to the proof of Proposition 3. □

Proof. Proposition 5

Condition 1) ensures that Ω is split up in disjoint subsets. If Condition 2) holds, there is at least one action $\hat{a}$ for each subset that maximizes the Receiver's expected utility that is such that the Sender does not want to deviate to another action. This means that no player wants to deviate. $\square$

Proof. Theorem 4

The first condition is necessary to have disjoint sets for partial revelation by cheap-talk and verifiable messages. The action $\hat{a}$ that satisfies condition 2) maximizes the Receiver's utility on the remaining states and by definition of $\boldsymbol{\Omega}^{vI}$ enforces the Sender to use the verifiable message in the states in Ω^{vI}. In the states in Ω^{ct} the Sender has no incentive to deviate to another cheap-talk message by the definition of $\boldsymbol{\Omega}^{vI}$. $\square$

Proof. Theorem 5

Conditions 1) and 2) make sure that Ω is completely split up into disjoint subsets. In each subset Ω_j^{div} the Sender sends a different cheap-talk message, so that the Receiver knows in which subset he is. The Receiver maximizes his expected utility in each of these subsets by $\hat{a}_j$. If Condition 4) holds, the Sender sends a verifiable message in all states in Ω^{vI} and has no incentive to deviate to any $\hat{a}_j$. $\square$

Proof. Proposition 6

By Condition 4) the Sender prefers sending cheap-talk over sending a verifiable message outside the interval $[\underline{\omega}, \overline{\omega}]$. Therefore it is correct that $\hat{a}$ is maximizing the Receiver's utility outside that interval.

Condition 1) ensures that for $\underline{\omega}$ the Sender sends the verifiable message. It remains to show that she does so for the rest of the interval as well. For $\omega_i \in [\underline{\omega}, \overline{\omega}]$ we need $u_S(a_R^*(\omega_i), \omega_i) - u_S(\hat{a}, \omega_i) > c$. As in the proof of Proposition 1 we show that $u_S(a_R^*(\omega_i), \omega_i) - u_S(\hat{a}, \omega_i) > u_S(a_R^*(\underline{\omega}), \underline{\omega}) - u_S(\hat{a}, \underline{\omega})$. By the first condition the result then follows. Starting from the left side:

$$\begin{aligned} u_S(a_R^*(\omega_i), \omega_i) - u_S(\hat{a}, \omega_i) &\overset{3)}{\geq} u_S(a_R^*(\omega_{i-1}), \omega_i) - u_S(\hat{a}, \omega_i) \\ &\overset{1)}{\geq} u_S(a_R^*(\omega_{i-1}), \omega_{i-1}) - u_S(\hat{a}, \omega_{i-1}) \end{aligned}$$

Repeating these steps until we reach $\underline{\omega}$ yields the result.

We can use the increasing difference property, because by assumption $a_R^*(\underline{\omega}) > \hat{a}$ holds.

$\square$

Proof. Proposition 7

The proof follows the same steps as the previous one, but it might help to rewrite the definition of increasing differences: $\forall a' \geq a, \omega' \geq \omega$:

$$u_S(a', \omega') - u_S(a, \omega') \geq u_S(a', \omega) - u_S(a, \omega)$$
$$\Leftrightarrow u_S(a, \omega) - u_S(a', \omega) \geq u_S(a, \omega') - u_S(a', \omega')$$

Compared to the proof of the previous proposition, this time the steps go up from ω_i to $\overline{\omega}$, using $\hat{a} > a_R^*(\overline{\omega})$. □

Proof. Theorem 6

The possible existence of fully revealing equilibria with just one type of message sent follows from the conditions imposed in Theorem 1 and Theorem 2. Assume that the Sender sends a cheap-talk message just in $\hat{w}$ and uses the verifiable message in all other states. The argumentation for sending cheap-talk in several states or intervals will be the same. The Sender has an incentive to use the verifiable message if $u_S(a_R^*(\omega), \omega) - c > u_S(a_R^*(\hat{\omega}), \omega)$. So for the states close to $\hat{\omega}$ we get:

$$u_S(a_R^*(\hat{\omega} \pm \epsilon), \hat{\omega} \pm \epsilon) - c > u_S(a_R^*(\hat{\omega}), \hat{\omega} \pm \epsilon) \tag{3}$$

For $\epsilon \to 0$ and the continuity of u_S and a_R^* this is equivalent to:

$$u_S(a_R^*(\hat{\omega}), \hat{\omega}) - c > u_S(a_R^*(\hat{\omega}), \hat{\omega})$$

This then leads to $c < 0$, which is clearly a contradiction. So under this assumptions it is not possible that there is a fully revealing equilibrium where the Sender uses both message types. □

Proof. Theorem 7

Assume that 1) or 2) do not hold, the problem is the same as described in equation (3), which requires negative costs c.

Let us assume that 3) does not hold, then the Sender deviates when the real state is in the interval $[\underline{\omega}, \overline{\omega}]$ and so there cannot be full revelation. □

Proof. Theorem 8

The proof is analogue to the proof of Theorem 7, using the discontinuity of u_S instead of a_R^*. □

Proof. Lemma 1

Assume that the Receiver answers every cheap-talk message with $\hat{\omega}$.

The utility of the Sender for any state ω is given by:

u_S("verifiable message") $= -(-b(\omega))^2 - c$

u_S("cheap-talk message") $= -(\hat{\omega} - \omega - b(\omega))^2 = -\left[(\hat{\omega} - \omega)^2 - 2(\hat{\omega} - \omega) \cdot b(\omega) + (b(\omega))^2\right]$

So the Sender will use the verifiable message if and only if:

$$\begin{aligned} & -(-b(\omega))^2 - c &> -\left[(\hat{\omega}) - \omega)^2 - 2(\hat{\omega} - \omega) \cdot b(\omega) + (b(\omega))^2\right] \\ \Leftrightarrow \quad & -2b(\hat{\omega} - \omega) &> -(\hat{\omega} - \omega)^2 + c \end{aligned}$$

Case 1: $\omega > \hat{\omega}$

$$\begin{aligned} \Leftrightarrow \quad & -2b &< -(\hat{\omega} - \omega) + \tfrac{c}{\hat{\omega} - \omega} \\ \Leftrightarrow \quad & b &> \tfrac{\hat{\omega} - \omega}{2} - \tfrac{c}{2(\hat{\omega} - \omega)} \end{aligned}$$

Case 2: $\hat{\omega} > \omega$

$$\begin{aligned} \Leftrightarrow \quad & -2b &> -(\hat{\omega} - \omega) + \tfrac{c}{\hat{\omega} - \omega} \\ \Leftrightarrow \quad & b &< \tfrac{\hat{\omega} - \omega}{2} - \tfrac{c}{2(\hat{\omega} - \omega)} \end{aligned}$$

□

Proof. Lemma 2

Increasing Differences mean that the following condition have to hold $\forall a' \geq a, \omega' \geq \omega$:

$$\begin{aligned} & u_S(a', \omega') - u_S(a, \omega') &\geq u_S(a', \omega) - u_S(a, \omega) \\ \Leftrightarrow \quad & -(a' - \omega' - b(\omega'))^2 + (a - \omega' - b(\omega'))^2 &\geq -(a' - \omega - b(\omega))^2 + (a - \omega - b(\omega))^2 \\ \Leftrightarrow \quad & a^2 - (a')^2 + 2(b(\omega') + \omega')(a' - a) &\geq a^2 - (a')^2 + 2(b(\omega) + \omega)(a' - a) \\ \Leftrightarrow \quad & b(\omega') + \omega' &\geq b(\omega) + \omega \end{aligned}$$

This condition is clearly satisfied if $b(\omega)$ is increasing, because $\omega' \geq \omega$ holds. □

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