Externalities in Economies with Endogenous Sharing Rules

Philippe Bich, Rida Laraki

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Abstract

Endogenous sharing rules were introduced by Simon and Zame [16] to model payoff indeterminacy in discontinuous games. They prove the existence in every compact strategic game of a mixed Nash equilibrium and an associated sharing rule. We extend their result to economies with externalities [1] where, by definition, players are restricted to pure strategies. We also provide a new interpretation of payoff indeterminacy in Simon and Zame's model in terms of preference incompleteness.

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JEL Classification: C02, C62, C72, D50.

1 Introduction

The model of strategic games with endogenous sharing rules was introduced by Simon and Zame [16]. Formally, it is a $(N+1)$-tuple $\mathcal{G} = ((X_i)_{i\in N}, \mathcal{U})$, where N is the set[1] of players, X_i is the strategy set of player $i \in N$, and $\mathcal{U}$ is a multivalued function from $X := \prod_{i\in N} X_i$ to $\mathbf{R}^N$ with nonempty values. The set $\mathcal{U}(x) \subset \mathbf{R}^N$ can be interpreted as the universe of payoff possibilities, given the strategy profile $x \in X$. When $\mathcal{U}(x) = \{(u_i(x))_{i\in N}\}$ is a singleton for every $x \in X$, $\mathcal{G}$ reduces to a usual strategic game, u_i being the payoff function of player i.

Simon and Zame [16] provide conditions that guarantee the existence of a *solution* for $\mathcal{G}$, i.e., existence of a selection $q = (q_i)_{i\in N}$ of $\mathcal{U}$ (a *sharing rule* of $\mathcal{U}$), together with a mixed Nash equilibrium $m^* = (m_i^*)_{i\in N}$ of the game $G = ((X_i)_{i\in N}, (q_i)_{i\in N})$.

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[1]For simplicity, we use the same letter N for the set of players or the number of players.

The concept of sharing rule gives rise to many interpretations. Imagine a designer who must determine who wins an indivisible object in some auction including tie-breaking rules. In that case, selections of $\mathcal{U}$ represent admissible auction rules, and a solution can be seen as a mechanism and a Nash equilibrium of the induced game. Another motivation comes from the payoff indeterminacy that many economic models exhibit: for example, several producers have to choose, each, a location in an area where a continuum of consumers are uniformly distributed. Assume each consumer goes to the closest location. Then payoffs are not well defined when some producers choose the same location: indeed, any division of consumers between the producers is plausible. Simon and Zame's result guarantees the existence of a market sharing rule under which the discontinuous game played by the producers admits a mixed Nash equilibrium.

In this note, we prove existence of a Simon and Zame "solution" in economies with externalities (also called generalized games). This is a general equilibrium model, introduced by Arrow and Debreu [1], in which players play in pure strategies and each player admissible set of strategies is constrained by the strategies chosen by the opponents. For example, in exchange economies, consumers are limited by their budget constraint, which depends on the price vector, itself depending on consumers' demands.

Our second contribution is an interpretation of sharing rules indeterminacy in terms of *preference incompleteness*. As Aumann [2] argues: "*of all axioms of utility theory, the completeness axiom is perhaps the most questionable*". Following this seminal paper, many extensions of equilibrium models to incomplete preferences have been investigated, either for continuous preferences [9, 14], or discontinuous ones [6, 12, 17]. In this note, we will assume that the ambiguity generated by the indeterminacy of payoffs creates incompleteness in the preferences. This permits to associate to every economy with externalities and endogenous sharing rules an economy with externalities and incomplete and discontinuous preferences. We prove that, in general, this economy does not possess a Nash equilibrium, but it is possible to complete the preferences in a *weak* sense to restore the existence of an equilibrium.

2 Economies with Endogenous Sharing Rules

An Economy with externalities and endogenous sharing rules $\mathcal{E}$ is a pair $\mathcal{E} = (\mathcal{G}, \mathcal{B} = (\mathcal{B}_i)_{i\in N})$ where $\mathcal{G} = ((X_i)_{i\in N}, \mathcal{U})$ is a game with endogenous sharing rules, and $\mathcal{B}_i$ is a multivalued mapping from X_{-i} to X_i with a closed graph and nonempty convex values (i.e., a Kakutani-type mapping).

Definition 1 *A solution of $\mathcal{E}$ is a pair (x^*, q), where $q = (q_i)_{i\in N}$ is a selection of $\mathcal{U}$, and $x^* \in X$ is a generalized Nash equilibrium of $((X_i)_{i\in N}, (q_i)_{i\in N}, \mathcal{B})$, i.e.:*
(i) For every $i \in N$, $x_i^* \in \mathcal{B}_i(x_{-i}^*)$.
(ii) For every $x_i \in \mathcal{B}_i(x_{-i}^*)$, $q_i(x_i, x_{-i}^*) \le q_i(x^*)$.

Consider the following assumptions:
*A*1: X is a convex and compact subset of a Hausdorff and locally convex topological

vector space;
$A2$: $\mathcal{U}$ is bounded;
$A3$: The graph of $\mathcal{E}$, defined by $\Gamma := \{(x,v) : v \in \mathcal{U}(x) \text{ and } x_i \in \mathcal{B}_i(x_{-i}) \text{ for every } i \in N\}$, is closed;
$A4$: $\mathcal{U}$ admits a selection $u = (u_i)_{i\in N}$, such that each u_i is quasiconcave in player i's strategy.

Theorem 2 *Any economy with externalities and endogenous sharing rules satisfying* $A1$ *to* $A4$ *admits a solution.*

This can be related to Simon and Zame existence result [16]: they prove the existence of a solution in mixed strategies in every strategic games under $A1$, $A2$, $A3$ and convexity of $\mathcal{U}(x)$ for every $x \in X$. Theorem 2 proves the existence of a solution in pure strategies, even when the strategies of each player are constrained by the strategies of his opponent. In strategic games where $\mathcal{B}_i(x_{-i}) = X_i$ for every $x_{-i} \in X_{-i}$ and every $i \in I$, we get the existence of a solution à la Simon-Zame in pure strategies. This was an open question in Jackson et al. [10] and was solved recently in Bich and Laraki [5] by using as a tool Reny 's [13] better-reply security condition. But we shall see that adding externalities makes the proof more complex. Indeed, the result relies on a recent condition for Nash equilibrium existence in discontinuous games by Barelli and Meneghel [3].

3 Applications

3.1 Incomplete Preferences

Let us give an interpretation of Theorem 2 in terms of incomplete preferences. If $\mathcal{G} = ((X_i)_{i\in N}, \mathcal{U})$ is a game with endogenous sharing rules, then we can define the following preorders[2] on X.

Definition 3 *We say that* $y \in X$ *is* $\mathcal{U}$*-preferable to* $x \in X$ *for player* i*, denoted* $x \lesssim_i y$*, if and only if* $u_i(x) \leq u_i(y)$ *for every selection*[3] u *of* $\mathcal{U}$*.*

When x and y are distinct, $x \lesssim_i y$ is equivalent to $\sup \mathcal{U}_i(x) \leq \inf \mathcal{U}_i(y)$, where $\mathcal{U}_i(x)$ denotes the projection of $\mathcal{U}(x) \subset \mathbf{R}^N$ on the i-th component. In short, $x \lesssim_i y$ if and only if y is at least as good as x, whatever the indeterminacy of payoffs modeled by $\mathcal{U}$. Formally, to every economy with externalities $(\mathcal{G}, (\mathcal{B}_i)_{i\in N})$, one can associate an economy with externalities and incomplete preferences $\mathcal{E} = ((X_i)_{i\in N}, (\lesssim_i)_{i\in N}, (\mathcal{B}_i)_{i\in N})$, where the preorders $\lesssim_i$ are derived from $\mathcal{U}$ as described above.

[2] A preorder is a reflexive and transitive binary relation.

[3] Every preorder $\lesssim$ on X admits a multi-utility representation (see [11]), that is there exists a family $(v_j)_{j\in J}$ of real-valued functions defined on X such that: $x \lesssim y \Leftrightarrow$ for every $j \in J$, $v_j(x) \leq v_j(y)$. Thus, there is no loss of generality in working with a cardinal multi-representation.

It is then standard to define a generalized Nash equilibrium of $\mathcal{E}$ as a profile $x \in \Pi_{i\in N}\mathcal{B}_i(x_{-i})$ such that there is no player $i \in N$ and no deviation $y_i \in \mathcal{B}_i(x_{-i})$ with[4] $x \lnsim_i (y_i, x_{-i})$. The following example proves that, in general, $\mathcal{E}$ fails to have a generalized Nash equilibrium, even if the initial game $\mathcal{G}$ satisfies assumptions $A1$ to $A4$.

Example 4 *Consider a strategic game with endogenous sharing rules and two players. The strategy spaces are $X_1 = X_2 = [0,1]$. The endogenous sharing rules are defined by $\mathcal{U}(x_1, x_2) = (1 - x_1(1 - x_2), 1 - (1 - x_1 - x_2)^2)$ if $(x_1, x_2) \neq (0,1)$ and $\mathcal{U}(0,1) = \{(-1,1),(1,1)\}$. This satisfies assumption A1 to A4. In particular, any selection u of $\mathcal{U}$ satisfies the quasiconcavity requirement A4. As described above, this defines a game with incomplete preferences $\mathcal{E} = ((X_i)_{i=1,2}, (\lesssim_i)_{i=1,2})$. Clearly, for player 2, the unique best-response to x_1 is $x_2 = 1 - x_1$. Thus, for every $x_1 > 0$, (x_1, x_2) is not a Nash equilibrium of $\mathcal{E}$, since it would imply $x_2 = 1 - x_1 < 1$, but then the only best-response of player 1 is $x_1 = 0$, a contradiction. Thus, the only candidate to be a Nash equilibrium is $(0,1)$, but it is not, since $(0,1) \lnsim_1 (\varepsilon, 1)$ for every $\varepsilon \in]0,1]$. Indeed, $1 = \sup \mathcal{U}_1(0,1) \leq \inf \mathcal{U}_1(\varepsilon, 1) = 1$ and $1 = \sup \mathcal{U}_1(\varepsilon, 1) > \inf \mathcal{U}_1(0,1) = -1$. Hence, $\mathcal{E}$ has no Nash equilibria. In particular, it is not generalized correspondence secure (see [8]), a condition that would imply the existence of a Nash equilibrium of $\mathcal{E}$.*

Thus, one cannot apply recent generalized Nash existence results to $\mathcal{E}$ (e.g., Yannelis, He [17] or Carmona and Podzeck[5] [8]) simply because the game may fail to have a Nash equilibrium. We now study the possibility of restoring existence after some completion of the preferences. Recall that a completion of the preorder $\lesssim_i$ defined on X is a complete preorder $\lesssim'_i$ on X such that:

(i) $\forall (x, y) \in X^2$, $x \lesssim_i y \Rightarrow x \lesssim'_i y$;

(ii) $\forall (x, y) \in X^2$, $x \lnsim_i y \Rightarrow x \lnsim'_i y$.

When the preorders $\lesssim_i$, $i \in N$, are defined from $\mathcal{U}$ as above, then for every selection q of $\mathcal{U}$, one can define a q-completion of $\lesssim_i$ as the complete preorder $\lesssim^q_i$ on X such that: $x \lesssim^q_i y \Leftrightarrow q_i(x) \leq q_i(y)$. This is a weak completion of $\lesssim_i$, in the sense that it satisfies property (i) but not property (ii). This is because $x \lnsim_i y$ is defined by: $v_i(x) \leq v_i(y)$ for every selection v of $\mathcal{U}$, and $w_i(x) < w_i(y)$ for at least one selection of $\mathcal{U}$, and this may not imply $q_i(x) < q_i(y)$.

Corollary 5 *Consider an economy with externalities and endogenous sharing rule which satisfies assumptions A1 to A4, and let $\lesssim_i$ be the preorders associated to $\mathcal{U}$ as described above. Then there exists a q-completions $\lesssim^q_i$ of the preorders $\lesssim_i$ ($i \in N$) for some selection q of $\mathcal{U}$, such that $((X_i)_{i\in N}, (\lesssim^q_i)_{i\in N}, (\mathcal{B}_i)_{i\in N})$ has a generalized Nash equilibrium $x^* \in X$.*

[4]Here, $\lnsim_i$ denotes the strict preorder associated to $\lesssim_i$, that is: for every $(x, y) \in X^2$, $x \lnsim_i y$ if and only if $x \lesssim_i y$ and not $(y \lesssim_i x)$.

[5]Remark that conversely, our paper does not generalize the existence results of these two papers.

Remark 6 *As said above, the completion above may not preserve the strict order. More precisely, it is possible (though not automatic) that the generalized Nash equilibrium $x^* \in X$ and the selection q in Corollary 5 satisfy $u_i(x) \leq u_i(y)$ for every selection u of $\mathcal{U}$ with at least one strict inequality, although $q_i(x) = q_i(y)$. It is not surprising, since the endogenous sharing rule q can be seen as "summarizing actions taken by unseen agents whose behavior is not modelled explicitely" (see [16]). This unmodeled behaviour implies that the new sharing rule q can really change the preferences of the players (at least at indeterminacy points), and there is no reason why the associated q-completion would preserve the strict order.*

Anyway, a possibility to preserve the strict order is to strengthen the definition of the preorders $\lesssim_i$ as follows. Say that $y \in X$ is $\mathcal{U}$-strongly preferable to $x \in X$ for player i, denoted $x \ll_i y$, if and only if $u_i(x) < u_i(y)$ for every selection u of $\mathcal{U}$. This leads to a new notion of profitable deviation for player i (more restrictive than this defined by $\lesssim_i$), thus to a weaker notion of Nash equilibrium. Under assumptions A1 to A4, the existence of a Nash equilibrium for such preorders $\ll_i$ is a consequence of Shafer-Sonnenschein's Theorem (see [15]). Indeed, if $y_i \in P_i(x) := \{y_i \in X_i : (x_i, x_{-i}) \ll_i (y_i, x_{-i})\}$, then, by definition of $\ll_i$, and because $\mathcal{U}$ has a closed graph, we get $x' \ll (y'_i, x'_{-i})$ for every (x', y'_i) in some neighborhood of (x, y_i). This proves that P_i has an open graph. Moreover, $x_i \notin coP_i(x)$ for every $i \in I$ (from assumption A4). Consequently, we can apply Shafer-Sonnenschein's Theorem to get the existence of $x^* \in X$ such that $P_i(x^*) = \emptyset$ for every $i \in N$, i.e. x^* is a Nash equilibrium in the game defined by the preorders $\ll_i$. This can be extended to the case of an economy with endogenous sharing rules.

3.2 Generalized Games with Discontinuous Payoffs

For every generalized game $G = ((X_i)_{i \in N}, (u_i)_{i \in N}), \mathcal{B})$ where each utility function u_i is assumed to be bounded and quasiconcave with respect to x_i, we can restore existence of a generalized Nash equilibrium by changing the payoff functions at discontinuity points under the constraint that the graph of the new game remains in the closure of the graph of the original game. More precisely, we can construct new payoff functions $q = (q_i)_{i \in N}$ such that:

(a) the economy with externalities $G' = ((X_i)_{i \in N}, (q_i)_{i \in N}), \mathcal{B})$ admits a generalized Nash equilibrium x^*.

(b) for every $y \in X$ with $y_i \in \mathcal{B}_i(y_{-i})$ for every $i \in N$, there is a sequence $(y^n)_{n \in \mathbb{N}}$ converging to y such that $y_i^n \in \mathcal{B}_i(y_{-i}^n)$ for every $i \in N$ and $q(y) = \lim_{n \to +\infty} u(y^n)$;

This extends the sharing rule existence result in [5] (Theorem 2). The proof is a direct consequence of Theorem 2. Indeed, for every profile $y \in X$, define $\mathcal{U}(y)$ to be the set of limit points of $(u(y^n))_{n \in \mathbb{N}}$ for all possible sequences $(y^n)_{n \in \mathbb{N}}$ converging to y and such that $y_i^n \in \mathcal{B}_i(y_{-i}^n)$ for all $i \in N$. Clearly, $\mathcal{U}$ satisfies all the assumptions A1 to A4. Consequently, from Theorem 2, there is a solution (x, q), which satisfies conditions (a) and (b) above.

3.3 Exchange Economies

Consider n consumers and m commodities. The initial endowment e_i of consumer i is assumed to be an interior point in $\mathbf{R}^m_+$. Consumer i's consumption set is equal to $X_i = \{x_i \in \mathbf{R}^m_+ : x_i \leq \sum_{j \in N} e_j + (1, ..., 1)\}$.

Following the interpretation of subsection 3.1, consumer's incomplete preferences are assumed to be represented by a multivalued function[6] $\mathcal{U}_i$ from X_i to $\mathbf{R}_+$ with a closed graph, nonempty bounded values on every compact set and admitting at least one quasi-concave selection u_i. An example, for $m = 2$, could be:

$$(1) \quad \mathcal{U}_1(x_1, x_2) = \begin{cases} x_1 + x_2 & \text{if } x_1 + x_2 < 2, \\ x_1 + x_2 + 1 & \text{if } x_1 + x_2 > 2, \\ [x_1 + x_2, x_1 + x_2 + 1] & \text{if } x_1 + x_2 = 2. \end{cases}$$

In this economy, there is a bonus of 1 unit if the consumer has a sufficient quantity of goods, because he may have a substantial benefit if he has more than some minimal level. Moreover, consumer 1's payoff is indetermined when $x_1 + x_2 = 2$, and there are many ways to complete the preferences .

Under the above assumptions, there exists a selection $(q_i)_{i \in N}$ of $(\mathcal{U}_i)_{i \in N}$ satisfying (a), (b) and (c) below:

(a) the economy $\{X_i, q_i, e_i\}_{i \in N}$ admits a walrasian equilibrium $(x^*, p^*) \in \prod_{i \in N} X_i \times \Delta(\mathbf{R}^m_+)$, that is:[7]

(1) $\sum_{i \in N} x_i^* \leq \sum_{i \in N} e_i$, and

(2) x_i^* maximizes the utility function q_i of agent i on his budget set $B_i(p^*) = \{x_i \in X_i : p^* \cdot (x_i - e_i) \leq 0\}$.

(b) for every $x_i \in B_i(p^*)$, there is a sequence $(x_i^n, p^n)_{n \in \mathbb{N}}$ converging to (x_i, p^*), with $x_i^n \in B_i(p^n)$, and such that $\lim_{n \to +\infty} u_i(x_i^n) = q_i(x_i)$.

(c) for every $x_i \in X_i$ and $x_i \notin B_i(p^*)$, $q_i(x_i) = u_i(x_i)$.

Conditions (b) and (c) guarantee that the payoff function q_i is not too far from u_i (modifications occur only at discontinuity points that are inside the budget set). In particular, $u_i(x_i) = q_i(x_i)$ if u_i is continuous at x_i. The proof can be found in appendix B. Let us illustrate the result with an example.

[6]Here, to simplify the exposition, we do not allow externalities, that is $\mathcal{U}_i$ depends only of player i's strategies.

[7]The set $\Delta(\mathbf{R}^m_+)$ denotes the unit simplex of $\mathbf{R}^m_+$.

Example 7 *Consider the following walrasian economy with externalities and discontinuous payoffs:* $m = 2$, $e_1 = e_2 = (1, 1)$ *and* $\mathcal{U}_1 = \mathcal{U}_2$ *defined as above in (1). The following payoff functions* u_1 *and* u_2 *are quasiconcave selections of* $\mathcal{U}_1 = \mathcal{U}_2$*:*

$$u_1(x_1, x_2) = u_2(x_1, x_2) = \begin{cases} x_1 + x_2 & \text{if } x_1 + x_2 < 2, \\ x_1 + x_2 + 1 & \text{if } x_1 + x_2 > 2, \\ \frac{3x_1}{2} + x_2 & \text{if } x_1 + x_2 = 2 \end{cases}$$

However, the exchange economy defined by u_1 *and* u_2 *has no walrasian equilibrium. Indeed, suppose* $p = (p_1, p_2)$ *is an equilibrium price vector. If* $p_1 \leq p_2$*, then no consumer demands* x_1*, if* $p_1 > p_2$ *no consumer demands* x_2*. Now, if we consider the selection:*

$$q_1(x_1, x_2) = q_2(x_1, x_2) = \begin{cases} x_1 + x_2 & \text{if } x_1 + x_2 \leq 2, \\ x_1 + x_2 + 1 & \text{if } x_1 + x_2 > 2 \end{cases}$$

Then $x^* = (1, 1), p^* = (1, 1)$ *is a walrasian equilibrium.*

4 Appendix A: Proof of Theorem 2

The proof consists in several steps: first, we turns G into an auxiliary discontinuous strategic game G'. Second (steps 2 and 3), we prove the existence of a relaxed Nash equilibrium of G'. This is used to construct in step 4 a sharing rule solution of G' that satisfies some desirable properties. Finally, such a sharing rule solution is used to build a solution of G. This methodology follows the one developed in [5] and [4] to prove existence of Nash, approximate and sharing rule equilibria in discontinuous games. But it is more complicated because of the externalities. Importantly, the existence results contained in steps 3 and 4 are valid for any quasiconcave compact discontinuous strategic game $((X_i)_{i \in N}, (u_i)_{i \in N})$.

By assumption, $\mathcal{U}$ admits a single-valued selection $\phi = (\phi_i)_{i \in I}$ where each ϕ_i is quasiconcave in player i's strategy.

Step 1. Associate to G *a discontinuous game* G'*.*

Following an idea of Reny [13], we associate to the economy with externalities $G = ((X_i)_{i \in N}, (\phi_i)_{i \in N}), \mathcal{B})$ a strategic game G' as follows. Because $\mathcal{U}$ is bounded, there exists $\Lambda \in \mathbf{R}$ such that $\phi_i(x) \geq \Lambda + 1$ for every $i \in N$ and every profile $x \in X$. The game G' has N players. For every $i \in N$, strategy set of player i is X_i, and his payoff is

$$u_i(x) = \begin{cases} \phi_i(x) & \text{if } x_i \in \mathcal{B}_i(x_{-i}), \\ \Lambda & \text{otherwise.} \end{cases}$$

These new payoff functions are also quasiconcave.

Step 2. Generalized regularization of payoff functions of G'*.*

Throughout this proof, for every $i \in N$, $x \in X$, and U in $\mathcal{V}(x_{-i})$ (the set of open subsets of X_{-i}), denote by $W_U(x_i, x_{-i})$ the set of Kakutani-type[8] multivalued mappings d_i from U to X_i such that $x_i \in d_i(x_{-i})$ for every $x_{-i} \in U$. Let $\underline{\underline{u_i}} : X \to \mathbf{R}$ be the following regularization[9] of the utility function u_i

(2) $\underline{\underline{u_i}}(x) := \sup_{U \in \mathcal{V}(x_{-i})} \sup_{d_i \in W_U(x)} \inf_{x'_{-i} \in U, x'_i \in d_i(x'_{-i})} u_i(x')$.

Remark that $\underline{\underline{u_i}}(x) \leq u_i(x)$ for every $x \in X$, since in the infimum above one can take $x' = x$.

Step 3. Existence of a refined Reny solution of G'.
Let us prove that there exists a pair $(x^*, v^*) \in \overline{\Gamma}$ (where $\Gamma := \{(x, u(x)) : x \in X\}$) such that:

(3) $\forall i \in N, \sup_{x_i \in X_i} \underline{\underline{u_i}}(x_i, x^*_{-i}) \leq v^*_i$.

Such pair (x^*, v^*) refines the *Reny solution concept* introduced in [5]. When u_i is continuous for every $i \in N$, x^* is a Nash equilibrium and $v^* = u(x^*)$ is the associated payoff vector.

By contradiction, assume that there is no such pair, and let us prove that G' is generalized better-reply secure. Recall that G' is generalized better-reply secure (Barelli and Meneghel [3]) if whenever $(x, v) \in \overline{\Gamma}$ and x is not a Nash equilibrium, there exists a player i and a triple $(d_i, V_{x_{-i}}, \alpha_i)$, where $V_{x_{-i}}$ is an open neighborhood of x_{-i}, d_i is a Kakutani-type multivalued function from $V_{x_{-i}}$ to X_i and $\alpha_i > v_i$ is a real number such that for every x'_{-i} in $V_{x_{-i}}$ and $x'_i \in d_i(x'_{-i})$, one has $u_i(x'_i, x'_{-i}) \geq \alpha_i$.

For, consider $(x, v) \in \overline{\Gamma}$ such that x is not a Nash equilibrium. By assumption, (x, v) does not satisfy inequality (3), thus there exists some player $i \in N$ such that $\sup_{y_i \in X_i} \underline{\underline{u_i}}(y_i, x_{-i}) > v_i$. From the definition of $\underline{\underline{u_i}}$, there is $\varepsilon > 0$, $U \in \mathcal{V}(x_{-i})$, $d_i \in W_U(x)$ such that for every $x'_{-i} \in U$ and every $x'_i \in d_i(x'_{-i})$, $u_i(x'_i, x'_{-i}) \geq v_i + \varepsilon$: this implies generalized better-reply security. Consequently, from Barelli and Meneghel [3], since G' is generalized better-reply secure, it admits a Nash equilibrium. But this is a contradiction, since if $x \in X$ is a Nash equilibrium, $(x, u(x))$ satisfies inequality (3) (because $\underline{\underline{u_i}}(x) \leq u_i(x)$ for every $x \in X$). By contradiction, this proves the existence of $(x^*, v^*) \in \overline{\Gamma}$ satisfying inequality (3).

Step 4. Existence of a sharing rule solution of G'.
We now prove that there exists some new payoff functions $(q_i)_{i \in I}$ and a pure Nash equilibrium $x^* \in X$ of $G'' = ((X_i)_{i \in N}, (q_i)_{i \in N})$, with the additional properties:
(i) for every i and $d_i \in X_i$, $q_i(d_i, x^*{}_{-i}) \geq \underline{\underline{u_i}}(d_i, x^*{}_{-i})$.

[8] A Kakutani-type multivalued mapping is a multivalued mapping with nonempty convex values and a closed graph.

[9] This function was introduced by Carmona (see [7]).

(ii) For every $y \in X$, there exists some sequence (y^n) converging to y such that $u(y^n)$ converges to $q(y)$.

For every $i \in N$, denote by $\underline{\underline{\mathcal{S}_i}}(y)$ the space of sequences $(y^n)_{n\in\mathbf{N}}$ of X converging to y such that $\lim_{n\to+\infty} u_i(y^n) = \underline{\underline{u_i}}(y)$. Then, define $q : X \to \mathbf{R}^N$ by

$$q(y) = \begin{cases} v^* & \text{if } y = x^*, \\ \text{any limit point of } u(x^n)_{n\in\mathbf{N}} & \text{if } y = (d_i, x^*_{-i}) \text{ for some } i \in N, d_i \neq x^*_i, (x^n)_{n\in\mathbf{N}} \in \underline{\underline{\mathcal{S}_i}}(d_i, x^*_{-i}), \\ q(y) = u(y) & \text{otherwise.} \end{cases}$$

Since $(x^*, v^*) \in \overline{\Gamma}$, and by definition of q, condition (ii) above is satisfied at x^*. Clearly, by definition, it is also satisfied at every y different from x^* for at least two components, and finally also at every (d_i, x^*_{-i}) with $d_i \neq x^*_i$ (for some $i \in N$), from the definition of $q(d_i, x^*_{-i})$ in this case. Condition (i) is true at every y different from x^* for at least two components (from $\underline{\underline{u_i}} \leq u_i$), is true at every (d_i, x^*_{-i}) with $d_i \neq x^*_i$ by definition, and is finally true at x^* from inequality (3). This ends the proof of Step 4.

Step 5. Existence of a solution of G.
Now, we finish the proof of Theorem 2. Take $d_i \in \mathcal{B}_i(x^*_{-i}) \neq \emptyset$. For every x'_{-i} in some neighborhood of x^*_{-i} and every $x'_i \in \mathcal{B}_i(x'_{-i})$, we have, by definition, $u_i(x'_i, x'_{-i}) = \phi_i(x'_i, x'_{-i}) \geq \Lambda + 1$. Since $\mathcal{B}_i$ is a Kakutani-type mapping, this implies, by definition, $\underline{\underline{u_i}}(d_i, x^*_{-i}) \geq \Lambda + 1$ (where $\underline{\underline{u_i}}$ is the regularization of u_i, defined in the beginning of this proof). Thus, from condition (i) in step 4 above, we get

(4) $\quad \forall d_i \in \mathcal{B}_i(x^*_{-i}),\ q_i(d_i, x^*_{-i}) \geq \underline{\underline{u_i}}(d_i, x^*_{-i}) \geq \Lambda + 1.$

Since x^* is a Nash equilibrium of G'', we have:

$$\forall i \in N,\ q_i(x^*) \geq \sup_{d_i \in X_i} q_i(d_i, x^*_{-i}) \geq \Lambda + 1.$$

From condition (ii) in step 4 above, there is a sequence (x^n) converging to x^* such that $u(x^n)$ converges to $q(x^*)$. Since $q_i(x^*) \geq \Lambda+1$ for every $i \in N$, we cannot have $u_i(x^n) = \Lambda$ for n large enough. Consequently, from the definition of u_i, we get $u_i(x^n) = \phi_i(x^n)$ and $x^n_i \in \mathcal{B}_i(x^n_{-i})$ for n large enough. Passing to the limit, we get $x^*_i \in \mathcal{B}_i(x^*_{-i})$ for every $i \in I$ (because $\mathcal{B}_i$ has a closed graph). A similar argument can be applied to any $(y_i, x^*_{-i}) \in X$ for which $y_i \in \mathcal{B}_i(x^*_{-i})$: there is a sequence (x^n) converging to (y_i, x^*_{-i}) such that $u(x^n)$ converges to $q(y_i, x^*_{-i})$. Since $q_i(y_i, x^*_{-i}) \geq \Lambda + 1$ (from inequality (4)), we cannot have $u_i(x^n) = \Lambda$ for n large enough. Consequently, $u_i(x^n) = \phi_i(x^n)$ and $x^n_i \in \mathcal{B}_i(x^n_{-i})$ for n large enough. In particular, since ϕ is a selection of U and since U has a closed graph, we get

(5) $\quad \forall y_i \in \mathcal{B}_i(x^*_{-i}),\ q(y_i, x^*_{-i}) \in U(y_i, x^*_{-i}).$

Now, define $\tilde{q}(y_i, x^*_{-i}) = q(y_i, x^*_{-i})$ whenever $y_i \in \mathcal{B}_i(x^*_{-i})$ for some $i \in N$, and $\tilde{q}(y) = \phi(y)$ elsewhere. The proof that x^* is a equilibrium of $((X_i)_{i\in N}, (\tilde{q}_i)_{i\in N}, B)$ is a straightforward consequence of x^* being a Nash equilibrium of $((X_i)_{i\in N}, (q_i)_{i\in N})$. Last, we have to prove that $\tilde{q}(y) \in U(y)$ for every $y \in X$. This is clear at $y = (y_i, x^*_{-i})$ for $y_i \in \mathcal{B}_i(x^*_{-i})$, from (5) above. For others y, we have $\tilde{q}(y) = \phi(y) \in U(y)$ by definition. This ends the proof of Theorem 2.

5 Appendix B: proof of the statements in Section 3.3

From the exchange economy, define an economy with externalities and discontinuous payoffs $(\mathcal{G}, \mathcal{B})$ as follows:

1. There are $(N+1)$ players.
2. For $i = 1, ..., N$, player i's convex compact strategy space is $X_i = \{x_i \in \mathbf{R}^m_+ : x_i \leq \sum_{i=1}^N e_i + (1, ..., 1)\}$ and his payoff function is u_i.
3. The strategy space of player $(N+1)$ (called the auctioneer) is $X_{N+1} = \Delta(\mathbf{R}^m_+)$, and his payoff function is $v_{N+1}(x, p) = p. \sum_{i\in N}(x_i - e_i)$.
4. Last, define the strategy correspondences as follows: for every $i \in N$, $\mathcal{B}_i(x, p) = B_i(p) = \{x_i \in X_i : p \cdot x_i \leq p \cdot e_i\}$, and finally define $\mathcal{B}_{N+1}(x, p) = X_{N+1}$.

Following section 3.2, this economy has a solution $(x^*, p^*, \tilde{q})$. This means that:

1. For every $(x, p) \in \prod_i X_i \times \Delta(\mathbf{R}^m_+)$ such that $x_i \in B_i(p)$ for every $i \in N$, there is a sequence $(x^n, p^n)_{n\in\mathbb{N}}$ converging to (x, p) such that $x^n_i \in B_i(p^n)$ for every $i \in N$ and $\tilde{q}_i(x, p) = \lim_{n\to+\infty} u_i(x^n_i)$. In particular, from the continuity of v_{N+1}, $\tilde{q}_{N+1}(x, p) = v_{N+1}(x, p) = p. \sum_{i\in N}(x_i - e_i)$.

2. (i) For every $i \in N$, $x^*_i \in B_i(p^*)$.
   (ii) For every $i \in N$, for every $x_i \in B_i(p^*)$, $\tilde{q}_i(x_i, x^*_{-i}, p^*) \leq \tilde{q}_i(x^*, p^*)$.
   (iii) For every $p \in \Delta(\mathbf{R}^m_+)$, $p. \sum_{i\in N}(x^*_i - e_i) \leq p^*. \sum_{i\in N}(x^*_i - e_i)$.

Let us now define $q_i(x_i) := \tilde{q}_i(x_i, x^*_{-i}, p^*)$ for every $x_i \in B_i(p^*)$, and $q_i(x_i) = u_i(x_i)$ otherwise. From 1 and 2 above, there is a sequence $(x^n, p^n)_{n\in\mathbb{N}}$ converging to (x_i, x^*_{-i}, p^*) such that $x^n_i \in B_i(p^n)$ for every $i \in N$ and

$$(6) \quad \tilde{q}_i(x_i, x^*_{-i}, p^*) = \lim_{n\to+\infty} u_i(x^n_i) = q_i(x_i).$$

Thus condition (b) and (c) in section 3.3 hold. Let us prove that condition (a) also holds, that is, (x^*, p^*) is a walrasian equilibrium of the economy with payoff functions $(q_i)_{i\in N}$.

First, assume that we do not have $\sum_{i\in N}(x_i^* - e_i) \leq 0$. Then, let us define $p = (p(1), ..., p(k), ..., p(m)) \in \Delta(\mathbf{R}_+^m)$ with $p(k) = 0$ when $\sum_{i\in N}(x_i^* - e_i)(k) \leq 0$ (where $\sum_{i\in N}(x_i^* - e_i)(k)$ denotes k-component of $\sum_{i\in N}(x_i^* - e_i)$), and $p(k) = \lambda. \sum_{i\in N}(x_i^* - e_i)(k)$ otherwise (where $\lambda > 0$ is a normalization coefficient that insures that $p \in \Delta(\mathbf{R}_+^m)$). By definition, we get $p. \sum_{i\in N}(x_i^* - e_i) > 0$, thus from (iii) above, $p^*. \sum_{i\in N}(x_i^* - e_i) > 0$. But from condition (i) above, the budget constraint yields $p^*(x_i^* - e_i) \leq 0$ for every $i \in N$, and summing these inequalities, we get $p^*. \sum_{i\in N}(x_i^* - e_i) \leq 0$, a contradiction. This proves $\sum_{i\in N}(x_i^* - e_i) \leq 0$.

From (ii) above, for every $x_i \in B_i(p^*)$, we have $q_i(x_i) = \tilde{q}_i(x_i, x_{-i}^*, p^*) \leq \tilde{q}_i(x^*, p^*) = q_i(x_i^*)$. Thus, for every $i \in N$, x_i^* maximizes q_i in $B_i(p^*)$, which ends the proof.

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