A Historiographical Exploration of Āryabahṭa's verses on Indeterminate Problems of the First Degree

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Sphere, Historiography Conference

25/10/2016

The Pulveriser (*kuṭṭaka)* Pulverising *kuṭṭakāra*)

Āryabhaṭa’s (b. 476) *Āryabhaṭīya* (499)

Bhāskara (629) Parameśvara (ca. 1360-1431)

Brahmagupta (b. 598) ’s

Brāhmasphuṭasiddhānta (628)

Mahāvīra (ca. 850) *Gaṇitasārasaṃgrāha*

Lalla (ca. 7th-9th century) *Siddhāntatilaka*

Bhāskara (b.1114)

Siddhāntaśiromāṇi; Līlāvatī, Bījagaṇita

The Pulveriser (*kuṭṭaka, kuṭṭakāra*)

Kern 1874
Rodet 1879
Kaye 1907-1908
Mazumdar 1911
Sengupta 1927
Clark 1930

Āryabhaṭa's (b. 476) *Āryabhaṭīya* (499)

Bhāskara (629) Parameśvara (ca. 1360-1431)

Ganguli, Datta (& Singh) 1929-1935

Brahmagupta (b. 598) 's

Brāhmasphuṭasiddhānta (628)

Mahāvīra (ca. 850) *Gaṇitasārasaṃgrāha*

Colebrooke 1817

Lalla (ca. 7th-9th century) *Siddhāntatilaka*

Stratchey 1813
Taylor 1816

Bhāskara (b.1114)

Siddhāntaśiromāṇi; Līlāvatī, Bījagaṇita

Āryabhaṭa's (b. 476) *Āryabhaṭīya* (499) (Ab)

Bhāskara commentary (629) BAB

and my translation

Ab.2.32. adhikāgrabhāgahāraṃ chindyād ūnāgrabhāgahāreṇa|
śeṣaparasparabhaktaṃmatiguṇam agrāntare kṣiptam||
Ab.2.33ab. adhopariguṇitam antyayugūnāgracchedabhājite śeṣam|

Ab.2.32. One should reduce the divisor which is a large number (*adhikāgrabhāgahāra*) <and the dividend> by a divisor which is a small number (*ūnāgrabhāgahāra*). The mutual division of the remainders <is made continuously. The last remainder> having a clever <quantity> for multiplier and added in the inside of a number (*agrāntara*) <is divided by the last divisor>||
Ab.2.33ab. The one above is multiplied by the one below, and increased by the last. When <the remaining upper quantity> is divided by the divisor which is a small number, the remainder is <the pulveriser. When the lower one remaining is divided by the dividend the quotient is produced.>|

$$y = \frac{ax \pm c}{b}$$

BAB.2.32-33 « Pulverizer without remainder »

BAB.2.32-33.Ex5. Eight multiplied by what <is sought> increased by six, divided by thirteen should give an exact division. What is the multiplier? And what is the quotient?||

aṣṭau kenābhyastāḥ ṣaḍrūpayutā hṛtāḥ trayodaśabhiḥ |
dadyuḥ śuddhaṃ bhāga ko guṇakāraḥ kimāptam ca || 5 ||

nyāsaḥ

Setting:

8 → the dividend

13 → the divisor

6 → the interior number

$$y = \frac{8x+6}{13}$$

y: the quotient, *apta, labdha*

x: the pulveriser, the multiplier

$$y = \frac{ax+c}{b}$$

The dividend is eight, the divisor is thirteen, the inside number is six.

bhājyo'aṣṭau, bhāgahāras trayodaśa, agrāntaraṃ ṣaṭ |

BAB.2.32-33 « Pulverizer without remainder »

$$y = \frac{8x+6}{13}$$

Procedure: The divisor and dividend quantities are reduced by unity

karaṇam – bhājyabhāgahārarāśī rūpeṇāpavartitau

8 dividend

13 divisor

this has the shape of a fraction

``The mutual division of the remainders <is made continuously>", what results is

"śeṣaparasparabhakta" iti jātam

$$\frac{13}{8} = 1 + \frac{5}{8}$$

1 $\quad \frac{8}{5} = 1 + \frac{3}{5}$

1 $\quad \frac{5}{3} = 1 + \frac{2}{3}$

1 $\quad \frac{3}{2} = 1 + \frac{1}{2}$

1

q_1
q_2
q_3
q_4

$$\frac{b}{a} = q_1 + \frac{r_1}{a}$$

$$\frac{a}{r_1} = q_2 + \frac{r_2}{r_1}$$

$$\frac{r_1}{r_2} = q_3 + \frac{r_3}{r_2}$$

$$\frac{r_2}{r_3} = q_4 + \frac{r_4}{r_3}$$

The remainder of the mutual division is 1
2

parasparabhaktaśeṣam

Another way of looking at these “substitutions”

is to consider them as a continuous fraction

$$\frac{b}{a} = q_1 + \frac{1}{q_2 + \frac{1}{q_3 + \frac{1}{q_4}}}$$

$$y = \frac{8x+6}{13} \qquad y = \frac{ax+c}{b} \qquad x = y + x_1 \qquad x = q_1 y + x_1$$

$$x = \frac{13y-6}{8} \qquad x = \frac{by-c}{a} \qquad x_1 = \frac{5y-6}{8} \qquad x_1 = \frac{r_1 y - c}{a}$$

$$\frac{13}{8} = 1 + \frac{5}{8} \qquad \frac{b}{a} = q_1 + \frac{r_1}{a} \qquad y = \frac{8x_1+6}{5} \qquad y = \frac{ax_1+c}{r_1}$$

$$\frac{8}{5} = 1 + \frac{3}{5} \qquad \frac{a}{r_1} = q_2 + \frac{r_2}{r_1} \qquad y = x_1 + y_2 \qquad y = q_2 x_1 + y_2$$

$$y_2 = \frac{3x_1+6}{5} \qquad y_2 = \frac{r_2 x_1 + c}{r_1}$$

1

1

$$x_1 = \frac{5y_2-6}{3} \qquad x_1 = \frac{r_1 y_2 - c}{r_2}$$

1

1

$$\frac{5}{3} = 1 + \frac{2}{3} \qquad \frac{r_1}{r_2} = q_3 + \frac{r_3}{r_2} \qquad x_1 = 5y_2 + x_2 \qquad x_1 = q_3 y_2 + x_2$$

$$x_2 = \frac{2y_2-6}{3} \qquad x_2 = \frac{r_3 y_2 - c}{r_2}$$

$$y_2 = \frac{3x_2+6}{2} \qquad y_2 = \frac{r_2 x_2 + c}{r_3}$$

$$\frac{3}{2} = 1 + \frac{1}{2} \qquad \frac{r_2}{r_3} = q_4 + \frac{r_4}{r_3} \qquad y_2 = x_2 + y_3 \qquad y_2 = q_4 x_2 + y_3$$

$$y_3 = \frac{x_2+6}{2} \qquad y_3 = \frac{r_4 x_2 + c}{r_3}$$

BAB.2.32-33 « Pulverizer without remainder »

``Having a clever <quantity> for multiplier and added to the inside number"; will this one quantity multiplied by what <is sought>, when one has added six units <to it>, give an exact division by two?

"matiguṇam agrāntare kṣiptam" itiy ayam eko rāśiḥ kena guṇitaḥ ṣaḍrūpāṇi prakṣipya dvābhyāṃ śuddhaṃ bhāgaṃ dāsyatīti

$$l = \frac{k+6}{2}$$

The clever <quantity> is two, 2; <one> is multiplied by the clever <quantity>, what results is $\begin{matrix} 2 \\ 2 \end{matrix}$ k

matiḥ dve 2, matyā guṇitaṃ jātam

This is increased by six units $\begin{matrix} 8 \\ 2 \end{matrix}$

etat ṣaḍrūpayutam

The quotient is four units, 4. All of these are in due order,

labdham rūpacatuṣkam 4 | etat sarvam yathākrameṇa

1	q_1
1	q_2
1	q_3
1	q_4
2	k
4	l

``The one above is multiplied by the one below and increased by the last'', what results is 22
14

"adhaupariguṇitam antyayuk" iti jātam

1	1x2+4=6	1	1x6+2=8	1	1x8+6=14	1	1x14+8=22	22
1		1		1		14		14
1		1		8		8		
1		6		6				
2		2						
~~4~~								

x

y

q_1		q_1		q_1		q_1	$q_1 y + x_1$
q_2		q_2		q_2	$q_2 x_1 + y_2$	y	$x = q_1 y + x_1$
q_3		q_3	$q_3 y_2 + x_2$	x_1	$y = q_2 x_1 + y_2$	x_1	
q_4	$q_4 x_2 + y_3$	y_2	$x_1 = q_3 y_2 + x_2$	y_2			
x_2	$y_2 = q_4 x_2 + y_3$	x_2					
y_3							

$$y = \frac{8x+6}{13}$$

$$14 = \frac{(22\times 8)+6}{13}$$

``**When <the result of this procedure> is divided by the divisor which is a small number, the remainder**", the remainders of the divisions <of the upper and lower quantities respectively> by the divisor and dividend which is a small number [= is reduced] are placed 9

6

$$\frac{22}{13} = 1 + \frac{9}{13}$$

$$\frac{14}{8} = 1 + \frac{6}{8}$$

This is the pulveriser and the quotient.

$$6 = \frac{(8\times 9)+6}{13}$$

$$y = \frac{8x+6}{13}$$

If (x_0,y_0) is the smallest solution of the equation,
then all solutions have the shape (x_i,y_i) were, for i integer

$$x_i = x_0 + 13i$$

$$y_i = y_0 + 8i$$

$$x_i = x_0 + bi$$

$$y_i = y_0 + ai$$

BAB.2.32-33 « Pulverizer without remainder »

To solve problems of the type

$$y = \frac{ax \pm c}{b}$$

Bhāskara also, when higher numbers are involved solves first

$$y = \frac{ax \pm 1}{b}$$

Solutions of this problem are then multiplied by c to find solutions of the above

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Ab.2.32-33. adhopariguṇitam antyayugūnāgracchedabhājite śeṣam|
adhikāgracchedaguṇam dvicchedāgram adhikāgrayutam||

Ab.2.32. One should divide the divisor of the greater remainder (*adhikāgrabhāgahāra*) by the divisor of the smaller remainder (*ūnāgrabhāgahāra*). The mutual division <of the previous divisor> by the remainder <is made continuously. The last remainder> having a clever <quantity> for multiplier is added to the difference of the <initial> remainders <and divided by the last divisor>.||
Ab.2.33. The one above is multiplied by the one below, and increased by the last. When <the result of this procedure> is divided by the divisor of the smaller remainder. The remainder, having the divisor of the greater remainder for multiplier, and increased by the greater remainder is the <quantity that has such> remainders for two divisors||

$$N = ax + R_1$$

$$N = by + R_2$$

$$y = \frac{ax \pm c}{b}$$

Edward Strachey (1774–1832)

CHAP. V *.

BIJA GANITA:

OR THE

ALGEBRA

OF THE

HINDUS.

BY

EDWARD STRACHEY,

OF THE

EAST INDIA COMPANY's BENGAL CIVIL SERVICE.

LONDON:

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* The rules given in this chapter are in effect the same as those which have been given by the modern European Algebraists for the solution of indeterminate problems of the first degree. Compare them with the process by continued fractions.

THE

PAÑCHASIDDHÂNTIKÂ

THE ASTRONOMICAL WORK

OF

VARÂHA MIHIRA.

THE TEXT, EDITED WITH AN ORIGINAL COMMENTARY IN SANSKRIT AND AN ENGLISH TRANSLATION AND INTRODUCTION

BY

G. THIBAUT, PH. D.

AND

MAHÂMAHOPÂDHYÂYA SUDHÂKARA DVIVEDÎ.

PRINTED BY E. J. LAZARUS AND CO., AT THE MEDICAL HALL PRESS, BENARES.

1889.

TO

F. MAX MÜLLER K. M.

A TOKEN

OF ADMIRATION

AND

REGARD.

वर्षायुते धृतिघ्ने	वर्षायुते धृतिघ्ने
नववसुगुणरसरसाः स्युरधिमासाः ।	नववसुगुणरसरसाः स्युरधिमासाः ।
सावित्रे शरनवखे-	सावित्रे शरनवखे-
द्रि[या]र्णवाशास्तिथिप्रलयाः ॥ १४ ॥	न्द्रियार्णवाशास्तिथिप्रलयाः ॥ १४ ॥
रोमकयुगमर्केंद्वो-	रोमकयुगमर्केन्द्वो-
र्वर्षाण्याकाशपंचवसुपक्षाः ।	र्वर्षाण्याकाशपञ्चवसुपक्षाः ।
खेंद्रियदिशो धिमासाः	खेन्द्रियदिशोऽधिमासाः
स्वकृतविषयाष्टयप्रलयाः ॥ १५ ॥	स्वरकृतविषयाष्टयः प्रलयाः ॥ १५ ॥
युगवर्षमासपिंडं	युगवर्षमासपिण्डं
रविमानं साधिमासकं चांद्रं ।	रविमानं साधिमासकं चान्द्रम् ।
अवमविहीनं सावन-	अवमविहीनं सावन-
मैंदवमब्दान्वितं चार्चं ॥ १६ ॥	मैन्दवमब्दान्वितं चार्चम् ॥ १६ ॥
मुनियमयमद्रियुक्ते	मुनियमयमद्रियुक्ते
द्युगणे शून्यद्विपंचयमभक्ते ।	द्युगणे शून्यद्विपञ्चयमभक्ते ।
प्रतिराश्यखर्तुदहनै-	प्रतिराशि खर्तुदहनै-
र्लब्धं वर्षाणि पातानि ॥ १७ ॥	र्लब्धं वर्षाणि यातानि ॥ १७ ॥
तानि प्रपन्नसहिता-	तानि प्रपन्नसहिता-
न्यग्निगुणान्यब्धिवर्जिता हरेत् ।	न्यग्निगुणान्यङ्घ्रिवर्जितानि हरेत् ।
सप्तभिरेवं शेषे	सप्तभिरेवं शेषं
वषाधिपतिः क्रमात्सूर्यात् ॥ १८ ॥	वर्षाधिपतिः क्रमात् सूर्यात् ॥ १८ ॥
त्रिंशद्गुक्ते मासाः	त्रिंशद्गुक्ते मासाः
प्रपन्नसहिता द्विसंगुणाः कार्याः ।	प्रपन्नसहिता द्विसङ्गुणाः कार्याः ।
सप्तोद्धृतावशेषे	सप्तोद्धृतावशेषे
मासाधिपतिस्तथैवार्कात् ॥ १९ ॥	मासाधिपतिस्तथैवार्कात् ॥ १९ ॥
सप्तोद्धृते दिनेश-	सप्तोद्धृते दिनेश-
स्त्रिगुणेध्येकश्वहोरादिः ।	स्त्रिगुणो व्येको युतश्च होराभिः ।
पंचघ्ने सप्तहृते	पञ्चघ्नः सप्तहृतो
विज्ञेया कालहोरेशाः ॥ २० ॥	विज्ञेयः कालहोरेशः ॥ २० ॥
वर्षाधिपश्चतुर्थे	वर्षाधिपश्चतुर्थो
मासाधिपतिस्तथानतो योन्यः ।	मासाधिपतिस्तथा तृतीयोऽन्यः ।

१४. धृतिपे· °वखेन्द्रिया ग्रा – १५. °केन्दोर्व° °पञ्चयेस्तुपक्षाः °धिमासाः स्यात्कतविषयाष्टयः – १६. युगवर्षणां सपिण्डं तार्चं – १७. गतिराश्च°– १८ °सहितान्यधिवर्जितानि हरेत्· °पतिक्र° –· १९. प्रमवसहिताः °थैवार्घ्यात् – २० धिज्ञेयकायहोरेषः –

$=\frac{७}{२२८}$ अस्मादनुपातेनाधिमासानयनमुपपन्नम् । तत एकस्मिन् चान्द्रदिने क्षयाहमानम् $=\frac{१६५४७}{५०५७५००}=\frac{११}{७०३}$ स्वल्पान्तरात्, ततोऽनुपातेनावमानयनमुपपन्नं भवति । मनुशरा ग्रन्थारम्भे गुणखसप्तहारयोग्यः क्षेप इत्युपपन्नं सर्वम् । अत्रेदमवधेयं यदाचार्येण स्वल्पान्तराद्ग्रन्थारम्भकालिकोऽधिमासक्षेपस्त्यक्त एव यतो न तत्र "रोमकसूर्यौ द्युगणात्खतिथिघ्नादि"त्यादिनाऽधिमासक्षेपाभावः सिध्यति ॥ विशेषं तच्छ्लोकव्याख्याने कथयिष्य इति ।

११–१६ इदानीमन्येषां चतुर्णां सिद्धान्तानां मतानुसारेणाधिमासादिमानं तत्तन्मतीयाहर्गणसाधनोपयोग्याह । दिग्घ्नासाष्टेत्यादि ।

भट्टोत्पलमतानुसारेण पौलिशसौरयोः सौरवर्षादिमानं समानमेव केवलं नाम्ना भेदः । अर्थात् पराशरादिमते यत् सौरमानं तस्य पुलिशाचार्येण सावनसंज्ञा कृता । एवं सावनस्य सौरसंज्ञा कृता । परन्तु यद्येवं स्वीकृत्य पौलिशमतेनाधिमासादयः साध्यन्ते तर्हि, आचार्योक्तप्रकारेण महान् भेदो ह्युत्पद्यते । अतः ११–१३ श्लोकानामद्यावधि याथातथ्येनाशयो मनसि नायातस्तथापि भट्टोत्पललिखितपौलिशमतानुसारेणैकस्मिनधिमासे सौरदिवसानामासन्नमानानि ११ – १३ श्लोकशोधनोपयोगीनि लिख्यन्ते ।

$$९७६\frac{४३३६}{६६३८९}=९७६+\cfrac{१}{१५+\cfrac{१}{३+\cfrac{१}{४+\cfrac{१}{१+\cfrac{१}{२+\frac{१}{६६}}}}}}$$

अस्मादासन्नमानानि, ९७६, ९७६ $\frac{१}{१५}$, ९७६ $\frac{३}{४६}$, ९७६ $\frac{१३}{१९९}$, ९७६ $\frac{१६}{२४५}$, ९७६ $\frac{४५}{६८९}$,

एवमेकस्मिन्सौरवर्षेऽधिमासमानम् ।

$$\frac{६६३८९}{१८००००}+\cfrac{१}{२+\cfrac{१}{१+\cfrac{१}{२+\cfrac{१}{२+\cfrac{१}{६+\cfrac{१}{२+\cfrac{१}{१+\cfrac{१}{१+\cfrac{१}{३+\cfrac{१}{३+\cfrac{१}{१+\cfrac{१}{३+\cfrac{१}{१+\frac{१}{३}}}}}}}}}}}}}$$

तत आसन्नमानानि

$\frac{१}{२}$, $\frac{१}{३}$, $\frac{३}{८}$, $\frac{७}{१९}$, $\frac{४५}{१२२}$, $\frac{९७}{२६३}$, $\frac{१४२}{३८५}$, $\frac{२३९}{६४८}$, $\frac{८५९}{२३२९}$, $\frac{२८१६}{७६३५}$, $\frac{३६७५}{९९६४}$, $\frac{१३८४१}{३७५२७}$, $\frac{१७५१६}{४७४९१}$,

आर्य्यभटीये गणितपादः

LEÇONS DE CALCUL D'ÂRYABHATA,

PAR

M. LÉON RODET.

EXTRAIT DU JOURNAL ASIATIQUE.

PARIS.
IMPRIMERIE NATIONALE.

M DCCC LXXIX.

Rodet, Leon. "Leçons de Calcul d'Āryabhāṭa." Journal Asiatique 7 (1879): 394–434.

Le morceau suivant a été ajouté ici, à la demande et aux frais de l'auteur, mais n'a pas paru dans le *Journal asiatique*.

Rodet 1879 p. 8

"Enfin en ce qui concerne la possibilité d'emprunts fait par Âryabhaṭa à l'enseignement mathématique des Grecs, je laisse de côté pour le moment cette étude, qui exigera des recherches historiques un peu trop longue pour figurer ici. Il s'agira en effet d'établir avec le plus de certitude qu'il sera possible, jusqu'à quelle époque on peut admettre que l'influence grecque se soit fait sentir à Pâṭaliputra; puis quel était à cette époque l'état des connaissances mathématiques des Grecs: deux points non moins difficiles à éclairicir l'un que l'autre, vue le peu de documents qui nous sont parvenus sur l'histoire de l'Inde d'une part, sur l'histoire des mathématiques chez les Grecs avant l'école d'Alexandrie d'autre part."

Concerning

Rodet 1879 p. 7 (pdf p. 6)

On peut dire qu'au point de vue scientifique Âryabhaṭa était plus avancé que son commentateur, lequel imbu des enseignements de l'école d'*Ujjayinî*, et en particulier de Bhâskara, qu'il cite fréquemment, n'a pas compris ou, si l'on aime mieux, n'a pas reconnu les *formules parlées* d'Âryabhaṭa.

We can say that scientifically Âryabhaṭa was more advanced that his commentator, who was imbued with the teachings of the *Ujjayinî* school, in particular those of Bhâskara, whom he quotes often, hasn't understood, or if you'd rather, hasn't recognized the *spoken formulas/expressions* of Âryabhaṭa.

Rodet 1879 p. 15 (pdf. 14)

Divide the denominator of the greatest **temporary value** by the denominator of the smallest value; the remainders divide each other successively [and the quotients are placed one below the other]: an **arbitrary factor** is chosen. The one below is multiplied by the one above and added to the last [continuously going up, then] this is exhausted by the denominator of the smallest **temporary value**: the remainder multiplied by the denominator of the greatest [is the corrective part] that is added to the greatest **temporary value** to obtain the value for both denominators [together].

agra	lit. “remainder”	*temporary value*	*valeur tempora*
mati	lit. “clever”	*arbitrary factor*	*facteur arbitraire*

Rodet 1879 p. 15 (pdf. 14)

Divide the denominator of the greatest temporary value by the denominator of the smallest value; the remainders divide each other successively [and the quotients are placed one below the other]: an arbitrary factor is chosen. The one below is multiplied by the one above and added to the last [continuously going up, then] this is exhausted by the denominator of the smallest temporary value: the remainder multiplied by the denominator of the greatest [is the corrective part] that is added to the greatest temporary value to obtain the value for both denominators [together].

$$y = \frac{ax+c}{b},\ z = \frac{cx-g}{f}$$

Rodet 1879 p. 42

Âryabhaṭa, whom we have seen likes to give general solutions, furnishes here the
solve in integers two simultaneous equations

$$y = \frac{ax+c}{b},\ z = \frac{cx-g}{f}$$

or to take the numerical example given by the commentator

$$8x + 29y = 4\ 17x + 45z = 7$$

such that for a same value of *x*, both

$$y = \frac{ax-c}{b}\quad z = \frac{cx-g}{f}$$

should be integers.

Parameśvara’s Example:

rāśau vasughne navadasrabhakte
śeṣaś caturbhis tulitas tathāsmin|
atyaṣṭinighne śaravedabhakte
śeṣo 'dritulyo budha kas sa rāśiḥ||

O clever one, what is the quantity that when multiplied by eight and divided by twenty-nine, has a remainder equal to four and then when multiplied by seventeen and divided by forty-five has a remainder equal to seven?|

$$\frac{8x}{29} = y + \frac{4}{29} \text{ and } \frac{17x}{45} = z + \frac{7}{45} \qquad \text{or}$$

$$8x = 29y + 4 \text{ and } 17x = 45z + 7$$

compare with Rodet’s

$$8x + 29y = 4 \text{ and } 17x + 45z = 7$$

Rodet 1879 p. 42

Ce problème est une des questions favorites des algébristes indiens, à tel point que Brahmagupta, qui lui avait donné le nom de कुट्टक *kuṭṭaka* "broyeur", a pris ce mot pour titre de son chapitre qui traite, non seulement du problème en question, mais de toute l'algèbre: semblant vouloir dire par là que tout le calcul algébrique n'a qu'un but, celui d'amener à la solution dudit problème. Bhâskara a fait figurer le chapitre qui le concerne et dans sa *Lîlâvatî* (arithmétique) et dans son *Vîjagaṇita* (algèbre).

This problem is one of the favorite questions of Indian algebraists, to the point were Brahmagupta gave the name कुट्टक *kuṭṭaka* "pulveriser", for his whole chapter which not only treats this problem but also all of algebra: he seems to mean that all algebraical computation has for only aim the solution of this problem. Bhâskara includes this chapter in his *Lîlâvatî* (arithmetics) and in his *Vîjagaṇita* (algebra).

Rodet 1879 p. 46

J'ignore sur quelle autorité s'est répandue, parmi les historiens des Mathématiques, la croyance que les Indiens résolvaient le problème qui nous occupe par le moyen des *fractions continues*. Ni le calcul d'Âryabhaṭa, ni celui de Bhâskara, que je viens de citer l'un et l'autre, n'autorisent pourtant une semblable opinion.

इत्य् आर्यभटीये गणिपादो द्वितीयः समाप्तः

ERRATUM.

Au dernier alinéa des *Notes* le Lecteur est prié de substituer le suivant :

Le calcul de Bhâskara revient, comme divers auteurs, du reste, l'ont déjà reconnu, à celui de la curieuse *fraction continue*

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{3+\cfrac{1}{0+\cfrac{0}{-4}}}}}}$$

dont les 0 doivent être traités comme des chiffres quelconques, sauf application des règles spéciales aux "opérations avec zéro" *kha-shadvidham* (V. mon *Alg. d'Al-kharizmi* p. 23, & Colebrooke. *Algebra*). On arrive, par ce moyen original & assurément fort ingénieux, à faire disparaître la dernière réduite & à multiplier les deux termes de l'avant-dernière par -4 ($\pm c$ de $ax-by=c$) sans y rien ajouter : résultat que nous obtenons en arrêtant en route notre calcul conduit en sens inverse de celui de Bhâskara.

George Rusby Kaye (1866 -1929)

17. **Notes on Indian Mathematics. No. 2.—Āryabhaṭa.**

By G. R. Kaye, *Bureau of Education, Simla.*

Kaye, George Rusby. “Notes on Indian Mathematics.” Journal of the Asiatic Society of Bengal 4 (1908): 111–41.

Kaye 1908 p. 135

32-33 The greater original divisor is divided by the lesser original divisor and the rest divide one another. An assumed number together with the original difference is thrown in. The lower is multiplied by the upper and the last added. Divide by the smaller first divisor and multiply the remainder by the larger first divisor and add the original larger remainder for the final result.

agra lit. "remainder" *original/first*

mati lit. "clever" *assumed number*

"This rule is not expressed at all clearly and is difficult to translate into unambiguous mathematical language; nevertheless its general aim is obvious. It is a rule for the solution of indeterminate equations of some such form as (Ax+C)/B=y"

Kaye 1908 p. 135

"It is not our business here to give an exposition of the general theory of indeterminate equations, but rather to attempt to trace their history up to the time of Āryabhaṭa. Even a cursory examination of the mathematics of our author will convince any one familiar with the subject that Aryabhaṭa (sic) was not the inventor of the method under consideration; and a closer investigation establishes this conclusion beyond all doubt. "

"A diligent search through Hindu works has failed to bring to ligh any of those orderly processes by which such a complicated theorem as this is bound to be preceded; but we do find the necessary preliminary notions abundantly set forth by Greek writers"

Kaye 1908 p. 135

"The fundamental process involved in the method given by Āryabhaṭa is contained in the first and second propositions of the seventh book, and the second and third of the tenth book of Euclid. The results of these propositions translated into Algebraic notation [footnote] give us the following indeterminate equations: AL-BM=1 and AL'-BM'=g.

footnote between p. 135 and 136

The question has often been asked, had Euclid any substitute for Algebra? If not, his skill, as shown particularly in the tenth book was marvellous (sic). Whether or not Euclid employed some sort of algebraic smybolism (sic), we know that the later Alexandrian scholars did, and we also know that they translated Euclid's proposition into their new symbolism. (See Gow, 83 and 104)

Kaye 1908

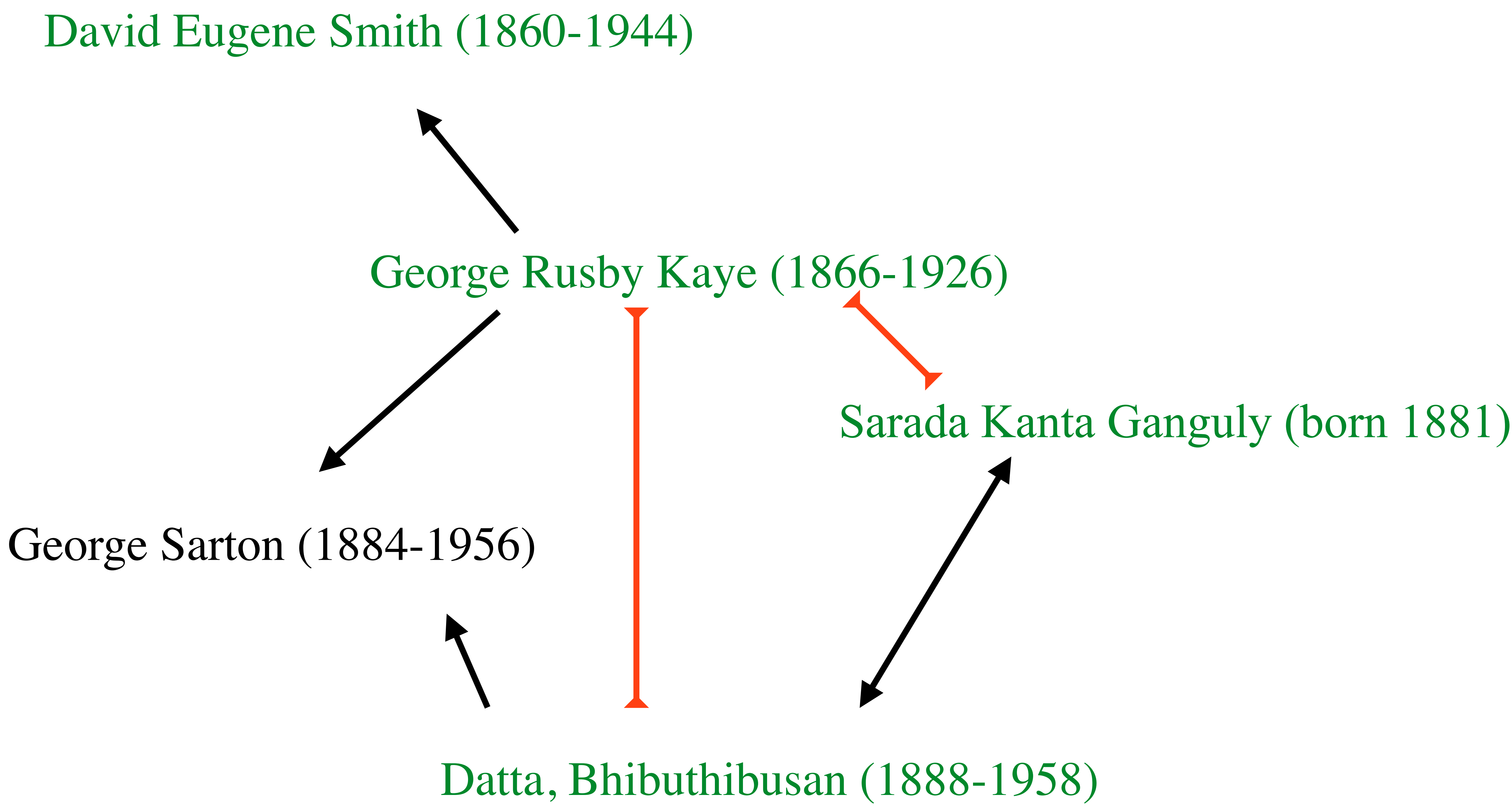
"These results, which are vaguely embodied in Āryabhaṭa's rule, may, of course, be put in a perfectly general form."

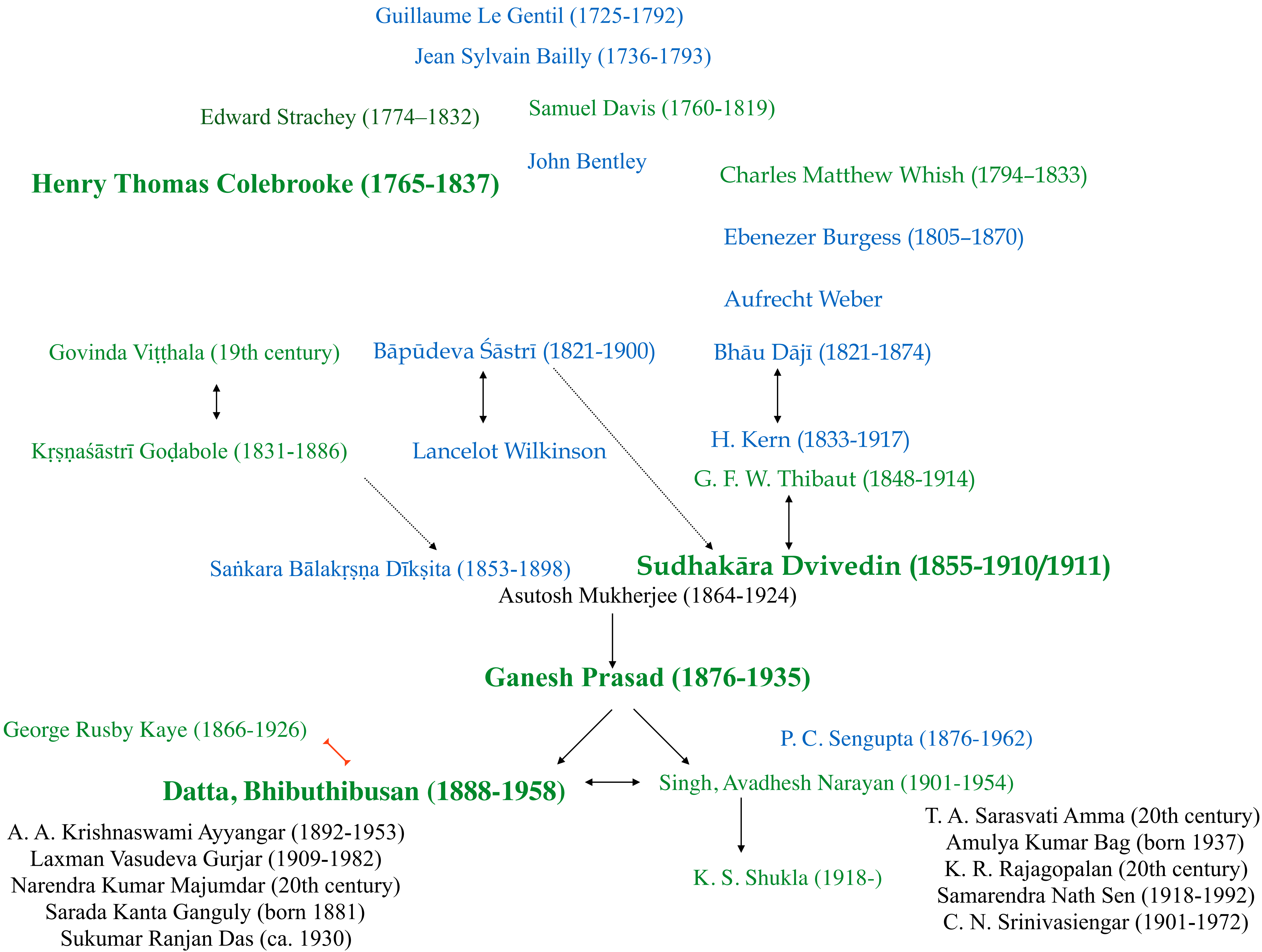
"Although there is ample evidence in Greek mathematics as to the existence of the preliminary notions necessary for the evolution of the particular rule under consideration, yet we nowhere find in extant Greek works the rule itself applied in just this manner. On the other hand we do find that the Greeks carried the treatment of indeterminate equations much further than did Āryabhaṭa, and there is no doubt that they were able to manipulate indeterminates of the first degree in the manner indicated in the rule of Āryabhaṭa."

"Diophantus lived about A. D. 300-350, and Hypatia, who wrote a commentary on the works of Diophantus, was murdered by these quarrelsome Alexandrian Christians in A. D. 415. Āryabhaṭa was born in A. D. 476.”

"It is in connection with questions on the calendar that the most useful applications of indeterminate equations of the first degree arise. The following example in a very marked manner illustrates many points of Āryabhaṭa's rule that at first seemed inexplicable:-"

"Although Āryabhaṭa's rule is by no means unambiguous in parts, yet the working of the above problem agrees so closely with it that there is no doubt that the rule is intended for similar examples."


David Eugene Smith (1860-1944)
George Rusby Kaye (1866-1926)
Sarada Kanta Ganguly (born 1881)
George Sarton (1884-1956)
Datta, Bhibuthibusan (1888-1958)



Guillaume Le Gentil (1725-1792)
Jean Sylvain Bailly (1736-1793)
Edward Strachey (1774–1832)
Samuel Davis (1760-1819)
John Bentley
Henry Thomas Colebrooke (1765-1837)
Charles Matthew Whish (1794–1833)
Ebenezer Burgess (1805–1870)
Aufrecht Weber
Govinda Viṭṭhala (19th century)
Bāpūdeva Śāstrī (1821-1900)
Bhāu Dājī (1821-1874)
Kṛṣṇaśāstrī Goḍabole (1831-1886)
Lancelot Wilkinson
H. Kern (1833-1917)
G. F. W. Thibaut (1848-1914)
Saṅkara Bālakṛṣṇa Dīkṣita (1853-1898)
Sudhakāra Dvivedin (1855-1910/1911)
Asutosh Mukherjee (1864-1924)
Ganesh Prasad (1876-1935)
George Rusby Kaye (1866-1926)
P. C. Sengupta (1876-1962)
Datta, Bhibuthibusan (1888-1958)
Singh, Avadhesh Narayan (1901-1954)
A. A. Krishnaswami Ayyangar (1892-1953)
Laxman Vasudeva Gurjar (1909-1982)
Narendra Kumar Majumdar (20th century)
Sarada Kanta Ganguly (born 1881)
Sukumar Ranjan Das (ca. 1930)
K. S. Shukla (1918-)
T. A. Sarasvati Amma (20th century)
Amulya Kumar Bag (born 1937)
K. R. Rajagopalan (20th century)
Samarendra Nath Sen (1918-1992)
C. N. Srinivasiengar (1901-1972)

Sarada Kanta Ganguly (born 1881)

Ganguly, Sarada Kanta. "Bhāskarācārya and Simultaneous Indeterminate Equations of the First Degree." Bulletin of the Calcutta Mathematical Society 17 (1926): 89–98.

———. "Was Aryabhata Indebted to the Greeks for His Alphabetic System of Expressing Numbers?" Bulletin of Cal. Math. Soc. 17, no. 4 (1926): 195–202.

n of the Calcutta Mathematical Society XVIII, no. 2 (1927).

———. "The Source of the Indian Solution of the so-Called Pellian Equation." Bulletin of the Calcutta Mathematical Society XIX, no. 4 (1928).

Ganguly, S.K. "Alphabetical System of Aryabhaṭa." Bulletin of the Calcutta Mathematical Society XVII, no. 4 (1926): 66–67.

Gānguli, S. "On the Modern Place-Value Notation in the Aryabhatiyam." American Mathematical Monthly XXVII (1927).

———. "The Elder Âryabhata and the Modern Arithmetical Notation." The American Mathematical Monthly 34, no. 8 (1927): 409–15.

———. "Notes on Indian Mathematics. A Criticism of George Rusby Kaye's Interpretation." ISIS 12, no. 1 (1929): 132–45.

———. "The Elder Aryabhata's Value of π." The American Mathematical Monthly 37, no. 1 (1930): 16–22. doi:10.2307/2299981.

Ganguli, Saradakanta. "India's Contribution to the Theory of Indeterminate Equations of the First Degree." Journal of the Indian Mathematical Society Notes and Questions, no. 9 (1932 1931): 110–20; 129–42; 153–68.

Ganguli, Saradakanta. "India's Contribution to the Theory of Indeterminate Equations of the First Degree." *Journal of the Indian Mathematical Society Notes and Questions*, no. 9 (1932 1931): 110–20; 129–42; 153–68.

The object of this paper is to trace the development of the theory of indeterminate equations of the first degree in India and to show thereby that the above remark of Mr. Kaye cannot be supported by facts, and that the Indian treatment of indeterminate equations of the first degree was original and not influenced by Chinese or Greek writers.

Ganguli 1931 p. 114

"Hence Greek contributions to this is theory (of indeterminate equations of the first degree) is practically nil. Thus there is no justification whatseover, for Mr Kaye's dogmatic statement that "there is no doubt that they (i.e. the Greeks) were able to manipulate indeterminates of the first degree in the manner indicated in the rule of Āryabhaṭa."

"The honour of being the first to give a general solution of the equation Ax-By=C of which the above equation is a particular case must, therefore, go to the elder Āryabhaṭa (b. 476 A. D.) whose rule for finding a number which leaves given remainders on being divided by *two* given divisors contains the general solution. Āryabhaṭa's problem in indeterminate analysis appears to be exactly similar to the one given by Sun-Tsu. Āryabhaṭa considers only *two* divisors, while Sun-Tsu contemplates any number of divisors. This difference may, at first sight, appear to be of no importance. But it is fundamental. Accordingly Āryabhaṭa's solution cannot be extended so as to give a solution of Sun-Tsu's problem. Yet the latter solution depends on the former. "

Gangly 1931 p. 120

Thus we arrive at the following interpretation of Āryabhaṭa's rule quoted above:

"To find a number which, when divided by two given numbers, leaves given remainders, divide the divisor corresponding to the greater remainder by the divisor corresponding to the smaller. The divisor in this operation of division should then be divided by the remainder given by this operation. This process of mutual division should be continued so long as the remainder does not vanish. The quotients of mutual division should be set down, one below another in a vertical line in the order in which they are obtained. Set down any assumed integer under it. Then multiply the last quotient of mutual division by the assumed integer and add to the product the difference between the given remainders. Continue this process of multiplying a lower number by the one just above it and adding to the product the number just under it. In each case the lowest number added should be rejected just after the operation of addition so that the next upper number in the resulting column may, in its turn, be the lowest in position. Finally two numbers will be obtained which are the quotients of division of the required number by the given divisors, the upper number corresponding to the given smaller given remainder. The remainder thus obtained, being multiplied by the given divisor corresponding to the greater of the given remainders and then increased by the greater given remainder, gives the least number answering to the two given divisors and the the two given remainders. The implication is that the least number satisfying the given conditions can also be obtained by multiplying the remainder, obtained as the result of division of the upper number by the divisor corresponding to the greater given remainder, by the divisor corresponding to (the) smaller given remainder and then adding the smaller given remainder to the product."

Ganguli 1931 p. 116

$$N = Ax + R_1 \text{ and } N = By + R_2 \ (1)$$

$$Ax + R_1 = By + R_2 \ (2)$$

Let R_2 be greater than R_1 and let us write C for $R_2 - R_1$. Then this equation may be written as

$$Ax = By + C \ (3)$$

When an indeterminate equation is reduced to the form of (3) in which only one term, and that involving an unknown quantity [e.g. A in (3)] is called the *divisor* as it divides the right-hand member of the equation without remainder. The co-efficient of the other unknown quantity on the other side is called the *dividend.* In equation (3) B is called the dividend. The dividend is sometimes called the *multiplier*. The term independent of the unknown quantities is called the *kṣepa* (what is thrown into or away from, i.e., what is added to or subtracted from) which, therefore, may be positive or negative. In equation (3) C is a *kṣepa*. For the sake of convenience we shall give the name dividend-side or divisor-side to a side of an equation of the form (3) according as the absolute term does or does not occur on the side."

p. 163 Enough has been written to show that all the different Indian methods of solution of the equation ax=by+c have grown out of the elder Āryabhaṭa's method which follows, by simple and natural steps, from the first four simple rules and does not depend on Euclid's method of finding the greatest common divisor of two numbers or of proving by that process that two numbers have no common factor. "

"The Chinese probems of "hundred hens" and a similar problem given by Mahāvīra led ultimately to equations of the form ax +by=c. Mahāvīra's problem might or might not have been suggested by the Chinese problems. But it is extremely doubtful if Mahāvīra's rule for the solution of such problems was borrowed from China."

Datta, Bhibuthibusan (1888-195

Datta, Bibhutibhusan. “Elder Āryabhaṭa’s Rule for the Solution of Indeterminate Equations of the First Degree.” Bulletin of the Calcutta Mathematical Society XXIV, no. 1 (1932): 19–36.

Datta, Bibhutibhushan & Avadesh Narayan, Singh. History of Hindu Mathematics. Vol. 2. Lahore: Motilal Banarsidass, 1935.

Datta 1932

"The above rule is somewhat obscure inasmuch as all the operations in the process of solving the equation by=ax+c or ax=by+c, have not been described fully and clearly. So it is liable to give rise to a good deal of misunderstandings and controversy. Amongst the modern writings on the subject, the translations of Rodet and Kaye are wrong and worthless. So we shall neglect their methods as they are bound to be incorrect. For the same reason we may discard the interpretations of N. K. Mazumdar and Heath, based as they are admittedly on the mistranslations of Kaye. It may be noted that Kaye himself has later on qualified his translation as being unsatisfactory. P.C. Sengupta's interpretation is admittedly based on the rule of Brahmagupta. W. E. Clark has explained the rule on the basis of Parameśvara's interpretation and Brahmagupta's method. They have thus followed a path safer than that of an independent attept to explain a truly enigmatical rule. According to their point of

p. 21 view the problem aimed at is to find a number which being multiplied by a given number and increased or decreased by another given number will leave no remainder when divided by a given divisor. But strictly speaking the letters of the rule clearly show that its obvious object is to define a method for the solution of a problem the kind stated before, though it can be explained to supply a method for a problem of the kind just mentioned. The Sanskrit commentators are of one mind on this point.

Datta 1932

S. K. Ganguly has arrived at an interpretation of the rule by working out a numerical example and has consequently made a rather free translation of it. But as will be presently seen that too is restricted. Above all it is different from what has been implied by the author.

The Source of the Indian Solution of the so-Called Pellian Equation.”
Bulletin of the Calcutta Mathematical Society XIX, no. 4 (1928).