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A One-Sector Optimal Growth Model in Which Consuming Takes Time *

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Abstract

This article establishes a growth model in which consumption takes time. The agent faces a time constraint, i.e; her/his available amount of time must be optimally share between consuming time and working time. By using a dynamic programming argument, it is proved that the optimal capital sequences are monotonic and have property that converges to steady state.
We also compare this model to the one agent growth model with elastic labor. We obtain that
(i) When the quantity of time to consume one unit of consumption increases, the agent devotes less time for labour.
(ii) When the quantity of time to consume one unit of consumption is smaller than the threshold, it is better for the economy to spend time to consume than to enjoy leisure. We have more time for labour. This implies more output and more consumption. We reverse the situation when the quantity of time to consume one unit of consumption is larger than the threshold. We give an example to illustrate this result.
Finally, if both models have the same technology which is of constant returns to scale, then they have the same ratios capital stock per head and consumption per head.

Keywords. time consuming model, allocation of time, elastic labour, leisure, value function.

1 Introduction

The standard economic models are based on assumptions either that consumption is instantaneous, i.e. does not take time or that the economic agent has a sufficiently large amount of time to consume any finite bundle of commodities. In the reality, none of these assumptions is plausible. Consumption takes time and every agent has only a finite amount of time to allocate between working and consumption, or between consumption and leisure. Note further that the act of consumption is broadly interpreted to include search, purchase, preparation, and consumption.

The consumption time constraint was first formally introduced by Gossen [3] and much later generalized by Becker [1] in the famous theory of time allocation. Tran-Nam and Pham [8] assume that the quantity of time to consume one unit of consumption is constant. The sum of the time devoted for consumption and the one devoted for labour is bounded by the available amount of time of the agent. Le Van and al. [7] extend this model to a general equilibrium model with many goods and many heterogeneous economic agents.

In this paper, we establish a growth model with one agent, in which consumption is itself time consuming. At each period, the agent must share her/his available amount of time between the consuming time and the working time. We obtain that this model produces results which are similar to those obtained from a single-sector growth model with elastic labor, see e.g. [6]. We prove the monotonicity of the optimal capital sequences and prove the existence of optimal steady state. In our model, the felicity of the agent is only based on her/his stream of consumptions. But, as we wrote above, she/he must make a trade-off between her/his consuming time and her/his working time. In the one-sector growth model with elastic labor, the consumption does not take time. But the felicity of the agent depends on the streams of consumptions and of leisures. The trade off between consumption and labour is formalized by the utility function taking into account consumption and leisure.

Some interesting results arise when we compare this model to the one agent growth model with elastic labor. We obtain that
(i) When the quantity of time to consume one unit of consumption increases, the agent devotes less time for labour.
(ii) When the quantity of time to consume one unit of consumption is smaller than the threshold, it is better for the economy to spend time to consume than to enjoy leisure. We have more time for labour. This implies more output and more consumption. We reverse the situation when the quantity of time to consume one unit of consumption is larger than the threshold. We give an example to illustrate this result.

The remainder of this paper is organized as follows. Section 2 describes the model. Section 3 characterizes the indirect utility function. Section 4 is devoted to the study of the value function. Section 5 deals with the properties of optimal paths. We compare our model with the one-agent growth model with elastic labor in Section 6. Some long proofs are gathered in the Appendix.

2 The Model

We consider an economy populated by consumers which are identical in all respects. Preferences are represented by the function

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

where u is the instantaneous utility function and $\beta \in (0, 1)$ is the discount factor.

The capital stock and consumption of consumer at period t are denoted by k_t and c_t respectively. We assume that the initial capital stock k_0 and labour endowment L (measured in time units) are exogenous.

In this economy there is one good produced by a single firm using as inputs physical capital (k_t) and labour (l_t).

$$Y_t = F(k_t, l_t)$$

The dynamics of the sequence of capital stocks:

$$k_{t+1} = (1 - \delta)k_t + I_t$$

where I_t is investment and $\delta \in [0, 1]$ is the rate of depreciation of capital.

In each period , consumers face resource constraint:

$$c_t + I_t \leq Y_t$$

Let $a > 0$ denote $a > 0$ the technological coefficient associated with this good. It is the quantity of time to consume one unit of good. At period t, the consumer purchases the quantity c_t of good and spends ac_t of time to transform these consumption goods to the "final" consumption goods that the consumer wishes to enjoy. The consumer also spends l_t of time to work in firm. The time constraint is given by

$$ac_t + L_t \leq L$$

The social planner wants to maximize the global utility of the consumer:

$$(P) \qquad \max_{(c_t,k_{t+1},l_t)} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

under the constraints:

$$c_t + k_{t+1} - (1-\delta)k_t \leq F(k_t, l_t) \tag{1}$$

$$ac_t + l_t \leq L \tag{2}$$

$$c_t \geq 0, l_t \geq 0, k_{t+1} \geq 0 \tag{3}$$

$$k_0 \geq 0 \text{ is given}$$

We next specify the properties assumed for the preferences and the technology.

The Assumptions

Assumption P1. $k_0 \geq 0, L > 0, a > 0$

Assumption P2. *The utility function $u(c)$ is continuous, strictly increasing, strictly concave, $u(0) = 0$.*

Assumption P3. *u is continuous differentiable on $\mathbb{R}_+$, $\lim_{c \to 0} u'(c) = +\infty$.*

Assumption P4. *The production function $F(k,l)$ is concave, strictly increasing for $k > 0, l > 0$, $F(k,0) = F(0,l) = 0$.*

Assumption P5. *$F(k,l)$ is twice continuously differentiable on $\mathbb{R}^2_{++}$. In addition, $\lim_{k \to +\infty} F_1(k,L) = 0$, and for all $k > 0$: $\lim_{l \to 0} F_2(k,l) = +\infty$.*

Assumption P6. $\lim_{k \to 0} F_1(k,L) > \frac{1}{\beta} - 1 + \delta$.

3 Characterization of the Indirect Utility Function

Consider problem (P). The two following lemmas claim that the budget constraint and the time constraints are binding at optimal.

Lemma 3.1. *Assume* **P1** *and* **P2**. *If $(c_t^*, k_{t+1}^*, l_t^*)$ is an optimal solution of (P) then, for any $t \geq 0$:*

$$ac_t^* + l_t^* = L. \tag{4}$$

Proof. If $ac_t^* + l_t^* < L$, we can increase both c_t^* and l_t^* such that

$$a(c_t^* + \epsilon_c) + (l_t^* + \epsilon_l) \leq L.$$

and

$$(c_t^* + \epsilon_c) + k_{t+1}^* - (1-\delta)k_t^* \leq F(k_t^*, l_t^* + \epsilon_l).$$

In this case we have $u(c_t^* + \epsilon_c) > u(c_t^*)$ contradiction with the optimal solution of (c_t^*, l_t^*). □

Then problem (P) is equivalent to

$$(P') \qquad \max_{(k_{t+1}, l_t)} \sum_{t=0}^{\infty} \beta^t u\left(\frac{L - l_t}{a}\right)$$

under the constraints: $(\forall t \geq 0)$

$$0 \leq k_{t+1} \leq F(k_t, l_t) + (1-\delta)k_t + \frac{l_t - L}{a}$$
$$0 \leq l_t \leq L$$
$$k_0 \geq 0 \text{ is given}$$

Lemma 3.2. *Assume* **P1** *and* **P2**. *If* $(c_t^*, k_{t+1}^*, l_t^*)$ *is an optimal solution to* (P) *then, for any* $t \geq 0$*:*

$$l_t^* > 0 \tag{5}$$
$$k_{t+1}^* = F(k_t^*, l_t^*) + (1-\delta)k_t^* + \frac{l_t^* - L}{a} \tag{6}$$
$$(1-\delta)k_t^* - \frac{L}{a} \leq k_{t+1}^* \leq F(k_t^*, L) + (1-\delta)k_t^* \tag{7}$$

Proof. See appendix. □

To make further analysis we will study the value function. Working in this direction, we define the technology set as following:

Let
$$\Gamma(k) = \{k' : \max\{0, (1-\delta)k - \frac{L}{a}\} \leq k' \leq F(k, L) + (1-\delta)k\}$$
and
$$\text{Graph}\Gamma = \{(k, k') : k \geq 0, k' \in \Gamma(k)\}.$$
Given $k_0 \geq 0$, denote by $\Pi(k_0)$ the set of feasible capital sequences from k_0, i.e.

$$\Pi(k_0) = \{\mathbf{k} = (k_0, k_1, \ldots, k_t, \ldots) : k_{t+1} \in \Gamma(k_t)\ \forall t \geq 0\}.$$

Lemma 3.3. *Under assumptions* **P4, P5** *and* **P6***,*

i) there exists $\bar{k} \geq 0$ *such that for any* $k' \in \Gamma(k)$ *then* $k' \leq \max(\bar{k}, k)$*.*

ii) The correspondence $\Gamma : \mathbb{R}_+ \to \mathbb{R}_+$ *is continuous, with non-empty, compact convex values. Moreover* $0 \in \Gamma(0)$*.*

iii) The set $\Pi(k_0)$ *is compact for the product topology.*

Proof. See appendix.

□

Definition 3.1. *For* $(k, y) \in Graph(\Gamma)$ *we define the indirect utility function* V *by:*

$$V(k, y) = \max_{(c,l)} u(c)$$
$$s.t. \quad c + y \leq F(k, l) + (1 - \delta)k$$
$$ac + l \leq L$$
$$c \geq 0; \quad 0 \leq l \leq L$$

The following proposition establishes the properties of the indirect utility function.

Proposition 3.1. *Assume* **P1-P6**.

1. *Assume* $y \in int(\Gamma(k))$. *Let* $c^*(k, y)$ *and* $l^*(k, y)$ *denote the solution to the problem defined in 3.1. We have* (c^*, l^*) *is unique with* $c^* > 0$ *and* $l^* \in (0, L)$.

2. V *is differentiable in* $int(Graph\Gamma)$ *and*

$$\frac{\partial V(k, y)}{\partial k} = \lambda[F_1(k, l^*) + (1 - \delta)]$$
$$\frac{\partial V(k, y)}{\partial y} = -\lambda$$

where $\lambda = \dfrac{u'(c^*)}{1 + aF_2(k, l^*)}$.

3. V *is increasing in its first variable and decreasing in its second one.*

4. V *is strictly concave. Moreover, there exists* $A \geq 0, B \geq 0$ *such that* $V(k, y) \leq A + B(k + y)$.

Proof. See appendix.

□

Corollary 3.1. *Assume* **P1-P6**. *For each* $(k, y) \in Graph\Gamma$, *we can rewrite the function* V *as following:*

$$V(k, y) = u\left(\frac{L - l}{a}\right)$$

with:

$$y = F(k, l) + (1 - \delta)k + \frac{l - L}{a}.$$

Corollary 3.2. *Problem* (P) *is equivalent to:*

$$(P_2) \quad \max_{\Pi(k_0)} \sum_{t=0}^{\infty} \beta^t V(k_t, k_{t+1}), \quad k_0 \geq 0 \text{ is given}$$

Corollary 3.3. *Assume* **P1-P6**. *For every* $k_0 \geq 0$, *the sum* $\sum_{t=0}^{+\infty} \beta^t V(k_t, k_{t+1})$ *exists and is finite-valued.*

Proof. From proposition 3.1 and lemma 3.3, there existss $A \geq 0, B \geq 0, K \geq 0$:

$$\begin{aligned} V(k_t, k_{t+1}) &\leq A + B(k_t + k_{t+1}), \quad \forall t \geq 0 \\ &\leq A + 2BK \\ \Rightarrow \sum_{t=0}^{\infty} \beta^t V(k_t, k_{t+1}) &\leq (A + 2BK) \sum_{t=0}^{\infty} \beta^t < +\infty \end{aligned}$$

□

4 Characterization of the Value Function

Definition 4.1. *For* $\mathbf{k} \in \Pi(k_0)$, *we denote*

$$w(\mathbf{k}) = \sum_{t=0}^{+\infty} \beta^t V(k_t, k_{t+1}).$$

Let

$$\begin{aligned} W(k_0) &= \max\{w(\mathbf{k}) : \mathbf{k} \in \Pi(k_0)\} \\ &= \max_{\mathbf{k} \in \Pi(k_0)} \sum_{t=0}^{+\infty} \beta^t V(k_t, k_{t+1}) \end{aligned}$$

be the Value Function of the optimal growth problem (P).

Lemma 4.1. *Under assumptions* **P1-P6***,*
i) The function w *is continuous,*
ii) The value function W *is strictly concave, continuous.*

Proof. i) Let $\{k^n\}$ be a sequence of $\Pi(k_0)$ converging to $k \in \Pi(k_0)$. From lemma 3.3, for all $\varepsilon > 0$ there existss $T_0 > 0$ for all $T \geq T_0$:

$$\sum_{t=T}^{+\infty} \beta^t V(k_t^n, k_{t+1}^n) < \varepsilon, \qquad \sum_{t=T}^{+\infty} \beta^t V(k_t, k_{t+1}) < \varepsilon$$

Fix T. Because of continuity of V, there existss $N_0 \geq 0$, for all $N \geq N_0$:

$$|V(k_t^n, k_{t+1}^n) - V(k_t, k_{t+1})| < \varepsilon \frac{1-\beta}{1-\beta^T}$$

Then we have, for any $\varepsilon > 0$, any $N \geq N_0$:

$$\begin{aligned} |w(k^n) - w(k)| &= \left| \sum_{t=0}^{+\infty} \beta^t V(k_t^n, k_{t+1}^n) - \sum_{t=0}^{+\infty} \beta^t V(k_t, k_{t+1}) \right| \\ &\leq \sum_{t=T}^{+\infty} \beta^t V(k_t^n, k_{t+1}^n) + \sum_{t=T}^{+\infty} \beta^t V(k_t, k_{t+1}) + \sum_{t=0}^{T-1} \beta^t \left| V(k_t^n, k_{t+1}^n) - V(k_t, k_{t+1}) \right| \\ &\leq 2\varepsilon + \sum_{t=0}^{T} \beta^t \varepsilon \frac{1-\beta}{1-\beta^T} = 3\varepsilon \end{aligned}$$

this proves the continuity of w.

ii) Given $k_0 \geq 0, k'_0 \geq 0, k_0 \neq k'_0$. there exist sequences $k = (k_0, k_1, \ldots), k' = (k'_0, k'_1, \ldots)$ such that

$$W(k_0) = \sum_{t=0}^{+\infty} \beta^t V(k_t, k_{t+1}); \quad W(k'_0) = \sum_{t=0}^{+\infty} \beta^t V(k'_t, k'_{t+1})$$

Let $\lambda \in [0,1]$. Denote $k^\lambda = \lambda k + (1-\lambda)k' = (\lambda k_t + (1-\lambda)k'_t)_t$. Since $k \in \Pi(k_0), k' \in \Pi(k'_0)$ we have for all $t \geq 0$:

$$\begin{aligned} (1-\delta)k_t - \frac{L}{a} &\geq k_{t+1} \geq F(k_t, L) + (1-\delta)k_t \\ (1-\delta)k'_t - \frac{L}{a} &\geq k'_{t+1} \geq F(k'_t, L) + (1-\delta)k'_t \end{aligned}$$

this implies

$$\lambda k_{t+1} + (1-\lambda)k'_{t+1} \geq (1-\delta)[\lambda k_t + (1-\lambda)k'_t] - \frac{L}{a}$$

and

$$\begin{aligned} \lambda k_{t+1} + (1-\lambda)k'_{t+1} &\leq \lambda F(k_t, L) + (1-\lambda)F(k'_t, L) + (1-\delta)[\lambda k_t + (1-\lambda)k'_t] \\ &\leq F(\lambda k_t + (1-\lambda)k'_t, L) + (1-\delta)[\lambda k_t + (1-\lambda)k'_t] \end{aligned}$$

hence $k^\lambda \in \Pi(\lambda k_0 + (1-\lambda)k'_0)$. We have:

$$\begin{aligned} W(\lambda k_0 + (1-\lambda)k'_0) &\geq \sum_{t=0}^{+\infty} \beta^t V(k_t^\lambda, k_{t+1}^\lambda) \\ &= \sum_{t=0}^{+\infty} \beta^t V(\lambda k_t + (1-\lambda)k'_t, \lambda k_{t+1} + (1-\lambda)k'_{t+1}) \\ &> \sum_{t=0}^{+\infty} \beta^t [\lambda V(k_t, k_{t+1}) + (1-\lambda)V(k'_t, k'_{t+1})] \\ &= \lambda \sum_{t=0}^{+\infty} \beta^t V(k_t, k_{t+1}) + (1-\lambda) \sum_{t=0}^{+\infty} \beta^t V(k'_t, k'_{t+1}) \\ &= \lambda W(k_0) + (1-\lambda)W(k'_0) \end{aligned}$$

We have proved the concavity of W. Since W is concave on $[0, +\infty)$, it is continuous in $(0, +\infty)$. We will show that W is continuous at 0.

It is easy to see that $\Pi(0) = 0$ and $W(0) = 0$. Let k_0 converge to zero, $(k_t^*)_{t=0}^\infty \in \Pi(k_0)$ and

$$W(k_0) = \sum_{t=0}^{\infty} V(k_t^*, k_{t+1}^*)$$

For any $\varepsilon > 0$ and for T large enough we have

$$\sum_{t=T}^{\infty} V(k_t^*, k_{t+1}^*) < \frac{\varepsilon}{2}$$

On the other hand, since $V(k_t^*, .)$ is decreasing we have

$$\begin{aligned} V(k_t^*, k_{t+1}^*) &\leq V(k_t^*, 0) \\ &= \max\{u(\frac{L-l}{a}):\ F(k_t^*, l) + (1-\delta)k_t^* + \frac{l-L}{a} \geq 0\} \\ &\leq u(F(k_t^*, L) + (1-\delta)k_t^*) \end{aligned}$$

Denote $\gamma(k) = F(k, L) + (1-\delta)k$ and reminder that $k_{t+1}^* \leq \gamma(k_t^*)$ we have:

$$\begin{aligned} V(k_0, k_1^*) &\leq u(\gamma(k_0)) \\ V(k_1^*, k_2^*) &\leq V(\gamma(k_0), k_2^*) \leq u(\gamma^2(k_0)) \\ V(k_t^*, k_{t+1}^*) &\leq u(\gamma^{t+1}(k_0)) \quad \forall t \geq 0 \end{aligned}$$

Hence

$$W(k_0) \leq \sum_{t=0}^{T-1} u(\gamma^{t+1}(k_0)) + \sum_{t=T}^{\infty} V(k_t^*, k_{t+1}^*)$$

using the continuity of V and γ, these exists a neighborhood of 0 such that

$$|u(\gamma^{t+1}(k_0)) - u(\gamma^{t+1}(0))| < \frac{\varepsilon}{2T},\ \forall t \leq T$$

$$\Rightarrow \sum_{t=0}^{T-1} u(\gamma^{t+1}(k_0)) < \frac{\varepsilon}{2}$$

then

$$W(k_0) < \varepsilon$$

implies that W is continuous at 0.

□

Proposition 4.1. *Under assumptions* **P1-P6***, there exists a unique solution to problem (P)*

Proof. Problem (P) is equivalent to the maximization of a continuous function $w(.)$ over a compact set $\Pi(k_0)$, and therefore it admits a solution. From lemma 3.1, the function V is strictly concave, and then implies that the solution is unique. □

Proposition 4.2. *Under assumptions* **P1-P6***, the value function W satisfies the Bellman equation:*

$$W(k_0) = \max_{k' \in \Gamma(k_0)} \{V(k_0, k') + \beta W(k')\}$$

Proof. Let $k^* = (k_0, k_1^*, k_2^*, \ldots, k_t^*, \ldots)$ satisfy

$$W(k_0) = \max_{k \in \Pi(k_0)} \sum_{t=0}^{+\infty} \beta^t V(k_t, k_{t+1}) = \sum_{t=0}^{+\infty} \beta^t V(k_t^*, k_{t+1}^*)$$

We have:

$$\begin{aligned}
W(k_0) &= V(k_0, k_1^*) + \sum_{t=1}^{+\infty} \beta^t w(k_t^*, k_{t+1}^*) \\
&= V(k_0, k_1^*) + \beta W(k_1^*, k_2^*, \ldots, k_t^*, \ldots) \\
&\leq V(k_0, k_1^*) + \beta W(k_1^*) \\
&\leq \max_{k \in \Gamma(k_0)} \{V(k_0, k) + \beta W(k)\}
\end{aligned}$$

In the other hand, let $\bar{k_1} \in \Gamma(k_0)$, there exists $(\bar{k_2}, \bar{k_3}, \ldots) \in \Pi(\bar{k_1})$ such that

$$W(\bar{k_1}) = \max_{k \in \Pi(\bar{k_1})} \sum_{t=0}^{+\infty} \beta^t V(k_t, k_{t+1}) = \sum_{t=1}^{+\infty} \beta^{t-1} V(\bar{k}_t, \bar{k}_{t+1})$$

We have:

$$\begin{aligned}
V(k_0, \bar{k_1}) + \beta W(\bar{k_1}) &= V(k_0, \bar{k_1}) + \beta \sum_{t=1}^{+\infty} \beta^{t-1} V(\bar{k}_t, \bar{k}_{t+1}) \\
&= \sum_{t=0}^{+\infty} \beta^{t-1} V(\bar{k}_t, \bar{k}_{t+1}) \\
&= w(k_0, \bar{k_1}\bar{k_2}, \ldots) \\
&\leq W(k_0) \\
\Rightarrow \max_{k \in \Gamma(k_0)} \{V(k_0, k) + \beta W(k)\} &\leq W(k_0)
\end{aligned}$$

Hence

$$W(k_0) = \max_{k' \in \Gamma(k_0)} \{V(k_0, k') + \beta W(k')\}$$

□

5 Properties of Optimal Paths

To establish additional properties of the optimal solution, we need to impose following assumption.

Assumption P7. $F_{12}(k, l) \geq 0$ *for all* $k > 0,\ 0 < l < L$.

The results in the following lemma are crucial in establishing important properties of optimal paths.

Lemma 5.1. *Under assumptions* **P1-P7**, V *is twice differentiable and*

$$\frac{\partial^2 V(k, y)}{\partial y \partial k} > 0$$

Proof. See appendix. □

Proposition 5.1. *Under assumptions* **P1-P7**, *the optimal capital sequence* $(k_t^*)_t$ *from* k_0 *is monotonic.*

Proof. [1]

Consider the Value Function

$$W(k_0) = \max_{k_{t+1} \in \Gamma(k_t)} \sum_{t=0}^{\infty} \beta^t V(k_t, k_{t+1}), \quad k_0 \text{ is given}$$

Suppose $k_0 < k_0'$ and let (k_t) and (k_t') be optimal paths for k_0, k_0' respectively.

The Bellman equation gives:

$$\begin{array}{rcl} W(k_0) & = & V(k_0, k_1) + \beta W(k_1) \geq V(k_0, k_1') + \beta W(k_1') \\ W(k_0') & = & V(k_0', k_1') + \beta W(k_1') \geq V(k_0', k_1) + \beta W(k_1) \end{array}$$

This implies

$$V(k_0, k_1) + V(k_0', k_1') \geq V(k_0, k_1') + V(k_0', k_1) \tag{8}$$

Since the indirect utility function is twice continuously differentiable we can rewrite as following:

$$V(k_0, k_1') = \int_{k_1}^{k_1'} \frac{\partial V(k_0, y)}{\partial y} dy + V(k_0, k_1)$$

$$V(k_0', k_1) = \int_{k_1'}^{k_1} \frac{\partial V(k_0', y)}{\partial y} dy + V(k_0', k_1')$$

Substituting to (8) we then have

$$\int_{k_1}^{k_1'} \frac{\partial V(k_0, y)}{\partial y} dy + \int_{k_1'}^{k_1} \frac{\partial V(k_0', y)}{\partial y} dy \leq 0$$

implies

$$0 \leq \int_{k_1}^{k_1'} \left[\frac{\partial V(k_0', y)}{\partial y} - \frac{\partial V(k_0, y)}{\partial y} \right] dy = \int_{k_1}^{k_1'} \left[\int_{k_0}^{k_0'} \frac{\partial^2 V(x, y)}{\partial x \partial y} dx \right] dy$$

Hence

$$\int_{k_1}^{k_1'} \int_{k_0}^{k_0'} \frac{\partial^2 V(x, y)}{\partial x \partial y} dx dy \geq 0$$

[1] We adapt the proof given in [2] of Benhabib and Nishimura

From lemma 5.1 we have $\dfrac{\partial^2 V(x,y)}{\partial x \partial y} > 0$, this implies $k_1 \leq k'_1$. Setting $k_0 = k_{t-1}, k'_0 = k_t$ for all $t \geq 1$, we then receive that the optimal path (k_t) is monotonic.

□

Proposition 5.2. *Assume* **P1-P7**. *If* $\mathbf{k}^* = (k^*_t)_{t=0}^{\infty}$ *is optimal and satisfies* $(k^*_t, k^*_{t+1}) \in int(Graph\Gamma)$, $\forall t$, *then* $\mathbf{k}^*$ *satisfies the Euler equation:*

$$V_2(k^*_t, k^*_{t+1}) + \beta V_1(k^*_{t+1}, k^*_{t+2}) = 0, \ \forall t \geq 0$$

where V_1, V_2 *denote the derivatives of* V *with respect to the first and the second variables.*

Proof. Let $\mathbf{k}^*$ is optimal. Denote:

$$E(k_{t+1}) = V(k^*_t, k_{t+1}) + \beta V(k_{t+1}, k^*_{t+2})$$

We have E is differentiable and

$$E'(k_{t+1}) = V_2(k^*_t, k_{t+1}) + \beta V_1(k_{t+1}, k^*_{t+2})$$

Since $(k^*_t, k^*_{t+1}) \in int(\text{Graph}\Gamma)$, $(k^*_{t+1}, k^*_{t+2}) \in int(\text{Graph}\Gamma)$, there existss a neighborhood of k^*_{t+1} such that $(k^*_t, y) \in int(\text{Graph}\Gamma)$, $(y, k^*_{t+2}) \in int(\text{Graph}\Gamma)$ for all y in this neighborhood. We define the sequence

$$z = (k_0, k^*_1, \ldots, k^*_t, y, k^*_{t+1}, \ldots) \in \Pi(k_0)$$

Because of optimality of k^* we have $W(k^*) \geq W(z)$, this implies:

$$\begin{aligned} \sum_{t=0}^{+\infty} \beta^t V(k^*_t, k^*_{t+1}) &\geq \sum_{t=0}^{+\infty} \beta^t V(z_t, z_{t+1}) \\ \Rightarrow \beta^t V(k^*_t, k^*_{t+1}) + \beta^{t+1} V(k^*_{t+1}, k^*_{t+2}) &\geq \beta^t V(y^*_t, y) + \beta^{t+1} V(y, k^*_{t+2}) \\ \Rightarrow V(k^*_t, k^*_{t+1}) + \beta V(k^*_{t+1}, k^*_{t+2}) &\geq V(y^*_t, y) + \beta V(y, k^*_{t+2}) \\ \Rightarrow E(k^*_{t+1}) &\geq E(y) \quad \text{for every } y \text{ in a neighborhood of } k^*_{t+1} \end{aligned}$$

It means that k^*_{t+1} is a maximum in this neighborhood. Hence $E'(k^*_{t+1}) = 0$. The result then follows. □

Proposition 5.3. *Assume* **P1-P7**. *If* $\mathbf{k}^* = (k^*_t)_{t=0}^{\infty} \in \Pi(k_0)$ *satisfies all of the following conditions:*

(i) $(k^*_t, k^*_{t+1}) \in int(Graph(\Gamma))$, $\forall t \geq 0$

(ii) Euler equation: $V_2(k^*_t, k^*_{t+1}) + \beta V_1(k^*_{t+1}, k^*_{t+2}) = 0$, $\forall t \geq 0$

(iii) Transversality condition: $\lim_{t\to\infty} \beta^t V_1(k^*_t, k^*_{t+1}) \cdot k^*_t = 0$

then $\mathbf{k}^*$ *is optimal.*

Proof. Let $\mathbf{k}^* \in \Pi(k_0)$ satisfies the conditions (i)-(iii). Let $k \in \Pi(k_0)$. We will prove that

$$\Delta := \sum_{t=0}^{\infty} \beta^t V(k_t^*, k_{t+1}^*) - \sum_{t=0}^{\infty} \beta^t V(k_t, k_{t+1}) \geq 0$$

Indeed, by the concavity and differentability of V we have:

$$\begin{aligned}
\Delta_T &:= \sum_{t=0}^{T} \beta^t V(k_t^*, k_{t+1}^*) - \sum_{t=0}^{T} \beta^t V(k_t, k_{t+1}) \\
&\geq \sum_{t=0}^{T} \beta^t \Big[V_1(k_t^*, k_{t+1}^*) \cdot (k_t^* - k_t) + V_2(k_t^*, k_{t+1}^*) \cdot (k_{t+1}^* - k_{t+1}) \Big] \\
&= \sum_{t=0}^{T-1} \beta^t \Big[V_2(k_t^*, k_{t+1}^*) + \beta V_1(k_{t+1}^*, k_{t+2}^*) \Big] (k_{t+1}^* - k_{t+1}) + \beta^T V_2(k_T^*, k_{T+1}^*) \cdot (k_{T+1}^* - k_{T+1})
\end{aligned}$$

since the Euler equation holds, then:

$$\begin{aligned}
\Delta_T &\geq \beta^T V_2(k_T^*, k_{T+1}^*) \cdot (k_{T+1}^* - k_{T+1}) \geq \beta^T V_2(k_T^*, k_{T+1}^*) \cdot k_{T+1}^* \\
&= -\beta^{T+1} V_1(k_{T+1}^*, k_{T+2}^*) \cdot k_{T+1}^*
\end{aligned}$$

hence

$$\Delta = \lim_{T \to \infty} \Delta_T \geq 0$$

implies that $\mathbf{k}^*$ is optimal. □

Corollary 5.1. *Assume* **P1-P7**. *Let* $\mathbf{k}^*$ *is optimal path of capital, then the optimal path of labour is the sequence* $\mathbf{l}^* = (l_t^*)_{t \geq 0}$ *where* l_t^* *is unique solution of the equation:*

$$k_{t+1}^* = F(k_t^*, l_t^*) + (1 - \delta) k_t^* + \frac{l_t^* - L}{a}$$

The final step in this section is to establish the convergence of the optimal solution $(\mathbf{c}^*, \mathbf{k}^*, \mathbf{l}^*)$ to a unique non trivial optimal steady state (c^s, k^s, l^s). The following proposition excludes convergence to a trivial steady state.

Proposition 5.4. *Assume* **P1-P7**. *Let* $k_0 > 0$ *and denote by* $\mathbf{k}^*$ *the optimal capital sequence from* k_0. *Then,*

i) $k_t^* > 0, \ \forall t \geq 0$.

ii) The sequence $\mathbf{k}^*$ *cannot converge to zero.*

Proof. See appendix.

□

We know that the optimal sequence $\mathbf{k}^*$ is monotonic and bounded, so it converges to some k^s. Associated with this k^s, there existss (c^s, l^s) that solves the maximization problem defined in definition 3.1. Taking the limits in Euler equation, we get

$$V_2(k^s, k^s) + \beta V_1(k^s, k^s) = 0.$$

Substituting the expressions for V_1 and V_2 from proposition 3.1 we obtain

$$F_1(k^s, l^s) + (1 - \delta) = \frac{1}{\beta}.$$

Lemma 5.2. *Assume* **P1-P7**. *Let* (k^s, l^s) *satisfy*

$$F_1(k^s, l^s) + (1 - \delta) = \frac{1}{\beta}$$

Define the sequence $\mathbf{k}^s$ *by:*

$$\mathbf{k}^s = (k_t^s)_{t\geq 1} : \; k_t^s = k^s \; \forall t \geq 1$$

Then $k^s \in \Gamma(k^s)$ *and the sequence* $\mathbf{k}^s$ *is optimal.*

Proof. Since $F_1(k^s, l^s) = \frac{1}{\beta} - 1 + \delta < F_1(0, L)$ then $k^s > 0, l^s < L$. This implies $k^s > (1-\delta)k^s - \frac{L}{a}$. Since $F_1(k^s, l^s) - \delta = \frac{1}{\beta} - 1 > 0$, it is easy to see that $k^s < F(k^s, l^s) + (1-\delta)k^s < F(k^s, L) + (1-\delta)k^s$. Hence $k^s \in \Gamma(k^s)$. Moreover $k^s \in int(\Gamma(k^s))$.

By the definition of V we have

$$V(k_t, k_{t+1}) = u\left(\frac{L - l_t}{a}\right)$$

where

$$k_{t+1} = F(k_t, l_t) + (1 - \delta)k_t + \frac{l_t - L}{a}$$

we get

$$\frac{dl_t}{dk_t} = -[F_1(k_t, l_t) + (1 - \delta)]\frac{dl_t}{dk_{t+1}}$$

then

$$V_1(k_{t+1}^s, k_{t+2}^s) = -\frac{1}{\beta}V_2(k_t^s, k_{t+1}^s) \;\Rightarrow\; V_2(k_t^s, k_{t+1}^s) + \beta V_1(k_{t+1}^s, k_{t+2}^s) = 0$$

It is obvious that

$$\lim_{t\to\infty} \beta^t V_1(k_t^s, k_{t+1}^s) \cdot k_t^s = [V_1(k^s, k^s) \cdot k^s] \lim_{t\to\infty} \beta^t = 0.$$

From proposition 5.3 the result then follows. □

Corollary 5.2. *Assume* **P1-P7**. *Let* (c^s, k^s, l^s) *denote the unique nontrivial steady state. It satisfies*

$$\begin{aligned} F_1(k^s, l^s) &= \delta + \frac{1}{\beta} - 1 && (9) \\ c^s + \delta k^s &= F(k^s, l^s) && (10) \\ ac^s + l^s &= L && (11) \end{aligned}$$

Then for all $k_0 > 0$, *the optimal solution* $(\mathbf{c}^*, \mathbf{k}^*, \mathbf{l}^*)$ *converges to* (c^s, k^s, k^s) *when* t *goes to* $+\infty$.

6 Comparison with Growth Models with Elastic Labour

We have shown that problem (P) is equivalent to

$$(P') \qquad \max_{(k_{t+1}, l_t)} \sum_{t=0}^{\infty} \beta^t u\left(\frac{L - l_t}{a}\right)$$

under the constraints: $(\forall t \geq 0)$

$$0 \leq k_{t+1} \leq F(k_t, l_t) + (1-\delta)k_t + \frac{l_t - L}{a}$$

$$0 \leq l_t \leq L$$

$$k_0 \geq 0 \text{ is given}$$

In growth models with elastic labor, the social planning problem determines a consumption-leisure allocation and production sequence: [2]

$$(Q) \qquad \max_{(c_t, k_{t+1}, \mathfrak{L}_t)} \sum_{t=0}^{\infty} \beta^t u^Q(c_t, \mathfrak{L}_t)$$

under the constraints:

$$c_t + k_{t+1} - (1-\delta)k_t \leq F(k_t, l_t) \qquad (12)$$

$$\mathfrak{L}_t + l_t \leq L \qquad (13)$$

$$c_t \geq 0, l_t \geq 0, \mathfrak{L} \geq 0, k_{t+1} \geq 0 \qquad (14)$$

$$k_0 \geq 0 \text{ is given}$$

where $\mathfrak{L}_t$ denotes the quantity of leisure which the consumer spends in period t.

If the assumptions used for the problem (Q) are similar with those of problem (P), in particular $u_1^Q(0, x) = +\infty, u_2^Q(x, 0) = +\infty$ for all $x > 0$, Le Van and Vailakis [6] impose an additional assumptions which are $\frac{u_{11}^Q}{u_1^Q} \leq \frac{u_{12}^Q}{u_2^Q}$ and the production function is of constant

[2]For problem (Q), we use the results from the working paper [6] of Cuong Le Van and Yiannis Vailakis entitle "Existence of Competitive Equilibrium in a Single-Sector Growth Model with Elastic Labor"

returns to scale.
It is easy to see that problem (Q) can be written as

$$(Q') \qquad \max_{(k_{t+1},l_t)} \sum_{t=0}^{\infty} \beta^t u^Q \left(F(k_t,l_t) + (1-\delta)k_t - k_{t+1}, L - l_t\right)$$

under the constraints: $(\forall t \geq 0)$

$$0 \leq k_{t+1} \leq F(k_t,l_t) + (1-\delta)k_t$$

$$0 \leq l_t \leq L$$

$k_0 \geq 0$ is given

For problem (Q), we define the correspondence:

$$\Gamma^Q(k) = \{y \in \mathbb{R}_+ : 0 \leq y \leq (1-\delta)k + F(k,L)\}$$

and the technology set

$$\text{Graph}(\Gamma^Q) = \{(k,y) \in \mathbb{R}_+^2 : y \in \Gamma^Q(k)\}$$

Let $(k,y) \in \text{Graph}(\Gamma^Q)$, define the indirect utility function V^Q by:

$$V^Q(k,y) = \max u^Q(c, L-l)$$
$$\text{s.t. } c + y \leq (1-\delta)k + F(k,l)$$
$$c \geq 0, 0 \leq l \leq L$$

- Since u is strictly concave, F is concave, the problem above has a unique solution l, given (k,y). We write $l = l^*(k,y)$.

The results are similar with those of problem (P):

- V^Q is continuous at any $(k,y) \in \text{Graph}(\Gamma^Q)$ with $k > 0$.
- $V^Q(k,y)$ is increasing in k, decreasing in y and strictly concave in (k,y).
- The set of feasible capital sequences:

$$\Pi^Q(k_0) = \{\mathbf{k} \in (\mathbb{R}_+)^\infty : k_{t+1} \in \Gamma^Q(k_t), \forall t \geq 0\}$$

  is compact for the product topology

For $k_0 \geq 0$, define by $W^Q : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ the value function, i.e:

$$W^Q(k_0) = \max_{\mathbf{k} \in \Pi^Q(k_0)} \sum_{t=0}^{\infty} \beta^t V^Q(k_t, k_{t+1})$$

The value function W^Q has the same properties as the value function W of problem (P):

- W^Q is strictly increasing, strictly concave.
- W^Q is upper semicontinuous. By adding the consumption that $u^Q(0,l) = 0$ then W^Q is continuous.
- The value function W^Q solves the Bellman equation, i.e

$$\forall k_0 \geq 0, \quad W^Q(k_0) = \max\{V^Q(k_0, k_1) + \beta W^Q(k_1) : k_1 \in \Gamma^Q(k_0)\}$$

We also have the following results for problem (Q).

- Let $k_0 \geq 0$. The optimal capital sequence $\mathbf{k}^*$ from k_0 is monotonic.
- There exists a unique nontrivial steady state (c^Q, k^Q, l^Q) that satisfies

$$F_1(k^Q, l^Q) + (1-\delta) = \frac{1}{\beta} \quad \text{and} \tag{15}$$

$$c^Q = F(k^Q, l^Q) - \delta k^Q \tag{16}$$

$$l^Q = l^*(k^Q, k^Q) \tag{17}$$

- For all $k_0 > 0$, the optimal solution $(\mathbf{c}^*, \mathbf{k}^*, \mathbf{l}^*)$ has the property that converges to (c^Q, k^Q, l^Q) when t converges to $+\infty$.

We see at the steady state, the system of equations which determine the values of the capital, labour and consumption differ only by the equations giving the value of labour, i.e. equation (11) for model (P') and equation (17) for model (Q'). We also observe that k^Q, l^Q, c^Q depend only on the unit investment cost $\zeta = \delta + \frac{1}{\beta} - 1$.

Lemma 6.1. *Assume* **P1-P7***. If a converges to 0 then l^s converges to L. If a converges to $+\infty$ then l^s converges to 0.*

Proof. Consider first the equations (9), (10), (11) of the steady state of problem (P'). We claim that $a\frac{c^s}{l^s}$ converges to 0 when a converges to 0. Indeed, from (10) we get

$$c^s \leq F(k^s, l^s) \leq A$$

since $l^s \leq L$ and we know that the feasible path of capitals is uniformly bounded above. Suppose l^s converges to 0 when a converges to 0. In this case, from (11), c^s must converge to $+\infty$ contradicting the result that c^s is bounded above. Hence, there exists $\hat{l} > 0$ such that $l^s \geq \hat{l}$ when a converges to 0. We obtain that, for any a small enough, we have $a\frac{c^s}{l^s} \leq a\frac{A}{\hat{l}}$. Now let a go to zero to obtain the claim.
Equation (11) can be written as

$$l^s(1 + a\frac{c^s}{l^s}) = L$$

Hence when a converges to 0, l^s converges to L
We now prove that when $a \to +\infty$ then $l^s \to 0$. Suppose the contrary $l^s \to \hat{l} \in (0, L]$. From (11), we have $\frac{ac^s}{l^s} \to \frac{L}{\hat{l}} - 1 \geq 0$. This implies $c^s \to 0$. Equation (9) implies $k^s \to \hat{k} > 0$ and $F_1(\hat{k}, \hat{l}) = \delta + \frac{1}{\beta} - 1 > \delta$. Equation (10) implies $\delta\hat{k} = F(\hat{k}, \hat{l})$. In this case we must have $F_1(\hat{k}, \hat{l}) < \delta$ contradicting $F_1(\hat{k}, \hat{l}) = \delta + \frac{1}{\beta} - 1 > \delta$. Therefore, $l^s \to 0$.

□

Proposition 6.1. *Assume* **P1-P7**. *Then*
(1) The steady state value of labour l^s is a continuous decreasing function of a
(2) There exists a threshold $\hat{a}$ for a such that if $a < \hat{a}$ then $l^s > l^Q$ and if $a > \hat{a}$ then $l^s < l^Q$.

Proof. (1) Consider the equations (9), (10), (11) of the steady state of problem (P'). Differentiating these equations with respect to c^s, k^s, l^s and a gives:

$$F_{11}(k^s, l^s)dk^s + F_{12}(k^s, l^s)dl^s = 0 \tag{18}$$

$$dc^s + \delta dk^s = F_1(k^s, l^s)dk^s + F_2(k^s, l^s)dl^s \tag{19}$$

$$c^s da + adc^s + dl^s = 0 \tag{20}$$

Substituting dk^s from (18) and $F_1(k^s, l^s)$ from (9) into (19):

$$\begin{aligned} dc^s &= (F_1(k^s, l^s) - \delta)dk^s + F_2(k^s, l^s)dl^s \\ &= -(\frac{1}{\beta} - 1)\frac{F_{12}(k^s, l^s)}{F_{11}(k^s, l^s)}dl^s + F_2(k^s, l^s)dl^s \end{aligned}$$

Equation (20) implies

$$\frac{dl^s}{da} = -\frac{c^s}{1 + a\left[F_2(k^s, l^s) - (\frac{1}{\beta} - 1)\dfrac{F_{12}(k^s, l^s)}{F_{11}(k^s, l^s)}\right]} \tag{21}$$

One can see that the right hand side of the above equation is negative.
(2) From lemma (6.1) we can conclude that there exists a threshold $\hat{a}$ for a such that if $a < \hat{a}$ then $l^s > l^Q$ and if $a > \hat{a}$ then $l^s < l^Q$. □

Suppose now that the production function is of constant returns to scale. Then we obtain
for model (P'):

$$\frac{c^s}{l^s} = F(\frac{k^s}{l^s}, 1) - \delta\frac{k^s}{l^s} \tag{22}$$

$$F_1(\frac{k^s}{l^s}, 1) = \delta + \frac{1}{\beta} - 1 \tag{23}$$

$$l^s(1 + a\frac{c^s}{l^s}) = L \tag{24}$$

while for model (Q'):

$$\begin{aligned} \frac{c^Q}{l^Q} &= F(\frac{k^Q}{l^Q},1) - \delta\frac{k^Q}{l^Q} & (25)\\ F_1(\frac{k^Q}{l^Q},1) &= \delta + \frac{1}{\beta} - 1 & (26)\\ l^Q &= l^*(k^Q,k^Q) & (27)\end{aligned}$$

We see that the ratios capital per capita $(\frac{k^s}{l^s})$, $(\frac{k^Q}{l^Q})$ are equal. Hence the ratios consumptions per capita $(\frac{c^s}{l^s})$, $(\frac{c^Q}{l^Q})$ are also equal. Define $\rho_{kl} = \frac{k^s}{l^s} = \frac{k^Q}{l^Q}$, $\rho_{cl} = \frac{c^s}{l^s} = \frac{c^Q}{l^Q}$. These ratios depend only on the investment cost $\zeta = (\delta + \frac{1}{\beta} - 1)$. This leads to the following proposition

Proposition 6.2. *Assume as before* **P1-P7**. *Assume moreover the production function F of both problems (P) and (Q) is of constant returns to scale. Then, at the steady state, both models have the same ratio capital per capita and the same ratio consumption per capita.*

But equations (24), (27) show that the values of l^s, l^Q may differ.

Proposition 6.3. *Assume* **P1-P7** *and the production function F of both problems (P) and (Q) is of constant returns to scale. Then there exists $\hat{a}$ which depends only on ζ such that:*

$$a < \hat{a} \Leftrightarrow l^s > l^Q, a = \hat{a} \Leftrightarrow l^s = l^Q$$

Proof. Let $(k,y) \in \text{Graph}(\Gamma^Q)$ and consider again the indirect utility function V^Q defined by:

$$\begin{aligned} V^Q(k,y) = \max u^Q(c, L-l)\\ \text{s.t. } c + y \le (1-\delta)k + F(k,l)\\ c \ge 0, 0 \le l \le L\end{aligned}$$

This problem can be written, when $l > 0$

$$\begin{aligned} V^Q(k,y) = \max u^Q(l(\frac{c}{l}), L-l)\\ \text{s.t. } \frac{c}{l} + \frac{y}{l} \le (1-\delta)\frac{k}{l} + F(\frac{k}{l},1)\\ c \ge 0, 0 \le l \le L\end{aligned}$$

At the optimum, we have

$$\frac{c}{l} + \frac{y}{l} = (1-\delta)\frac{k}{l} + F(\frac{k}{l},1)$$

Suppose we know the ratios $(\frac{y}{l}, \frac{k}{l})$ and hence $\frac{c}{l}$. This problem becomes simply

$$\max_{0\le l\le L} u^Q\big(l\big((1-\delta)\frac{k}{l} + F(\frac{k}{l},1) - \frac{y}{l}\big), L-l\big)$$

There exists a unique solution $l = l^{**}(\frac{y}{l}, \frac{k}{l}))$. Therefore, at the steady state, $y = k = k^Q$ and since the ratio $(\frac{k^Q}{l^Q})$ is determined by (26), we have $l^Q = l^{**}(\rho_{kl}, \rho_{kl})$ and depends only on ζ. Define $\hat{a} = \frac{L-l^Q}{\rho_{cl} l^Q}$. Then

$$a < \hat{a} \Leftrightarrow l^s > l^Q, a = \hat{a} \Leftrightarrow l^s = l^Q$$

□

The economic interpretation of propositions 6.1 and 6.3 is the following.
(i) When the quantity of time to consume one unit of consumption increases, the agent devotes less time for labour.
(ii) When the quantity of time to consume one unit of consumption is smaller than the threshold, it is better for the economy to spend time to consume than to enjoy leisure. We have more time for labour. This implies more output and more consumption. We reverse the situation when the quantity of time to consume one unit of consumption is larger than the threshold.

The following example illustrates clearly this result.

Example. Consider an economy with instantaneous utility function of the following form: $u(c) = c^\alpha,\ \alpha \in (0,1)$. Technology is Cobb-Douglas, $F(k,l) = k^\gamma l^{1-\gamma}$, where $\gamma \in (0,1)$. For simplicity assume that $\delta = 1$ (full depreciation) and $L = 1$.

It is easy to check that the steady state of problem (P) of this economy is:

$$c^s = \frac{\tau^\gamma - \tau}{a(\tau^\gamma - \tau) + 1}$$
$$k^s = \frac{\tau}{a(\tau^\gamma - \tau) + 1}$$
$$l^s = \frac{1}{a(\tau^\gamma - \tau) + 1}$$

where $\tau = (\beta\gamma)^{\frac{1}{1-\gamma}}$.

To compare with problem (Q), consider the instantaneous utility function $u^Q(c, \mathfrak{L}) = c^\alpha + \mathfrak{L}^\alpha,\ \alpha \in (0,1)$. The steady state is:

$$c^Q = \frac{\eta(\tau^\gamma - \tau)}{\tau^\gamma - \tau + \eta}$$
$$k^Q = \frac{\tau\eta}{\tau^\gamma - \tau + \eta}$$
$$l^Q = \frac{\eta}{\tau^\gamma - \tau + \eta}$$
$$\mathfrak{L}^Q = \frac{\tau^\gamma - \tau}{\tau^\gamma - \tau + \eta}$$

where $\tau = (\beta\gamma)^{\frac{1}{1-\gamma}},\ \eta = [(1-\gamma)\tau^{\gamma}]^{\frac{1}{1-\alpha}}$.

One can easily check that if $a < 1/\eta$ then then $l^s > l^Q$ and if $a > 1/\eta$ then $l^s < l^Q$.

Appendix

Proof of the Lemma 3.2

We claim that there exists $t \geq 0$ such that $l_t^* > 0$. Indeed, if $l_t^* = 0$ for all $t \geq 0$ then the constraint (1) becomes:

$$\frac{L}{a} + k_{t+1}^* - (1-\delta)k_t^* \leq 0 \quad \Leftrightarrow \quad k_{t+1}^* \leq (1-\delta)k_t^* - \frac{L}{a} \quad \Rightarrow \quad k_{t+1}^* \leq k_t^*$$

The sequence $(k_t^*)_t$ is decreasing and bounded below by 0, so k_t^* converges to $k^* \geq 0$ and satisfies:

$$k^* \leq (1-\delta)k^* - \frac{L}{a} \quad \Rightarrow \quad k^* \leq -\frac{L}{a\delta} < 0$$

This contradiction shows that there existss $l_t^* > 0$.

Assume $(c_t^*, k_{t+1}^*, l_t^*)$ is an optimal solution of (P) and

$$k_{t+1}^* < F(k_t^*, l_t^*) + (1-\delta)k_t^* + \frac{l_t^* - L}{a}$$

If $l_t^* > 0$, we can reduce l_t^* by a small amount $\varepsilon > 0$ such that $l_t^* - \varepsilon \geq 0$ and

$$k_{t+1}^* \leq F(k_t^*, l_t^* - \varepsilon) + (1-\delta)k_t^* + \frac{l_t^* - \varepsilon - L}{a}$$

Then $((c_t^* + \frac{\varepsilon}{a}), (l_t^* - \varepsilon), (k_{t+1}^*))$ satisfies all constraints and

$$\sum \beta^t u(\frac{L - (l_t^* - \varepsilon)}{a}) > \sum \beta^t u(\frac{L - l_t^*}{a})$$

contradiction with the optimality of (l_t^*). So equation (6) is satisfied with any t such that $l_t^* > 0$.

We will now prove that (6) holds for every $t \geq 0$. In contrary, suppose that there existss t_0 such that (6) does not bind. It implies $l_{t_0}^* = 0$. Without loss of generality, we can assume that $l_0^* = 0,\ l_1^* > 0$. We have:

$$k_1^* < F(k_0^*, l_0^*) + (1-\delta)k_0^* + \frac{l_0^* - L}{a} \tag{28}$$

$$k_2^* = F(k_1^*, l_1^*) + (1-\delta)k_1^* + \frac{l_1^* - L}{a} \tag{29}$$

We can increase k_1 from (28) such that:

$$(k_1^* + \varepsilon_k) \leq F(k_0^*, l_0^*) + (1-\delta)k_0^* + \frac{l_0^* - L}{a}, \quad \varepsilon_k > 0$$

From (29) we have:

$$k_2^* < F(k_1^* + \varepsilon_k, l_1^*) + (1-\delta)(k_1^* + \varepsilon_k) + \frac{l_1^* - L}{a}$$

Since $l_1^* > 0$, we can decrease l_1^* such that:

$$k_2^* \leq F(k_1^* + \varepsilon_k, (l_1^* - \varepsilon_l)) + (1-\delta)(k_1^* + \varepsilon_k) + \frac{(l_1^* - \varepsilon_l) - L}{a}, \quad \varepsilon_l > 0, l_1^* - \varepsilon_l > 0$$

Consider $(\bar{k}_t, \bar{l}_t)$:

$$\bar{k}_t = \begin{cases} k_1^* + \varepsilon_k, & t = 1 \\ k_t^*, & t \neq 1 \end{cases} \qquad \bar{l}_t = \begin{cases} l_1^* - \varepsilon_l, & t = 1 \\ l_t^*, & t \neq 1 \end{cases}$$

then $(\bar{k}_t, \bar{l}_t)$ satisfies all the constraints of problem (P') and

$$\sum_{t=0}^{\infty} \beta^t u \left(\frac{L - \bar{l}_t}{a} \right) > \sum_{t=0}^{\infty} \beta^t u \left(\frac{L - l_t}{a} \right)$$

a contradiction with the optimality of (k_t^*, l_t^*).

Hence at the optimum

$$k_{t+1}^* = F(k_t^*, l_t^*) + (1-\delta)k_t^* + \frac{l_t^* - L}{a}$$

we obtain:

$$k_{t+1}^* \leq F(k_t^*, L) + (1-\delta)k_t^* + \frac{L - L}{a} = F(k_t^*, L) + (1-\delta)k_t^*$$
$$k_{t+1}^* \geq F(k_t^*, 0) + (1-\delta)k_t^* + \frac{0 - L}{a} = (1-\delta)k_t^* - \frac{L}{a}$$

□

Proof of the Lemma 3.3

i) We consider equation (the unknown is k):

$$F(k, L) + (1-\delta)k = k \quad \Leftrightarrow \quad F(k, L) = \delta k \tag{30}$$

Because of concavity of F and the assumption $F_1(\infty, L) = 0$, we have two cases:

- Case 1. The equation (30) has unique solution $\bar{k} = 0$. Then we have

$$F(k, L) + (1-\delta)k \leq k \quad \Rightarrow \quad k' \leq k$$

• Case 2. The equation (30) has two solutions $\{0, \bar{k} > 0\}$. If $k \leq \bar{k}$ then $k' \leq \bar{k}$. If $k > \bar{k}$ then $k' < k$. We always have $k' \leq \max(\bar{k}, k)$

ii) We have for all $k \geq 0$: $(1-\delta)k \in \Gamma(k)$, so $\Gamma(k) \neq \emptyset$. It is easy to see that $\Gamma(k)$ is compact convex. Moreover

$$\Gamma(0) = \{k' \geq 0 : -\frac{L}{a} \leq k' \leq 0\} = \{0\}$$

Fix $k \geq 0$. For any $k' \in \Gamma(k)$ and for any sequence $\{k_n\} \subset \mathbb{R}_+$ converging to k, we consider sequence $\{k'_n\}$:

$$k'_n = \frac{F(k_n, L) + \frac{L}{a}}{F(k, L) + \frac{L}{a}} \left[k' - (1-\delta)k - \frac{L}{a} \right] + (1-\delta)k_n - \frac{L}{a}$$

One can easily check that $k'_n \to k'$ and

$$(1-\delta)k_n - \frac{L}{a} \leq k'_n \leq F(k_n, L) + (1-\delta)k_n \quad \Rightarrow k'_n \in \Gamma(k_n)$$

It shows that the correspondence Γ is lower semi-continuous.

On the other hand, for any sequence $\{k_n\} \subset \mathbb{R}_+$ converging to k, for any sequence $\{k'_n\}$ with $k'_n \in \Gamma(k_n), \forall n$, we have

$$(1-\delta)k_n - \frac{L}{a} \leq k'_n \leq F(k_n, L) + (1-\delta)k_n$$

Because of assumption P6, there exists $\bar{k} \geq 0$ such that $k'_n \leq \max(\bar{k}, k_n)$. The sequence $\{k_n\}$ is converging and hence bounded. This implies that the sequence $\{k'_n\}$ is also bounded. Hence it has a sub-sequence $\{k'_{n_m}\}$ converging to k'. It is easy to see that $k' \in \Gamma(k)$. Then the correspondence Γ is upper semi-continuous.

The result then follows.

iii) From the first part of this lemma we have, for $k = (k_0, k_1, \ldots) \in \Pi(k_0)$:

$$\begin{aligned} & k_1 \leq \max(\bar{k}, k_0) \\ & k_2 \leq \max(\bar{k}, k_1) \leq \max(\bar{k}, k_0) \\ & \ldots \end{aligned}$$

Denote $K = \max(\bar{k}, k_0)$ we have $k_t \leq K, \forall t \geq 0$. Hence the set $\Pi(k_0)$ is bounded.

Because of continuity of Γ we can easily check that $\Pi(k_0)$ is compact.

□

Proof of the Proposition 3.1

1) The Kuhn-Tucker first-order conditions are:

$$u'(c^*) - \lambda - a\mu + \xi_1 = 0 \tag{31}$$
$$\lambda F_2(k, l^*) - \mu + \xi_2 = 0 \tag{32}$$
$$\lambda \geq 0,\ \lambda[F(k, l^*) + (1-\delta)k - c^* - y] = 0 \tag{33}$$
$$\mu \geq 0,\ \mu[L - ac^* - l] = 0 \tag{34}$$
$$\xi_1 \geq 0,\ \xi_1 c^* = 0 \tag{35}$$
$$\xi_2 \geq 0,\ \xi_2 l^* = 0 \tag{36}$$

The Inada condition of utility function implies that $c^* > 0$, and hence $\xi_1 = 0$. The Inada condition on labor's marginal productivity implies that $l^* > 0$, $\xi_2 = 0$. The strict increasingness of V and F imply that $\lambda > 0$ and $\mu > 0$. From (34) we then have $l^* \in (0, L)$. The uniqueness of solution follows from the strict concavity of u.

2) Consider

$$\begin{aligned} \mathfrak{L} = u(c) \quad &+ \quad \lambda[F(k,l) + (1-\delta)k - c - y] \\ &+ \quad \mu[L - ac - l] \end{aligned}$$

We have

$$u'(c^*) - \lambda - a\mu = 0 \tag{37}$$
$$\lambda F_2(k, l^*) - \mu = 0 \tag{38}$$
$$c^* + y = F(k, l^*) + (1-\delta)k \tag{39}$$
$$ac^* + l^* = L \tag{40}$$

Differentiating the above equations with respect to k and y gives:

$$\begin{aligned} &u''dc^* - d\lambda - ad\mu = 0 \\ &F_2 d\lambda + \lambda[F_{12}dk + F_{22}dl^*] - d\mu = 0 \\ &dc^* + dy - [F_1 dk + F_2 dl^*] - (1-\delta)dk = 0 \\ &adc^* + dl^* = 0 \end{aligned}$$

Writting these equations in a matrix form we get:

$$\underbrace{\begin{pmatrix} u'' & 0 & -1 & -a \\ 0 & \lambda F_{22} & F_2 & -1 \\ 1 & -F_2 & 0 & 0 \\ a & 1 & 0 & 0 \end{pmatrix}}_{A} \begin{pmatrix} dc^* \\ dl^* \\ d\lambda \\ d\mu \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -\lambda F_{12} & 0 \\ F_1 + 1 - \delta & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} dk \\ dy \end{pmatrix}$$

The determinant of matrix A:

$$det(A) = (1 + aF_2)^2$$

Since the production function F is strictly increasing then $det(A) > 0$, hence A is invertible. It implies that $c^*(k,y), l^*(k,y), \lambda(k,y), \mu(k,y)$ are continuously differentiable in a neighborhood of (k,y).

By the Envelope Theorem we have:

$$\frac{\partial V(k,y)}{\partial k} = \frac{\partial \mathfrak{L}}{\partial k}(c^*, l^*, \lambda, \mu) = \lambda[F_1(k,l^*) + 1 - \delta]$$
$$\frac{\partial V(k,y)}{\partial y} = \frac{\partial \mathfrak{L}}{\partial y}(c^*, l^*, \lambda, \mu) = -\lambda$$

From (37) and (38) we have

$$\lambda = \frac{u'(c^*)}{1 + aF_2(k,l^*)}; \quad \mu = \frac{u'(c^*)F_2(k,l^*)}{1 + aF_2(k,l^*)}$$

3) The results follow 2).

4) It is easy to see that GraphΓ is a convex set.

Given $(k_1, k_1') \in \mathbb{R}^2_+, (k_2, k_2') \in \mathbb{R}^2_+$, we have

$$V(k_1, k_1') = u(\frac{L - l_1}{a}),\ k_1' = F(k_1, l_1) + (1-\delta)k_1 + \frac{l_1 - L}{a}$$

$$V(k_2, k_2') = u(\frac{L - l_2}{a}),\ k_2' = F(k_2, l_2) + (1-\delta)k_2 + \frac{l_2 - L}{a}$$

For $\lambda \in [0,1]$, define $k_\lambda = \lambda k_1 + (1-\lambda)k_2$; $k_\lambda' = \lambda k_1' + (1-\lambda)k_2'$. We have $(k_\lambda, k_\lambda') \in$ GraphΓ and

$$V(k_\lambda, k_\lambda') = u(\frac{L - l_\lambda}{a}),\ k_\lambda' = F(k_\lambda, l_\lambda) + (1-\delta)k_\lambda + \frac{l_\lambda - L}{a}$$

this implies

$$\lambda k_1' + (1-\lambda)k_2' = F(\lambda k_1 + (1-\lambda)k_2, l_\lambda) + (1-\delta)(\lambda k_1 + (1-\lambda)k_2) + \frac{l_\lambda - L}{a}$$

then

$$F\Big(\lambda k_1 + (1-\lambda)k_2,\ \lambda l_1 + (1-\lambda)l_2\Big) + \frac{\Big(\lambda l_1 + (1-\lambda)l_2\Big) - L}{a} \geq F\Big(\lambda k_1 + (1-\lambda)k_2, l_\lambda\Big) + \frac{l_\lambda - L}{a}$$

$$\Rightarrow l_\lambda \leq \lambda l_1 + (1-\lambda)l_2$$

Using the strictly concavity of V we then have:

$$\begin{aligned}
V\Big(\lambda k_1 + (1-\lambda)k_2,\ \lambda k_1' + (1-\lambda)k_2'\Big) &= u\Big(\frac{L-l_\lambda}{a}\Big)\\
&\geq u\Big(\frac{L-(\lambda l_1 + (1-\lambda)l_2)}{a}\Big)\\
&= u\Big(\lambda\frac{L-l_1}{a} + (1-\lambda)\frac{L-l_2}{a}\Big)\\
&> \lambda u\Big(\frac{L-l_1}{a}\Big) + (1-\lambda)u\Big(\frac{L-l_2}{a}\Big)\\
&= \lambda V(k_1,k_1') + (1-\lambda)V(k_2,k_2')
\end{aligned}$$

In other words, we have proved that V is strictly concave.

Moreover, because V is non-negative then there exists $A \geq 0, B \geq 0$ such that $V(k,k') \leq A + B(k+k')$

□

Proof of the Lemma 5.1

Using the argument applied in proof of the proposition 3.1, by tedious computations, we get:

$$\frac{\partial\lambda}{\partial k} = \frac{1}{(1+aF_2)^2}\left[-a\lambda(1+aF_2)F_{12} + (a^2\lambda F_{22} + u'')(F_1 + 1 - \delta)\right]$$

Consider the expression in parentheses. Observe that the concavity of F and strict concavity of u imply $(a^2\lambda F_{22} + u'') < 0$. Then the second term is strictly negative. By the assumption P7 we have that the first term is strictly negative. Hence

$$\frac{\partial\lambda}{\partial k} < 0$$

It follows that

$$\frac{\partial^2 V(k,y)}{\partial y\partial k} = -\frac{\partial\lambda}{\partial k} > 0$$

□

Proof of the Proposition 5.4

i) Let $k_0 > 0$ and assume that $k_1^* = 0$. We have

$$c_0^* = F(k_0, l_0^*) + (1-\delta)k_0 > 0,\ l_0^* = L - ac_0^*$$

Since $k_1^* = 0$ then

$$c_1^* + k_2^* = F(k_1^*, l_1^*) + (1-\delta)k_1^* = 0$$

implies $c_1^* = 0$ and $k_2^* = 0$. Hence for all $t \geq 1$ we have $c_t^* = 0, k_t^* = 0, l_t^* = L$.

Consider the feasible path $(\bar{\mathbf{c}}, \bar{\mathbf{k}}, \bar{\mathbf{l}})$, defined as follows:

$$\begin{aligned}
&\bar{c}_0 = c_0^* - \varepsilon;\ \bar{c}_1 = F(\bar{k}_1, \bar{l}_1) + (1-\delta)\bar{k}_1, \\
&\bar{k}_1 = F(k_0, \bar{l}_0) - F(k_0, l_0^*) + \varepsilon \\
&\bar{c}_t = 0;\ \bar{k}_t = 0 \quad \forall t \geq 2 \\
&\bar{l}_t = L - a\bar{c}_t \quad \forall t \geq 0.
\end{aligned}$$

The concavity of F implies that

$$\begin{aligned}
\bar{k}_1 &\geq F_2(k_0, \bar{l}_0)[\bar{l}_0 - l_0^*] + \varepsilon \\
&= F_2(k_0, \bar{l}_0)[a(c_0^* - \bar{c}_0)] + \varepsilon \\
&= F_2(k_0, \bar{l}_0)[a\varepsilon] + \varepsilon \\
&\geq \varepsilon.
\end{aligned}$$

Define

$$\Delta_\varepsilon := \sum_{t=0}^{\infty} \beta^t u(\bar{c}_t) - \sum_{t=0}^{\infty} \beta^t u(c_t^*)$$

The concavity of u implies that

$$\begin{aligned}
\Delta_\varepsilon &\geq u'(\bar{c}_0)[\bar{c}_0 - c_0^*] + \beta u'(\bar{c}_1)[\bar{c}_1 - c_1^*] \\
&= -\varepsilon u'(\bar{c}_0) + \beta \bar{c}_1 u'(\bar{c}_1) \\
&\geq -\varepsilon u'(\bar{c}_0) + \beta(1-\delta)\bar{k}_1 u'(\bar{c}_1) \\
&\geq \varepsilon \Big[\beta(1-\delta)u'(\bar{c}_1) - u'(c_0^* - \varepsilon)\Big]
\end{aligned}$$

As $\varepsilon \to 0, \bar{c}_1 \to 0$ and $\beta(1-\delta)u'(\bar{c}_1) \to +\infty$, while $u'(c_0^* - \varepsilon) \to u'(c_0^*) < +\infty$. Hence we can chose ε small enough such that $\Delta_\varepsilon > 0$, a contradiction with the optimal of $(\mathbf{c}^*, \mathbf{k}^*, \mathbf{l}^*)$. It follows that $k_1^* > 0$. Working by induction, we can proof that $k_t^* > 0,\ \forall t \geq 0$.

ii) Let $k_0 > 0$ and assume that $\mathbf{k}^*$ is optimal with $k_t^* \to 0$. Since $\mathbf{k}^*$ is monotonic, it follows that $\mathbf{k}^*$ is decreasing. Moreover there existss a date T_1, such that for all $t > T_1$: $(1-\delta)k_t^* < \frac{L}{a}$, and then $k_{t+1}^* > (1-\delta)k_t^* - \frac{L}{a}$.

From assumption **P6**, for k small enough we have

$$F_1(k, L) + (1-\delta) > \frac{1}{\beta} > 1 \;\Rightarrow\; F(k, L) + (1-\delta)k > k$$

This implies that there exists a date T_2, such that for all $t > T_2$:

$$0 < k_{t+1}^* < F(k_t^*, L) + (1-\delta)k_t^*$$

We then have a date T, such that $k^*_{t+1} \in \text{int}(\Gamma(k^*_t)),\ \forall t > T$. With this optimal solution $\mathbf{k}^*$, there exist associated sequences $(\mathbf{c}^*, \mathbf{l}^*)$ for consumption and labor, such that $(\mathbf{k}^*, \mathbf{c}^*, \mathbf{l}^*)$ is a solution to problem (P). From lemma 3.2

$$k^*_{t+1} = F(k^*_t, L) + (1-\delta)k^*_t + \frac{l^*_t - L}{a}$$

It follows that l^*_t converges to L.

For any $t > T$, Euler equation gives:

$$\begin{aligned} V_2(k^*_t, k^*_{t+1}) + \beta V_1(k^*_{t+1}, k^*_{t+2}) = 0 \\ \Leftrightarrow\ -\lambda_t + \beta\lambda_{t+1}[F_k(k^*_{t+1}, l^*_{t+1}) + (1-\delta)] = 0 \\ \Leftrightarrow \beta[F_k(k^*_{t+1}, l^*_{t+1}) + (1-\delta)] = \frac{\lambda_t}{\lambda_{t+1}} \end{aligned}$$

Since (k^*_{t+1}, l^*_{t+1}) converges to $(0, L)$, there existss a date T' such that

$$\frac{\lambda_t}{\lambda_{t+1}} > 1,\ \forall t > T'.$$

On the other hand, Bellman equation gives:

$$W(k^*_t) = V(k^*_t, k^*_{t+1}) + \beta W(k^*_{t+1})$$

it implies that

$$\begin{aligned} W'(k^*_{t+1}) &= V_1(k^*_{t+1}, k^*_{t+2}) \\ &= -\frac{1}{\beta}V_2(k^*_t, k^*_{t+1}) \\ &= \frac{1}{\beta}\lambda_t \\ W'(k^*_{t+2}) &= \frac{1}{\beta}\lambda_{t+1} \end{aligned}$$

Since $k^*_{t+2} \leq k^*_{t+1}$, the concavity of W implies that $\lambda_{t+1} \geq \lambda_t$, a contradiction. Hence $\mathbf{k}^*$ cannot converge to zero.

□

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