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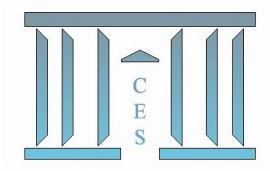

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Equilibrium of a production economy with noncompact attainable allocations set*

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Abstract

In this paper, we consider a production economy with an unbounded attainable set where the consumers may have non-complete nontransitive preferences. To get the existence of an equilibrium, we provide an asymptotic property on preferences for the attainable consumptions and we use a combination of nonlinear optimization and fixed point theorem on truncated economies together with an asymptotic argument. We show that this condition holds true if the set of attainable allocations is compact or, when preferences are representable by utility functions, if the set of attainable individually rational utility levels is compact. This assumption generalizes the CPP condition of Allouch (2002) and covers the example of Page et al. (2000) when the attainable utility levels set is not compact. So we extend the previous existence results with non compact attainable sets in two ways by adding a production sector and considering general preferences.

Keywords: production economy, non compact attainable allocations, quasi-equilibrium, nonlinear optimization.

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1 Introduction

Since the seventies, with the exception of the seminal paper of Mas-Colell [14] and a first paper of Shafer-Sonnenschein [18], equilibrium for a finite dimensional standard economy is commonly proved using explicitly or implicitly equilibrium existence for the associated abstract economy (see [3], [9], [8], [12], [16], [17]) in which agents are the consumers, the producers and an hypothetical additional agent, the Walrasian auctioneer. Moreover, in exchange economies, it is well-known that the existence of equilibrium with consumption sets that are not bounded from below requires some non-arbitrage conditions (see [13], [20], [5], [4], [6], [7], [2]). In [7], it is shown that these conditions imply the compactness of the individually rational utility level set, which is clearly weaker than assuming the compactness of the attainable allocation, and the authors prove an existence result of an equilibrium under this last condition.

The purpose of our paper is to extend this result to finite dimensional production economies with non-complete, non-transitive preferences, which may not be representable by a utility function. Furthermore, we also allow preferences to be other regarding in the sense that the preferred set of an agent depends on the consumption of the other consumers. We posit the standard assumptions about the closedness, the convexity and the continuity on the consumption side as well as on the production side of the economy like in Florenzano [9] and a survival assumption. We only consider quasi-equilibrium and we refer to the usual interiority of initial endowments or irreducibility condition to get an equilibrium from a quasi-equilibrium (see for example Florenzano [9] section 3.2).

The non-compactness of the attainable sets appears naturally in an economy with financial markets and short-selling. Using the Hart's trick [13], we can reduce the problem to a standard exchange economy when the financial markets are frictionless. But, if there are some transaction costs, intermediaries like clearing house mechanisms or other kind of frictions, this method is no more working and we then need to introduce a production sector to encompass these frictions. That is why we add in this paper a production sector, which is also justified if we want to analyze a stock market where the payments of an asset depend on the production plan of a firm.

Considering non-complete, non-transitive preferences allows us to deal

with Bewley preferences where the agents have several criterions and a consumption is preferred to another one only if all criterions are improved. Such preferences are not representable by utility functions. They appear naturally in financial models where the objectives is to minimize the risk according to some consistent measures.

Our main contribution is to provide a sufficient condition (H3) to replace the standard compactness of the attainable allocation set, which is suitably written to deal with general preferences. More precisely, we assume that for each sequence of attainable consumptions, there exists an attainable consumption where the preferred consumptions can be approximated by preferred consumptions of the elements of the sequence. Actually, we also restrict our attention to the attainable allocation, which are individually rational, in a sense adapted to the fact that preferences may not be transitive. The formulation of our assumption is in the same spirit as the CPP condition of Allouch [1].

We prove that our condition is satisfied when the attainable set is compact and when preferences are represented by utility functions and the set of attainable individually rational utility levels is compact. So, our result extends the previous ones in the literature. Our asymptotic assumption is weaker than the CPP condition within the framework considered by Allouch where preferences are supposed to be transitive with open lower sections.

To compare our work with the contribution of Won and Yannelis [21], we provide an asymmetric assumption (EWH3) for exchange economies which is less demanding for one particular consumer. We are not please with this assumption since the fundamentals of the economy are symmetric and there is no reason to treat a consumer differently from the others. Won and Yannelis condition and the (EWH3) are not comparable and both of them cover the example of Page et al [15]. Nevertheless, neither of these conditions covers Example 3.1.2 of Won and Yannelis. So, there is room for further works to provide a symmetric assumption covering both examples.

We also remark that our condition deals only with feasible consumptions and not with the associated productions. So, our condition can be identically stated for an exchange economy or for a production economy. This means that even, if there exists non compact feasible productions, an equilibrium still exists if the attainable consumption set remains compact. In other words, the key problem comes from the behaviour of the preferences for large consumptions and not from the geometry of the productions sets at infinity.

To prove the existence of a quasi-equilibrium, we use several tricks borrowed from various authors. Using a truncated economy in order to apply

a fixed point theorem to an artificial compact economy is an old trick as in the first equilibrium proofs. We apply our assumption on the asymptotic behaviour of preferences to a sequence of quasi-equilibrium allocations in growing associated truncated economies. We prove that the attainable consumption given by Assumption (H3) is a quasi-equilibrium consumption of the original economy. The originality of the proof is mainly contained in this last section.

2 The Model

In this paper, we consider the private ownership economy:

$$\mathcal{E} = (\mathbb{R}^L, (X_i, P_i, \omega_i)_{i\in I}, (Y_j)_{j\in J}, (\theta_{ij})_{(i,j)})$$

where L is a finite set of goods, so that $\mathbb{R}^L$ is the commodity space and the price space. I is a finite set of consumers, each consumer i has a consumption set $X_i \subset \mathbb{R}^L$ and an initial endowment $\omega_i \in \mathbb{R}^L$. The tastes of this consumer are described by a preference correspondence $P_i : \prod_{k\in I} X_k \to X_i$. $P_i(x)$ represents the set of strictly preferred consumption to $x_i \in X_i$ given the consumption $(x_k)_{k\neq i}$ of the other consumers. J is a finite set of producers and $Y_j \subset \mathbb{R}^L$ is the set of possible productions of firm $j \in J$. For each i and j, θ_{ij} is the portfolio of shares of the consumer i on the profit of the producer j. The θ_{ij} are nonnegative and for every $j \in J$, $\sum_{i\in I} \theta_{ij} = 1$. These shares together with their initial endowment determine the wealth of each consumer.

Definition 2.1. *An allocation* $(x, y) \in \prod_{i\in I} X_i \times \prod_{j\in J} Y_j$ *is called attainable if:*

$$\sum_{i\in I} x_i = \sum_{j\in J} y_j + \sum_{i\in I} \omega_i.$$

We denote by $\mathcal{A}(\mathcal{E})$ *the set of attainable allocations.*

In this paper, we are only dealing with the existence of quasi-equilibrium. We refer to the large literature on irreducibility, which provides sufficient conditions for a quasi-equilibrium to be an equilibrium. The simplest one is the interiority of the initial endowments linked with the possibility of inaction for the producers.

Definition 2.2. *A quasi-equilibrium of the private ownership economy is a pair of an allocation* $((\bar{x}_i)_{i\in I}, (\bar{y}_j)_{j\in J}) \in \prod_{i\in I} X_i \times \prod_{j\in J} Y_j$ *and a non-zero price vector* $\bar{p} \neq 0$*, such that:*

(a) (Profit maximization): for every $j \in J$*, for every* $y_j \in Y_j$*,* $\bar{p} \cdot y_j \leq \bar{p} \cdot \bar{y}_j$*,*

(b) (Quasi-demand): for each $i \in I$*,* $\bar{p} \cdot \bar{x}_i \leq \bar{p} \cdot \omega_i + \bar{p} \cdot (\sum_{j\in J} \theta_{ij} \bar{y}_j)$ *and* $x_i \in P_i(\bar{x}) \Rightarrow \bar{p} \cdot x_i \geq \bar{p} \cdot \bar{x}_i$

(c) (Attainability): $\sum_{i\in I} \bar{x}_i = \sum_{i\in I} \omega_i + \sum_{j\in J} \bar{y}_j$*.*

Notice that, in view of Condition (c), Condition (b) can be rephrased as

$$\text{for every } i \in I, \bar{p} \cdot \bar{x}_i = \bar{p} \cdot \omega_i + \bar{p} \cdot (\sum_{j\in J} \theta_{ij} \bar{y}_j) \text{ and } [x_i \in P_i(\bar{x}) \Rightarrow \bar{p} \cdot x_i \geq \bar{p} \cdot \bar{x}_i]$$

Before stating the assumptions considered on $\mathcal{E}$, let us introduce some notations:

- $\omega = \sum_{i\in I} \omega_i$ is the total initial endowment;
- $Y = \sum_{j\in J} Y_j$ is the total production set;
- $\hat{X} = \{x \in \prod_{i\in I} X_i : \exists y \in Y : \sum_{i\in I} x_i = \omega + y\}$ is the set of all attainable consumption allocations;
- $\hat{Y} = \{y \in Y : \exists x \in \prod_{i\in I} X_i : \sum_{i\in I} x_i = \omega + y\}$ is the attainable total production set.

In this paper, we consider the following hypothesis:

Assumption (H1) For every $i \in I$

(a) X_i is a non-empty, closed, convex subset of $\mathbb{R}^L$;

(b) [irreflexivity]$\forall x \in \prod_{i\in I} X_i$, $x_i \notin \text{co}P_i(x)$ (the convex hull of $P_i(x)$);

(c) [lower semicontinuous] $P_i : \prod_{k\in I} X_k \to X_i$ is lower semicontinuous;

(d) $\omega_i \in X_i - \sum_{j\in J} \theta_{i,j} Y_j$, i.e. there exists $(\underline{x}_i, (\underline{y}_{i,j})) \in X_i \times \prod_{j\in J} Y_j$ such that $\underline{x}_i = \omega_i + \sum_{j\in J} \theta_{i,j} \underline{y}_{i,j}$;

(e) For each $x \in \hat{X}$, one has $P_i(x) \neq \emptyset$.

Assumption (H2) Y is a non-empty, closed and convex subset of $\mathbb{R}^L$.

To overcome the fact that we do not assume local non-satiation but only non-satiation, we introduce the definition of "augmented preferences" as in Gale and Mas-Collel ([10], [11]). We can avoid the use of augmented preferences if Assumption (H1)(e) is replaced by x_i belongs to the closure of $P_i(x)$.

$$\hat{P}_i(x) = \{x_i' \in X_i | x_i' = \lambda x_i + (1-\lambda)x_i'', 0 \leq \lambda < 1, x_i'' \in \mathrm{co}P_i(x)\},$$

Assumption (H3) For all sequence $((x_i^\nu))$ of $\hat{X}$ such that for all i, $\underline{x}_i \in \overline{\hat{P}_i(x^\nu)^c}$, there exists a subsequence $((x_i^{\varphi(\nu)})) \in \hat{X}$ and $(\bar{x}_i) \in \hat{X}$ such that for all i, for all $\xi_i \in \hat{P}_i(\bar{x})$, there exists an integer ν_1 and a sequence $(\xi_i^{\varphi(\nu)})_{\nu \geq \nu_1}$ convergent to (ξ_i) such that for all $\nu \geq \nu_1$, for all $i \in I$, $\xi_i^{\varphi(\nu)} \in \hat{P}_i(x^{\varphi(\nu)})$.

Closedness and convexity are standard assumptions on consumptions and productions sets. They imply in particular that commodities are perfectly divisible. Assumption (H1)(c) is a weak continuity assumption on preferences. Assumption (H1)(b), i.e. the irreflexivity, is made on the sets co$P_i(x)$ to avoid to assume the convexity of the preference correspondences P_i. Assumption (H1)(d) implies that using his own shares in the productive system, consumer i can survive without participating in any exchange. This implies no trader will be allowed to starve no matter what the prices are. It also insures that the set $\mathcal{A}(\mathcal{E})$ is nonempty. Usually, in exchange economy, this assumption is merely written as $\omega_i \in X_i$, which corresponds to $\omega_i = \underline{x}_i$ and $\underline{y}_{i,j} = 0$ for all j. Assumption (H1)(e) assumes, for every i, the insatiability of the ith consumer at any point of his attainable consumption set.

Assumption (H3) is an attempt to weaken the compactness assumption on the global attainable set $\mathcal{A}(\mathcal{E})$. A large literature tackles this question by considering what is called a non-arbitrage condition (see for example [2], [4], [6], [7]). Our work is much in the spirit of Dana et al. [6, 7] considering a compact set of attainable utility levels as generalized by Allouch [1]. But, we remove the transitivity assumption on preferences like in Won and Yannelis [21]. We discuss into details the relationships with these contributions in the last section of the paper.

We assume that for each sequence of attainable consumptions, there exists an attainable consumption where the preferred consumptions can be approximated by preferred consumptions of the elements of the sequence. Indeed, the element $\bar{x}$ of $\hat{X}$ is not necessarily a cluster point of the sequence (x^ν) but any element strictly preferred to $\bar{x}$ by any agent is approachable by a sequence of elements strictly preferred to $(x^{\varphi(\nu)})$. This condition imposes some restriction on the asymptotic behaviour of the preferences for attainable allocations in the sense that some preferred elements remain at a finite distance of the origin even if the allocation is very far.

Note that the productions are not considered in Assumption (H3). So, only the total production set matters since it determines the attainable consumptions. The fact that some unbounded sequences of individual productions can be attainable does not prevent the existence of an equilibrium as long as the total production set is not modified.

Example 2.1. *We present an example of an exchange economy where Assumption (H3) is satisfied while the attainable set is not bounded and the preference correspondences are not representable by utility functions. Then we extend it to a production economy with a class of productions sets. Let us consider an exchange economy with two commodities A and B and two consumers.*

The consumption sets are given by

$$X_1 = X_2 = \{(a, b) \in \mathbb{R}^2 | a + b \geq 0\}$$

The attainable allocations set $\mathcal{A}(\mathcal{E})$ of the economy is then

$$\mathcal{A}(\mathcal{E}) = \{((a, b), (\omega_A - a, \omega_B - b)) | 0 \leq a + b \leq \omega_A + \omega_B\}$$

where (ω_A, ω_B) with $\omega_A + \omega_B > 0$ denotes the global endowment. The set $\mathcal{A}(\mathcal{E})$ is clearly non compact.

We consider the following continuous function $\Pi : X_i \to \mathbb{R}^2$ defined by:

$$\Pi(a, b) = (\frac{1}{2} + \frac{a - b}{(|a - b| + 1)(a^2 + b^2 + 2)}, \frac{1}{2} + \frac{b - a}{(|a - b| + 1)(a^2 + b^2 + 2)}).$$

The preference correspondence is the same for the two consumers and it is defined by $P_i : X_1 \times X_2 \to X_i$

$$P_i((a_1, b_1), (a_2, b_2)) = \{(\alpha, \beta) \in X_i | \Pi(a_i, b_i) \cdot (\alpha, \beta) > \Pi(a_i, b_i) \cdot (a_i, b_i)\}$$

One easily checks that Assumption (H1) is satisfied by the preference relations since Π is continuous so P_i has an open graph and $\Pi(a, b) \gg (0, 0)$ so that the local non-satiation holds true everywhere.

We remark that if (a_i^ν, b_i^ν) *is a sequence of* X_i *such that* $\|(a_i^\nu, b_i^\nu)\|$ *converges to* $+\infty$ *and* $a_i^\nu + b_i^\nu$ *converges to a finite limit* c*, then* $\Pi(a_i^\nu, b_i^\nu)$ *converges to* $(1/2, 1/2)$ *and* $\Pi(a_i^\nu, b_i^\nu) \cdot (a_i^\nu, b_i^\nu)$ *converges to* $\lim_\nu (1/2)(a_i^\nu + b_i^\nu) = \frac{c}{2}$*.*

Let $((a_1^\nu, b_1^\nu), (a_2^\nu, b_2^\nu))$ *be a sequence of* $\mathcal{A}(\mathcal{E})$*. If it has a bounded subsequence, then this subsequence has a cluster point* $((\bar{a}_1, \bar{b}_1), (\bar{a}_2, \bar{b}_2))$*. Then the desired property of Assumption (H3) holds true thanks to the fact that the preference correspondences have an open graph. See the proof of Proposition 5.1 (i).*

If the sequence is unbounded, we remark that the sequences $(a_1^\nu + b_1^\nu)$ *and* $(a_2^\nu + b_2^\nu)$ *belongs to* $[0, \omega_A + \omega_B]$ *and for all* ν*,* $a_1^\nu + b_1^\nu + a_2^\nu + b_2^\nu = \omega_A + \omega_B$*. So, there exists a subsequence* $((a_1^{\varphi(\nu)}, b_1^{\varphi(\nu)}), (a_2^{\varphi(\nu)}, b_2^{\varphi(\nu)}))$ *such that the sequences* $(a_1^{\varphi(\nu)} + b_1^{\varphi(\nu)})$ *and* $(a_2^{\varphi(\nu)} + b_2^{\varphi(\nu)})$ *converges respectively to* $c \in [0, \omega_A + \omega_B]$ *and to* $\omega_A + \omega_B - c$*. Let us consider the attainable allocation* $((\bar{a}_1 = c/2, \bar{b}_1 = c/2), (\bar{a}_2 = (\omega_A + \omega_B - c)/2, \bar{b}_2 = (\omega_A + \omega_B - c)/2))$*. We remark that* $\Pi(\bar{a}_1, \bar{b}_1) = \Pi(\bar{a}_2, \bar{b}_2) = (1/2, 1/2)$ *and* $\Pi(\bar{a}_1, \bar{b}_1) \cdot (\bar{a}_1, \bar{b}_1) = (1/2)(\bar{a}_1 + \bar{b}_1) = c/2$ *and* $\Pi(\bar{a}_2, \bar{b}_2) \cdot (\bar{a}_2, \bar{b}_2) = (1/2)(\bar{a}_2 + \bar{b}_2) = (\omega_A + \omega_B - c)/2$*. Let* $i = 1, 2$ *and* $(a_i, b_i) \in X_i$ *such that* $(a_i, b_i) \in P_i((\bar{a}_1, \bar{b}_1), (\bar{a}_2, \bar{b}_2))$*. From the definition of* P_i*, one deduces that* $(1/2)(a_i + b_i) > (1/2)(\bar{a}_i + \bar{b}_i) = (1/2) \lim_{\nu\to\infty} (a_i^{\varphi(\nu)} + b_i^{\varphi(\nu)}) = \lim_{\nu\to\infty} \Pi(a_i^{\varphi(\nu)}, b_i^{\varphi(\nu)}) \cdot (a_i^{\varphi(\nu)}, b_i^{\varphi(\nu)})$*. Furthermore, since* $\Pi(a_i^{\varphi(\nu)}, b_i^{\varphi(\nu)})$ *converges to* $(1/2, 1/2)$*,* $(1/2)(a_i + b_i) = \lim_{\nu\to\infty} \Pi(a_i^{\varphi(\nu)}, b_i^{\varphi(\nu)}) \cdot (a_i, b_i)$*. Consequently, for* ν *large enough,* $\Pi(a_i^{\varphi(\nu)}, b_i^{\varphi(\nu)}) \cdot (a_i, b_i) > \lim_{\nu\to\infty} \Pi(a_i^{\varphi(\nu)}, b_i^{\varphi(\nu)}) \cdot (a_i^{\varphi(\nu)}, b_i^{\varphi(\nu)})$*, which means that* $(a_i, b_i) \in P_i((a_1^{\varphi(\nu)}, b_1^{\varphi(\nu)}), (a_2^{\varphi(\nu)}, b_2^{\varphi(\nu)}))$*, so the desired property in Assumption (H3) holds true.* □

We now consider a finite collection of production sets $(Y_j)_{j\in J}$ *of* $\mathbb{R}^2$ *such that* $Y = \sum_{j\in J} Y_j$ *is closed, convex, contains* 0 *and* $y_A + y_B \leq 0$ *for all* $(y_A, y_B) \in Y$*. Let us consider the production economy where the consumption sector is as above, the production sector is described by* $(Y_j)_{j\in J}$ *and the portfolio shares* (θ_{ij}) *are any ones satisfying the standard conditions. One easily checks that Assumption (H3) is satisfied by this production economy since the attainable consumption set is smaller or equal to the one of the exchange economy.*

The main result of this paper is the following existence theorem of a quasi-equilibrium for a production economy.

Theorem 2.1. *Under Assumptions* $(H1), (H2)$ *and* $(H3)$*, there exists a quasi-equilibrium of the economy* $\mathcal{E}$*.*

3 Preliminary results

First, we show that some properties of the preference correspondences P_i are still true for $\hat{P}_i$.

Proposition 3.1. *Assume that for all i, X_i is convex*

(i) If P_i is lower semicontinuous on $\prod_{i\in I} X_i$, then the same is true for $\hat{P}_i$.

(ii) $\hat{P}_i(x)$ has convex values. Furthermore, if for all $x_i \in X_i$, $x_i \notin coP_i(x)$, then $x_i \notin \hat{P}_i(x)$.

Proof.

(i) Let $x \in \prod_{i\in I} X_i$ and V an open subset of X_i such that

$$V \cap \hat{P}_i(x) \neq \emptyset.$$

Then, there exists $\xi_i \in \hat{P}_i(x) \cap V$, which means that $\xi_i = \lambda x_i + (1-\lambda)\zeta_i$ for some $\lambda \in [0,1[$, $\zeta_i \in \text{co}P_i(x)$. Let $\epsilon > 0$ such that $B(\xi_i, \epsilon) \subset V$. Since the correspondence P_i is lower semicontinuous, then coP_i is lower semicontinuous (see [9], page 154). Consequently, there exists a neighborhood W of x in $\prod_{i\in I} X_i$ such that

$$x' \in W \Rightarrow \text{co}P_i(x') \cap B(\zeta_i, \epsilon) \neq \emptyset.$$

Thus, for all $x' \in W$, there exists $\zeta_i' \in \text{co}P_i(x') \cap B(\zeta_i, \epsilon)$. Let W' such that

$$W' = \{x' \in W | \|x_i' - x_i\| < \epsilon\}.$$

Let $x' \in W'$ and $\xi_i' = \lambda x_i' + (1-\lambda)\zeta_i'$, then $\xi_i' \in \hat{P}_i(x')$

$$\|\xi_i' - \xi_i\| \leq \lambda\|x_i' - x_i\| + (1-\lambda)\|\zeta_i' - \zeta_i\| < \epsilon.$$

Then, one gets $\xi_i' \in B(\xi_i, \epsilon) \subset V$. Hence, $\xi_i' \in \hat{P}_i(x') \cap V$, which proves the lower semi-continuity of $\hat{P}_i$.

(ii) Let $x \in \prod_{i\in I} X_i$ and $z_i, z_i' \in \hat{P}_i(x)$ such that $z_i = x_i + \lambda(\xi_i - x_i)$ and $z_i' = x_i + \beta(\xi_i' - x_i)$ for some $\lambda, \beta \in]0,1]$ and $\xi_i, \xi_i' \in \text{co}P_i(x)$. For $\alpha \in]0,1[$, we have:

$$\begin{aligned}\alpha z_i + (1-\alpha)z_i' &= x_i + \alpha\lambda\xi_i + (1-\alpha)\beta\xi_i' - [\alpha\lambda x_i + (1-\alpha)\beta x_i] \\ &= x_i + \alpha\lambda\xi_i + (1-\alpha)\beta\xi_i' - [\alpha\lambda + (1-\alpha)\beta]x_i \\ &= x_i + \gamma(\xi_i'' - x_i).\end{aligned}$$

where $\gamma = \alpha\lambda+(1-\alpha)\beta$ and $\xi_i'' = \frac{\alpha\lambda}{\gamma}\xi_i+\frac{(1-\alpha)\beta}{\gamma}\xi_i'$. One easily checks that $\gamma \in]0,1]$ since $\lambda, \beta \in]0,1]$ and $\xi_i'' \in \mathrm{co}P_i(x)$. Then, $\alpha z_i+(1-\alpha)z_i' \in \hat{P}_i(x)$ which means that $\hat{P}_i$ has convex values.

We prove by contraposition the irreflexivity. Let us suppose that $x_i \in \hat{P}_i(x)$ for some i, then $x_i = \lambda x_i + (1-\lambda)x_i'$ with $\lambda \in [0,1[$ and $x_i' \in \mathrm{co}P_i(x)$. Hence, we have $x_i = x_i' \in \mathrm{co}P_i(x)$ which contradicts Assumption (H1)(b). □

Now, we consider the following economy

$$\mathcal{E}' = (\mathbb{R}^L, (X_i, \hat{P}_i, \omega_i)_{i\in I}, (Y_j')_{j\in J}, (\theta_{ij})_{(i,j)})$$

where the preference correspondences are replaced by the augmented preference correspondences and the production sets are replaced by their closed convex hull, that is for each j, $Y_j' = \overline{\mathrm{co}}Y_j$.

Lemma 3.1. *Under assumption (H2), the economies $\mathcal{E}$ and $\mathcal{E}'$ have the same total production set so the same attainable consumption set $\hat{X}$.*

Proof. Let $Y' = \sum_{j\in J} Y_j'$.

It is clear that $Y \subset Y'$. Conversely, $Y' = \sum_{j\in J} \overline{\mathrm{co}}Y_j \subset cl(\sum_{j\in J} \mathrm{co}Y_j)$, see [19] (Corollary 6.6, page 48). Since the convex hull of a sum is the sum of the convex hulls, one gets

$$Y' = \sum_{j\in J} \bar{\mathrm{co}}Y_j \subset \mathrm{cl}(\sum_{j\in J} \mathrm{co}Y_j) = \mathrm{cl}(\mathrm{co}(\sum_{j\in J} Y_j)) = \overline{\mathrm{co}}Y.$$

Since Y is a non-empty closed, convex subset of $\mathbb{R}^L$, then $\overline{\mathrm{co}}Y = Y$. Hence $Y = Y'$. □

Proposition 3.2. *If $((\bar{x}_i), (\bar{\zeta}_j), \bar{p})$ is a quasi-equilibrium of $\mathcal{E}'$, then there exists $\bar{y} \in \prod_{j\in J} Y_j$ such that $((\bar{x}_i), (\bar{y}_j), \bar{p})$ is a quasi-equilibrium of $\mathcal{E}$.*

Proof. Let $((\bar{x}_i), (\bar{\zeta}_j), \bar{p})$ be a quasi-equilibrium of $\mathcal{E}'$. So, $\sum_{j\in J} \bar{\zeta}_j \in \sum_{j\in J} Y'_j$. By Lemma 3.1, $\sum_{j\in J} Y'_j = Y$. Consequently, there exists $\bar{y} \in \prod_{j\in J} Y_j$ such that $\sum_{j\in J} \bar{\zeta}_j = \sum_{j\in J} \bar{y}_j$. Hence $\sum_{i\in I} \bar{x}_i = \omega + \sum_{j\in J} \bar{y}_j$. In other words, Condition (c) of Definition 2.2 is satisfied.

Moreover, one can remark that $\bar{y}_j \in Y'_j$ for every j. Consequently, $\bar{p} \cdot \bar{y}_j \leq \bar{p} \cdot \bar{\zeta}_j$. But since $\sum_{j\in J} \bar{\zeta}_j = \sum_{j\in J} \bar{y}_j$, one gets $\bar{p} \cdot \bar{y}_j = \bar{p} \cdot \bar{\zeta}_j$.

We now show that condition (a) is satisfied. Let $j \in J$ and $y_j \in Y_j$. Then, $y_j \in Y'_j$, so, $\bar{p} \cdot y_j \leq \bar{p} \cdot \bar{\zeta}_j = \bar{p} \cdot \bar{y}_j$. Hence, $\bar{p} \cdot y_j \leq \bar{p} \cdot \bar{y}_j$ and Condition (a) of Definition 2.2 is satisfied.

Last, we show that condition (b) is satisfied. Since $\bar{p} \cdot \bar{\zeta}_j = \bar{p} \cdot \bar{y}_j$ for all $j \in J$, we have, $\bar{p} \cdot \bar{x}_i \leq \bar{p} \cdot \omega_i + \sum_{j\in J} \theta_{i,j} \bar{p} \cdot \bar{y}_j$ for all i. Now, let $i \in I$ and $x_i \in X_i$ such that $x_i \in P_i(\bar{x})$. Since $P_i(\bar{x}) \subset \hat{P}_i(\bar{x})$, $\bar{p} \cdot x_i \geq \bar{p} \cdot \bar{x}_i$. □

4 Existence of quasi-equilibria

In this section we consider the economy $\mathcal{E}'$ as defined above. We have seen in the previous section that we can deduce the existence of a quasi-equilibrium of $\mathcal{E}$ from a quasi-equilibrium of $\mathcal{E}'$.

In what follow, we will consider Assumptions (H1'), (H2') whose correspond to (H1), (H2) but adapted to $\mathcal{E}'$ and the asymptotic assumption (WH3). In the previous section, we have shown that (H1') and (H2') are satisfied by $\mathcal{E}'$ if Assumptions (H1), (H2) are satisfied by $\mathcal{E}$ and (WH3) is weaker than (H3).

Assumption (H1') For every $i \in I$

(a) X_i is a non-empty closed, convex subset of $\mathbb{R}^L$;

(b) [irreflexivity] $\forall x \in \prod_{i\in I} X_i$, $x_i \notin \hat{P}_i(x)$;

(c) [lower semicontinuous] $\hat{P}_i : \prod_{k\in I} X_k \to X_i$ is lower semicontinuous and convex valued;

(d) $\omega_i \in X_i - \sum_{j\in J} \theta_{i,j} Y'_j$, i.e. there exists $(\underline{x}_i, (\underline{y}_{i,j})) \in X_i \times \prod_{j\in J} Y'_j$ such that $\underline{x}_i = \omega_i + \sum_{j\in J} \theta_{i,j} \underline{y}_{i,j}$;

(e) For each $x \in \hat{X}$, one has $\hat{P}_i(x) \neq \emptyset$ and for all $\xi_i \in \hat{P}_i(x)$, for all $t \in]0, 1]$, $t\xi_i + (1-t)x_i \in \hat{P}_i(x)$.

Assumption (H2') For each $j \in J$, Y'_j is a closed, convex subset of $\mathbb{R}^L$.

To prepare the discussion on the relationships with the paper of Won and Yannelis, we consider the following weakening of Assumption (H3). If A is a subset of $\mathbb{R}^L$, cone$A$ is the cone spanned by A.

Assumption (WH3) There exists a consumer i_0 such that, for all sequence $((x_i^\nu))$ of $\hat{X}$ such that for all i, $\underline{x}_i \in \overline{\hat{P}_i(x^\nu)^c}$, there exists a subsequence $((x_i^{\varphi(\nu)})) \in \hat{X}$ and $(\bar{x}_i) \in \hat{X}$ such that for all i, for all $\xi_i \in \hat{P}_i(\bar{x})$, there exists an integer ν_1 and a sequence $(\xi_i^{\varphi(\nu)})_{\nu \geq \nu_1}$ convergent to (ξ_i) such that for all $\nu \geq \nu_1$,

$\xi_{i_0}^{\varphi(\nu)} \in \text{cone}[\hat{P}_{i_0}(x^{\varphi(\nu)}) - \bar{x}_{i_0}^{\varphi(\nu)}] + \bar{x}_{i_0}^{\varphi(\nu)}$

for all $i \neq i_0$,

$\xi_i^{\varphi(\nu)} \in \hat{P}_i(x^{\varphi(\nu)})$.

Assumption (WH3) is clearly weaker than (H3) since

$$\hat{P}_{i_0}(x^{\varphi(\nu)}) \subset \text{cone}[\hat{P}_{i_0}(x^{\varphi(\nu)}) - \bar{x}_{i_0}^{\varphi(\nu)}] + \bar{x}_{i_0}^{\varphi(\nu)}$$

But this assumption exhibits the drawback of being asymmetric. That is why we did not emphasise it before since we think that further works should provide an even weaker but symmetric assumption. We provide more comments in the last section when we discuss the link with the work of Won and Yannelis.

We now state the existence result of a quasi-equilibrium for a finite private ownership economy satisfying Assumptions (H1'), (H2') and (WH3).

Theorem 4.1. *If Assumptions (H1'), (H2') and (WH3) are satisfied, then there exists a quasi-equilibrium of the economy $\mathcal{E}'$.*

The idea of the proof is as follows: we first truncate consumption and production sets with a closed ball with a radius large enough; following an idea of Bergstrom [3], we modify the budget sets in such a way that it will coincides with the original ones when the price belongs to the unit sphere; then, by applying the well known result of Gale and Mas-Colell - Bergstrom about the existence of maximal elements to a suitable family of lower semi-continuous correspondences, we obtain a sequence $((x^\nu), (y^\nu), p^\nu)$ such that $((x^\nu), (y^\nu))$ is an attainable allocation of the economy $\mathcal{A}(\mathcal{E}')$, p^ν belongs to the unit ball of $\mathbb{R}^L$, the domain of admissible prices, the producers maximize the profit over the truncated production sets and the consumptions are maximal elements of the preferences on the truncated consumption sets but with a

relaxed budget constraint; from Assumption (WH3) and the compactness of the price set, we obtain a subsequence $(x^{\varphi(\nu)}, y^{\varphi(\nu)}, p^{\varphi(\nu)})$ and an element $(\bar{x}, \bar{y}, \bar{p})$ such that the preferences at this point are close to the preferences at $x^{\varphi(\nu)}$ for ν large enough and $p^{\varphi(\nu)}$ converges to $\bar{p}$; finally, we prove that $(\bar{x}, \bar{y}, \bar{p})$ is a quasi-equilibrium of $\mathcal{E}'$. Note that the difficulty of the limit argument comes from the fact that $(\bar{x}, \bar{y})$ is not necessarily the limit of $(x^{\varphi(\nu)}, y^{\varphi(\nu)})$.

4.1 The fixed-point argument

From Assumption (H1')(d), let us fix $\underline{x}_i \in X_i$ and $\underline{y}_{i,j} \in Y'_j$ such that $\underline{x}_i = \omega_i + \sum_{j\in J} \theta_{i,j}\underline{y}_{i,j}$ for every $i \in I$. Let $\bar{B}^\nu$ be the closed ball with center 0 and radius ν with ν large enough so that ω, $\underline{x}_i$, $\underline{y}_{i,j}$ and ω_i belong to B^ν, the interior of $\bar{B}^\nu$, for all i, j. We consider the truncated economy obtained by replacing agent's consumption set by $X_i^\nu = X_i \cap \bar{B}^\nu$ for all $i_0 \neq i$, and $X_{i_0}^\nu = X_i \cap \bar{B}^{(\sharp I+\sharp J)\nu}$. The production set becomes $Y_j^\nu = Y'_j \cap \bar{B}^\nu$ and the augmented preference correspondences are $\hat{P}_i^\nu = \hat{P}_i \cap B^\nu$ and for $i_0 \neq i$, $\hat{P}_{i_0}^\nu = \hat{P}_{i_0} \cap B^{(\sharp I+\sharp J)\nu}$. The closed unit ball $\bar{B} = \{x \in \mathbb{R}^L : \|x\| \leq 1\}$ will be the price set. The truncation of X_{i_0} is chosen in such a way that if $(x, y) \in \prod_{i\in I} X_i^\nu \times \prod_{j\in J} Y_j^\nu$ is feasible, that is, $\sum_{i\in I} x_i = \omega + \sum_{j\in J} y_j$, then x_{i_0} belongs to the open ball $B^{(\sharp I+\sharp J)\nu}$.

We now consider the economy

$$\mathcal{E}^\nu = \left(\mathbb{R}^L, (X_i^\nu, \hat{P}_i^\nu, \omega_i)_{i\in I}, (Y_j^\nu)_{j\in J}, (\theta_{i,j})_{(i\in I, j\in J)}\right)$$

where the consumption and production sets are compact.

Remark 4.1. For all i, the correspondence $\hat{P}_i^\nu$ is lower semi-continuous. Indeed, $\hat{P}_i^\nu$ is the intersection of the lower semi-continuous correspondence $\hat{P}_i$ and the constant correspondence B^ν (or $B^{(\sharp I+\sharp J)\nu}$), which has on open graph.

Remark 4.2. With the above remark and since $\bar{B}^\nu$ is convex and closed, note that the compact economy $\mathcal{E}^\nu$ satisfies Assumption (H1') but the non satiation of preferences at attainable allocations and Assumption (H2'). Furthermore, Y_j^ν is now compact.

Since each Y_j^ν is compact, we can define for every $p \in \bar{B}$ the profit function

$$\pi_j^\nu(p) = \sup p.Y_j^\nu = \sup\{p.y_j : y_j \in Y_j^\nu\}$$

and the wealth of consumer i is defined by:

$$\gamma_i^\nu(p) = p.\omega_i + \sum_{j\in J} \theta_{ij}\pi_j^\nu(p).$$

Note that the function $\pi_j^\nu : \bar{B} \to \mathbb{R}$ is continuous since it is finite and convex.

In what follows, we will use the following notations for simplicity

- $Z^\nu = \prod_{i\in I} X_i^\nu \times \prod_{j\in J} Y_j^\nu \times \bar{B}$ and $z = (x, y, p)$ denotes a typical element of Z^ν
- $\hat{\gamma}_i^\nu(z) = \gamma_i^\nu(p) + \frac{1-\|p\|}{\sharp I}$
- $\widetilde{\gamma}_i^\nu(z) = \max\{\hat{\gamma}_i^\nu(z), \frac{1}{2}[\hat{\gamma}_i^\nu(p) + p \cdot x_i]\}$

Remark 4.3. Note that $p \cdot x_i > \widetilde{\gamma}_i^\nu(z) > \hat{\gamma}_i^\nu(z)$ when $p \cdot x_i > \hat{\gamma}_i^\nu(z)$ and $\widetilde{\gamma}_i^\nu(z) = \hat{\gamma}_i^\nu(z)$ when $p \cdot x_i \leq \hat{\gamma}_i^\nu(z)$.

Let now $N = I \cup J \cup \{0\}$ be the union of the set of consumers I indexed by i, the set of producers J indexed by j and an additional agent 0 whose function is to react with prices to a given total excess demand.

For all $i \in I$, we define the correspondences $\alpha_i^\nu : Z^\nu \to X_i^\nu$ and $\widetilde{\beta}_i^\nu : Z^\nu \to X_i^\nu$ as follows.

$$\alpha_i^\nu(z) = \{\xi_i \in X_i^\nu : p \cdot \xi_i \leq \hat{\gamma}_i^\nu(z)\}$$
$$\widetilde{\beta}_i{}^\nu(z) = \{\xi_i \in X_i^\nu : p \cdot \xi_i < \widetilde{\gamma}_i^\nu(z)\}$$

From the construction of the extended budget set, one checks that for all i, the consumption $\underline{x}_i$ belongs to $\widetilde{\beta}_i{}^\nu(z)$ if $x_i \notin \alpha_i^\nu(z)$. Indeed, from (H1')(d),

$$\underline{x}_i = \omega_i + \sum_{j\in J} \theta_{i,j}\underline{y}_{i,j}$$

since $x_i \notin \alpha_i{}^\nu(z)$, $p \cdot x_i > \hat{\gamma}_i^\nu(z)$ and $\widetilde{\gamma}_i^\nu(z) > \hat{\gamma}_i^\nu(z)$. Furthermore

$$\begin{aligned} p \cdot \underline{x}_i &= p \cdot \omega_i + \sum_{j\in J} \theta_{i,j} p \cdot \underline{y}_{i,j} \\ &\leq p \cdot \omega_i + \sum_{j\in J} \theta_{i,j}\pi_j^\nu(p) \\ &\leq \hat{\gamma}_i^\nu(z) \\ &< \widetilde{\gamma}_i^\nu(z) \end{aligned}$$

which means that $\underline{x}_i$ belongs to $\widetilde{\beta}_i^{\nu}(z)$. Furthermore, since $\widetilde{\gamma}_i^{\nu}$ is continuous, the correspondence $\widetilde{\beta}_i^{\nu}$ has an open graph in $Z^{\nu} \times X_i^{\nu}$.

Now, for $i \in I$, we consider the mapping ϕ_i^{ν} defined from Z^{ν} to X_i^{ν} by:

$$\phi_i^{\nu}(z) = \begin{cases} \widetilde{\beta}_i^{\nu}(z) & \text{if } x_i \notin \alpha_i^{\nu}(z) \\ \widetilde{\beta}_i^{\nu}(z) \cap \hat{P}_i^{\nu}(x) & \text{if } x_i \in \alpha_i^{\nu}(z) \end{cases}$$

For $j \in J$, we define ϕ_j^{ν} from Z^{ν} to Y_j^{ν} by:

$$\phi_j^{\nu}(z) = \{y_j' \in Y_j^{\nu} \mid p \cdot y_j < p \cdot y_j'\},$$

and the mapping ϕ_0^{ν} from Z^{ν} to $\bar{B}$ is defined by:

$$\phi_0^{\nu}(z) = \{q \in \bar{B} \mid (q-p) \cdot (\sum_{i \in I} x_i - \omega - \sum_{j \in J} y_j) > 0\}$$

Now we will apply to Z^{ν} and the correspondences $(\phi_i)_{i \in I}^{\nu}, (\phi_j)_{j \in J}^{\nu}, \phi_0^{\nu}$ the well known theorem of Gale and Mas-Colell [11]. We will actually use the Bergstrom version of this theorem in [3], which is more adapted to our setting.

Theorem 4.2. *(Gale and Mas-Colell - Bergstrom) For each $k = 1, \cdots, \bar{k}$, let Z_k be a nonempty, compact, convex subset of some finite dimensional Euclidean space. Given $Z = \prod_{k=1}^{\bar{k}} Z_k$, let for each k, $\phi_k : Z \to Z_k$ be a lower semicontinuous correspondences satisfying for all $z \in Z$, $z_k \notin co\phi_k(z)$. Then there exists $\bar{z} \in Z$ such that for each $k = 1, \cdots, \bar{k}$:*

$$\phi_k(\bar{z}) = \emptyset \tag{1}$$

For the correspondences $(\phi_j^{\nu})_{j \in J}$ and ϕ_0^{ν}, one easily checks that they are convex valued, irreflexive and lower semi-continuous since they have an open graph.

We now check that for all $i \in I$, the correspondence ϕ_i^{ν} satisfies the assumption of Theorem 4.2. We first remark that ϕ_i^{ν} has convex valued since $\widetilde{\beta}_i^{\nu}$ and $\hat{P}_i$ are so. We now check the irreflexivity. If $x_i \in \alpha_i^{\nu}(z)$, then, from Assumption (H1')(b), $x_i \notin \hat{P}_i(x)$, so $x_i \notin \phi_i^{\nu}(x)$ since $\phi_i^{\nu}(x) \subset \hat{P}_i(x)$. If $x_i \notin \alpha_i^{\nu}(z)$, then from Remark 4.3, $p \cdot x_i > \widetilde{\gamma}_i^{\nu}(z)$, so $x_i \notin \widetilde{\beta}_i^{\nu}(z) = \phi_i^{\nu}(z)$.

For the lower semi-continuity, let V be an open set and z such that $\phi_i^{\nu}(z) \cap V \neq \emptyset$. If $x_i \notin \alpha_i^{\nu}(z)$, then $p \cdot x_i > \hat{\gamma}_i^{\nu}(z)$. Since $\hat{\gamma}_i^{\nu}$ is continuous, there exists a neighborhood W of z such that for all $z' \in W$, $p' \cdot x_i' > \hat{\gamma}_i^{\nu}(z')$. Since $\widetilde{\beta}_i^{\nu}$ has an open graph, there existe a neighborhood W' of z such that for all $z' \in W'$,

$\widetilde{\beta}_i^\nu(z') \cap V \neq \emptyset$. So, for all $z' \in W \cap W'$, $\phi_i^\nu(z') \cap V \neq \emptyset$ and consequently, ϕ_i^ν is lower semi-continuous at z. If $x_i \in \alpha_i^\nu(z)$, we first remark that $\widetilde{\beta}_i^{\,\nu} \cap \hat{P}_i^\nu$ is lower semicontinuous as an intersection of a lower semicontinuous correspondence with an open graph correspondence. So, there exists a neighborhood W of z such that for all $z' \in W$, $\widetilde{\beta}_i^{\,\nu}(z') \cap \hat{P}_i^\nu(x') \cap V \neq \emptyset$. This implies that $\widetilde{\beta}_i^{\,\nu}(z') \cap V \neq \emptyset$. Hence, in both cases, $x_i' \in \alpha_i^\nu(z')$ or $x_i' \notin \alpha_i^\nu(z')$, $\phi_i^\nu(z') \cap V \neq \emptyset$ from the definition of ϕ_i^ν. Thus ϕ_i^ν is also lower semi-continuous at z in this case.

From Theorem 4.2 , there exists $\bar{z}^\nu = (\bar{x}^\nu, \bar{y}^\nu, \bar{p}^\nu) \in Z^\nu$ such that, for all $k \in N$

$$\phi_k^\nu(\bar{z}^\nu) = \emptyset \tag{2}$$

As already noticed, since for all $i \in I$, $\underline{x}_i \in \widetilde{\beta}_i^\nu(\bar{z}^\nu)$ and $\phi_i^\nu(\bar{z}^\nu) = \emptyset$, we conclude from the definition of ϕ_i^ν that

$$\begin{cases} \bar{p}^\nu \cdot \bar{x}_i^\nu \leq \hat{\gamma}_i^\nu(\bar{z}^\nu) \\ \widetilde{\beta}_i^\nu(\bar{z}^\nu) \cap \hat{P}_i^\nu(\bar{x}^\nu) = \emptyset \end{cases} \tag{3}$$

Furthermore, from Remark 4.3, one deduces that $\widetilde{\gamma}_i^\nu(\bar{z}^\nu) = \hat{\gamma}_i^\nu(\bar{z}^\nu)$.

In addition, for all $j \in J$, since $\phi_j^\nu(\bar{z}^\nu) = \emptyset$, we deduce that:

$$\forall y_j \in Y_j^\nu, \bar{p}^\nu \cdot y_j \leq \bar{p}^\nu \cdot \bar{y}_j^\nu = \pi_j^\nu(\bar{p}^\nu), \tag{4}$$

and since $\phi_0^\nu(\bar{z}^\nu) = \emptyset$,

$$\forall p \in \bar{B}, p \cdot (\sum_{i \in I} \bar{x}_i^\nu - \omega - \sum_{j \in J} \bar{y}_j^\nu) \leq \bar{p}^\nu \cdot (\sum_{i \in I} \bar{x}_i^\nu - \omega - \sum_{j \in J} \bar{y}_j^\nu) \tag{5}$$

We now prove that $(\sum_{i \in I} \bar{x}_i^\nu - \omega - \sum_{j \in J} \bar{y}_j^\nu) = 0$. Indeed, if not, it follows from (5) that $\bar{p}^\nu$ belongs to the boundary of $\bar{B}$, that is $\|\bar{p}^\nu\| = 1$ and $\bar{p}^\nu \cdot (\sum_{i \in I} \bar{x}_i^\nu - \omega - \sum_{j \in J} \bar{y}_j^\nu) > 0$. Now, by (3) and (4), for all i, $\bar{p}^\nu \cdot \bar{x}_i^\nu \leq \hat{\gamma}_i^\nu(\bar{z}^\nu) = \gamma_i^\nu(\bar{z}^\nu) = \bar{p}^\nu \cdot \omega_i + \sum_{j \in J} \theta_{i,j} \bar{p}^\nu \cdot \bar{y}_j^\nu$. Summing up over $i \in I$ these inequalities, one gets, $\bar{p}^\nu \cdot (\sum_{i \in I} \bar{x}_i^\nu - \omega - \sum_{j \in J} \bar{y}_j^\nu) \leq 0$, which yields a contradiction. We thus have proved that $(\bar{x}^\nu, \bar{y}^\nu) \in \mathcal{A}(\mathcal{E}^\nu)$.

Remark 4.4. Since $(\bar{x}^\nu, \bar{y}^\nu)$ is feasible, we deduce that $\bar{x}_{i_0}^\nu$ belongs to the open ball $B^{(\sharp I + \sharp J)\nu}$. From Assumption (H1')(e), $\hat{P}_{i_0}(\bar{x}^\nu)$ is nonempty and for all $\xi_{i_0} \in \hat{P}_{i_0}(\bar{x}^\nu)$ and for all $t \in]0,1]$, $t\xi_{i_0} + (1-t)\bar{x}_{i_0}^\nu \in \hat{P}_{i_0}(\bar{x}^\nu)$. For t small enough, $t\xi_{i_0} + (1-t)\bar{x}_{i_0}^\nu$ belongs to $B^{(\sharp I + \sharp J)\nu}$, so to $\hat{P}_{i_0}^\nu(\bar{x}^\nu)$. From (3), $\bar{p}^\nu \cdot (t\xi_{i_0} + (1-t)\bar{x}_{i_0}^\nu) \geq \hat{\gamma}_{i_0}^\nu(\bar{z}^\nu)$. At the limit when t tends to 1, knowing from (3) that $\bar{p}^\nu \cdot \bar{x}_{i_0}^\nu \leq \hat{\gamma}_{i_0}^\nu(\bar{z}^\nu)$, one gets

$$\bar{p}^\nu \cdot \bar{x}_{i_0}^\nu = \hat{\gamma}_{i_0}^\nu(\bar{z}^\nu) \tag{6}$$

from which one deduces that

$$\forall \xi_{i_0} \in \hat{P}_{i_0}(\bar{x}^\nu), \bar{p}^\nu \cdot \xi_{i_0} \geq \hat{\gamma}_{i_0}^\nu(\bar{z}^\nu) \tag{7}$$

4.2 The limit argument

We first show that we can apply Assumption (WH3) to the sequence $((\bar{x}_i^\nu))$ built in the previous sub-section. We have already proved that $\bar{x}^\nu$ is attainable in the truncated economy $\mathcal{E}^\nu$, so it is also attainable in the economy $\mathcal{E}'$. It remains to show that $\underline{x}_i \in \overline{\hat{P}_i(\bar{x}^\nu)^c}$ for all i.

There are two cases. First, if $\bar{p}^\nu \cdot \underline{x}_i < \hat{\gamma}_i^\nu(\bar{z}^\nu)$, which means that $\underline{x}_i \in \widetilde{\beta}_i^\nu(\bar{z}^\nu)$, then, from (3), $\underline{x}_i \notin \hat{P}_i^\nu(\bar{x}^\nu) = \hat{P}_i(\bar{x}^\nu) \cap B^\nu$. Since $\underline{x}_i \in B^\nu$ as ν has been chosen large enough, one deduces that $\underline{x}_i \notin \hat{P}_i(\bar{x}^\nu)$ and therefore $\underline{x}_i \in \overline{\hat{P}_i(\bar{x}^\nu)^c}$.

If $\bar{p}^\nu \cdot \underline{x}_i \geq \hat{\gamma}_i^\nu(\bar{z}^\nu)$, as $\underline{x}_i \in \widetilde{\beta}_i^\nu(\bar{z}^\nu)$ and $\hat{\gamma}_i^\nu(\bar{z}^\nu) = \widetilde{\gamma}_i^\nu(\bar{z}^\nu)$, we actually have the equality $\bar{p}^\nu \cdot \underline{x}_i = \hat{\gamma}_i^\nu(\bar{z}^\nu)$. We remark that $\hat{\gamma}_i^\nu(\bar{z}^\nu) = \gamma_i^\nu(\bar{z}^\nu) + \frac{1-\|\bar{p}^\nu\|}{\sharp I} = \bar{p}^\nu \cdot \underline{x}_i = \bar{p}^\nu \cdot (\omega_i + \sum_{j\in J} \theta_{i,j} \underline{y}_{i,j}) \leq \gamma_i^\nu(\bar{z}^\nu)$. So, $\|\bar{p}^\nu\| = 1$. By contradiction, we prove that $\underline{x}_i \in \overline{\hat{P}_i(\bar{x}^\nu)^c}$. Indeed, if not, $\underline{x}_i \in \operatorname{int} \hat{P}_i(\bar{x}^\nu)$ and there exists $\rho > 0$ such that $B(\underline{x}_i, \rho) \subset \hat{P}_i(\bar{x}^\nu)$ and $B(\underline{x}_i, \rho) \subset B^\nu$. Since $\bar{p}^\nu \neq 0$, there exists $\xi_i^\nu \in B(\underline{x}_i, \rho)$ such that $\bar{p}^\nu \cdot \xi_i^\nu < \bar{p}^\nu \cdot \underline{x}_i = \hat{\gamma}_i^\nu(\bar{z}^\nu)$ and this contradicts (3) since $\xi_i^\nu \in B(\underline{x}_i, \rho) \subset \hat{P}_i^\nu(\bar{x}^\nu)$.

We now consider the subsequence $((\bar{x}_i^{\varphi(\nu)}))$ of $\hat{X}$ and $((\bar{x}_i)) \in \hat{X}$ as given by Assumption (WH3). From the definition of $\hat{X}$, there exists $(\bar{y}_j) \in \prod_{j\in J} Y_j'$ such that $\sum_{i\in I} \bar{x}_i = \sum_{i\in I} \omega_i + \sum_{j\in J} \bar{y}_j$. Since $\bar{B}$ is compact, we can assume without any loss of generality that the sequence $(\bar{p}^{\varphi(\nu)})$ converges to $\bar{p} \in \bar{B}$.

Now let $(y_j) \in \prod_{j\in J} Y_j'$, $(\xi_i) \in \prod_{i\in I} \hat{P}_i(\bar{x})$ and $\lambda \in [0, 1[$. Such (ξ_i) exists from Assumption (H1')(e). Furthermore, from the definition of the extended preferences, note that $\xi_i^\lambda = \lambda \bar{x}_i + (1-\lambda)\xi_i \in \hat{P}_i(\bar{x})$.

By (WH3), there exists an integer ν_1 and a sequence $(\xi_i^{\varphi(\nu)})_{\nu\geq\nu_1}$ convergent to ξ_i^λ such that for all $\nu \geq \nu_1$, $\xi_{i_0}^{\varphi(\nu)} \in \operatorname{cone}\{\hat{P}_{i_0}(\bar{x}^{\varphi(\nu)}) - \bar{x}_{i_0}^{\varphi(\nu)}\} + \bar{x}_{i_0}^{\varphi(\nu)}$ and for all $i \neq i_0$, $\xi_i^{\varphi(\nu)} \in \hat{P}_i(\bar{x}^{\varphi(\nu)})$. Since the sequence $(\xi_i^{\varphi(\nu)})_{\nu\geq\nu_1}$ is convergent, it is bounded and for ν large enough, for all $i \neq i_0$, $\xi_i^{\varphi(\nu)}$ belong to B^ν, so $\xi_i^{\varphi(\nu)} \in \hat{P}_i^\nu(\bar{x}^{\varphi(\nu)})$, $\forall \nu \geq \nu_1$. We deduce from (3) that $\xi_i^{\varphi(\nu)} \notin \widetilde{\beta}_i^\nu(\bar{z}^{\varphi(\nu)})$, that is, $\bar{p}^\nu \cdot \xi_i^{\varphi(\nu)} \geq \widetilde{\gamma}_i^\nu(\bar{z}^\nu) = \hat{\gamma}_i^\nu(\bar{z}^\nu)$. Using the same argument as in Remark 4.4,

one deduces that $\bar{p}^\nu \cdot \bar{x}_i^{\varphi(\nu)} = \hat{\gamma}_i^\nu(\bar{z}^\nu)$. So, from Remark 4.4, for all $i \in I$,

$$\bar{p}^\nu \cdot \bar{x}_i^{\varphi(\nu)} = \bar{p}^{\varphi(\nu)} \cdot \omega_i + \sum_{j \in J} \theta_{i,j} \bar{p}^{\varphi(\nu)} \cdot \bar{y}_j^{\varphi(\nu)} + \frac{1 - \|\bar{p}^{\varphi(\nu)}\|}{\sharp I}$$

Summing over i these inequalities and knowing that $(\bar{x}^{\varphi(\nu)}, \bar{y}^{\varphi(\nu)})$ is a feasible allocation, we conclude that $\|\bar{p}^{\varphi(\nu)}\| = 1$ and at the limit, $\|\bar{p}\| = 1$.

For all $i \neq i_0$,

$$\bar{p}^{\varphi(\nu)} \cdot \xi_i^{\varphi(\nu)} \geq \gamma_i^\nu(\bar{z}^\nu) = \bar{p}^{\varphi(\nu)} \cdot \omega_i + \sum_{j \in J} \theta_{i,j} \bar{p}^{\varphi(\nu)} \cdot \bar{y}_j^{\varphi(\nu)}$$

and for i_0, there exists $\alpha \geq 0$ and $\zeta_{i_0}^{\varphi(\nu)} \in \hat{P}_{i_0}(\bar{x}^{\varphi(\nu)})$ such that,

$$\bar{p}^{\varphi(\nu)} \cdot \xi_{i_0}^{\varphi(\nu)} = \bar{p}^{\varphi(\nu)} \cdot [\bar{x}_{i_0}^{\varphi(\nu)} + \alpha(\zeta_{i_0}^{\varphi(\nu)} - \bar{x}_{i_0}^{\varphi(\nu)})]$$

From (6) and (7), $\bar{p}^{\varphi(\nu)} \cdot \bar{x}_{i_0}^{\varphi(\nu)} = \hat{\gamma}_{i_0}^\nu(\bar{z}^\nu) = \gamma_{i_0}^\nu(\bar{z}^\nu)$ and $\bar{p}^{\varphi(\nu)} \cdot \zeta_{i_0}^{\varphi(\nu)} \geq \hat{\gamma}_{i_0}^\nu(\bar{z}^\nu) = \gamma_{i_0}^\nu(\bar{z}^\nu)$, so, since $\alpha \geq 0$, one concludes that

$$\bar{p}^{\varphi(\nu)} \cdot \xi_{i_0}^{\varphi(\nu)} \geq \gamma_{i_0}^\nu(\bar{z}^\nu) = \bar{p}^{\varphi(\nu)} \cdot \omega_{i_0} + \sum_{j \in J} \theta_{i_0,j} \bar{p}^{\varphi(\nu)} \cdot \bar{y}_j^{\varphi(\nu)}$$

For ν large enough, for all $j \in J$, $y_j \in \bar{B}^\nu$. So, $(y_j) \in \prod_{j \in J} Y_j^\nu$, and from (4), one gets

$$\bar{p}^{\varphi(\nu)} \cdot \xi_i^{\varphi(\nu)} \geq \bar{p}^{\varphi(\nu)} \cdot \omega_i + \sum_{j \in J} \theta_{i,j} \bar{p}^{\varphi(\nu)} \cdot y_j \tag{8}$$

In particular, for $(\bar{y}_j) \in \prod_{j \in J} Y_j'$, one gets

$$\bar{p}^{\varphi(\nu)} \cdot \xi_i^{\varphi(\nu)} \geq \bar{p}^{\varphi(\nu)} \cdot \omega_i + \sum_{j \in J} \theta_{i,j} \bar{p}^{\varphi(\nu)} \cdot \bar{y}_j \tag{9}$$

Passing to the limit in (8) and (9), we obtain:

$$\bar{p} \cdot \xi_i^\lambda \geq \bar{p} \cdot \omega_i + \sum_{j \in J} \theta_{i,j} \bar{p} \cdot y_j \tag{10}$$

and

$$\bar{p} \cdot \xi_i^\lambda \geq \bar{p} \cdot \omega_i + \sum_{j \in J} \theta_{i,j} \bar{p} \cdot \bar{y}_j \tag{11}$$

The two above inequalities hold true for any $i \in I$, $\xi_i \in \hat{P}_i(\bar{x})$, $\lambda \in [0,1[$ and $(y_j) \in \prod_{j\in J} Y'_j$. Knowing that $(\bar{x},\bar{y})$ is an attainable allocation, we will show that $(\bar{x},\bar{y},\bar{p})$ is a quasi-equilibrium of the economy $\mathcal{E}'$, which completes the proof.

When λ goes to 1 in (10) and (11), one gets

$$\bar{p}\cdot\bar{x}_i \geq \bar{p}\cdot\omega_i + \sum_{j\in J}\theta_{i,j}\bar{p}\cdot y_j \tag{12}$$

and

$$\bar{p}\cdot\bar{x}_i \geq \bar{p}\cdot\omega_i + \sum_{j\in J}\theta_{i,j}\bar{p}\cdot\bar{y}_j \tag{13}$$

Since $(\bar{x},\bar{y})$ is a feasible allocation, summing over i the inequalities in (13), one deduces that

$$\bar{p}\cdot\bar{x}_i = \bar{p}\cdot\omega_i + \sum_{j\in J}\theta_{i,j}\bar{p}\cdot\bar{y}_j \tag{14}$$

Taken $\lambda = 0$ in (11), we obtain for all $i \in I$, for all $\xi_i \in \hat{P}_i(\bar{x})$,

$$\bar{p}\cdot\xi_i \geq \bar{p}\cdot\omega_i + \sum_{j\in J}\theta_{i,j}\bar{p}\cdot\bar{y}_j \tag{15}$$

So, the quasi-demand condition (b) of Definition 2.2 is satisfied.

Finally, from (12) and (13), for all $(y_j) \in \prod_{j\in J} Y'_j$, one gets

$$\bar{p}\cdot\omega_i + \sum_{j\in J}\theta_{i,j}\bar{p}\cdot y_j \leq \bar{p}\cdot\omega_i + \sum_{j\in J}\theta_{i,j}\bar{p}\cdot\bar{y}_j \tag{16}$$

Summing over i, we get

$$\sum_{j\in J}\bar{p}\cdot y_j \leq \sum_{j\in J}\bar{p}\cdot\bar{y}_j$$

For any $j \in J$, applying this inequality to $y' \in \prod_{j\in J} Y'_j$ defined by $y'_{j'} = \begin{cases} y_j & \text{if } j' = j \\ \bar{y}'_j & \text{if } j' \neq j \end{cases}$, it readily follows that

$$\bar{p}\cdot y_j \leq \bar{p}\cdot\bar{y}_j \tag{17}$$

which means that the profit maximization condition (a) of Definition 2.2 is also satisfied. □

5 Relationship with the literature

In this section, we compare Assumption (H3) with other conditions in the literature on the existence of equilibrium with non compact attainable sets. We show that Assumption (H3) is weaker than the compactness of the set of individually rational and attainable allocations or utility levels and the CPP condition of Allouch. We also explain the relationships with the condition of Won and Yannelis.

5.1 Compactness of the attainable utility set

The following proposition shows that Assumption (H3) is weaker than the compactness of $\mathcal{A}(\mathcal{E})$ or U the attainable utility set. We use the following assumption on preferences as in Allouch.

Assumption (H4) The utility function u_i is lower semi-continuous and strictly quasi-concave, that is, for all $(x_i, z_i) \in X_i \times X_i$ with $u_i(z_i) > u_i(x_i)$ then $u_i(\lambda x_i + (1-\lambda)z_i) > u_i(x_i)$ for all $\lambda \in [0, 1[$.

If P_i is represented by a utility function u_i satisfying Assumption (H4), i.e.:

$$P_i(x) = \{x_i' \in X_i : u_i(x_i') > u_i(x_i)\}$$

then, $P_i(x) = \hat{P}_i(x)$, for all $x \in \prod_{i\in I} X_i$. If the preferences of all consumers are represented by a utility function, the set of attainable utility level U is defined as:

$$U = \{(\mathrm{v}_1, \mathrm{v}_2, ..., \mathrm{v}_m) \in \mathbb{R}^I_+ : \exists x \in \hat{X} \text{ s.t. } u_i(\underline{x}_i) \leq \mathrm{v}_i \leq u_i(x_i)\},$$

In an exchange economy with the survival assumption $\omega_i \in X_i$ for all i, the set U is just the set of individually rational attainable consumptions.

Proposition 5.1. *Under Assumption (H1),*

(i) If $\mathcal{A}(\mathcal{E})$ is compact, then (H3) is satisfied.

(ii) If the preferences of all consumers are represented by a utility function satisfying Assumption (H4) and if U is compact, then Assumption (H3) is satisfied.

The proof of this proposition as a consequence of the lower semicontinuity of preferences for (i) and the utility representation for (ii) is left to the reader.

5.2 Comparison with the CPP condition of Allouch

We recall the following definition of the CPP condition considered by Allouch [1].

Definition 5.1. *The economy $\mathcal{E}$ satisfies the CPP condition if for every sequence $((x_i^\nu))$ of $\hat{X}$, there exists a subsequence $((x_i^{\varphi(\nu)})) \in \hat{X}$, an element $(\xi_i) \in \hat{X}$ and a sequence $(\xi_i^{\varphi(\nu)})_{\nu \geq \nu_1}$ convergent to ξ_i with $\xi_i^{\varphi(\nu)} \in \hat{P}_i(x^{\varphi(\nu)})$, for all ν.*

Beside this assumption, Allouch also assumes that the preference relations are transitive, have open lower-section and that the augmented preferences are equal to the preferences. Assumption (H3) and the CPP condition have the same flavour, but the transitivity allows to consider a unique sequence $(\xi_i^{\varphi(\nu)})$ whereas Assumption (H3) needs a sequence for each preferred element.

Proposition 5.2. *Let us assume that the preference relations are transitive, have open lower-section and are equal to the augmented preferences. Then if the CPP condition is satisfied, Assumption (H3) holds true.*

The proof is a direct consequence of the transitivity of preferences and the open lower section, which allow to get the desired property under the CPP condition.

5.3 Comparison with Won and Yannelis work

To compare our contribution to the one of Won and Yannelis [21], we restrict our attention to an exchange economy. Indeed, the initial endowments ω_i are used as a reference point on the budget line and there is no equivalent consumption in a production economy. The frameworks and the basic assumptions are quite similar and we focused our attention on the asymptotic condition corresponding to our Assumption (H3). To state it, we borrow the following notations from [21]. For $x \in \prod_{i \in I} X_i$, for all $i \in I$, $r_i(x) = \max\{\|x_k\| \mid k \neq i\}$ and $\bar{B}(0,r)$ denotes the closed ball of center 0 and radius r. We now state Assumption (B7a) of Won and Yannelis.

Assumption (B7a) There exists a consumer $i_0 \in I$ such that for all sequence $((x_i^\nu))$ of $\hat{X}$ with $\omega_i \in \overline{\hat{P}_i(x^\nu)^c}$ for all i and for all ν, there exists a subsequence $((x_i^{\varphi(\nu)}))$ and a sequence $(y^{\varphi(\nu)})$ convergent to a point $y \in \hat{X}$, such that, for all ν,

$P_{i_0}(y^{\varphi(\nu)}) \subset \text{cone}[P_{i_0}(x^{\varphi(\nu)}) - \{\omega_{i_0}\}] + \{\omega_{i_0}\}$

and for all $i \neq i_0$,

$P_i(y^{\varphi(\nu)}) \cap \bar{B}(0, r_{i_0}(x^{\varphi(\nu)})) \subset \text{cone}[P_i(x^{\varphi(\nu)}) \cap \bar{B}(0, r_{i_0}(x^{\varphi(\nu)})) - \{\omega_i\}] + \{\omega_i\}$.

We first remark that Assumption (H3) does not require the sequence $(y^{\varphi(\nu)})$ and the inclusion of the associated preferred set, or a truncation of it, in a set generated by the preferred set of $x^{\varphi(\nu)}$. Indeed, our assumption has the flavour of the CPP condition of Allouch.

Note that the use of the cone operator enlarges the set $[P_{i_0}(x^{\varphi(\nu)})] - \{\omega_{i_0}\}]$ or $[P_i(x\{\omega_{i_0}\}) \cap \bar{B}(0, r_{i_0}(x\{\omega_{i_0}\})) - \{\omega_i\}]$, so the condition is weaker than assuming $P_{i_0}(y^{\varphi(\nu)}) \subset P_{i_0}(x^{\varphi(\nu)})$ and $P_i(y^{\varphi(\nu)}) \cap \bar{B}(0, r_{i_0}(x^{\varphi(\nu)})) \subset P_i(y^{\varphi(\nu)}) \cap \bar{B}(0, r_i(x^{\varphi(\nu)}))$ for all $i \neq i_0$. Note that, thanks to the lower semi-continuity of the preferences, Assumption (H3) is weaker than assuming the existence of the convergent sequence $(y^{\varphi(\nu)})$ and the inclusion $P_i(y^{\varphi(\nu)}) \subset P_i(x^{\varphi(\nu)})$. So, at this stage, the two assumptions are not comparable.

The major advantage of Assumption (B7a) comes from the fact that it is satisfied by the example Page et al [15] where an equilibrium exists with an non compact set of attainable individually rational utility level. We easily check that this example satisfies the following asymmetric weakening of Assumption (H3) in the framework of an exchange economy:

Assumption (EWH3) There exists a consumer $i_0 \in I$, such that for all sequence $((x_i^\nu))$ of $\hat{X}$ such that for all i, $\omega_i \in \overline{\hat{P}_i(x^\nu)^c}$, there exists a subsequence $((x_i^{\varphi(\nu)})) \in \hat{X}$ and $(\bar{x}_i) \in \hat{X}$ such that for all i, for all $\xi_i \in \hat{P}_i(\bar{x})$, there exists an integer ν_1 and a sequence $(\xi_i^{\varphi(\nu)})_{\nu \geq \nu_1}$ convergent to ξ_i with, for all $\nu \geq \nu_1$,

$\xi_{i_0}^{\varphi(\nu)} - \omega_{i_0} \in \text{cone}[\hat{P}_{i_0}(x^{\varphi(\nu)}) - \omega_{i_0}]$

and for all $i \neq i_0$,

$\xi_i^{\varphi(\nu)} \in \hat{P}_i(x^{\varphi(\nu)})$.

We did not consider and emphasise this assumption previously since its asymmetry is an hint that there is still room for improvements to get a still weaker and symmetric assumption. We can easily adapt the proof of Section 4 to check that assumption (EWH3) is sufficient for the existence of quasi-equilibrium in exchange economies.

Finally, we discuss Example 3.1.2 of Won and Yannelis. Clearly, Assumption (EWH3) does not cover this example.The authors claim that this example satisfies their weaker assumption (B7). The argument is based on the fact that there is no equilibrium in the truncated economy except the no-trade

one with the two consumptions equal to 0 and any positive price. Actually, it seems to us that the price $p = (0, 1)$ associated to the consumptions $x_1 = (r, 0)$ and $x_2 = (-r, 0)$ is an equilibrium when the first agent has a truncated budget set $\bar{B}(0, r)$. In that case, the set $P_1(x) \cap \bar{B}(0, r_2(x))$ is empty, so is the set $G_2(x)$ with the notation of the paper. Consequently, finding an assumption covering Example 3.1.2 of [21] is still an open challenge.

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