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OPTION VALUATION WITH IG-GARCH MODEL AND AN U-SHAPED PRICING KERNEL.

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Abstract

Empirical and theoretical studies have attempted to establish the U-shape of the log-ratio of conditional risk-neutral and physical probability density functions. The main subject of this paper is to question the use of such a U-shaped pricing kernel to improve option pricing performances. Starting from the so-called Inverse Gaussian GARCH model (IG-GARCH), known to provide semi-closed form formulas for classical European derivatives when an exponential affine pricing kernel is used, we build a new pricing kernel that is non-monotonic and that still has this remarkable property. Using a daily dataset of call options written on the $S\&P500$ index, we compare the pricing performances of these two IG-GARCH models proving, in this framework, that the new exponential U-shaped stochastic discount factor clearly outperforms the classical exponential affine one. What is more, several estimation strategies including options or VIX information are tested taking advantage of the analytical tractability of these models.

Keywords: Option valuation, Pricing kernel, VIX, IG-GARCH, S&P500.

JEL Classification: G12; G16; G22; G52.

1. Introduction

In the financial literature, ARCH/GARCH models, introduced by [28] and [10], have gained widespread acceptance over the last few decades to model heteroscedasticity of asset returns emerging as one of the most popular and flexible discrete time alternative to continuous time diffusions because the endogenous parametric specification of the volatility makes it possible to estimate the joint dynamics of returns and volatility only using time series of returns. From this seminal step, GARCH models have been extended in various directions to cope, in particular, with asymmetry properties (see e.g. [51] for a recent survey). Recently, [24] was the first to provide a strong theoretical framework, the so-called Local Risk Neutral Valuation Relationship (LRNVR), to price contingent claims when the underlying dynamics is given by a GARCH model with Gaussian innovations. While this approach outperforms the [9] benchmark it is restricted to Gaussian innovations and the prices are obtained using Monte Carlo simulations.

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Following the preceding methodology, [40] considered a new conditionally-normal GARCH-like volatility updating scheme able to cope with skews in option prices. Moreover, they derived an interesting semi-closed form expression for call option prices making the pricing of such financial products fast and compatible with calibration estimation methods at a reasonable computational cost. Nevertheless, as this model is conditionally Gaussian, it usually fails to capture the short term behavior of equity option smiles. In fact, it is now well documented ([17], p.41) that Gaussian innovations can not take into account all the mass in the tails and the asymmetry that characterize the distribution of daily log-returns even if an asymmetric GARCH filter is applied[1].

During the last decade (see [17], Chap.2 for a recent survey), researchers have intensively investigated the way to extend the Duan's option pricing model to incorporate in GARCH residuals the skewness and leptokurtosis observed in financial datasets. In general, such a choice is motivated by equilibrium arguments (see [2]) and/or by its compatibility with the myriad of possible candidates for the distribution. An important contribution in this direction was the work of [48] in which the authors used for the first time, in the GARCH setting, the conditional Esscher transform introduced in [12] to price European options using a shifted Gamma distribution. This approach is equivalent (see [33]) to considering a special parametric form for the pricing kernel (exponential affine of the log-returns) and allows for explicit and tractable risk-neutral dynamics in many situations[2]. The flexibility of the exponential affine parametrization is probably one of its main advantages with respect to its natural competitors as the generalized LRNVR of [25] (see also [49] and [50]) or the extended Girsanov principle of [29] (see also [1] and [3]). Nevertheless, in spite of their differences, all the preceding specifications coincide with the LRNVR in the Gaussian framework and depend on a single stochastic parameter related to the equity risk premium and uniquely determined by the martingale constraints.

The choice of an interesting characterization for the pricing kernel is an old topic (see [47], [11]. [13], [22], [46] and [38] among others) that often leads to parametric forms that are monotonic functions of the log-returns ([47], [35], [36], [31] and [12]). However, many recent empirical studies suggest evidences against the monotonicity assumption ([7], [5], [53], [15] and [6]). In the GARCH setting, two approaches have been proposed to overcome this problem and take into account market and volatility risks: [43] introduced an extension of the classical Esscher transform, including a quadratic term in the pricing kernel while [21] proposed a variance dependent pricing kernel (see also [4] for a slight different approach compatible with non-affine models).

We propose in this paper an extension of the so-called Inverse Gaussian GARCH (IG-GARCH) model of [18] where the authors provide a new particular affine GARCH structure with Inverse Gaussian innovations to take into account conditional skewness. Using the pricing kernel derived from the conditional Esscher transform, they obtained the risk neutral dynamics, depending only on historical parameters, which gives rise to a closed-form option pricing for-

[1] Concerning asymmetric volatility responses, refer to the EGARCH model introduced in [44], the GJR GARCH model of [32], the APARCH model developed in [23], as well as the TGARCH studied in [52].

[2] See among others, [18] for the Inverse Gaussian distribution, [1] for the mixture of Gaussian distributions, [16] for the Generalized hyperbolic distribution.

mula as in [40].

The main idea is to use here an extended and non-monotonic version of the exponential affine pricing kernel, particularly well-adapted to the Inverse Gaussian distribution, in order to increase the flexibility of the link between the historical and the risk-neutral distributions while preserving the tractability of the model. In fact, even in the case of our[3] new pricing kernel, closed-form expressions remain available for European call options and the VIX index[4]. Thus, it is possible to combine, at a reasonable computational cost, historical returns dynamics with options or VIX information in the estimation process to build more accurate joint likelihood as explained in [20] and [42].

Finally, we perform a GMM test to check the validity of each pricing kernel with respect to the martingale conditions and present a comparative analysis of in-sample and out-of-sample pricing performances of the IG-GARCH model associated with both exponential affine and exponential U-shaped pricing kernels and estimated using options or VIX information. We compute the Implied Volatility Root Mean Square (IVRMSE) for each model to evaluate and compare the pricing errors. This empirical study provides strong evidences indicating that the exponential U-shaped pricing kernel is clearly superior in approximating the price of options written on the $S\&P500$ for the concerned period. What is more, we show, in this framework, that an estimation strategy based on returns-VIX information provides very interesting pricing errors at a low computational cost because expensive calibration on options can be bypassed.

The remainder of the paper is organized as follows. The next section defines and develops the theoretical framework giving, in particular, the risk neutral dynamics under the two different pricing kernels and the associated closed form expressions for option prices and the VIX index. We present in Section 3 the methods of estimation based on different joint maximum likelihood. The numerical results are contained in Section 4. More precisely, we describe the returns, VIX and options datasets on the $S\&P500$ used in the paper, we perform a GMM test to validate the martingale conditions and, finally, provide the in and out of sample pricing performances. Concluding remarks are given in Section 5.

2. The stock price dynamics and the stochastic discount factors

This section presents the theoretical framework of the present paper. Our study uses as a core model the inverse Gaussian GARCH (IG-GARCH) model of [18] known to cope with conditional skewness as well as conditional heteroskedasticity and a leverage effect. First, let us briefly remind the main lines of this approach that will be used in the following as a keystone to price options written on the $S\&P500$ index using different pricing kernels.

[3]The new form of the pricing kernel has been inspired by the work of [43] that introduce a second order Esscher transform particularly well-adapted to the Gaussian (or mixture of Gaussian) case. In the IG-GARCH setting, the idea is to replace in the pricing kernel, the quadratic term of [43] by an hyperbolic one that is more suitable for our choice of distribution.

[4]The VIX expresses the market expectations of the 30-day volatility implied in equity index options.

2.1. The stock price dynamics under the physical probability measure $\mathbb{P}$

We consider a discrete time economy with a time horizon $T \in \mathbb{N}^*$ consisting of a risk-free zero-coupon bond (associated to the risk free rate r expressed on a daily basis and supposed to be constant) and a stock (the risky asset). Following [18], we assume that, under the physical probability measure $\mathbb{P}$, the logarithm of the returns of the stock price process $(S_t)_{t\in\{0,\dots,T\}}$ fulfills

$$\begin{cases} Y_{t+1} = \log\left(\frac{S_{t+1}}{S_t}\right) & = & r + \nu h_{t+1} + \eta y_{t+1} \\ h_{t+1} & = & w + bh_t + cy_t + a\frac{h_t^2}{y_t} \end{cases} \tag{2.1}$$

where the $(y_t)_{t\in\{1,\dots,T\}}$ are random variables generating an information filtration denoted by $(\mathcal{F}_t)_{t\in\{0,\dots,T\}}$ where $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $(\mathcal{F}_t = \sigma(y_u; 1 \leq u \leq t))_{t\in\{1,\dots,T\}}$. Moreover, we suppose that, given $\mathcal{F}_{t-1}$, y_t follows an Inverse Gaussian distribution with degree of freedom $\delta_t = \frac{h_t}{\eta^2}$[5]. Classically, the moment generating function[6] of the pair $(y_t, \frac{1}{y_y})$ can be expressed as :

$$\mathbb{E}\left[e^{\theta y_t + \frac{\phi}{y_t}}\right] = \frac{\delta_t}{\sqrt{\delta_t^2 - 2\phi}} e^{\left[\delta_t - \sqrt{(\delta_t^2 - 2\phi)(1-2\theta)}\right]} \tag{2.2}$$

from which we deduce that

$$E[Y_t \mid \mathcal{F}_{t-1}] = r + (\nu + \frac{1}{\eta})h_t, \; Var[Y_t \mid \mathcal{F}_{t-1}] = h_t$$

and

$$Cov[Y_t - Y_{t-1}, h_{t+1} - h_t \mid \mathcal{F}_{t-1}] = Cov[Y_t, h_{t+1} \mid \mathcal{F}_{t-1}] = (\frac{c}{\eta} - \eta^3 a)h_t.$$

In particular, h_t is the conditional variance of the log-returns and 2.1 may be considered as a GARCH-type model of conditional volatility accommodating with asymmetric volatility responses. We refer the reader to [18] for an in-depth discussion on the statistical characteristics of this process.

To conclude the presentation of the historical dynamics, let us remind one of the key feature of the IG-GARCH model (that may be seen in this way as a skewed analogous of the [40] model): the historical conditional moment generating function of $log(S_T)$ may be expressed using backward recursive equations.

Proposition 2.1. *(See [18] Appendix A) Given $\mathcal{F}_t$, the moment generating function under $\mathbb{P}$ of $log(S_T)$ is characterized by :*

$$\mathbb{G}^{\mathbb{P}}_{log(S_T)|\mathcal{F}_t}(\phi) = \mathbb{E}\left[S(T)^{\phi} \mid \mathcal{F}_t\right] = S(t)^{\phi} exp\left[A(t) + B(t)\left(w + bh_t + cy_t + a\frac{(h_t)^2}{y_t}\right)\right]$$

with $A(T) = B(T) = 0$ and

$$\begin{array}{l} A(t) = A(t+1) + \phi r + wB(t+1) - \frac{1}{2}\log(1 - 2a(\eta)^4 B(t+1)) \\ B(t) = bB(t+1) + \phi\nu + (\eta)^{-2} - (\eta)^{-2}\sqrt{(1 - 2a(\eta)^4 B(t+1))(1 - 2cB(t+1) - 2\phi\eta)}. \end{array}$$

[5]There exist in the literature different parametrizations of the Inverse Gaussian distribution, in this paper, definition and properties of the Inverse Gaussian distribution are presented along the lines of [41] and [8], in particular, the associated density function is given by the one parameter family : $\mathbf{1}_{\{y>0\}}\frac{\delta}{\sqrt{2\pi y^3}}e^{-\left(\sqrt{y}-\delta/\sqrt{y}\right)^2/2}$ where $\delta \in \mathbb{R}_+^*$.

[6]Having option pricing in mind, the existence and the simple expression of the moment generating of the Inverse Gaussian distribution will be fundamental to use the so-called Esscher transform (and the variant presented in this paper) to specify stochastic discount factors.

This property of the conditional moment generating function will be used in the option pricing analysis to obtain prices using the fast Fourier transform methodology.

2.2. Two stochastic discount factors and the related risk-neutral dynamics

When we have option pricing in mind, conditional distributions of returns and volatility specifications are not the only issue one should pay attention to. In fact the use of realistic discrete time volatility structures and continuous distributions gives rise to incompleteness and equivalent martingale measures are not unique in general. It is classically known that in the discrete time setting the construction of such a probability measure is equivalent to the specification of a one-period stochastic discount factor process (see for example [17], Chap. 3.2.2). The purpose of this section is to present two approaches compatible with the dynamics introduced in 2.1 in order to obtain tractable risk-neutral processes. The first one, due to [12], first applied in the GARCH setting by [48], is based on the conditional extension of the [30] transform used by [31] to price contingent claims in continuous time. The second and new one, inspired by the second order Esscher transform introduced by [43] for Gaussian GARCH models, induces more flexibility in the definition of the stochastic discount factor and permits to obtain different realistic shapes.

2.2.1. The exponential affine stochastic discount factor

The conditional Esscher transform introduced by [12] has been a major innovation in the discrete time financial literature providing a flexible framework to price European derivatives. In the GARCH setting it has been combined, with empirical successes, with various families of distributions such as Gaussian jumps in [26] and [27], the mixture of Gaussian distributions in [1] or the Generalized Hyperbolic distributions in [16]. This approach is equivalent (see [33]) to considering a stochastic discount factor that is exponential affine of the log-returns[7]:

$$\forall t \in \{1, \cdots, T\}, \qquad M_t^{ess} = e^{\theta_t Y_t + \varepsilon_t},$$

where θ_t and ε_t are $\mathcal{F}_{t-1}$-measurable random variables that may be uniquely obtained, under mild conditions, from the pricing equations[8]

$$\begin{cases} \mathbb{E}_{\mathbb{P}}\left\{e^{r} M_t^{ess} \mid \mathcal{F}_{t-1}\right\} = 1 \\ \mathbb{E}_{\mathbb{P}}\left\{e^{Y_t} M_t^{ess} \mid \mathcal{F}_{t-1}\right\} = 1. \end{cases} \tag{2.3}$$

The equivalent martingale measure associated to $(M_t^{ess})_{t\in\{1,\cdots,T\}}$ is denoted by $\mathbb{Q}^{ess}$ and in the framework of the IG-GARCH model 2.1 introduced in the preceding section we obtain the following proposition that perfectly describes the risk-neutral dynamics under $\mathbb{Q}^{ess}$:

Proposition 2.2. *(See [18] Appendix B) Assuming that the process* $(Y_t)_t$ *is defined by* 2.1*, then,*

a) $\forall t \in \{1, \cdots, T\}$*, the system* (2.3) *admits a unique solution* $(\theta_t^*, \varepsilon_t^*)$ *characterized by :*

$$\begin{aligned} \theta_t^* &= \theta^* = \frac{1}{2}\left[\eta^{-1} - \frac{1}{\nu^2\eta^3}\left[1 + \frac{\nu^2\eta^3}{2}\right]^2\right] \\ \varepsilon_t^* &= -r(\theta^* + 1) - \theta^*\nu h_t - \left[\delta_t\left(1 - \sqrt{(1 - 2\theta^*\eta)}\right)\right]. \end{aligned}$$

[7]This exponential affine restriction of the stochastic discount factor is equivalent to the assumption (12) of [18]

[8]The equations are derived by applying the pricing formula to the risk-free and risky assets.

b) Under $\mathbb{Q}^{ess}$, the process $(Y_t)_t$ is again an IG-GARCH model with changed parameters :

$$\begin{cases} Y_{t+1} = \log\left(\frac{S_{t+1}}{S_t}\right) & = & r + \nu^* h^*_{t+1} + \eta^* y^*_{t+1} \\ h^*_{t+1} & = & w^* + b^* h^*_t + c^* y^*_t + a^* \frac{(h^*_t)^2}{y^*_t} \end{cases} \tag{2.4}$$

$$\text{where} \qquad \begin{aligned} \nu^* &= \nu\left(\frac{\eta^*}{\eta}\right)^{-\frac{3}{2}}, \quad y^*_{t+1} = y_{t+1}\left(\frac{\eta^*}{\eta}\right)^{-1}, \\ w^* &= w\left(\frac{\eta^*}{\eta}\right)^{\frac{3}{2}}, \quad c^* = c\left(\frac{\eta^*}{\eta}\right)^{\frac{5}{2}}, \quad a^* = a\left(\frac{\eta^*}{\eta}\right)^{-\frac{5}{2}}, \end{aligned}$$

with $\eta^* = \dfrac{\eta}{1 - 2\theta^*\eta}$ and where, given $\mathcal{F}_{t-1}$, y^*_t follows an Inverse Gaussian distribution with degree of freedom $\delta^*_t = \frac{h^*_t}{(\eta^*)^2}$.

We remark, from the preceding proposition, that the conditional dynamics under $\mathbb{Q}^{ess}$ is the same as under the historical probability with changed parameters and that the risk-neutral conditional variance can be expressed as $h^*_{t+1} = (\eta^*/\eta)^{\frac{3}{2}} h_{t+1}$[9]. One important empirical consequence for the pricing of European call and put options is that proposition 2.1 remains valid under $\mathbb{Q}^{ess}$, thus semi-closed form formulas will be available for prices.

Even if the assumption of an exponential-affine stochastic discount factor is well theoretically justified in the literature, in particular in equilibrium pricing models (see [2]), it is not the only issue to obtain arbitrage-free price processes that derive from the pricing equations (2.3). Thus, in the next subsection, we are going to see how to extend the exponential affine pricing kernel M^{ess}_t in order to increase the flexibility of the link between the historical and the risk-neutral distributions while preserving the tractability of the model.

2.2.2. The exponential U-shaped stochastic discount factor

We derive in this subsection the risk-neutral dynamics of the IG-GARCH model using an exponential U-shaped pricing kernel that extends the classical conditional Esscher transform. Inspired by the second order Esscher transform recently introduced by [43] in the Gaussian setting, we include the term $\dfrac{\rho_t}{y_t}$ in the specification of M^{ess}_t to be able to generate an exponential U-shaped function:

$$\forall t \in \{0, \cdots, T\}, \qquad M^{Ushp}_t = e^{\theta_t Y_t + \varepsilon_t + \frac{\rho_t}{y_t}}$$

where θ_t, ε_t and ρ_t are $\mathcal{F}_{t-1}$ measurable random variables[10]. Under the risk-neutral probability $\mathbb{Q}^{Ushp}$ associated to $(M^{Ushp}_t)_{t\in\{1,\cdots,T\}}$, the overall dynamics of the log-return is, once again similar the historical one:

[9]Contrary to what happens for Gaussian GARCH models, the IG-GARCH framework is able to cope with the well-known stylized fact that the risk-neutral variance is in general greater than the historical one.

[10]From (2.1), we obtain $M^{Ushp}_t = e^{\theta_t \eta y_t + \frac{\rho_t}{y_t} + \varepsilon_t + \theta_t(r + \nu h_t)}$. In the empirical exercise performed in section 4, we obtain, independently of the estimation process, $\eta < 0$, $\theta_t < 0$ and $\rho_t > 0$. Thus, $\underset{y_t\to 0^+}{lim} M^{Ushp}_t = \underset{y_t\to +\infty}{lim} M^{Ushp}_t = +\infty$ and M^{Ushp}_t is an U-shaped function of y_t.

Proposition 2.3. *(See Appendix) $\forall t \in \{1, \cdots, T\}$, if we assume a constant proportional wedge between h_t and h_t^* (i.e $h_t^*/h_t = \pi$) the dynamics of Y_t under $\mathbb{Q}^{Ushp}$ is of the form:*

$$\begin{cases} Y_{t+1} = \log\left(\frac{S_{t+1}}{S_t}\right) &= r + \nu^* h_{t+1}^* + \eta^* y_{t+1}^* \\ h_{t+1}^* &= w^* + b h_t^* + c^* y_t^* + a^* \dfrac{(h_t^*)^2}{y_t^*} \end{cases} \tag{2.5}$$

where

$$\nu^* = \frac{\nu}{\pi}, \quad w^* = w\pi, \quad c^* = \frac{c\pi\eta^*}{\eta}, \quad a^* = \frac{a\eta}{\pi\eta^*},$$

$$\eta^* = \sqrt[3]{\frac{\pi^2}{\nu^2}\left(-1 + \sqrt{1 + \frac{8\nu}{27\pi}}\right)} + \sqrt[3]{\frac{\pi^2}{\nu^2}\left(-1 - \sqrt{1 + \frac{8\nu}{27\pi}}\right)},$$

and where, given $\mathcal{F}_t$, y_{t+1}^* follows an IG distribution with degree of freedom $\delta_{t+1}^* = \dfrac{h_{t+1}^*}{(\eta^*)^2}$.

As before, we obtain a similar IG-GARCH structure for the risk-neutral dynamics of the log returns. Of course, if $\forall t \in \{1, \cdots, T\}$, we impose $\rho_t^* = 0$, we recover the result of the proposition 2.2. Nevertheless, the risk-neutral dynamics given by proposition 2.2 only depends on the initial historical set of parameters while the dynamics presented in proposition 2.3 introduces a risk-neutral parameter π. Thus, the first model may be estimated from returns using a conditional version of the classical maximum likelihood (ML) estimation while an extra information (based on option prices) is needed for the estimation of the second one. In the two next subsections we show how to include this extra information in an efficient way in the estimation strategy. More precisely, we show that for the two risk-neutral IG-GARCH models we have numerically efficient closed form expressions not only for the price of European call options but also for the VIX index at any time.

2.3. Pricing European call options using Fast Fourier Transform (FFT)

It is well known from the pioneering work of [39] that the price of European call options may be expressed using the risk neutral conditional moment generating function of $log(S_T)$ (see also [17], p. 184): if $\mathbb{Q}$ is an arbitrary equivalent martingale measure, we have

$$\begin{aligned} e^{-r(T-t)} E_{\mathbb{Q}}[(S_T - K)_+ \mid \mathcal{F}_t] &= \frac{S_t}{2} + \frac{e^{-r(T-t)}}{\pi} \int_0^{+\infty} \mathcal{R}e\left[\frac{K^{-i\phi}\mathbb{G}^{\mathbb{Q}}_{log(S_T)|\mathcal{F}_t}(i\phi+1)}{i\phi}\right] d\phi \\ &- K e^{-r(T-t)}\left(\frac{1}{2} + \frac{1}{\pi}\int_0^{+\infty} \mathcal{R}e\left[\frac{K^{-i\phi}\mathbb{G}^{\mathbb{Q}}_{log(S_T)|\mathcal{F}_t}(i\phi)}{i\phi}\right] d\phi\right). \end{aligned}$$

Even though this formula prevents to use slow Monte-Carlo methods to approximate the price process, two important numerical issues stay. First, $\mathbb{G}^{\mathbb{Q}}_{log(S_T)|\mathcal{F}_t}$ has to be computed effectively, second, finding the price necessitates univariate numerical integration. For the first point, the IG-GARCH model is particularly well designed because proposition 2.1 (combined with the two preceding risk-neutral dynamics) provides an interesting backward recursive approach. For the second point, the answer is given by [14] that offer a powerful strategy based on the Fast Fourier Transform (FFT) to compute option prices efficiently for a full range of strikes and a given maturity[11]. In the empirical part, this approach will be used to estimate parameters directly from option prices minimizing an appropriate loss function.

To conclude this section, we provide, for the IG-GARCH model and the two preceding specifications of the stochastic discount factor, a closed-form expression for the one month risk-neutral expectation of the integrated variance to integrate information on VIX without costly computations.

2.4. Pricing using VIX information

In a recent paper, [37] (see also [45]) derived implied VIX formulas that may be deduces from Gaussian GARCH models combined with the so-called [24] Local Risk Neutral Valuation Relationship. What is more, they proposed a joint likelihood estimation methodology including returns and VIX data that was used in [42] to improve pricing performances. The aim of this subsection is to derive analogous formulas and estimation tools for the IG-GARCH model.

The VIX index may be seen as the fair-value strike for a 21-business days variance swap and is known as the fear index. From 2003, the VIX relies on the concept of static replication using all calls and puts with valid quotes, and thus it does not subjected to a specific option pricing model. Nevertheless, in discrete time and in the absence of jumps, it can be written as

$$\frac{1}{\tau}\left(\frac{VIX_t}{100}\right)^2 = \frac{1}{T_c}\sum_{j=1}^{T_c}\mathbb{E}_{\mathbb{Q}}\left[h^*_{t+j} \mid \mathcal{F}_t\right] \tag{2.6}$$

where $\tau = 250$, $T_c = 21$, $\mathbb{Q}$ is an equivalent martingale measure[12] and h^*_t the conditional and risk-neutral daily variance. Concerning the IG-GARCH model, from the risk neutral dynamics in 2.4, 2.5 and using iterative properties of the conditional expectation, the expected conditional variance $\mathbb{E}_{\mathbb{Q}}\left[h^*_{t+j} \mid \mathcal{F}_t\right]$ can be expressed as a linear combination of the conditional spot variance h^*_{t+1} and the unconditional variance h^*_0, weighted by $(\psi^*)^{j-1}$:

$$\mathbb{E}_{\mathbb{Q}}\left[h^*_{t+j} \mid \mathcal{F}_t\right] = h^*_{t+1}\left[\psi^*\right]^{j-1} + h^*_0\left[1 - (\psi^*)^{j-1}\right]$$

where the variance persistence $\psi^* = b + \frac{c^*}{(\eta^*)^2} + a^*(\eta^*)^2$ and $h^*_0 = \frac{w^* + a^*(\eta^*)^4}{1-\psi^*}$ only depend on the risk-neutral parameters of the model[13]. Thus, we easily obtain (see the proof in the Appendix)

$$\frac{1}{\tau}\left(\frac{VIX_t}{100}\right)^2 = h^*_{t+1}\frac{1-(\psi^*)^{T_c}}{(1-\psi^*)\,T_c} + h^*_0\left(1 - \frac{1-(\psi^*)^{T_c}}{(1-\psi^*)\,T_c}\right). \tag{2.7}$$

[11]For the sake of brevity, we refer the reader to [17], p. 137, where a detailed algorithm is proposed with the associated R source code also used in the present paper.

[12]In this section, we implicitly suppose that $\mathbb{Q}$ derives from the one period stochastic discount factor processes defined in sections 2.2.1 and 2.2.2

[13]For the IG-GARCH model, the risk-neutral parameters are simple functions of the historical ones in the case of an exponential affine stochastic discount factor while they are functions of the historical parameters and π under $\mathbb{Q}^{Ushp}$.

3. Estimation of parameters

In the literature, there exist different methods for the estimation of GARCH parameters, the most popular one being the conditional version of the classical (the) Maximum Likelihood Estimation (MLE). In fact, once the GARCH volatility structure and the innovations' density are specified, the conditional log-likelihood based on return observations is in general easy to express and historical parameters are obtain using optimization schemes. For the IG-GARCH model, the knowledge of historical parameters is sufficient to deduce the dynamics under $\mathbb{Q}^{ess}$ because risk-neutral parameters are functions of the historical ones. For the dynamics under $\mathbb{Q}^{Ushp}$, it is not a priori possible to extract the risk neutral parameter π from return data only. An additional information, based for example on options or the VIX index, has to be exploited. To make fair the comparison between the risk-neutral dynamics presented in this paper and to deeply use the technical flexibility of the IG-GARCH framework, we favor in our study joint estimation strategies using both return-option (see for example [20]) or return-VIX (see [42]) observations.

3.1. Joint MLE Estimation using option prices and asset returns

It is well-known that GARCH parameters may be efficiently extracted from option data, when semi-closed form formulas are available for call options prices, minimizing an appropriate loss function. In [40] or [18] the authors minimize the root mean square error between model and market option prices but as argued in [20] this criteria places a greater weight on expensive in-the-money and long-maturity options. To overcome this problem, the linear vega-approximation of implied volatility errors is a popular approach. We obtain estimates of the set of the risk neutral parameters, denoted by ϑ^*, minimizing the Implied Volatility Root Mean Square Error (IVRMSE)[14]:

$$\hat{\vartheta}^* = \arg\operatorname*{Min}_{\vartheta} IVRMSE(\vartheta) = \arg\operatorname*{Min}_{\vartheta} \sqrt{\frac{1}{N_{T_{Op}}}\sum_{t,i}\left(\frac{c_{i,t}(h_t^*;\vartheta) - \hat{c}_{i,t}}{\hat{V}_{i,t}}\right)^2}. \tag{3.8}$$

Here, n_t is the number of option contracts in the sample at time t and $N_{T_{Op}} = \sum_{t=1}^{T_{Op}} n_t$ where T_{Op} is the number of days in the options sample. $c_{i,t}(h_t^*;\vartheta)$ denotes the price of the $i-$th option at time t given by the model[15] while $\hat{c}_{i,t}$ is the price observed in the market. $\hat{V}_{i,t}$ is the Vega associated to $\hat{c}_{i,t}$ that is computed using the implied Black-Scholes volatility $\sigma_{i,t}$ obtained from the market price.

To avoid the distortion of parameters that may appears performing pure calibration exercises[16], we present in this subsection a joint MLE estimation using both option prices and asset returns to estimate the parameters of the model as explained in [20]. On the one hand, we need to build the log-likelihood function associated to the log-returns $(Y_1, ..., Y_T)$. Under IG

[14]The Implied Volatility Root Mean Square Error (IVRMSE) will be used in the empirical study to evaluate and compare the pricing performances of the models.

[15]This price is computed using the FFT methodology presented in section 2.3 and depends on the risk-neutral conditional volatility at time t, h_t^*, that is obtained from the log-returns and the risk-neutral GARCH updating rule initialized at its unconditional level.

[16]In fact, when calibrating model parameters, all the attention is focused on the minimization of the in-sample error. Thus, it is possible to overfit the options dataset and to produce poor out of sample pricing errors.

innovations, the conditional density function of Y_t given $(Y_1, ..., Y_{t-1})$ is given by :

$$f(Y_t \mid Y_1, ..., Y_{t-1}) = \frac{h(t)}{\sqrt{2\pi (Y_t - r - \nu h_t)^3}} e^{-\frac{1}{2}\left(\sqrt{\frac{Y_t - r - \nu h_t}{\eta}} - \frac{h_t}{\eta^2}\sqrt{\frac{\eta}{Y_t - r - \nu h_t}}\right)^2},$$

and the conditional log-likelihood is given by

$$\ln L_R = \sum_{t=1}^{T} \ln f(Y_t \mid Y_1, ..., Y_{t-1}) \tag{3.9}$$

that is a function of the historical parameters. On the other hand, in order to obtain the log-likelihood function associated to option data, we consider the Black-Scholes Vega weighted option valuation error:

$$\epsilon_{i,t} = \left(\frac{c_{i,t}(h_t^*; \vartheta^*) - \hat{c}_{i,t}}{\hat{V}_{i,t}}\right)$$

that is an approximation of the implied volatility error. Moreover, assuming that the errors $(\epsilon_{i,t})$ are independent and identically distributed Gaussian random variables the corresponding option log-likelihood can be written (see [19]) as :

$$\ln L_{Op} = -\frac{1}{2} \sum_{t=1}^{N_{T_{Op}}} \left[ln\left(\frac{1}{N_{T_{Op}}} \sum_{i=1}^{N_{T_{Op}}} \epsilon_{i,t}^2\right) + \frac{\epsilon_{i,t}^2}{\frac{1}{N_{T_{Op}}} \sum_{i=1}^{N_{T_{Op}}} \epsilon_{i,t}^2} \right] \tag{3.10}$$

Using both likelihoods in equations 3.9 and 3.10, the joint estimation of the parameters can be obtained by maximizing the joint log-likelihood function:

$$\hat{\vartheta}^* = \arg \underset{\vartheta}{\mathrm{Max}} \quad \frac{T + N_{T_{Op}}}{2} \frac{\ln L_R}{T} + \frac{T + N_{T_{Op}}}{2} \frac{\ln L_{Op}}{N_{T_{Op}}} \tag{3.11}$$

where T is the number of days in the returns sample, and $N_{T_{Op}}$ is the total number of option contracts[17].

3.2. Joint MLE Estimation using asset returns and VIX index

This subsection introduces a joint MLE estimation using both returns and the VIX index. In a recent paper, [37] proposed a joint likelihood estimation method that incorporates VIX information to capture, in GARCH estimation, the Variance Risk Premium. Their study is based on closed-form formulas for the VIX approximations associated to several Gaussian GARCH pricing models. These formulas, similar to the one obtained in the present paper for the affine IG-GARCH model, permit to compute efficiently the related log-likelihood from risk-neutral parameters. Using a similar approach [42] have implemented a joint maximum likelihood estimation using returns and VIX with auto-regressive disturbances to enhance the estimation

[17] We have $\vartheta^* = \{\nu, \omega, b, c, a, \eta\}$ in the case of the exponential affine stochastic discount factor and $\vartheta^* = \{\nu, \omega, b, c, a, \eta, \pi\}$ in the case of the exponential U-shaped one.

performances of the GARCH option pricing model at a reasonable computational cost. More precisely, in this latter study, the likelihood function on VIX is obtained considering the following model which introduced an error process with autoregressive disturbances:

$$\begin{cases} u_t &= \varrho u_{t-1} + e_t \\ u_t &= VIX_t^{Market} - VIX_t^{Model}(h_{t+1}^*; \vartheta) \end{cases} \tag{3.12}$$

where $(e_t)_t$ are independent and identically distributed centered Gaussian random variables with variance Σ and where $VIX_t^{Model}(h_{t+1}^*; \vartheta)$ is obtained from equation 2.7. Thus,

$$\ln L_{VIX} = -\frac{T}{2}\left(\ln(2\pi) + \ln(\Sigma(1-\varrho^2))\right) + \frac{1}{2}\left(\ln(1-\varrho^2)\right) - \frac{1}{2\Sigma}\left(u_1^2 + \sum_{t=2}^{T}\frac{(u_t - \varrho u_{t-1})^2}{1-\varrho^2}\right) \tag{3.13}$$

We combine this log-likelihood with the one associated to the log-returns in equation 3.9 to solve the joint likelihood optimization problem on returns and VIX as follows :

$$\bar{\vartheta}^* = \arg \underset{(\vartheta,\varrho)}{\mathrm{Max}} \left(\ln L_R + \ln L_{VIX}\right) \tag{3.14}$$

where $\bar{\vartheta}^* = (\vartheta^*, \varrho^*)$ and ϱ^* is the estimated value of the autoregressive parameter introduced above.

4. Empirical results

Based on the preceding theoretical results, this section examines the empirical pricing performances of the IG-GARCH models using the two different stochastic discount factors.

4.1. Data properties

To implement the previous joint maximum likelihood estimation strategies using VIX or options information we use in this paper several time series data. The first one is made of daily log-returns of the $S\&P500$ index and the associated CBOE VIX ranging from January 07, 1999 to December 31, 2010. The series of returns is computed from closing prices. Both the return and VIX series have 2718 daily observations available for our study. In *Table 1*, we provide the descriptive statistics of the $S\&P500$ Log-returns and VIX time series.

The second dataset is made of Wednesday's European call options written on the $S\&P500$ from the CBOE. It contains call option prices for a large range of moneynesses and maturities. The sample period extends from January 01, 2009 to December 31, 2010. Our sample consists of option contracts on 104 Wednesdays and we apply, as most of the empirical studies in the literature (see [40], [18] or [42]), the same filters as [5]. To empirically study the real option pricing performances of our models, we split up our option dataset into an in-sample and an out-of-sample, the models will be estimated with the returns-option strategy only using the in-sample data. The in-sample option data ranges from January 01, 2010 to December 31, 2010 and the out-of-sample data from January 01, 2009 to December 31, 2009. *Table 2* (resp. *Table 3*) reports the in-sample (resp. out of sample) summary statistics for option data: average price, average implied volatility and the number of contracts for each moneyness/maturity[18]

[18]We divide the option data into 18 categories according to either moneynesses and times to expiration. The moneyness is defined as the ratio between the forward price of the underlying asset and the option's strike price.

category. The in-sample contains 1332 contracts and the out-of-sample one 1533. Finally, for the risk-free rate, that is essential to implement pricing formulas, we use the daily 3 month U.S. Treasury bills (secondary market), obtained from the U.S. Federal Reserve website.

Table 4, contains the estimated parameters, as well as their standard errors, for the IG-GARCH model combined with the two different stochastic discount factors using the option-returns and the VIX-returns methodologies. All the parameters are statistically significant at conventional 5% significance levels. Instead of focusing on the individual parameter values of the models (that are, roughly speaking, quite stable across the estimation strategy and the choice of the pricing kernel) we can remark that the risk-neutral persistence is high for all the models, that the levels of annualized volatility are realistic and that the leverage effect is observed.

4.2. Testing the validity of the stochastic discount factors

Before to test more precisely the pricing performances of the IG-GARCH model, we propose, following [34], to question the consistency of the exponential affine and exponential U-shaped forms of the stochastic discount factor. In this way, we perform a Generalized Method of Moments (GMM) test based on the classical martingale conditions for the risky asset and the associated derivatives. In fact, when (M_t) is a one period stochastic discount factor we need to have

$$\begin{cases} \mathbb{E}_{\mathbb{P}}\left\{e^{Y_{t+1}} M_{t+1} \mid \mathcal{F}_t\right\} = 1 \\ \mathbb{E}_{\mathbb{P}}\left\{\frac{P_{t+1}(K,T)}{P_t(K,T)} M_{t+1} \mid \mathcal{F}_t\right\} = 1 \end{cases} \tag{4.15}$$

where $P_t(K,T)$ is the price at time t of a call option of strike K and maturity T. Thus, we test the null hypothesis $\mathbb{E}_{\mathbb{P}}\left\{e^{Y_{t+1}} M_{t+1} \mid \mathcal{F}_t\right\} = 1$[19] using the statistics

$$t_S = \frac{1}{T}\sum_{t=1}^{T}\left(M_{t+1}\frac{S_{t+1}}{S_t} - 1\right). \tag{4.16}$$

Under the null hypothesis, $t_S/\hat{\sigma}_T\sqrt{T}$ is asymptotically standard normal where $\hat{\sigma}_n$ is the Newey-West long-run sample variance estimate for $M_{t+1}\frac{S_{t+1}}{S_t} - 1$. The results are presented in *Table 5*: for each collection of estimated parameters (see *Table 4*), the statistics proposed in equation 4.16 is computed and compared to the 5% level critical values for standard normal distribution. We find that the null hypothesis is accepted for each stochastic discount factor and estimation methodology. More precisely, the values of the GMM test statistics obtained in *Table 5* are between -1.96 and 1.96 and the null hypothesis that the moment condition is equal to zero is not rejected at a 5% risk level. This preliminary analysis is not sufficient to discriminate both stochastic discount factors and estimation methodologies that are all compatible with the martingale restriction. In the next subsection, we investigate in details the related pricing performances.

[19] We perform a similar analysis to test the moment condition for the returns on the options for different moneynesses and different time to maturities. The results are presented in *Table 6*, *Table 7*, *Table 8* and *Table 9* with similar conclusions.

4.3. *Pricing performances*

Observing the general pricing performances reported at the bottom of *Table 4*, one might reach, without ambiguities, to the conclusion that, independently of the estimation method, the IG-GARCH model combined with an U-shaped pricing kernel provides a much better fit (in-sample and out-of-sample) than the classical exponential affine approach.

In fact, the in-sample implied volatility roots mean square error IVRMSE for the period 2009 with 1322 contracts is 0.0523 for the exponential affine SDF model using the joint MLE estimation with option-returns data, while the U-shaped SDF performs slightly better with an IVRMSE of 0.04215, which represents a 19.40% improvement as observed in *Table 11*. Analogous in-sample results are observed when estimating the models using the joint MLE with VIX-returns data, the IVRMSE is smaller when the U-shaped SDF is used: the IVRMSE for the exponential affine SDF is now 0.0544 versus 0.0440 for the U-shaped SDF, which represents a 19.11% improvement. We can also observe from *Table 12* to *Table 15* the values of the in-sample IVMRSE for different moneynesses and maturities. Thus, the in-sample analysis strongly favor the U-shaped specification. Concerning the choice of the estimation methodology, even if the results are quite similar, in terms of computational time, we can observe from *Table 10* that the results associated to the VIX approach are clearly faster to obtain than results from option prices.

The preceding conclusion is not really surprising because an extra parameter is introduced in our approach allowing for more flexibility in calibration exercises. Thus, it is now interesting to focus on the true test for a pricing model, the out of sample pricing performances for the period 2010 when the models are evaluated using the parameter estimates from the 2009 sample period. As observed in *Table 4*, when the model is estimated using option-returns information, the IVRMSE drops from 0,1285 to 0,09727 with the U-shaped pricing kernel which represents a 24.65% improvement. The same holds when VIX-returns observations are used to estimate the model with a 28.29% improvement. What is more, we can observe from *Table 16* to *Table 19* that this result is homogenous regarding moneynesses and time to maturities. It is now clear that the out of sample results largely confirm the in-sample ones, the IG-GARCH model provides better pricing performances when the U-shaped SDF is used to obtain risk-neutral dynamics.

5. Conclusion

In an important paper, [18] proposed an option pricing model based on an IG-GARCH process and the conditional Esscher transform to underline the importance of modelling conditional skewness. One of the main features feature of this approach is to provide, as in [40], semi-closed form formulas for call options but for non Gaussian innovations. Recently, the monotonicity of the stochastic discount factor (often supposed to be exponential affine of the log-returns) was discussed in the literature (see for example [19] and [43]) to favor U shapes. In this paper we have explored an extension of [18] using an U-shaped pricing kernel that increases the flexibility of the link between the historical and the risk-neutral distributions while preserving the tractability of the model. Our empirical results are clear, the in and out of sample pricing performances of the IG-GARCH are improved by the choice of this new pricing kernel. What is more, we show, in this framework, that an estimation strategy based on returns-VIX information provides very interesting pricing errors at a low computational cost because expensive calibration on options can be bypassed.

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Appendix: Proofs

Proposition 2.3. Let us first suppose that the pricing equations

$$\begin{cases} \mathbb{E}_{\mathbb{P}}\left\{e^r M_{t+1}^{Ushp} \mid \mathcal{F}_t\right\} = 1 \\ \mathbb{E}_{\mathbb{P}}\left\{e^{Y_{t+1}} M_{t+1}^{Ushp} \mid \mathcal{F}_t\right\} = 1 \\ \pi = \dfrac{h_{t+1}^*}{h_{t+1}} \end{cases} \tag{5.17}$$

have a unique solution denoted by $(\theta_{t+1}^*, \varepsilon_{t+1}^*, \rho_{t+1}^*)$. The preceding system can be expressed using the conditional moment generating of the pair (Y_{t+1}, y_{t+1}^{-1}) under $\mathbb{P}$:

$$\begin{cases} \mathbb{G}^{\mathbb{P}}_{(Y_{t+1}, y_{t+1}^{-1})|\mathcal{F}_t}(\theta_{t+1}^*, \rho_{t+1}^*) = e^{-r-\varepsilon_{t+1}^*} \\ \mathbb{G}^{\mathbb{P}}_{(Y_{t+1}, y_{t+1}^{-1})|\mathcal{F}_t}(\theta_{t+1}^* + 1, \rho_{t+1}^*) = e^{-\varepsilon_{t+1}^*} \\ \pi = \dfrac{h_{t+1}^*}{h_{t+1}}. \end{cases} \tag{5.18}$$

To obtain the dynamics under $\mathbb{Q}^{Ushp}$, we compute the risk-neutral conditional moment generating function of Y_{t+1}:

$$\mathbb{G}^{\mathbb{Q}^{Ushp}}_{Y_{t+1}|\mathcal{F}_t}(u) = \mathbb{E}_{\mathbb{Q}^{Ushp}}\left[e^{uY_{t+1}} \mid \mathcal{F}_t\right] = \mathbb{E}_{\mathbb{P}}\left[e^{uY_{t+1}} e^r M_{t+1}^{Ushp} \mid \mathcal{F}_t\right] = e^{r+\varepsilon_{t+1}^*} \mathbb{G}^{\mathbb{P}}_{(Y_{t+1}, y_{t+1}^{-1})|\mathcal{F}_t}(\theta_{t+1}^* + u, \rho_{t+1}^*).$$

Using the first equation in (5.18), we can express the risk-neutral moment generating function simply using the historical one:

$$\mathbb{G}^{\mathbb{Q}^{Ushp}}_{Y_{t+1}|\mathcal{F}_t}(u) = \frac{\mathbb{G}^{\mathbb{P}}_{(Y_{t+1}, y_{t+1}^{-1})|\mathcal{F}_t}(\theta_{t+1}^* + u, \rho_{t+1}^*)}{\mathbb{G}^{\mathbb{P}}_{(Y_{t+1}, y_{t+1}^{-1})|\mathcal{F}_t}(\theta_{t+1}^*, \rho_{t+1}^*)}.$$

Given $\mathcal{F}_t$, we know that y_{t+1} follows, under the historical probability $\mathbb{P}$, an IG distribution with degree of freedom $\delta_{t+1} = \frac{h_{t+1}}{\eta^2}$. Thus, using (2.2), we obtain

$$\mathbb{G}^{\mathbb{Q}^{Ushp}}_{Y_{t+1}|\mathcal{F}_t}(u) = \frac{\mathbb{G}^{\mathbb{P}}_{(Y_{t+1}, y_{t+1}^{-1})|\mathcal{F}_t}(\theta_{t+1}^* + u, \rho_{t+1}^*)}{\mathbb{G}^{\mathbb{P}}_{(Y_{t+1}, y_{t+1}^{-1})|\mathcal{F}_t}(\theta_{t+1}^*, \rho_{t+1}^*)} = e^{u(r+\nu h_{t+1})} \frac{e^{\left[\delta_{t+1} - \sqrt{(\delta_{t+1}^2 - 2\rho_{t+1}^*)(1-2(\theta_{t+1}^*+u)\eta)}\right]}}{e^{\left[\delta_{t+1}^* - \sqrt{(\delta_{t+1}^2 - 2\rho_{t+1}^*)(1-2\theta_{t+1}^*\eta)}\right]}}$$

and

$$\mathbb{G}^{\mathbb{Q}^{Ushp}}_{Y_{t+1}|\mathcal{F}_t}(u) = e^{[u(r+\nu h_{t+1})] + \delta_{t+1}^*\left[1 - \sqrt{1-2(u)\eta^*}\right]}$$

where $\eta^* = \dfrac{\eta}{1 - 2\theta_{t+1}^*\eta}$ [20] and $\delta_{t+1}^* = \sqrt{(\delta_{t+1}^2 - 2\rho_{t+1}^*)(1 - 2\theta_{t+1}^*\eta)}$. Therefore, we can write

[20] A priori, the parameter η^* depends on time through θ_{t+1}^* but as we are going to see below, θ_{t+1}^* is time independent.

$$Y_{t+1} = r + \nu h_{t+1} + \eta^* y^*_{t+1}$$

where, given $\mathcal{F}_t$, y^*_{t+1} follows an IG distribution with degree of freedom δ^*_{t+1}. In particular the risk neutral volatility at time $t+1$ fulfills $h^*_{t+1} = \eta^* \delta^*_{t+1}$ and we deduce from

$$Y_{t+1} = r + \nu h_{t+1} + \eta^* y^*_{t+1} = r + \nu h_{t+1} + \eta y_{t+1}$$

that $y_{t+1} = \frac{\eta^* y^*_{t+1}}{\eta^*}$. Thus, using that $\pi = \dfrac{h^*_{t+1}}{h_{t+1}}$, (2.1) gives

$$h^*_{t+1} = w^* + b h^*_t + c^* y^*_t + a^* \frac{(h^*_t)^2}{y^*_t}$$

where

$$w^* = w\pi, \quad c^* = \frac{c\pi\eta^*}{\eta}, \quad a^* = \frac{a\eta}{\pi\eta^*}.$$

To conclude the proof it only remains to express η^* using the historical parameters of the model and π. We start from

$$\delta^*_{t+1} = \frac{h^*_{t+1}}{(\eta^*)^2} = \sqrt{(\delta^2_{t+1} - 2\rho^*_{t+1})(1 - 2\theta^*_{t+1}\eta)}.$$

The martingale condition for the risky asset implies $\mathbb{G}^{\mathbb{Q}^{Ushp}}_{Y_{t+1}|\mathcal{F}_t}(1) = e^r$ from which we can extract ρ^*_{t+1} as a function of θ^*_{t+1}:

$$\rho^*_{t+1} = \frac{\delta^2_{t+1}}{2}\left[1 - \frac{\nu^2\eta^4}{(1 - 2\theta^*_{t+1}\eta)\left[1 - \left(\sqrt{1 - 2\eta^*}\right)\right]^2}\right].$$

Thus,

$$\frac{h^*_{t+1}}{(\eta^*)^2} = \frac{-\nu h_{t+1}}{1 - \sqrt{1 - 2\eta^*}}$$

and

$$\pi = \frac{-\nu}{\left[1 - \left(\sqrt{1 - 2\eta^*}\right)\right]}\left[\eta^*\right]^2.$$

Then, the parameter η^* is obtained as the solution of the following cubic equation:

$$(\eta^*)^3 + \frac{2\pi}{\nu}\eta^* + 2\frac{\pi^2}{\nu^2} = 0.$$

It is well known that this equation has a unique real solution if and only if[21]:

$$4\left(\frac{2\pi}{\nu}\right)^3 + 27\left(\frac{\sqrt{2}\pi}{\nu}\right)^4 > 0 \Leftrightarrow 27\pi > -8\nu.$$

[21]From the empirical values of the parameters obtained in Table 4, this condition is always fulfilled in our framework.

More precisely, we get

$$\eta^* = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

where $p = \frac{2\pi}{\nu}$ and $q = 2\frac{\pi^2}{\nu^2}$ and we can simplify this expression to obtain

$$\eta^* = \sqrt[3]{\frac{\pi^2}{\nu^2}\left(-1 + \sqrt{1 + \frac{8\nu}{27\pi}}\right)} + \sqrt[3]{\frac{\pi^2}{\nu^2}\left(-1 - \sqrt{1 + \frac{8\nu}{27\pi}}\right)}.$$

Finally, we may deduce from the preceding equality that

$$\theta^*_{t+1} = \frac{1}{2\eta} - \frac{1}{2\left[\sqrt[3]{\frac{\pi^2}{\nu^2}\left(-1 + \sqrt{1 + \frac{8\nu}{27\pi}}\right)} + \sqrt[3]{\frac{\pi^2}{\nu^2}\left(-1 - \sqrt{1 + \frac{8\nu}{27\pi}}\right)}\right]}$$

and that the pricing system (5.17) has a unique solution depending on the historical parameters and π. □

VIX as a function of the spot volatility (Section 2.4). Under both specifications of the pricing kernel, the risk-neutral dynamics of the IG-GARCH model may be written as

$$\begin{cases} Y_{t+1} = \log\left(\frac{S_{t+1}}{S_t}\right) & = & r + \nu^* h^*_{t+1} + \eta^* y^*_{t+1} \\ h^*_{t+1} & = & w^* + b^* h^*_t + c^* y^*_t + a^* \frac{(h^*_t)^2}{y^*_t} \end{cases}$$

where, given $\mathcal{F}_t$, y^*_{t+1} follows an IG distribution with parameter $\frac{h^*_{t+1}}{\eta^*}$ under the risk-neutral probability $\mathbb{Q}$. Thus[22],

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\left[h^*_{t+j} \mid \mathcal{F}_{t+j-2}\right] &= w^* + bh^*_{t+j-1} + \frac{c^*}{(\eta^*)^2} h^*_{t+j-1} + a^* \mathbb{E}_{\mathbb{Q}}\left[\frac{(h^*_{t+j-1})^2}{y^*_{t+j-1}} \mid \mathcal{F}_{t+j-2}\right] \\
&= w^* + \left[b + \frac{c^*}{(\eta^*)^2} + a^* (\eta^*)^2\right] h^*_{t+j-1} + a^* (\eta^*)^4 \\
&= h^*_{t+j-1}\psi^* + h^*_0 \left[1 - \psi^*\right]
\end{aligned}$$

where $\psi^* = b + \frac{c^*}{(\eta^*)^2} + a^* (\eta^*)^2$ is the variance persistence, and $h^*_0 = \frac{w^* + a^* (\eta^*)^4}{1 - \psi^*}$ is the unconditional volatility, under the risk-neutral probability. Now, using the tower property of the conditional expectation operator, the $j-$step ahead prediction of the risk-neutral volatility under the risk neutral measure is given by

$$\mathbb{E}_{\mathbb{Q}}\left[h^*_{t+j} \mid \mathcal{F}_t\right] = h^*_{t+1} \left[\psi^*\right]^{j-1} + h^*_0 \left[1 - (\psi^*)^{j-1}\right]$$

and (2.7) follows easily from (2.6). □

[22] Using the fact that an IG random variable Z with degree of freedom δ fulfills $E[\frac{1}{Z}] = \frac{1}{\delta} + \frac{1}{\delta^2}$.

Tables and figures

Table 1: Descriptive statistics of the $S\&P500$ and VIX datasets covering the period January 7, 1999-December 22, 2010.

	Number of observations	Min	Max	Mean	Std Dev	Skewness	Kurtosis
Price index	2718	676.53	1565.15	1182.75	190.14	−0.0959	−0.6909
Log returns	2718	−0.0947	0.1096	−0.0001	0.0139	−0.1214	7.3758
VIX index	2718	9.8900	80.8600	22.1859	9.6098	1.8853	5.6964
Log VIX	2718	−0.3506	0.4960	−0.0001	0.0613	0.5697	4.1682

Table 2: Properties of the in-sample options data (2009), the table shows the number of contracts, the average price, and the average implied volatility across moneynesses and times to maturities.

	$T < 60$	$60 \leqslant T \leqslant 180$	$T > 180$	**All**
Number of call option contracts :				
$0 < S/K < 0.975$	112	300	118	530
$0.975 < S/K < 1.00$	19	48	16	83
$1.00 < S/K < 1.025$	16	40	24	80
$1.025 < S/K < 1.05$	17	45	12	74
$1.05 < S/K < 1.075$	17	39	16	72
$1.075 < S/K$	90	267	126	483
All	271	739	312	1322
Average call price :				
$0 < S/K < 0.975$	9.213	27.174	48.402	28.263
$0.975 < S/K < 1.00$	30.933	63.110	89.545	61.196
$1.00 < S/K < 1.025$	43.823	71.326	96.972	70.707
$1.025 < S/K < 1.05$	57.645	87.631	105.020	83.432
$1.05 < S/K < 1.075$	74.122	97.937	124.152	98.737
$1.075 < S/K$	137.727	163.161	177.191	159.360
All	58.911	85.056	106.880	83.616
Average implied volatility from call options :				
$0 < S/K < 0.975$	0.246	0.245	0.251	0.247
$0.975 < S/K < 1.00$	0.273	0.268	0.282	0.275
$1.00 < S/K < 1.025$	0.262	0.261	0.264	0.262
$1.025 < S/K < 1.05$	0.279	0.272	0.274	0.275
$1.05 < S/K < 1.075$	0.297	0.277	0.272	0.282
$1.075 < S/K$	0.342	0.300	0.283	0.308
All	0.283	0.271	0.271	0.275

Table 3: Properties of the out of sample options data (2010), the table shows the number of contracts, the average price, and the average implied volatility across moneynesses and times to maturities.

	$T < 60$	$60 \leqslant T \leqslant 180$	$T > 180$	**All**
Number of call option contracts :				
$0 < S/K < 0.975$	89	311	168	568
$0.975 < S/K < 1.00$	21	55	28	104
$1.00 < S/K < 1.025$	20	56	30	106
$1.025 < S/K < 1.05$	18	48	25	91
$1.05 < S/K < 1.075$	20	54	24	98
$1.075 < S/K$	69	293	143	505
All	237	817	418	1472
Average call price :				
$0 < S/K < 0.975$	5.654	19.504	36.922	20.693
$0.975 < S/K < 1.00$	25.600	55.743	81.932	54.425
$1.00 < S/K < 1.025$	41.917	72.037	96.380	70.111
$1.025 < S/K < 1.05$	61.682	87.728	111.314	86.908
$1.05 < S/K < 1.075$	80.434	106.678	126.177	104.430
$1.075 < S/K$	146.301	178.813	196.600	173.905
All	60.264	86.751	108.221	85.079
Average implied volatility from call options :				
$0 < S/K < 0.975$	0.161	0.174	0.182	0.172
$0.975 < S/K < 1.00$	0.177	0.198	0.205	0.194
$1.00 < S/K < 1.025$	0.202	0.207	0.211	0.207
$1.025 < S/K < 1.05$	0.202	0.210	0.213	0.208
$1.05 < S/K < 1.075$	0.226	0.222	0.211	0.220
$1.075 < S/K$	0.260	0.235	0.228	0.241

Table 4: Estimated parameters for the IG model and the two stochastic discount factors.

Joint-Estimation	Returns-Option		Returns-VIX	
Model	M_t^{ess}	M_t^{Ushp}	M_t^{ess}	M_t^{Ushp}
Parameters :				
w	$6.5759\ e^{-06}$	$6.4945\ e^{-06}$	$4.0781\ e^{-06}$	$1.9930\ e^{-06}$
Stand.Dev	$(5.9687\ e^{-09})$	$(1.340\ e^{-07})$	$(6.953\ e^{-08})$	$(9.798\ e^{-08})$
b	$1.4019\ e^{-03}$	$5.4045\ e^{-01}$	$1.2642\ e^{-03}$	$6.4464\ e^{-01}$
Stand.Dev	$(3.5485\ e^{-06})$	$(1.165\ e^{-04})$	$(1.659\ e^{-04})$	$(2.301\ e^{-05})$
c	$5.1407\ e^{-05}$	$1.1031\ e^{-05}$	$5.0538\ e^{-05}$	$8.4884\ e^{-06}$
Stand.Dev	$(1.931\ e^{-03})$	$(4.318\ e^{-04})$	$(9.735\ e^{-09})$	$(6.968\ e^{-04})$
a	$3.0174\ e^{+03}$	$3.6359\ e^{+02}$	$3.3177\ e^{+03}$	$6.7459\ e^{+02}$
Stand.Dev	$(2.339\ e^{-04})$	$(1.853\ e^{-01})$	$(2.621\ e^{-01})$	$(1.883\ e^{-01})$
η	$-8.252\ e^{-03}$	$-5.0328\ e^{-03}$	$-8.2753\ e^{-03}$	$-5.03224\ e^{-03}$
Stand.Dev	$(2.450\ e^{-10})$	$(1.6027\ e^{-06})$	$(2.535\ e^{-04})$	$(4.392\ e^{-07})$
ν	$1.2122\ e^{+02}$	$1.9460\ e^{+02}$	$1.2122\ e^{+02}$	$1.9445\ e^{+02}$
Stand.Dev	$(7.1635\ e^{-03})$	$(8.0899\ e^{-02})$	$(7.509\ e^{-02})$	$(1.437\ e^{-02})$
π	−	1.2453	−	1.3138
Stand.Dev	−	$(5.0449\ e^{-02})$	−	$(1.341\ e^{-02})$
ϱ	−	−	$9.9585\ e^{-01}$	$9.9161\ e^{-01}$
Stand.Dev	−	−	$(1.118\ e^{-04})$	$(2.280\ e^{-04})$
Model Properties :				
Persistence	0.9649	0.9777	0.9702	0.9911
Annualized volatility	0.20358	0.24513	0.3281	0.2885
Leverage coefficient	−0.0042	−0.0026	−0.0043	−0.0020
Pricing performances :				
IVRMSE in sample (2009)	0.0523	0.04215	0.0544	0.0440
IVRMSE out of sample (2010)	0.1285	0.09727	0.1375	0.0986

For the returns-option strategy, the model is estimated using the log-returns dataset obtained from the closing prices of the $S\&P500$ between January 07, 1999 and December 31, 2009 and the in sample (2009) option contracts minimizing (3.11). For the returns-VIX one, the model parameters are obtained minimizing (3.14) using the log-returns and VIX data from January 07, 1999 to December 31, 2009.

Table 5: GMM tests for the estimated models to test the moment condition on returns

Estimation\ SDF	M_t^{ess}	M_t^{Ushp}
Returns-option	0.003113	0.002287
Returns-VIX	−0.031945	−0.001627

We compute the statistics t_S for the IG model both combined with the Esscher and the $U-$shaped stochastic discount factors. In each case, the model parameters are estimated using the Returns-option and the Returns-VIX strategies.

Table 6: GMM tests, desegregated by moneynesses and times to maturities, to test the moment condition on options for the IG model combined with M_t^{ess} and estimated using the returns-option strategy.

	$T < 60$	$60 \leqslant T \leqslant 180$	$T > 180$	**All**
$0 < S/K < 0.975$	−0.017258101	−0.0136872098	−0.07538249	−0.014646453
$0.975 < S/K < 1.00$	−0.039620826	0.0129173186	−0.09619293	−0.010465128
$1.00 < S/K < 1.025$	−0.129469574	−0.0456363351	−0.15851951	−0.055335364
$1.025 < S/K < 1.05$	0.003544071	0.0004430793	−0.14311745	−0.010701396
$1.05 < S/K < 1.075$	−0.108593938	−0.0242357128	−0.19067001	−0.045369457
$1.075 < S/K$	−0.041329272	−0.0214402374	−0.06120986	−0.020496088
All	−0.014862593	−0.0087567933	−0.04112527	−0.009686458

Table 7: GMM tests, desegregated by moneynesses and times to maturities, to test the moment condition on options for the IG model combined with M_t^{Ushp} and estimated using the returns-option strategy.

	$T < 60$	$60 \leqslant T \leqslant 180$	$T > 180$	**All**
$0 < S/K < 0.975$	−0.0004344452	0.0011770689	−0.04185127	−0.0034510578
$0.975 < S/K < 1.00$	0.0159685870	0.0320895639	−0.09320900	0.0095889416
$1.00 < S/K < 1.025$	−0.1495197908	0.0155962006	−0.04853216	−0.0108747469
$1.025 < S/K < 1.05$	0.0620291249	0.0281280309	−0.02678905	0.0173579275
$1.05 < S/K < 1.075$	−0.0952223260	0.0227545857	−0.02405072	−0.0005099153
$1.075 < S/K$	−0.0085787816	−0.0004703747	−0.02814252	−0.0047468849
All	−0.0020122851	0.0015137772	−0.01882751	−0.0017628326

Table 8: GMM tests, desegregated by moneynesses and times to maturities, to test the moment condition on options for the IG model combined with M_t^{ess} and estimated using the returns-VIX strategy.

	$T < 60$	$60 \leqslant T \leqslant 180$	$T > 180$	**All**
$0 < S/K < 0.975$	−0.017491375	−0.0138173612	−0.07569169	−0.014753708
$0.975 < S/K < 1.00$	−0.039761880	0.0129714548	−0.09639429	−0.010483069
$1.00 < S/K < 1.025$	−0.130020198	−0.0464413950	−0.15901562	−0.055659121
$1.025 < S/K < 1.05$	0.002894318	0.0001089329	−0.14438269	−0.011002377
$1.05 < S/K < 1.075$	−0.108907777	−0.0247088917	−0.19152152	−0.045649178
$1.075 < S/K$	−0.041522823	−0.0215391411	−0.06145062	−0.020580660
All	−0.014995412	−0.0088290586	−0.04128875	−0.009745392

Table 9: GMM tests, desegregated by moneynesses and times to maturities, to test the moment condition on options for the IG model combined with M_t^{Ushp} and estimated using the returns-VIX strategy.

	$T < 60$	$60 \leqslant T \leqslant 180$	$T > 180$	**All**
$0 < S/K < 0.975$	−0.004684884	−0.006401658	−0.06216133	−0.008551977
$0.975 < S/K < 1.00$	0.003587144	0.029941636	−0.09484246	0.005369085
$1.00 < S/K < 1.025$	−0.157457464	−0.020157752	−0.12658172	−0.044397168
$1.025 < S/K < 1.05$	0.050213884	0.021417297	−0.05051933	0.010863591
$1.05 < S/K < 1.075$	−0.100132796	0.001191484	−0.09977091	−0.023839605
$1.075 < S/K$	−0.017754327	−0.009652761	−0.04613641	−0.011797260
All	−0.005418894	−0.003197436	−0.03144532	−0.005322943

Table 10: Computation times (in hours) to estimate the IG model with the different estimation and risk-neutralization strategies

Estimation\ SDF	Returns-Option	Returns-VIX
M_t^{ess}	21.038	0.018 (66.8 *sec*)
M_t^{Ushp}	20.697	0.025 (92.4 *sec*)

Table 11: Comparison, based on the IVRMSE, of empirical pricing performances of the IG-GARCH model using M_t^{ess} or M_t^{Ushp}

Model	Returns-Option	Returns-VIX
IVRMSE (2009)	19.40%	19.11%
IVRMSE (2010)	24.65%	28.29%

For example, the value 19.40% represents the improvement (in percentage) of the pricing error for the IG-GARCH model estimated using the returns-option strategy when we use the U-shaped pricing kernel instead of the exponential affine one.

Table 12: In-sample IVRMSE, desegregated by moneynesses and time to maturities, using the returns-option estimates and M_t^{ess}.

	$T < 60$	$60 \leqslant T \leqslant 180$	$T > 180$	**All**
$0 < S/K < 0.975$	0.066110446	0.019277028	0.010511988	0.034037343
$0.975 < S/K < 1.00$	0.009949488	0.008492271	0.007982866	0.008755149
$1.00 < S/K < 1.025$	0.011276463	0.009500098	0.009308076	0.009825972
$1.025 < S/K < 1.05$	0.014463079	0.010109781	0.009355752	0.011153583
$1.05 < S/K < 1.075$	0.024128216	0.012097448	0.010811183	0.015578917
$1.075 < S/K$	0.153035884	0.054426843	0.024207488	0.078449833
All	0.098225453	0.035279455	0.017255154	0.052381679

Table 13: In-sample IVRMSE, desegregated by moneynesses and time to maturities, using the returns-VIX estimates and M_t^{ess}.

	$T < 60$	$60 \leqslant T \leqslant 180$	$T > 180$	**All**
$0 < S/K < 0.975$	0.07227001	0.021782920	0.01213094	0.037484197
$0.975 < S/K < 1.00$	0.01072347	0.009648779	0.00937507	0.009854289
$1.00 < S/K < 1.025$	0.01164509	0.010511643	0.01057072	0.010765267
$1.025 < S/K < 1.05$	0.01505117	0.011304843	0.01084214	0.012199214
$1.05 < S/K < 1.075$	0.02470578	0.013244516	0.01228675	0.016513078
$1.075 < S/K$	0.15466605	0.057546436	0.02677213	0.080467683
All	0.10085421	0.037658544	0.01924780	0.054454451

Table 14: In-sample IVRMSE, desegregated by moneynesses and time to maturities, using the returns-option estimates and M_t^{Ushp}.

	$T < 60$	$60 \leqslant T \leqslant 180$	$T > 180$	**All**
$0 < S/K < 0.975$	0.050523634	0.008753392	0.005154439	0.024263375
$0.975 < S/K < 1.00$	0.006403037	0.004431021	0.003908261	0.004866656
$1.00 < S/K < 1.025$	0.007428218	0.005296823	0.004840060	0.005664954
$1.025 < S/K < 1.05$	0.010003493	0.005720761	0.004867728	0.006836157
$1.05 < S/K < 1.075$	0.019205732	0.007315557	0.006293074	0.011175024
$1.075 < S/K$	0.130926105	0.040796905	0.016189430	0.064672512
All	0.082360855	0.025299360	0.011019208	0.042154206

Table 15: In-sample IVRMSE, desegregated by moneynesses and time to maturities, using the returns-VIX estimates and M_t^{Ushp}.

	$T < 60$	$60 \leqslant T \leqslant 180$	$T > 180$	**All**
$0 < S/K < 0.975$	0.058999947	0.007328361	0.004096775	0.027744223
$0.975 < S/K < 1.00$	0.008001028	0.005065256	0.004373989	0.005760217
$1.00 < S/K < 1.025$	0.008296737	0.005342967	0.004654264	0.005877029
$1.025 < S/K < 1.05$	0.011189721	0.006266320	0.006028062	0.007650846
$1.05 < S/K < 1.075$	0.018965308	0.007764575	0.006372408	0.011251906
$1.075 < S/K$	0.136322195	0.040442062	0.016504625	0.066618316
All	0.087460457	0.024930153	0.011067243	0.044095221

Table 16: Out of sample IVRMSE, desegregated by moneynesses and time to maturities, using the returns-option estimates and M_t^{ess}.

	$T < 60$	$60 \leqslant T \leqslant 180$	$T > 180$	**All**
$0 < S/K < 0.975$	0.15519418	0.09680432	0.02294874	0.09518761
$0.975 < S/K < 1.00$	0.01695327	0.01295548	0.01222848	0.01367695
$1.00 < S/K < 1.025$	0.01624557	0.01322850	0.01241347	0.01363293
$1.025 < S/K < 1.05$	0.02396978	0.01510924	0.01348541	0.01685302
$1.05 < S/K < 1.075$	0.03521808	0.01675262	0.01489015	0.02149575
$1.075 < S/K$	0.35368801	0.13933129	0.18258228	0.19441065
All	0.21368395	0.10288149	0.10798550	0.12859895

Table 17: Out of sample IVRMSE, desegregated by moneynesses and time to maturities, using the returns-VIX estimates and M_t^{ess}.

	$T < 60$	$60 \leqslant T \leqslant 180$	$T > 180$	**All**
$0 < S/K < 0.975$	0.17214693	0.10976845	0.02586650	0.10695164
$0.975 < S/K < 1.00$	0.01768244	0.01411207	0.01359858	0.01477300
$1.00 < S/K < 1.025$	0.01661613	0.01437070	0.01375944	0.01465533
$1.025 < S/K < 1.05$	0.02498530	0.01637078	0.01497219	0.01806735
$1.05 < S/K < 1.075$	0.03588704	0.01816551	0.01646552	0.02260656
$1.075 < S/K$	0.35961954	0.15016354	0.19984799	0.20509211
All	0.22133149	0.11286499	0.11826782	0.13758423

Table 18: Out of sample IVRMSE, desegregated by moneynesses and time to maturities, using the returns-option estimates and M_t^{Ushp}.

	$T < 60$	$60 \leqslant T \leqslant 180$	$T > 180$	**All**
$0 < S/K < 0.975$	0.10472832	0.062781860	0.014702687	0.062774694
$0.975 < S/K < 1.00$	0.01276129	0.008812444	0.007908489	0.009528476
$1.00 < S/K < 1.025$	0.01161843	0.009172681	0.008095419	0.009405722
$1.025 < S/K < 1.05$	0.01836763	0.010680433	0.009056761	0.012224422
$1.05 < S/K < 1.075$	0.02657740	0.011782546	0.009975221	0.015653134
$1.075 < S/K$	0.27586165	0.111926742	0.137738553	0.151784399
All	0.16243596	0.077588868	0.081220584	0.097276679

Table 19: Out of sample IVRMSE, desegregated by moneynesses and time to maturities, using the returns-VIX estimates and M_t^{Ushp}.

	$T < 60$	$60 \leqslant T \leqslant 180$	$T > 180$	**All**
$0 < S/K < 0.975$	0.11187002	0.071848201	0.013216852	0.069563609
$0.975 < S/K < 1.00$	0.01163551	0.007950823	0.007108631	0.008624016
$1.00 < S/K < 1.025$	0.01115946	0.008678356	0.007351717	0.008864643
$1.025 < S/K < 1.05$	0.01742428	0.009819353	0.008223511	0.011379420
$1.05 < S/K < 1.075$	0.02534974	0.011699993	0.009747100	0.015160497
$1.075 < S/K$	0.26477928	0.119580344	0.131830590	0.150986449
All	0.15877875	0.084364490	0.077669321	0.098602385