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Asymptotic value in frequency-dependent games: A differential approach.

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Abstract

We study the asymptotic value of a frequency-dependent zero-sum game following a differential approach. In such a game the stage payoffs depend on the current action and on the frequency of actions played so far. We associate in a natural way a differential game to the original game and although it presents an irregularity at the origin, we prove existence of the value on the time interval $[0, 1]$. We conclude, using appropriate approximations, that the limit of $\mathbf{V}_n$, as n tends to infinity exists and coincides with the value of the associated continuous time game. We extend the existence of the asymptotic value to discounted payoffs and we show that $\mathbf{V}_\lambda$, as λ tends to 0, converges to the same limit.

Keywords: stochastic game, frequency-dependent payoffs, continuous-time game, Hamilton-Jacobi-Bellman-Isaacs equation.

JEL Classification: C73 **AMS Classification**: 91A15 91A23 91A25

Introduction

In the context of repeated games, Smale (1980) introduces dynamics that take the past into account. *Frequency-dependent games* (FD games) are a class of dynamic games in which stage payoffs depend on the frequency of past actions. They have been introduced by Brenner and Witt (2003). Such games consist in the repetition at discrete moments, of an one-shot game in which the stage payoff functions depend on the choices of the players at the current stage, as well as on the relative frequencies of actions played at previous stages. Stage payoffs may be frequency-dependent over time because of several reasons. The actions undertaken by the players at each stage may generate externalities, which accumulate as the game unfolds. For instance, payoffs may change due to learning, habit formation, addiction,

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or satiation. The class of frequency-dependent games covers a wide variety of applications such as littering and pollution problems, the impact of human activities on species and more generally on the environment. An extensive review of this class of games and its applications can be found in Joosten et al. (2003).

This literature focused mostly on long term analysis of non zero-sum FD games and in particular, in the last reference the authors derived several folk-like theorems using however a non standard notion of equilibrium. In order to approach the Nash payoffs of a long game in any standard notion of equilibrium, the study of the zero-sum case seems to be necessary. It is mostly in this perspective that recently Contou-Carrère (2011), studies some aspects of FD games. The main insight of this work is that on one hand no uniform value exists even for an one-player game, on the other hand the asymptotic value exists although its convergence of is not uniform in the state variable. Precisely, the author considers a particular case of the littering game (Joosten (2004)), in which the decision-maker has two actions, one that litters the environment in some sense, and the other one that preserves it; the littering action is a dominant action, but the use of this action degrades the environment condition so that all future payoffs are decreased. The author proves that given a length $N \in \mathbb{N}^*$, the unique optimal strategy consists in playing the non littering action from stage $t = 1$ up to some stage $t^*(N)$ and then starting to play the littering action until the end of the game. Since the time of switching from one action to the other depends on the length of the game and due to uniqueness of the optimal strategy, the uniform value does not exist in the FD control problem. Nevertheless, the fraction of time $t^*(N)/N$ converges when N goes to infinity so that in particular the asymptotic value $\lim \mathbf{V}_n(z)$ exits and is independent on the initial state z, although this convergence is not uniform in the state.

In this paper, we are interested in the study of the value of a class of two-player zero-sum FD games with finite action sets I and J respectively. At each stage the two players choose simultaneously an action and the stage payoff depends on two variables: the current joint action and the frequency of past actions. We study the subclass of FD games with separable payoffs. This means that the stage payoff is the sum of two parts, one part is derived from the current actions and the other depends linearly on the frequency of the past actions. This game can be viewed as a stochastic game with countable state space, namely $\mathbb{N}^{I\times J}$ and deterministic transitions. The current state at the n-th stage is the *aggregate past matrix*, i.e., it reflects how many times each action profile has been selected in the previous $n-1$ stages. Player 1 maximizes and Player 2 minimizes the average payoff on the first n stages and the game is played under *perfect-monitoring* meaning that both players know the current state, as well as the entire history, i.e., the state visited and action pair played at each of the preceding stages. Since it is already known from the study of the one-player game that no uniform value exists, our main focus will be the existence of the asymptotic value of this game. We treat in parallel both the average and discounted cases. Note that the convergence being non-uniform in the state variable (see Contou-Carrère (2011)), we cannot rely on the Tauberian theorem of Ziliotto (2016) to deduce the existence of one of the limits from the existence of the other.

The traditional approach to the sequence $(\mathbf{V}_n)_{n\in\mathbb{N}^*}$ is through the study of the so-called recursive equation (Mertens et al. (2015)). However, as in the case of many repeated games, it seems difficult to derive the asymptotic behavior directly from this formula. Therefore, we switch to a differential approach in the sense that we associate to the FD game, a differential game played over $[0,1] \times \mathbb{R}^{I \times J}$. Indeed, by a heuristic reasoning it is possible to conjecture as a limit of the recursive equation, a hypothetical partial differential equation (PDE) that governs the evolution of the value. It turns out that this is precisely the Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation of some differential game and furthermore that the value of this continuous game is closely related to the value of our repeated game. However, an important difficulty arises due to an irregularity of the payoff function at the origin and it is precisely at the origin where our analysis has to be done. Everywhere but the origin, regularity conditions are satisfied by the payoff and dynamics functions and since Isaacs condition holds true, i.e., lower and upper hamiltonians coincide, from Evans and Souganidis (1984)[1] and Souganidis (1999), the differential game admits a value which is characterized as the unique viscosity solution in the space of bounded, continuous functions of the (HJBI) equation with a boundary condition. Despite the irregularity at the origin, we prove existence of the value in the differential game starting at $(0,0)$. In order to compare the values of the repeated game with that of the differential game, we proceed by discretization. The previously mentioned irregularity at the origin prevents the usual methods of approximations to apply. We adapt the methods of Souganidis (1999) for finite horizon differential games (see Bardi and Capuzzo-Dolcetta (2008) for infinite horizon [2]) so that they can fit our context. We prove that $\mathbf{V}_n$ as n tends to infinity, and $\mathbf{V}_\lambda$ as λ goes to zero, both converge to the same limit which is precisely the value of the differential game starting at the origin.

In the literature, the use of differential games to study the asymptotic value of a repeated game is not new. A differential approach first appeared in Vieille (1992) to study weak approachability. An approach similar to ours, has been proposed by Laraki (2002) to prove existence of the asymptotic value in n-stage and λ-discounted repeated games with incomplete information on one side. Cardaliaguet et al. (2012) achieve a transposition to discrete time games of the numerical schemes used to approximate the value function of differential games via viscosity solution arguments, presented in Barles and Souganidis (1991). The authors prove existence of the asymptotic value in absorbing, splitting and incomplete information games, where convergence is uniform in the state variable. Our approach differs from all this literature in the nature of the state space of the continuous game and chiefly in that, due to the irregularity at the origin in our setting, existence of the value in the continuous game is not straightforward. Since in our model, the state space is countable following their approach would lead us to an infinite dimensional state space in the associated differential game. As a consequence, the way we associate the differential game to the repeated game is quite different from theirs.

Structure of the paper. The remainder of the paper is organized as follows. In Section 1, we give the description of a two-player FD zero-sum game and we provide properties of the n-stage value function, which will be useful, in the sequel. In Section 2, starting from the recursive formula satisfied by the value, we heuristically derive a (PDE). Then, we define the associated differential game and prove existence of the value in the differential game played

[1]Existence of the value follows from the standard comparison and uniqueness theorems for viscosity solutions presented in Crandall and Lions (1983).

[2]The authors prove that under some regularity conditions on the payoff and dynamics functions, the discrete values converge to the values of the continuous time game as the mesh of the discretization tends to 0. These approximations do not converge in general if the value function is discontinuous.

over $[0,1]$ and starting at initial state 0. We then provide its uniformly and λ-discounted discretized versions. In Section 3, we conclude by identifying the value of the continuous time game, as the limit value of the n-stage and the λ-discounted FD games. Section 4 deals with perspectives and future work.

1 The FD game and some preliminary results

In this section, we provide the description of the model we study and some preliminary results.

1.1 Definitions

Let I, J be finite sets and denote the space of real matrices with $|I|$ rows and $|J|$ columns by $\mathcal{M}^{I\times J}$. Let $A = [a_{ij}]$ and H be two elements of $\mathcal{M}^{I\times J}$ and let $z_0 \in \mathcal{Z} := \mathbb{N}^{I\times J}$. An FD zero-sum repeated game with initial state z_0 is a dynamic game played by steps as follows:
At stage $t = 1, 2, ...,$ Player 1 and 2 simultaneously and independently choose an action in their own set of actions, $i_t \in I$ and $j_t \in J$ respectively. The stage payoff to Player 1 is equal to:

$$g_t := g(z_{t-1}, i_t, j_t) = a_{i_t j_t} + h(z_{t-1}),$$

where $z_t = z_0 + e_{i_1 j_1} + ... + e_{i_t j_t}$. The notation $(e_{ij})_{ij}$ stands for the canonical basis in $\mathbb{R}^{I\times J}$ and,

$$h(z) := \begin{cases} \left\langle H, \frac{z}{|z|} \right\rangle, & z \neq 0 \\ 0, & z = 0, \end{cases}$$

where $\langle \cdot, \cdot \rangle$ denotes the canonical inner product in $\mathbb{R}^{I\times J}$ and $|\cdot|$ stands for $\|\cdot\|_1$. The payoff of Player 2 is the opposite of that of Player 1. We assume perfect monitoring of past actions by both players.

1.2 The values of $\Gamma_N(z_0)$ and $\Gamma_\lambda(z_0)$

Several games are associated to this game form. Given $z_0 \in \mathcal{Z}$, for any $N \in \mathbb{N}^*$ and any $\lambda \in (0,1)$, we will be interested in the finite N-stage and λ-discounted games of initial state z_0, denoted by $\Gamma_N(z_0)$ and $\Gamma_\lambda(z_0)$. A play is given by $\omega := (i_t, j_t)_{t\in\mathbb{N}^*}$ and the induced payoff in the game of initial state z_0, is $\gamma_N(z_0, \omega) = (1/N)\sum_{t=1}^{N} g_t$, (resp. $\gamma_\lambda(z_0, \omega) = \sum_{t=1}^{\infty} \lambda(1-\lambda)^{t-1} g_t$). Note that, due to the nature of the transition in the state space, announcing the selected moves publicly also reveals the state variable to the players. Therefore, we will denote by $\mathbf{H}_t = \mathcal{Z} \times (I \times J)^{t-1}$ the set of histories at stage t and $\mathbf{H} = \cup_{t\geq 0}\mathbf{H}_t$ will denote the set of all histories. $\Delta(I)$ and $\Delta(J)$ are the sets of mixed moves of Player 1 and Player 2 respectively. A behavioral strategy for Player 1 is a family of maps $\sigma = (\sigma_t)_{t\geq 1}$, such that $\sigma_t : \mathbf{H}_t \to \Delta(I)$. Similarly, a behavioral strategy for Player 2 is a family of maps $\tau = (\tau_t)_{t\geq 1}$, where $\tau_t : \mathbf{H}_t \to \Delta(J)$. Σ and T denote the sets of behavioral strategies of Player 1 and Player 2, respectively. Given $z_0 \in \mathcal{Z}$, each strategy profile (σ, τ) induces a unique probability distribution $\mathbb{P}^{z_0}_{\sigma,\tau}$ on the set $\mathcal{Z} \times (I \times J)^\infty$ of plays (endowed with the σ-field generated by the cylinders). $\mathbb{E}^{z_0}_{\sigma,\tau}$ stands for the corresponding expectation.

We study the games $\Gamma_N(z_0)$ and $\Gamma_\lambda(z_0)$, where the payoff of Player 1 is given by $\gamma_N(z_0, \sigma, \tau) = \mathbb{E}^{z_0}_{\sigma,\tau}\big(\frac{1}{N}\sum_{t=1}^{N} g_t\big)$ and $\gamma_\lambda(z_0, \sigma, \tau) = \mathbb{E}^{z_0}_{\sigma,\tau}\left(\sum_{t=1}^{\infty} \lambda(1-\lambda)^{t-1} g_t\right)$ respectively. Existence of the value in $\Gamma_N(z)$ and $\Gamma_\lambda(z)$ in mixed strategies follows from the minmax theorem of von Neumann (1928) and since the game is played under perfect-recall by Kuhn's theorem the value can be achieved by using behavioral strategies. The N-stage and λ-discounted values are given by $\mathbf{V}_N(z_0) = \sup_{\sigma\in\Sigma} \inf_{\tau \in T} \gamma_N(z_0, \sigma, \tau)$ and $\mathbf{V}_\lambda(z_0) = \sup_{\sigma\in\Sigma} \inf_{\tau\in T} \gamma_\lambda(z_0, \sigma, \tau)$ respectively. By Mertens et al. (2015), given $(n, \lambda) \in \mathbb{N}^* \times (0, 1]$ and a state $z \in \mathcal{Z}$, $\mathbf{V}_n$ and $\mathbf{V}_\lambda$ satisfy the following recursive formulas:

$$(n+1)\mathbf{V}_{n+1}(z) = h(z) + \max_{u\in\Delta(I)} \min_{v\in\Delta(J)} \left(\sum_{i,j} u_i v_j \big(a_{ij} + n\mathbf{V}_n(z + e_{ij})\big)\right) \tag{1.1}$$

$$\mathbf{V}_\lambda(z) = \lambda h(z) + \max_{u\in\Delta(I)} \min_{v\in\Delta(J)} \left(\sum_{i,j} u_i v_j \left(\lambda a_{ij} + (1-\lambda)\mathbf{V}_\lambda(z + e_{ij})\right)\right) \tag{1.2}$$

In the remainder of this section, given $n \in \mathbb{N}^*$, we provide a formula for the value of the n-stage game. This formula is too complex to allow the study of the limit, nevertheless it sheds a light on the asymptotic behavior of the value.

Notation. We use the following notations:

- Given $t \in \mathbb{N}$, let Π_t denote the subset of the state space $\mathcal{Z}$ defined as follows:

$$\Pi_t = \{z \in \mathcal{Z} : |z| = t\}.$$

- We denote the max min operator by **val**.
- For any $(a, p) \in \mathbb{R}^*_+ \times \mathbb{N}^*$, we put :

$$\Lambda_p(a) := \frac{1}{a} + \frac{1}{a+1} \ldots + \frac{1}{a+p-1} = \sum_{k=0}^{p-1} \frac{1}{a+k}.$$

Proposition 1.1. *For all $n \in \mathbb{N}^*$, for all $t \in \mathbb{N}^*$, there exist $K_{n,t} \in \mathcal{M}^{I\times J}$ and $C_{n,t} \in \mathbb{R}$, such that for all $z \in \Pi_t$*

$$n\,\mathbf{V}_n(z) = \langle K_{n,t}, z\rangle + C_{n,t},$$

where $K_{n,t} = \Lambda_n(t)H$ and $C_{n,t} = \sum_{k=1}^{n-1} \mathbf{val}\big(A + \Lambda_{n-k}(t+k)H\big)$.

Proof. We proceed by induction on the variable n:
For $n = 1$, for any $t \in \mathbb{N}^*$ and any $z \in \Pi_t$:

$$\mathbf{V}_1(z) = \left\langle H, \frac{z}{|z|} \right\rangle + \max_{u \in \Delta(I)} \min_{v \in \Delta(J)} \left(\sum_{ij} u_i v_j a_{ij} \right).$$

Then, $\mathbf{V}_1(z) = \langle K_{1,t}, z \rangle + C_{1,t}$, where $K_{1,t} = \frac{H}{t}$ and $C_{1,t} = \mathbf{val}(A)$.
The recursive formula (1.1) and the induction hypothesis for $n = m$ implies, for all $z \in \Pi_t$:

$$\begin{aligned}(m+1)\mathbf{V}_{m+1}(z) &= \left\langle \frac{H}{t} + K_{m,t+1}, z \right\rangle + \max_{u \in \Delta(I)} \min_{v \in \Delta(J)} \left(\sum_{ij} u_i v_j \Big(a_{ij} + \langle K_{m,t+1}, e_{ij} \rangle + C_{m,t+1} \Big) \right) \\ &= \left\langle \frac{H}{t} + K_{m,t+1}, z \right\rangle + \mathbf{val}\Big(A + K_{m,t+1} \Big) + C_{m,t+1},\end{aligned}$$

the middle equality folows from the inner product properties and the fact that $\sum_{ij} u_i v_j = 1$, and the last one from the **val** operator properties. Hence,

$$(m+1)V_{m+1}(z) = \langle K_{m+1,t}, z \rangle + C_{m+1,t},$$

where $K_{m+1,t} = \frac{H}{t} + K_{m,t+1}$ and $C_{m+1,t} = \mathbf{val}\Big(A + K_{m,t+1} \Big) + C_{m,t+1}$, This concludes the proof of the assumption. The rest is routine algebra. Note that $\Lambda_{m+1,t} = \Lambda_{m,t} + \frac{1}{t}$. □

Corollary 1.2. *Let $\rho \in (0,1)$. If $N \to +\infty$ and $\frac{n}{N} \to \rho$, then $\lim K_{n,N-n} = -H \ln(1-\rho)$.*

Proof. Writing $\Lambda_n(N-n) = \sum_{k=1}^{N-1} \frac{1}{k} - \sum_{k=1}^{N-n-1} \frac{1}{k}$, the limit follows readily from the fact that the sequence $\sum_{k=1}^{n} \frac{1}{k} - \ln n$ converges to the Euler constant γ when n goes to infinity. □

2 A differential approach

In view of Corollary 1.2, it seems natural to introduce a continuous version of the dynamic game. To begin with, moving from the recursive formula obtained in (1.1), we heuristically derive a Partial Differential Equation (PDE) (Section 2.1). It turns out that the latter is precisely the (HJBI) equation of some differential game that we shall define.

For any $N \in \mathbb{N}^*$, we define the quotient state space and the uniform partition of $[0,1]$:

$$\mathcal{Q}_N := \left\{ q | q = \frac{z}{N}, z \in \mathcal{Z} \right\}, \qquad \mathcal{I}_N := \left\{ 0, \frac{1}{N}, ..., 1 \right\}.$$

2.1 The heuristic PDE and the differential game $\mathcal{G}(t,q)$

We define the function $\Psi_N : \mathcal{I}_N \times \mathcal{Q}_N \to \mathbb{R}$, such that

$$\Psi_N(t,q) := (1-t)\mathbf{V}_n(z), \tag{2.1}$$

where $z = Nq$ and $n = N(1-t)$. Ψ_N satisfies:

$$\begin{cases} \Psi_N(t,q) = \frac{h(q)}{N} + \max_{u\in\Delta(I)} \min_{v\in\Delta(J)} \left(\sum_{i,j} u_i v_j \left(\frac{a_{ij}}{N} + \Psi_N \left(t + \frac{1}{N}, q + \frac{e_{ij}}{N} \right) \right) \right), & t \in \mathcal{I}_N \setminus \{1\} \\ \Psi_N(1,q) = 0. \end{cases} \tag{2.2}$$

The first formula of (2.2), for any $t \in \mathcal{I}_N \setminus \{1\}$ can be written equivalently:

$$0 = h(q) + \max_{u\in\Delta(I)} \min_{v\in\Delta(J)} \left(\sum_{i,j} u_i v_j \left(a_{ij} + N \left(\Psi_N \left(t + \frac{1}{N}, q + \frac{e_{ij}}{N} \right) - \Psi_N(t,q) \right) \right) \right). \tag{2.3}$$

When $N \to +\infty$, we heuristically assume that there exists a sufficiently differentiable function $\Psi : [0,1] \times \mathbb{R}_+^{I\times J} \setminus \{0\} \to \mathbb{R}$ as the limit of Ψ_N, which will therefore satisfy the following (PDE) with boundary condition of (2.2):

$$\begin{cases} \frac{\partial \Psi}{\partial t}(t,q) + h(q) + \max_{u\in\Delta(I)} \min_{v\in\Delta(J)} \sum_{i,j} u_i v_j \left(a_{ij} + \frac{\partial \Psi}{\partial q_{ij}}(t,q) \right) = 0, & (t,q) \in [0,1) \times \mathbb{R}_+^{I\times J} \setminus \{0\}, \\ \Psi(1,q) = 0, \qquad q \in \mathbb{R}_+^{I\times J} \setminus \{0\}. \end{cases} \tag{2.4}$$

The differential game. Given $(t,q) \in [0,1] \times \mathbb{R}_+^{I\times J}$, we define a differential zero-sum game, denoted by $\mathcal{G}(t,q)$ starting at time t with initial state q. It consists of:

- The state space $\mathcal{Q} = \mathbb{R}_+^{I\times J}$.
- The time interval of the game $T = [t,1]$.
- Player 1 uses a measurable control $\tilde{u} : [t,1] \to \Delta(I)$ and his control space is $\mathcal{U}_t$. Player 2 uses a measurable control $\tilde{v} : [t,1] \to \Delta(J)$ and his control space is $\mathcal{V}_t$. For $t = 0$, we use the notation $\mathcal{U} := \mathcal{U}_0$ and $\mathcal{V} := \mathcal{V}_0$.
- If Player 1 uses $\tilde{u}$ and Player 2 uses $\tilde{v}$, then the dynamics in the state space is defined as follows:

$$\begin{cases} \frac{dq}{dt}(s) = \tilde{u}(s) \otimes \tilde{v}(s), & s \in (t,1), \\ q(t) = q. \end{cases} \tag{2.5}$$

  Clearly, the dynamics is driven by a bounded, continuous function, which is Lipschitz in q and thus (2.5) admits a unique solution.

- The running-payoff at time $s \in [t,1]$ that Player 1 receives from Player 2 is given by $g : \mathcal{Q} \times \Delta(I) \times \Delta(J) \to \mathbb{R}$ and defined as:

$$g(q,u,v) = h(q) + \left\langle u \otimes v, A \right\rangle \tag{2.6}$$

where,

$$h(q) := \begin{cases} \left\langle H, \frac{q}{|q|} \right\rangle, & q \neq 0 \\ 0, & q = 0. \end{cases}$$

It is easy to see that g is bounded by $\|H\|_\infty + \|A\|_\infty$ and since $q : [t,1] \to \mathcal{Q}$ is a differentiable function of time (see (2.5)), g is differentiable on $\mathcal{Q} \setminus \{0\}$.

- The payoff associated to the pair of controls $(\tilde{u}, \tilde{v}) \in \mathcal{U}_t \times \mathcal{V}_t$ that Player 2 pays to Player 1 at time 1 is given by:

$$G(t,q,\tilde{u},\tilde{v}) = \int_t^1 g(q(s), \tilde{u}(s), \tilde{v}(s))ds. \tag{2.7}$$

Following Varaiya (1967), Roxin (1969) and Elliott and Kalton (1972), we allow the players to update their controls using non-anticipative strategies. A non-anticipative strategy for Player 1 is a map $\alpha : \mathcal{V}_t \to \mathcal{U}_t$ such that for any time $\tilde{t} > t$,

$$\tilde{v}_1(s) = \tilde{v}_2(s) \qquad \forall s \in [t,\tilde{t}] \qquad \Rightarrow \qquad \alpha[\tilde{v}_1(s)] = \alpha[\tilde{v}_2(s)] \quad \forall s \in [t,\tilde{t}].$$

The definition of non-anticipative strategies for Player 2 is analogous. Denote by $\mathcal{A}_t$ and $\mathcal{B}_t$ the sets of non-anticipative strategies of the players respectively and let us put $\mathcal{A} := \mathcal{A}_0$ and $\mathcal{B} := \mathcal{B}_0$. With respect to this notion of strategies, the lower and upper values are defined as follows:

$$\begin{aligned} W^-(t,q) &:= \sup_{\alpha \in \mathcal{A}_t} \inf_{\tilde{v} \in \mathcal{V}_t} G(t,q,\alpha[\tilde{v}],\tilde{v}) \\ W^+(t,q) &:= \inf_{\beta \in \mathcal{B}_t} \sup_{\tilde{u} \in \mathcal{U}_t} G(t,q,\tilde{u},\beta[\tilde{u}]). \end{aligned}$$

When both functions coincide, we say that the game $\mathcal{G}(t,q)$ has a *value*, denoted by $W(t,q)$.

Following Cardaliaguet (2000) and Bardi and Capuzzo-Dolcetta (2008), the lower and upper hamiltonian functions of the game $\mathcal{G}(t,q)$, $\mathcal{H}^\pm : \mathcal{Q} \times \mathcal{Q} \to \mathbb{R}$ are given by:

$$\begin{aligned} \mathcal{H}^-(\xi,q) &= h(q) + \max_{u \in \Delta(I)} \min_{v \in \Delta(J)} \left\langle u \otimes v, A + \xi \right\rangle \\ \mathcal{H}^+(\xi,q) &= h(q) + \min_{v \in \Delta(J)} \max_{u \in \Delta(I)} \left\langle u \otimes v, A + \xi \right\rangle. \end{aligned} \tag{2.8}$$

Notation. In the sequel, $\mathcal{Q}^*$ stands for $\mathcal{Q} \setminus \{0\}$.

Given $(t, q) \in [0, 1] \times \mathcal{Q}^*$, since: (i) $\Delta(I)$ and $\Delta(J)$ are compact sets; (ii) the dynamics in the state space (2.5) and the running payoff (2.6) are bounded, continuous in all their variables and Lipschitz in the state variable q functions. From the minmax theorem in von Neumann (1928), it clearly follows that the Isaacs condition, i.e., $\mathcal{H}^- = \mathcal{H}^+$ holds true (see (2.8)). Hence, by Evans and Souganidis (1984) and Souganidis (1999), the differential game $\mathcal{G}(t, q)$, starting at time $t \in [0, 1)$ with initial state $q \in \mathcal{Q}^*$ admits a value, denoted by $W(t, q)$. Moreover, the authors characterize[3] the value by means of the (DPP). Namely, for all $(t, q) \in [0, 1) \times \mathcal{Q}^*$ and all $\delta \in (0, 1 - t]$, we have:

$$W(t,q) = \sup_{\alpha \in \mathcal{A}_t} \inf_{\tilde{v} \in \mathcal{V}_t} \left\{ \left\langle H, \int_t^{t+\delta} \frac{q(s)}{|q| + s} ds \right\rangle + \left\langle \int_t^{t+\delta} \alpha[\tilde{v}(s)] \otimes \tilde{v}(s) ds, A \right\rangle + W^* \right\}, \tag{2.9}$$

where $W^* := W\big(t + \delta, q(t + \delta)\big)$ with $q(t + \delta) = q + \int_t^{t+\delta} \alpha[\tilde{v}(s)] \otimes \tilde{v}(s) ds$.

Furthermore under the preceding assumptions, $W(t, q)$ is the unique solution in the space of real-valued, bounded, continuous functions defined over $[0, 1] \times \mathcal{Q}^*$ of the following (HJBI) equation:

$$\begin{cases} \frac{\partial W}{\partial t}(t,q) + \mathcal{H}\big(\nabla_q W(t,q), q\big) = 0, & (t,q) \in [0,1) \times \mathcal{Q}^*, \\ W(1,q) = 0, \quad q \in \mathcal{Q}^*. \end{cases} \tag{2.10}$$

where, $\mathcal{H} := \mathcal{H}^- = \mathcal{H}^+$ is the hamiltonian defined earlier. Consequently, one can identify the (PDE) obtained in (2.4) with the (HJBI) equations of (2.10).

2.2 Existence of the value in $\mathcal{G}(0, 0)$

In this section, we extend the results of the differential game $\mathcal{G}(t, q)$ over the set $[0, 1] \times \mathcal{Q}$. Namely, we show that $W(0, q)$ admits a limit as q tends to 0 and we further establish that such limit is the value of the game starting at $(0, 0)$, which therefore exists. The idea of the proof lies in the consideration of approximative optimal strategies in the game $\mathcal{G}(0, q)$ and to establish that they remain approximative optimal in the game $\mathcal{G}(0, 0)$.

Lemma 2.1. *Let $q \in \mathcal{Q}^*$ and $(\tilde{u}, \tilde{v}) \in \mathcal{U} \times \mathcal{V}$. Denote by $q(\cdot)$ and $\tilde{q}(\cdot)$ the trajectories with initial conditions $q(0) = q$ and $\tilde{q}(0) = 0$ obtained from (2.5). Then, for any $s \in [0, 1]$, we have: $|h(q(s)) - h(\tilde{q}(s))| \leq 2 \|H\|_\infty \frac{|q|}{|q(s)|}$.*

Proof. If $\tilde{q}(s) = 0$, we get $h(\tilde{q}(s)) = 0$ and thus, $|h(q(s)| \leq \|H\|_\infty$. For any $s \in [0, 1]$, such that $\tilde{q}(s) \neq 0$, since $q(s) \neq 0$, it is elementary that $\left| \frac{q(s)}{|q(s)|} - \frac{\tilde{q}(s)}{|\tilde{q}(s)|} \right| \leq 2 \frac{|q(s) - \tilde{q}(s)|}{|q(s)|}$. Since the controls of the players depend only on the time variable, by the ordinary differential

[3] In the literature, the values of the minorant and majorant games have been first characterized by means of (DPP) in Elliott and Kalton (1974).

equation (2.5), for all $s \in [0,1]$, $|q(s) - \tilde{q}(s)| = |q|$. Then, we obtain $\left|\frac{q(s)}{|q(s)|} - \frac{\tilde{q}(s)}{|\tilde{q}(s)|}\right| \leq 2\,\frac{|q|}{|q(s)|}$. Hence, $|h(q(s)) - h(\tilde{q}(s))| = \left|\left\langle H, \frac{q(s)}{|q(s)|} - \frac{\tilde{q}(s)}{|\tilde{q}(s)|}\right\rangle\right| \leq 2\,\|H\|_\infty\,\frac{|q|}{|q(s)|}$. $\square$

Lemma 2.2. *For any $\varepsilon > 0$, there exists $\eta \in (0, \frac{1}{4})$, such that for any $q \in \mathcal{Q}^*$ with $|q| = \eta$ and any $(\tilde{u}, \tilde{v}) \in \mathcal{U} \times \mathcal{V}$, we have:*

$$|G(0, q, \tilde{u}, \tilde{v}) - G(0, 0, \tilde{u}, \tilde{v}| < \varepsilon. \tag{2.11}$$

Proof. Let us put $G(q) := G(0, q, \tilde{u}, \tilde{v})$ and $G(0) := G(0, 0, \tilde{u}, \tilde{v})$ and we further denote by $q(\cdot)$ and $\tilde{q}(\cdot)$ the trajectories with initial conditions $q(0) = q$ and $\tilde{q}(0) = 0$ obtained from (2.5). Then,

$$|G(q) - G(0)| = \left|\int_0^1 \big(h\left(q(s)\right) - h\left(\tilde{q}(s)\right)\big)ds\right|.$$

For all $s \in [0,1]$, it holds true that $|q(s)| = |q| + s$. By Lemma 2.1, we get:

$$\begin{aligned}
|G(q) - G(0)| \leq 2\,\|H\|_\infty \left|\int_0^1 \frac{|q|}{|q| + s}ds\right| &= 2\,\|H\|_\infty\,|q| \left|\int_0^1 \frac{ds}{|q| + s}\right| \\
&= 2\,\|H\|_\infty \left|\Big(\ln\left(1 + |q|\right) - \ln\left(|q|\right)\Big)\right| |q| \\
&= 2\,\|H\|_\infty \left|\ln\left(\frac{1 + |q|}{|q|}\right)\right| |q|\,.
\end{aligned}$$

If $|q| < \frac{1}{4}$, we claim that $\big|\ln(1+|q|)\big| \leq |\ln(|q|)|$. Indeed, since $|q| < \frac{1}{4}$, we have $|q|(1+|q|) < 1$ and thus, $1 < 1 + |q| < \frac{1}{|q|}$, so that $0 \leq |\ln(1 + |q|)| < |\ln(|q|)|$. Then, for any $q \in \mathcal{Q}^*$ such that $|q| < \frac{1}{4}$, we have:

$$|G(q) - G(0)| \leq 4\,\|H\|_\infty \big|\ln(|q|)\big||q|.$$

To conclude, for any $\varepsilon > 0$, choose $\eta \in (0, \frac{1}{4})$, such that $\eta|\ln(\eta)| < \frac{\varepsilon}{4\|H\|_\infty + 1}$ and the result is immediate. $\square$

Theorem 2.3. *The game $\mathcal{G}(0, 0)$ has a value, $W(0, 0) = \lim\limits_{q \to 0} W(0, q)$.*

Proof. Let $\varepsilon > 0$ and fix $\eta \in (0, \frac{1}{4})$, such that $\eta|\ln(\eta)| < \frac{\varepsilon}{4(4\|H\|_\infty + 1)}$. Since the game $\mathcal{G}(0, q)$ admits a value for any $q \in \mathcal{Q}^*$, consider an $\frac{\varepsilon}{4}$-optimal non anticipative strategy for Player 1 in $\mathcal{G}(0, q)$, where $|q| = \eta$, i.e., a measurable function $\alpha(\cdot)$, such that for all $s \in [0, 1]$, $\alpha[\tilde{v}(s)] \in \mathcal{U}$. Then, for any $\tilde{v} \in \mathcal{V}$, we have:

$$G(0, 0, \alpha[\tilde{v}], \tilde{v}) > G(0, q, \alpha[\tilde{v}], \tilde{v}) - \frac{\varepsilon}{4} > W^-(0, q) - \frac{\varepsilon}{4} - \frac{\varepsilon}{4}$$

where the first inequality follows by Lemma 2.2 and the second inequality is due to $\alpha(\cdot)$ is an $\frac{\varepsilon}{4}$-optimal strategy. Reversing the roles of the players and following similar arguments we get that for any $\tilde{u} \in \mathcal{U}$, $G(0,0,\tilde{u},\beta[\tilde{u}]) < G(0,q,\tilde{u},\beta[\tilde{u}]) + \frac{\varepsilon}{4} < W^+(0,q) + \frac{\varepsilon}{4} + \frac{\varepsilon}{4}$, where $\beta(\cdot)$ is an $\frac{\varepsilon}{4}$-optimal non anticipative strategy of Player 2 in $\mathcal{G}(0,q)$. Since the value exists in $\mathcal{G}(0,q)$ for any $q \in \mathcal{Q}^*$, we have:

$$
\begin{aligned}
W^-(0,0) &= \sup_{\alpha \in \mathcal{A}} \inf_{\tilde{v} \in \mathcal{V}} G(0,0,\alpha[\tilde{v}],\tilde{v}) > W(0,q) - \frac{\varepsilon}{2} \\
W^+(0,0) &= \inf_{\beta \in \mathcal{B}} \sup_{\tilde{u} \in \mathcal{U}} G(0,0,\tilde{u},\beta[\tilde{u}]) < W(0,q) + \frac{\varepsilon}{2}
\end{aligned}
$$

and we therefore get $|W^-(0,0) - W^+(0,0)| < \varepsilon$, which proves existence of the value in $\mathcal{G}(0,0)$ since the inequality holds true for any positive ε. $\square$

2.3 The discretized game $\mathcal{G}_{\mathcal{P}}(t_0, q_0)$

In this section, we prove a coincidence result between an auxiliary function related to the value of the original game and the value of the associated differential game in which players are allowed to choose their actions only on the nodes of a suitable partition of the time interval. To that purpose, we next consider subdivisions of $[0,1]$ and we define a family of discretized games played on them.

- For all $t_0 \in [0,1)$, $\mathcal{P}$ stands for any countable subdivision of $[t_0,1]$ and if $\mathcal{P}$ is finite let $\omega_{\mathcal{P}}$ denote the number of intervals of such subdivision, otherwise we put $\omega_{\mathcal{P}} = \infty$.
- Given $N \in \mathbb{N}^*$, $\mathcal{P}_N = (t_k^N)_{0 \le k \le N}$, where $t_k^N := \frac{k}{N}$ stands for the uniform subdivision of $[0,1]$ in N intervals. We will also use the notation $\mathcal{P}_N = \left(t_n^N\right)_{0 \le n \le N}$ for $n = N - k$.
- Given $\lambda \in (0,1)$, $\mathcal{P}_\lambda = (t_k^\lambda)_{k \ge 0}$ stands for the induced by the discount factor λ, countable subdivision of $[0,1]$, such that $t_0^\lambda = 0$, $t_1^\lambda = \lambda$, $t_k^\lambda := \lambda + \ldots + \lambda(1-\lambda)^{k-1}$, for $k \ge 1$ and $t_\infty^\lambda = 1$.
- By $\pi_k := t_{k+1} - t_k$ is denoted the k-th increament and $|\mathcal{P}|$ stands for the mesh of the subdivision $\mathcal{P}$, i.e., $|\mathcal{P}| = \sup_k |\pi_k|$.

Given $\mathcal{P}$, for all $(t_0, q_0) \in [0,1) \times \mathcal{Q}$ we associate to $\mathcal{G}(t_0, q_0)$ a discrete time game adapted to the subdivision $\mathcal{P}$ denoted by $\mathcal{G}_{\mathcal{P}}(t_0, q_0)$. Such discrete time game starts at time t_0, has initial state $q_0 \in \mathcal{Q}$ and is repeated $\omega_{\mathcal{P}}$ times. At time $t_k \in \mathcal{P}$, both players observe the current state q_k and choose simultaneously and independently actions u_{k+1} and v_{k+1} in $\Delta(I)$ and $\Delta(J)$ respectively. The control sets are denoted by $\Delta(I)^{\omega_{\mathcal{P}}}$ and $\Delta(J)^{\omega_{\mathcal{P}}}$, indicating that players now choose piecewise constant functions defined over the $\omega_{\mathcal{P}}$-times cartesian product of their corresponding mixed strategy sets. We will use the notation $\hat{u} = (u_k)_{k=1}^{\omega_{\mathcal{P}}}$ and $\hat{v} = (v_k)_{k=1}^{\omega_{\mathcal{P}}}$. The state evolves according to:

$$
\begin{cases}
q_{k+1} = q_k + \pi_k u_{k+1} \otimes v_{k+1}, & k \ge 0, \\
q_0 = q.
\end{cases}
$$

At stage k, the expected payoff that Player 1 receives from Player 2 is given by:

$$g(q_{k-1}, u_k, v_k) = h(q_{k-1}) + u_k A v_k \tag{2.12}$$

and given $(\hat{u}, \hat{v}) \in \Delta(I)^{h_{\mathcal{P}}} \times \Delta(J)^{h_{\mathcal{P}}}$, the total payoff of the game is

$$G_{\mathcal{P}}(q_0, \hat{u}, \hat{v}) = \sum_{k=1}^{h_{\mathcal{P}}} \pi_{k-1} g(q_{k-1}, u_k, v_k). \tag{2.13}$$

Given $\mathcal{P}$, for all $(t_0, q_0) \in [0,1) \times \mathcal{Q}$, the game $\mathcal{G}_{\mathcal{P}}(t_0, q_0)$ admits a value. Following Friedman (1970), the value of the game denoted by $W_{\mathcal{P}}(t_0, q_0)$ is characterized by means of discrete version of the (HJBI) equations (2.10):

$$\begin{cases} W_{\mathcal{P}}(t_k, q_k) = \pi_k h(q) + \max_{u \in \Delta(I)} \min_{v \in \Delta(J)} \left(\langle \pi_k u \otimes v, A \rangle + W_{\mathcal{P}}\left(t_{k+1}, q_k + \pi_k u \otimes v\right) \right), \\ W_{\mathcal{P}}(1, q) = 0 \end{cases} \tag{2.14}$$

We will refer to this equation as the *discrete* Dynamic Programing Principle that will be abbreviated to (discrete DPP).

In what follows we compare the similar recursive formulas of $W_{\mathcal{P}}$ with Ψ_N and Ψ_λ (see 2.20) for discretizations $\mathcal{P}_N$ and $\mathcal{P}_\lambda$ respectively. The difference between them is essentially that in the discretized game $\mathcal{G}_{\mathcal{P}}(t_0, q_0)$, the play generated by pure strategies is deterministic and lives in $\mathbb{R}^{I \times J}$, whereas the play generated in the original game $\Gamma_N(z_0)$ (resp. $\Gamma_\lambda(z_0)$) in behavioral strategies is random and takes its values in a discrete subset of $\mathbb{R}^{I \times J}$.

2.4 Coincidence of Ψ_N and $W_{\mathcal{P}_N}$

We first prove that Ψ_N preserves a very similar property to the one satisfied by $\mathbf{V}_n$ in Proposition 1.1 and we then show $\Psi_N = W_{\mathcal{P}_N}$. Recall that the map $\Psi_N : \mathcal{P}_N \times \mathcal{Q}_N \to \mathbb{R}$ has been defined by (2.1) and is characterized by the recursive formula (2.2). It is then clear that an extension of Ψ_N to a map $\Psi_N : \mathcal{P}_N \times \mathcal{Q} \to \mathbb{R}$ is obtained if we define it by the same recursive formula and terminal condition; namely:

$$\begin{cases} \Psi_N\left(t_k^N, q\right) = \frac{h(q)}{N} + \max_{u \in \Delta(I)} \min_{v \in \Delta(J)} \left(\sum_{i,j} u_i v_j \left(\frac{a_{ij}}{N} + \Psi_N\left(t_{k+1}^N, q + \frac{e_{ij}}{N}\right)\right)\right), & 0 \le k \le N-1, q \in \mathcal{Q}, \\ \Psi_N(1, q) = 0, \quad k = N, \ q \in \mathcal{Q}. \end{cases} \tag{2.15}$$

Proposition 2.4. *Let $N \in \mathbb{N}^*$. There exists a collection $(k_{n,t}, c_{n,t}) \in \mathcal{M}^{I \times J} \times \mathbb{R}$ where $n \in \{0, ..., N\}, t \in \mathbb{R}_+$ such that for all $q \in \mathcal{Q}$, and $n \in \{0, ..., N\}$:*

$$\Psi_N(t_n^N, q) = \langle k_{n,|q|}, q \rangle + c_{n,|q|}. \tag{2.16}$$

The general terms of the sequence $(k_{n,t})$ are given for $t \in \mathbb{R}_+^*$ by:

$$k_{n,t} = \Lambda_{N-n}(Nt)H$$

Proof. For $t = 0$ we take by convention $k_{n,0} = 0$ for all $n \in \{0, ..., N\}$. Let $q \in \mathcal{Q}^*$, we proceed by backward induction on the variable n:
For $n = N$, $\Psi_N(1, q) = 0$ for any $q \in \mathcal{Q}^*$ and thus, one can take $k_{N,t} = 0$ and $c_{N,t} = 0$ for all $t > 0$. Assume the result is true for $n = m$, i.e., for all $q \in \mathcal{Q}^*$, there exist $k_{m,t} \in \mathcal{M}^{I\times J}$ and $c_{m,t} \in \mathbb{R}$, such that (2.16) is satisfied. For $n = m$, for all $q \in \mathcal{Q}^*$, we get from (2.15):

$$\begin{aligned}\Psi\left(t_m^N, q\right) &= \left\langle \frac{H}{N}, \frac{q}{|q|}\right\rangle + \max_{u\in\Delta(I)} \min_{v\in\Delta(J)} \left(\sum_{i,j} u_i v_j \left(\frac{a_{ij}}{N} + \left\langle k_{m+1,|q|+\frac{1}{N}}, q + \frac{e_{ij}}{N}\right\rangle + c_{m+1,|q|+\frac{1}{N}}\right)\right) \\ &= \left\langle \frac{H}{N|q|} + k_{m+1,|q|+\frac{1}{N}}, q\right\rangle + \max_{u\in\Delta(I)} \min_{v\in\Delta(J)} \left(\sum_{i,j} u_i v_j \left(\frac{a_{ij}}{N} + \left\langle k_{m+1,|q|+\frac{1}{N}}, \frac{e_{ij}}{N}\right\rangle\right)\right) + c_{m+1,|q|+\frac{1}{N}}.\end{aligned}$$

and thus (2.16) is satisfied if we put:

$$k_{m,t} = \tfrac{H}{Nt} + k_{m+1,t+\frac{1}{N}}$$
$$c_{m,t} = \tfrac{1}{N}\mathbf{val}\left(A + k_{m+1,t+\frac{1}{N}}\right) + c_{m+1,t+\frac{1}{N}}.$$

This ends the induction. □

Notation. Given $N \in \mathbb{N}^*$ and $q_0 \in \mathcal{Q}$, for all $t \in \mathcal{P}_N$, we define the subset of $\mathcal{Q}$:

$$\mathcal{Q}_N(t, q_0) = \{q \in \mathcal{Q} : |q| = |q_0| + t\}.$$

Proposition 2.5. *Given $N \in \mathbb{N}^*$ and $q_0 \in \mathcal{Q}$, for all $t \in \mathcal{P}_N$ and all $q \in \mathcal{Q}_N(t, q_0)$,*

$$\Psi_N(t, q) = W_{\mathcal{P}_N}(t, q).$$

Proof. Both functions share the same terminal condition, i.e $\Psi_N(1, q) = W_{\mathcal{P}_N}(1, q) = 0$, for all $q \in \mathcal{Q}$, (see (2.1), and characterization of $W_{\mathcal{P}_N}$ in terms of (discrete DPP)). Thus, it suffices to prove that Ψ_N and $W_{\mathcal{P}_N}$ satisfy the same recursive formula. To that purpose, fix $q_0 \in \mathcal{Q}$ and time $t = \frac{k}{N}$, where $k \in \{0, ..., N - 1\}$. From the (discrete (DPP)), it follows that for all $q \in \mathcal{Q}_N\left(\frac{k}{N}, q_0\right)$,

$$W_{\mathcal{P}_N}\left(\frac{k}{N}, q\right) = \frac{h(q)}{N} + \max_{u\in\Delta(I)} \min_{v\in\Delta(J)} \left(\left\langle \frac{u\otimes v}{N}, A\right\rangle + W_{\mathcal{P}_N}\left(\frac{k+1}{N}, q + \frac{u\otimes v}{N}\right)\right) \tag{2.17}$$

By (2.15), for any $k \in \{0, ..., N - 1\}$,

$$\Psi_N\left(\frac{k}{N}, q\right) = \frac{h(q)}{N} + \max_{u\in\Delta(I)} \min_{v\in\Delta(J)} \left(\sum_{ij} u_i v_j \left(\frac{a_{ij}}{N} + \Psi_N\left(\frac{k+1}{N}, q + \frac{e_{ij}}{N}\right)\right)\right),$$

where $q \in \mathcal{Q}_N\left(\frac{k}{N}, q_0\right)$. Equivalently:

$$\Psi_N\left(\frac{k}{N}, q\right) = \frac{h(q)}{N} + \max_{u\in\Delta(I)} \min_{v\in\Delta(J)} \left(\sum_{ij} u_i v_j \frac{a_{ij}}{N} + \sum_{ij} u_i v_j \Psi_n\left(\frac{k+1}{N}, q + \frac{e_{ij}}{N}\right)\right).$$

By Proposition 2.4, Ψ_N is affine in the state variable q and it thus follows:

$$\Psi_N\left(\frac{k}{N}, q\right) = \frac{h(q)}{N} + \max_{u\in\Delta(I)} \min_{v\in\Delta(J)} \left(\sum_{ij} u_i v_j \frac{a_{ij}}{N} + \Psi_n\left(\frac{k+1}{N}, \sum_{ij} u_i v_j \left(q + \frac{e_{ij}}{N}\right)\right)\right).$$

Hence, due to $\sum_{ij} u_i v_j = 1$, we get:

$$\Psi_N\left(\frac{k}{N}, q\right) = \frac{h(q)}{N} + \max_{u\in\Delta(I)} \min_{v\in\Delta(J)} \left(\left\langle \frac{u\otimes v}{N}, A\right\rangle + \Psi_N\left(\frac{k+1}{N}, q + \frac{u\otimes v}{N}\right)\right).$$

This in view of (2.17), proves that $\Psi_N = W_{\mathcal{P}_N}$. □

By Contou-Carrère (2011), the convergence is not uniform in the state variable and thus, we cannot use the Tauberian theorem of Ziliotto (2016) to obtain an immediate result on the convergence of the λ-discounted value, when λ tends to 0.

2.5 Coincidence of Ψ_λ with $W_{\mathcal{P}_\lambda}$

In the case of the n-stage value, coincidence between Ψ_N and $W_{\mathcal{P}_N}$ holds true since both functions admit the terminal value 0 and satisfy the same recursive equation due to the affine property of Ψ_N. To compare the value of the λ-discounted game and $W_{\mathcal{P}_\lambda}$, we follow a slightly different approach to show such coincidence since we deal with an infinite time horizon game.

Recall that the map $\mathbf{V}_\lambda : \mathcal{Z} \to \mathbb{R}$ has been characterised by the recursive formula (1.2). It is then clear that an extension of $\mathbf{V}_\lambda$ to a map $\mathbf{V}_\lambda : \mathcal{Q} \to \mathbb{R}$ is obtained if we define it by the same recursive formula; namely, for any $q \in \mathcal{Q}$,

$$\mathbf{V}_\lambda(q) = \lambda h(q) + \max_{u\in\Delta(I)} \min_{v\in\Delta(J)} \left(\sum_{i,j} u_i v_j \left(\lambda a_{ij} + (1-\lambda)\mathbf{V}_\lambda(q + e_{ij})\right)\right) \tag{2.18}$$

Given $\lambda \in (0,1)$, let us define the function $\Psi_\lambda : \mathcal{Q} \to \mathbb{R}$, such that

$$\Psi_\lambda(q) := \mathbf{V}_\lambda\left(\frac{q}{\lambda}\right). \tag{2.19}$$

By (2.18), Ψ_λ satisfies the following equation:

$$\Psi_\lambda(q) = \lambda h(q) + \max_{u\in\Delta(I)} \min_{v\in\Delta(J)} \left(\sum_{i,j} u_i v_j \left(\lambda a_{ij} + (1-\lambda)\Psi_\lambda(q + \lambda e_{ij})\right)\right) \tag{2.20}$$

By 2.14, for any $q \in \mathcal{Q}$, we get:

$$\begin{aligned} W_{\mathcal{P}_\lambda}(0,q) &= \lambda h(q) + \max_{u\in\Delta(I)} \min_{v\in\Delta(J)} \left\{ \lambda \sum_{ij} u_i v_j a_{ij} + W_{\mathcal{P}_\lambda}\left(\lambda, q+\lambda u\otimes v\right) \right\} \\ &= \lambda h(q) + \max_{u\in\Delta(I)} \min_{v\in\Delta(J)} \left\{ \lambda \sum_{ij} u_i v_j a_{ij} + (1-\lambda) W_{\mathcal{P}_\lambda}\left(0, q+\lambda u\otimes v\right) \right\}, \end{aligned} \tag{2.21}$$

where the last equation follows by stationarity of the discounted game $\mathcal{G}_{\mathcal{P}_\lambda}(t,q)$. In the sequel, we put $W_{\mathcal{P}_\lambda}(q) = W_{\mathcal{P}_\lambda}(0,q)$.

Notation. We use the following notations:

- The norm $\|\cdot\|_\infty$ of a bounded real-valued function f, defined on $\mathcal{Q}$, is

$$\|f\|_\infty = \sup_{q\in\mathcal{Q}} |f(q)|.$$

- Given $\lambda \in (0,1]$, for any $t \in \mathcal{P}_\lambda$, we define the following subset of $\mathcal{Q}$:

$$\mathcal{Q}_\lambda(t) = \{q \in \mathcal{Q} : |q| = t\}$$

- $\mathcal{F}_\mathcal{B}$ stands for the set of real-valued bounded functions f defined on $\mathcal{Q}$ with the norm $\|\cdot\|_\infty$. Clearly, $\mathcal{F}_\mathcal{B}$ is a Banach space.
- $\mathcal{L}_\mathcal{B}$ stands for the subspace of $\mathcal{F}_\mathcal{B}$, such that if $f \in \mathcal{L}_\mathcal{B}$ then there exist $K : \mathbb{R} \to \mathbb{R}^{I\times J}$ and $c : \mathbb{R} \to \mathbb{R}$, such that $f(q) = \langle K(|q|), q\rangle + c(|q|)$, for any $q \in \mathcal{Q}$.
- For all $f \in \mathcal{F}_\mathcal{B}$ and any $\lambda \in (0,1)$, we define the operator Θ_λ as follows:

$$\Theta_\lambda(f)(q) = \lambda\, h(q) + \max_{u\in\Delta(I)} \min_{v\in\Delta(J)} \left(\lambda \sum_{ij} u_i v_j a_{ij} + (1-\lambda) f(q+\lambda u\otimes v) \right) \tag{2.22}$$

  and clearly Θ_λ admits $W_{\mathcal{P}_\lambda}(\cdot)$ as unique fixed point.
- For all $f \in \mathcal{F}_\mathcal{B}$ and any $\lambda \in (0,1)$, we define the operator $\mathbf{T}_\lambda$ as follows:

$$\mathbf{T}_\lambda(f)(q) = \lambda\, h(q) + \max_{u\in\Delta(I)} \min_{v\in\Delta(J)} \left(\sum_{ij} u_i v_j \big(\lambda a_{ij} + (1-\lambda) f(q+\lambda e_{ij})\big) \right). \tag{2.23}$$

  Likewise, $\mathbf{T}_\lambda$ admits $\Psi_\lambda(\cdot)$ as unique fixed point.

Proposition 2.6. *If $f \in \mathcal{L}_\mathcal{B}$, then $\boldsymbol{T}_\lambda(f) \in \mathcal{L}_\mathcal{B}$ and $\boldsymbol{T}_\lambda(f) = \Theta_\lambda(f)$.*

Proof. Let $f \in \mathcal{L}_{\mathcal{B}}$. By definition of the operator $\mathbf{T}_\lambda$, it is easy to see that $\mathbf{T}_\lambda(f)$ is also bounded. For the rest of the proof, let $\lambda \in (0,1)$ and fix $t \in \mathcal{P}_\lambda$. Then, for any $q \in \mathcal{Q}_\lambda(t)$,

$$\mathbf{T}_\lambda(f)(q) = \lambda\, h(q) + \max_{u\in\Delta(I)} \min_{v\in\Delta(J)} \left(\sum_{ij} u_i v_j \big(\lambda a_{ij} + (1-\lambda) f(q+\lambda e_{ij})\big) \right)$$

$$= \lambda \left\langle H, \frac{q}{|q|} \right\rangle + \max_{u\in\Delta(I)} \min_{v\in\Delta(J)} \left(\sum_{ij} u_i v_j \big(\lambda a_{ij} + (1-\lambda)\left(\langle K(|q|+\lambda), q+\lambda e_{ij}\rangle + c(|q|+\lambda)\right)\big) \right).$$

Hence,

$$\mathbf{T}_\lambda(f)(q) = \left\langle \frac{\lambda H}{|q|} + (1-\lambda) K(|q|+\lambda), q \right\rangle +$$
$$+ \max_{u\in\Delta(I)} \min_{v\in\Delta(J)} \left(\sum_{ij} u_i v_j \left(\lambda a_{ij} + (1-\lambda)\left(\langle K(|q|+\lambda), \lambda e_{ij}\rangle + c(|q|+\lambda)\right)\right) \right).$$

It is easy to see that $\mathbf{T}_\lambda(f)(q) = \langle K(|q|), q\rangle + c(|q|)$, where

$$\begin{cases} K(|q|) = \frac{\lambda H}{|q|} + (1-\lambda) K(|q|+\lambda) \in \mathcal{M}^{I\times J} \\ c(|q|) = \max\limits_{u\in\Delta(I)} \min\limits_{v\in\Delta(J)} \left(\sum_{ij} u_i v_j \left(\lambda a_{ij} + (1-\lambda)\left(\langle K(|q|+\lambda), \lambda e_{ij}\rangle + c(|q|+\lambda)\right)\right) \right) \in \mathbb{R} \end{cases}$$

and it thus, clearly follows that $\mathbf{T}_\lambda(f) \in \mathcal{L}_{\mathcal{B}}$. By equations (2.22), (2.23), we conclude the proof of the Proposition. □

Corollary 2.7. *Fix $\lambda \in (0,1]$. Then, for any $q \in \mathcal{Q}$, we have $\Psi_\lambda(q) = W_{\mathcal{P}_\lambda}(q)$.*

Proof. By Proposition 2.6, since $f \equiv 0 \in \mathcal{L}_{\mathcal{B}}$, we get that for any $m \in \mathbb{N}^*$, $\mathbf{T}^m(0) = \Theta^m(0)$. It is easy to see that $\mathbf{T}_\lambda$ and Θ_λ are contracting operators and it thus follows $\lim_{m\to+\infty} \mathbf{T}_\lambda^m(0) = \lim_{m\to+\infty} \Theta_\lambda^m(0)$. In view of equations (2.22) and (2.23), we conclude that $\Psi_\lambda = W_{\mathcal{P}_\lambda}$. □

This result will allow us to use approximation schemes for differential games in the subsequent parts of the proof. Since the value of the original game is equal to that of the discretized approximated game, proving convergence of the value of the latter will prove the convergence of the value of the former. One difficulty arises however in applying approximation schemes, which lies in the irregularity of the differential game at the origin.

3 Existence of the limit value in $\Gamma_N(z)$

In this section, we provide the main result of the paper. We show that the asymptotic values in the N-stage and λ-discounted games exist and they are independent of the initial state z. We further prove that $\lim_{N\to+\infty} \mathbf{V}_N(z) = \lim_{\lambda\to 0} \mathbf{V}_\lambda(z) = W(0,0)$. We first provide some useful lemmas on the value of the original game $\mathbf{V}_N$ (resp. $\mathbf{V}_\lambda$) and the associated function Ψ_N (resp. Ψ_λ).

Lemma 3.1. *Let $\omega = (i_t, j_t)_{t\in\mathbb{N}^*}$ be a play and $(z_t)_{t\in\mathbb{N}}$ be the induced, by the initial position $z_0 = 0$, process in $\mathcal{Z}$. Then, for any $z \in \mathcal{Z}^*$, we have:*

- *for any $N \in \mathbb{N}^*$,*

$$|\gamma_N(z,\omega) - \gamma_N(0,\omega)| \leq \frac{2\,\|H\|_\infty\,|z|}{N}\left(\ln\left(\frac{|z|+N}{|z|}\right) + C\right)$$

  where $C = 2\sup_{N\in\mathbb{N}^*} \varepsilon(N)$ and $\varepsilon(N)$ is a function which goes to zero when N tends to infinity.

- *for any $\lambda \in (0,1]$, such that $\lambda z < 1$, there exists $N \in \mathbb{N}^*$, i.e., $N + |z| = \lfloor \frac{1}{\lambda} \rfloor$, such that we have:*

$$|\gamma_\lambda(z,\omega) - \gamma_\lambda(0,\omega)| \leq 2\,\|H\|_\infty\,\lambda|z|\left(\ln\left(\frac{1}{\lambda|z|}\right) + C^z_\lambda\right),$$

  where $C^z_\lambda = (1-\lambda)^{\lfloor\frac{1}{\lambda}\rfloor - |z|} + C$.

Proof. Fix $z \in \mathcal{Z}$ and $N \in \mathbb{N}^*$. Then, we have:

$$\begin{aligned}
\left|\sum_{t=1}^{N} h(z_t) - h(z+z_t)\right| &= \left|\left\langle H, \frac{z}{|z|}\right\rangle + \sum_{t=2}^{N}\left\langle H, \frac{z_t}{t} - \frac{z+z_t}{|z|+t}\right\rangle\right| \\
&= \left|\left\langle H, \frac{z}{|z|}\right\rangle + \sum_{t=2}^{N}\left\langle H, \frac{z_t(|z|+t) - t(z+z_t)}{t(|z|+t)}\right\rangle\right| \\
&\leq \left|\left\langle H, \frac{z}{|z|}\right\rangle\right| + \sum_{t=2}^{N}\left|\left\langle H, \frac{z_t\,|z| - tz}{t(|z|+t)}\right\rangle\right| \\
&\leq \|H\|_\infty\left(1 + \sum_{t=2}^{N}\frac{|z_t|\,|z| + t\,|z|}{t(|z|+t)}\right) \\
&\leq \|H\|_\infty\left(2|z|\sum_{t=1}^{N}\frac{1}{|z|+t}\right) \\
&\leq 2\,\|H\|_\infty\,|z|\left(\ln\left(\frac{|z|+N}{|z|}\right) + |\varepsilon(|z|+N)| + |\varepsilon(|z|)|\right),
\end{aligned}$$

where $\varepsilon(x)$ is a function that goes to 0 when x tends to infinity. We put $C := 2\sup_{N\in\mathbb{N}^*}\varepsilon(N)$ and we thus, conclude the proof of the assertion. To show the second assertion, in a similar way, we obtain:

$$|\gamma_\lambda(0) - \gamma_\lambda(z)| \le 2\lambda|z|\,\|H\|_\infty \sum_{t=1}^{\infty} \frac{(1-\lambda)^{t-1}}{|z|+t}$$

Define $N := \inf\{k \in \mathbb{N}^* : k + |z| + 1 > 1/\lambda\}$. Hence, we may write:

$$\begin{aligned}
|\gamma_\lambda(0) - \gamma_\lambda(z)| &\le 2\,\|H\|_\infty\,\lambda|z| \left(\sum_{t=1}^{N} \frac{1}{|z|+k} + (1-\lambda)^N \sum_{k=1}^{\infty} \lambda(1-\lambda)^k \right) \\
&\le 2\,\|H\|_\infty\,\lambda|z| \left(\ln\left(\frac{|z|+N}{|z|}\right) + |\varepsilon(|z|+N)| + |\varepsilon(|z|)| + (1-\lambda)^N \right) \\
&\le 2\,\|H\|_\infty\,\lambda|z| \left(\ln\left(\frac{1}{\lambda|z|}\right) + C + (1-\lambda)^{\lfloor \frac{1}{\lambda} \rfloor - |z|} \right),
\end{aligned}$$

which ends the proof of the Lemma. □

Proposition 3.2. *Let $z \in \mathcal{Z}$. If $\boldsymbol{V}_N(z)$ converges to some $\ell \in \mathbb{R}$ (resp. $\boldsymbol{V}_\lambda(z)$), then for any $\tilde{z} \in \mathcal{Z}$, $\boldsymbol{V}_N(\tilde{z})$ (resp. $\boldsymbol{V}_\lambda(\tilde{z})$) converges to the same limit ℓ.*

Proof. Given $N \in \mathbb{N}^*$ ($\lambda \in (0,1)$), fix $z \in \mathcal{Z}^*$ and let us consider the games $\Gamma_N(z)$ (resp. $\Gamma_\lambda(z)$) and $\Gamma_N(0)$ (resp. $\Gamma_\lambda(0)$). For a pair of behavioral strategies (σ,τ) we denote by $\mathbb{P}_{\sigma,\tau}$ the probability induced on $(I\times J)^N$, (resp. $(I\times J)^\infty$). Passing to expectations with respect to the probability $\mathbb{P}_{\sigma,\tau}$, by Lemma 3.1, we get:

$$\left|\gamma_N(0,\sigma,\tau) - \gamma_N(z,\sigma,\tau)\right| \le \frac{1}{N}\left(2\,\|H\|_\infty\,|z| \left(\ln\left(\frac{|z|+N}{|z|}\right) + C\right)\right).$$

Since the right hand term is independent of (σ,τ), by an easy argument:

$$\left|\mathbf{V}_N(z) - \mathbf{V}_N(0)\right| \le \frac{1}{N}\left(2\,\|H\|_\infty\,|z| \left(\ln\left(\frac{|z|+N}{|z|}\right) + C\right)\right).$$

The conclusion follows by remarking that the right hand side goes to zero when $N\to\infty$. For the rest of the proof, fix $\lambda \in (0,1)$. Likewise, by Lemma 3.1, we get:

$$|\gamma_\lambda(0,\sigma,\tau) - \gamma_\lambda(z,\sigma,\tau)| \le 2\,\|H\|_\infty\,\lambda|z| \left(\ln\left(\frac{1}{\lambda|z|}\right) + C_\lambda^z\right).$$

Since the right hand term is independent of (σ,τ), by an easy argument:

$$|\mathbf{V}_\lambda(z) - \mathbf{V}_\lambda(0)| \le 2\,\|H\|_\infty\,\lambda|z| \left(\ln\left(\frac{1}{\lambda|z|}\right) + C_\lambda^z\right).$$

Note that C_λ^z converges to $1/e$, as λ tends to 0. The result follows since the right hand side goes to zero when λ tends to zero. □

Next theorem provides the main result of the paper. Given $z \in \mathcal{Z}$, we first show that $\mathbf{V}_N(z)$ converges to $W(0,0)$ when N tends to infinity and we then prove that $\mathbf{V}_\lambda(z)$ converges to the same limit, when λ goes to 0.

Theorem 3.3. *For any $z \in \mathcal{Z}$, $\lim\limits_{N \to +\infty} \boldsymbol{V}_N(z) = \lim\limits_{\lambda \to 0} \boldsymbol{V}_\lambda(z) = W(0,0)$.*

Proof. We first prove that for any $z \in \mathcal{Z}$, $\lim_{N\to+\infty} \mathbf{V}_N(z) = W(0,0)$. To that purpose, fix $\varepsilon > 0$, choose $\eta \in (0, \frac{1}{4})$ such that,

$$\begin{cases} \eta < \frac{\varepsilon}{12(\|H\|_\infty(\ln(2)+C))} \\ \eta \ln(\eta) < \frac{\varepsilon}{12\|H\|_\infty + 1} \end{cases}$$

and in view of Theorem 2.3, we also require η to be such that for all $q \in \mathcal{Q}^*$ with $|q| \leq \eta$, $|W(0,q) - W(0,0)| < \frac{\varepsilon}{3}$. Choose some q_0 such that $|q_0| = \eta$. Assumptions on the strategy sets, the dynamics and running payoff functions of Theorem 4.4 in Souganidis (1999) are established in $\mathcal{G}\big(0, q_0\big)$. Accordingly, there exists $\delta > 0$, such that for all $|\mathcal{P}| < \delta$, the value $W_\mathcal{P}$ converges uniformly on every compact set of $\mathcal{Q}$ to W, as the mesh of the discretization $|\mathcal{P}|$ tends to 0. Fix $N_0 = \lfloor \frac{1}{\delta} \rfloor + 1$ and associate to $\mathcal{G}\big(0, q_0\big)$, for all $N \geq N_0$, a discrete time game adapted to the subdivision $\mathcal{P}_N$, denoted by $\mathcal{G}_{\mathcal{P}_N}\big(0, q_0\big)$. Then, $\big|W_{\mathcal{P}_N}(0, q_0) - W(0, q_0)\big| < \frac{\varepsilon}{3}$. From Proposition 2.5, $W_{\mathcal{P}_N}(0, q_0) = \Psi_N(0, q_0)$. By Lemma 3.1, for any $z \in \mathcal{Z}$, we have:

$$|\mathbf{V}_N(z) - \mathbf{V}_N(0)| \leq \frac{2\,\|H\|_\infty}{N} \left(|z| \left(\ln\left(\frac{|z| + N}{|z|} \right) + C \right) \right).$$

There exists $z_0 \in \mathcal{Z}$, such that $z_0 = \lfloor N q_0 \rfloor$ and since $|q_0| = \eta$, we have $|z_0| = N\eta - \rho$ for some $\rho \in (0,1)$. By definition $\Psi_N(0, q_0) = \mathbf{V}_N(\lfloor N q_0 \rfloor)$. Hence,

$$\big|\Psi_N(0, q_0) - \Psi_N(0,0)\big| \leq 2\,\|H\|_\infty \left(\eta - \frac{\rho}{N}\right) \left| \ln\left(\frac{\eta + 1 - \frac{\rho}{N}}{\eta - \frac{\rho}{N}} \right) + C \right|.$$

Since $\eta < \frac{1}{4}$, we have $\ln(1 + \eta - \rho/N) < \ln(2)$ and $(\eta - \rho/N)|\ln(\eta - \rho/N)| < \eta|\ln(\eta)|$. As a consequence, we get:

$$\big|\Psi_N(0, q_0) - \Psi_N(0,0)\big| \leq 2\eta\,\|H\|_\infty \left(\ln(2) + C + |\ln(\eta)|\right) < \frac{\varepsilon}{3}.$$

Therefore, for every integer $N \geq N_0$,

$$\begin{aligned} \big|\Psi_N(0,0) - W(0,0)\big| &\leq \big|\Psi_N(0,0) - \Psi_N(0, q_0)\big| + \big|\Psi_N(0, q_0) - W(0, q_0)\big| + \big|W(0, q_0) - W(0,0)\big| \\ &< \varepsilon. \end{aligned}$$

From (2.1), $\Psi_N(0,0) = \mathbf{V}_N(0)$. It follows that $\mathbf{V}_N(0) \to W(0,0)$ when $N \to \infty$. In view of Lemma 3.2, we conclude that for any $z \in \mathcal{Z}$, $\mathbf{V}_N(z)$ converges to $W(0,0)$ as N tends to infinity.

For the rest of the proof, fix $\varepsilon > 0$ and choose $\eta > 0$ such that we have:

$$\begin{cases} \eta < \frac{\varepsilon}{12(\|H\|_\infty + C_\lambda^z)} \\ \eta \ln(\eta) < \frac{\varepsilon}{12\|H\|_\infty + 1} \end{cases}$$

and in view of Theorem 2.3, we also require η to be such that for all $q \in \mathcal{Q}^*$ with $|q| \leq \eta$, $|W(0,q) - W(0,0)| < \frac{\varepsilon}{3}$. Likewise, choose some q_0, such that $|q_0| = \eta$. Since assumptions of Theorem 4.4 in Souganidis (1999) are established in $\mathcal{G}\big(0, q_0\big)$, fix $\lambda_0 := \delta$ and associate to $\mathcal{G}\big(0, q_0\big)$, for all $\lambda \leq \min(\lambda_0, \eta)$, a discrete time game adapted to the subdivision $\mathcal{P}_\lambda$, denoted by $\mathcal{G}_{\mathcal{P}_\lambda}\big(0, q_0\big)$. Then, $|W_{\mathcal{P}_\lambda}(q_0) - W(0, q_0)| < \frac{\varepsilon}{3}$. By Corollary 2.7, $W_{\mathcal{P}_\lambda}(q_0) = \Psi_\lambda(q_0)$. By Lemma 3.1, for any $z \in \mathcal{Z}$, we have:

$$|\mathbf{V}_\lambda(z) - \mathbf{V}_\lambda(0)| \leq 2\, \|H\|_\infty\, \lambda|z| \left(\ln\left(\frac{1}{\lambda|z|}\right) + C_\lambda^z\right).$$

There exists $z_0 \in \mathcal{Z}$, such that $z_0 = \lfloor \frac{q_0}{\lambda} \rfloor$ and since $|q_0| = \eta$, we have: $|z_0| = (\eta/\lambda) - \rho$ for some $\rho \in (0,1)$. By definition $\Psi_\lambda(q_0) = \mathbf{V}_\lambda\left(\lfloor \frac{q_0}{\lambda} \rfloor\right)$. It follows:

$$|\Psi_\lambda(q_0) - \Psi_\lambda(0)| \leq \|H\|_\infty\, 2(\eta - \lambda\rho) \left| \ln\left(\frac{1}{\eta - \lambda\rho}\right) + C_\lambda^z \right|.$$

Since $\eta \in (0, \frac{1}{4})$, we have $(\eta - \lambda\rho)|\ln(\eta - \lambda\rho)| < \eta|\ln(\eta)|$. It then follows:

$$|\Psi_\lambda(q_0) - \Psi_\lambda(0)| \leq 2\, \|H\|_\infty\, \eta(|\ln(\eta)| + C_\lambda^z) < \frac{\varepsilon}{3}.$$

Therefore, for any $\lambda \leq \min\{\lambda_0, \eta\}$,

$$\begin{aligned} \big|\Psi_\lambda(0) - W(0,0)\big| &\leq \big|\Psi_\lambda(0) - \Psi_\lambda(q_0)\big| + \big|\Psi_\lambda(q_0) - W(0, q_0)\big| + \big|W(0,q_0) - W(0,0)\big| \\ &< \varepsilon. \end{aligned}$$

From (2.19), $\Psi_\lambda(0) = \mathbf{V}_\lambda(0)$. It follows that $\mathbf{V}_\lambda(0) \to W(0,0)$ when $\lambda \to 0$. By Lemma 3.1, we conclude that for any $z \in \mathcal{Z}$, $\mathbf{V}_\lambda(z)$ converges to $W(0,0)$ as λ tends to zero, which completes the proof of the Theorem. $\square$

4 Conclusion and perspectives

In this paper, we study two-player zero-sum FD games and establish convergence of $\mathbf{V}_n$ and $\mathbf{V}_\lambda$ as n tends to infinity and λ goes to 0 respectively, to the value of the associated differential game starting at the origin, $W(0,0)$.

An interesting generalization of the existence result we provide in this paper, concerns a stage payoff function $g(z,i,j)$ where z stands for the average of past actions and (i,j) for the current actions, which is assumed to be linear in z. In such a model the impact of the past and that of present actions are not additive, but they combine together in some way.

Lastly, let us mention that since existence of the asymptotic value in the zero-sum case is established, a study of limits of Nash equilibria payoffs in non zero-sum FD games that leads to some Folk-like theorem now seems to be possible.

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