On the Measurement of Functional Income Distribution

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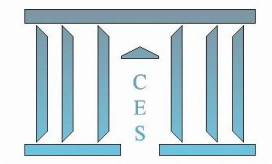

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On the Measurement of Functional Income Distribution*

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Abstract

The present work proposes a framework for the analysis of inequality in income composition. Hinging on the recent work by Ranaldi (2017), we here propose a metric of income-factor polarization, the *Income-Factor Polarization Index* $\mathscr{I}_f$, that captures the extent to which two income sources, notably profits and wages, are polarized across the distribution of income. We measure income-factor polarization as the distance between the *Polarization Curve for Income Source* and the *Zero-Polarization Curve*, suitably normalized. The former is the cumulative distribution of income source across the population with individuals being indexed by their income rank, and the latter is the benchmark of zero inequality in income composition, defined as the situation in which each individual has the same population share of profits and wages (Ranaldi, 2017). The greater the distance, the higher the polarization. We show that the index narrows down to a *single one* the *two conditions* for the rising share of capital income to increase overall income Gini introduced by Milanovic (2017), which is: $\mathscr{I}_f > 0$. The methodology is finally illustrated via an empirical application on the case of Italy.

JEL-Classification: C430, E250

Keywords: Income Composition Inequality, Income Distribution, Inequality, Polarization, Statistical Methodology

1 Introduction

The measurement of *income inequality* across people has attracted the attention of many scholars in the past. Among those, we recall Pareto (1896), Hoover (1936), Atkinson (1970), Theil (1979), Palma (2011), to name a few of them. Although the topic of inequality measurement is also at the core of the present analysis, our main focus is on a different type of inequality from that studied by the authors mentioned above. Indeed, we are here interested in *income composition inequality*. Precisely, instead

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of addressing the question: how unequally is *income* distributed across the population, we wonder: how unequally is the *composition of income* distributed across the population. To better grasp the meaning of inequality in income composition, consider the following simple example.

Let us suppose to analyze a society composed by two people only, A and B. We call y_A the income of A and y_B the income of B. The total income y, which is the sum of y_A and y_B ($y = y_A + y_B$), can be decomposed in profits, π, and wages, w: $y = \pi + w$. Let us suppose that $y_A = y_B$. In such scenario, A and B have the same income level, thus the total income is equally distributed across the population (i.e. the Gini coefficient is equal to zero). However, what can we say about the distribution of sources among the two people? If we suppose that $y_A = \pi$ and $y_B = w$ as in Kaldor (1955), we say that the two sources are *maximum-polarized* between A and B. Therefore, even though the total income is equally split between A and B, the two main actors play a completely different role in the economy: A produces goods[1], and B consumes them. On the contrary, if we suppose $y_A = \frac{1}{2}\pi + \frac{1}{2}w = y_B$, then the society registers *zero-polarization* of income sources, and the main actors' economic role changes considerably: both individuals are producers, and consumers at the same time.

Although the latter example oversimplifies reality, it can be helpful to understand both the rationale, and the purpose of our article. Indeed, there are many different reasons for studying income composition inequality. For example, if we know that income sources are fully polarized between two groups of people, we may have a better understanding not only of the underlying economic structure of the society in which they live, but also of each group's political interest. In fact, if the richest 10% of the population owned capital only, they could likely be unfavorable to a capital-tax increase. At the same time, if the bottom 90% of the population owned wages only, they could definitely prefer a capital-tax increase instead of a wage-tax increase. Along the same line, Atkinson argues that factor shares are not only concerned with depicting dissimilarities among individuals (Atkinson, 2009). In fact, the latter can also be seen as measures of similarities, in terms of interest groups. A specific group of people, whose income is mainly dependent on wages, for instance, may have different economic, as well as political, interests from the group of entrepreneurs, whose income is strongly conditioned by profit dynamics. Therefore, in a society where the income of an individual is likely to be jointly fostered by

[1]For simplicity, we are here assuming that profits comes from production of goods only.

different sources, the research topic in question is of utmost importance.

The current work, therefore, provides a methodology for the study of income composition inequality. Specifically, hinging on the recent work by Ranaldi (2017), in which the definition of *zero-polarization of income sources* is provided, and both the *Polarization Curve for Income Source* and the *Zero-Polarization Curve* are introduced, a new measure called the *Income-Factor Polarization Index*, $\mathscr{I}_f$, is presented in the paper. The index $\mathscr{I}_f$ is a metric that summarizes the polarization of two income sources, notably profits and wages, across the whole population. We measure income-factor polarization as the distance between the *Polarization Curve for Income Source* and the *Zero-Polarization Curve*, suitably normalized. The former is the cumulative distribution of income source across the population with individuals being indexed by their income rank, and the latter is the benchmark of zero inequality in income composition, defined as the situation in which each individual has the same population share of profits and wages. In addition to that, we show that the index narrows down to a *single one* the *two conditions* for the rising share of capital income to increase overall income Gini mentioned by Milanovic (2017), which is: $\mathscr{I}_f > 0$. This allows us to define such tool of analysis as an elasticity of inter-personal income Gini to changes in the factor share of income.

An increasing number of papers dealing with the relationship between income polarization and income sources has been registered "during the past twenty years" (Deutsch, Fusco, and Silber, 2013). The seminal papers by Esteban et al. (1994), and by Duclos et al. (2004) introduced the concept of income polarization as raised from the two notions of *identification* and *alienation*. Income polarization is, in the sense of Esteban et al. (1994), a synonym for income *clusterization*: a given population is *income-polarized* when we can identify a finite number of income means around which individual incomes are concentrated. When the focus is on two clusters only, then we talk about income bi-polarization (Foster and Wolfson, 1992, Wolfson, 1994), even though the mathematical formulation differs from that proposed by Esteban et al. Few papers attempted to study how various income sources contribute to either polarization and bi-polarization of income in the literature. Among those, we recall Araar (2008), Deutsch and Silber (2010) and Deutsch, Fusco, and Silber (2013). However, if these works' main objective is to study the contribution of income sources to either income polarization or bi-polarization (or to both), we here consider income-factor polarization in itself, independently from

income clusterization. Following on the previous illustrative example, it is straightforward to notice that zero inequality in income is, in the current framework, compatible with positive inequality in income composition.

The paper is structured as follows. Section 2 introduces the framework and the income-factor polarization index. Section 3 focuses on the particular case in which only two individuals (or groups) are considered, and on the mathematical properties of the simplified index. Section 4 empirically tests the index on the case of Italy between 1989 and 2014, and section 5 concludes.

2 The Methodology

2.1 Polarization Curves for Income Source

Assume we have a n-sized population, where each individual is endowed with income Y_i with $i = 1, \ldots, n$. We define each individual's income share as $y_i = \frac{Y_i}{Y}$, where $Y = \sum_{i=1}^{n} Y_i$ is the total income of the population. Total income is divided into two sources, profits (Π) and wages (W), so that $Y = \Pi + W$ and hence $y = \pi + w$, where $\pi = \frac{\Pi}{Y}$ and $w = \frac{W}{Y}$ are the profit and wage shares in income respectively. As in Ranaldi (2017), we decompose individual i's income as follows:

$$y_i = \alpha_i \pi + \beta_i w \tag{1}$$

where $\alpha_i = \frac{\Pi_i}{\Pi}$ and $\beta_i = \frac{W_i}{W}$ are the relative shares of profits and wages of individual i, such that $\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \beta_i = 1$, and Π_i and W_i represent i's total amount of profits and wages. Suppose that $y_i \leq y_{i+1}\ \forall i = 1, \ldots, n-1$ and $y_0 = 0$, so that individuals are indexed by their income rank. We define $p = \frac{i}{n}$ as the proportion of the population with income less than or equal to y_p, so that $p \in \mathbb{Q} := [0, 1]$. Let $\mathscr{L}(\mathbf{y}, p) = \sum_{j=1}^{i} y_j$, with $i = 1, \ldots, n$, be the *Lorenz Curve for Income* corresponding to the distribution $\mathbf{y}$.[2]

Following Ranaldi (2017), let $\mathscr{L}(\boldsymbol{\pi}, p) = \pi \sum_{j=1}^{i} \alpha_j$, with $i = 1, \ldots, n$, be the *Polarization Curve for Profits* corresponding to the distribution $\boldsymbol{\pi}$, and $\mathscr{L}(\mathbf{w}, p) = w \sum_{j=1}^{i} \beta_j$, with $i = 1, \ldots, n$, be the *Polarization Curve for Wages* corresponding to the distribution $\mathbf{w}$. The individuals are always indexed by their income rank, and not by their profit, or wage rank respectively. These two curves

[2] I am defining the Lorenz curve here as in Shorrocks (1983).

describes the distribution of two income sources across the population. Figure 1 provides us with a graphical representation of the two curves presented. For the analysis of income-source polarization a benchmark of zero polarization of income sources needs to be introduced.

Definition 2.1. *We say that two income sources are not polarized across a population when each individual has the same population shares of profits and wages; formally, when* $\frac{W_i}{\Pi_i} = \frac{w}{\pi}\ \forall i$*, or, equivalently, when* $\alpha_i = \beta_i\ \forall i$.[3]

In order to compare the polarization curves previously presented with the benchmark of zero polarization of income sources, we define the *Zero-Polarization Curve*, $\mathscr{L}^e(\boldsymbol{\kappa}, p)$, corresponding to the distribution $\boldsymbol{\kappa}$, as follows: $\mathscr{L}^e(\boldsymbol{\kappa}, p) = \kappa \sum_{j=1}^{i} y_j$, with $i = 1, \ldots, n$, and with $\kappa = \pi, w$. Note that the zero-polarization curve is a *scaled* version of the Lorenz curve for income. When the polarization curve for profits is below the zero-polarization curve it means that the profits are mainly concentrated in the hands of the rich, while when the curve is above the zero-polarization curve the reverse situation holds true, and the profits are mainly concentrated at the bottom of the income distribution.
The zero-polarization curve describes the distribution of income sources such that an increase in any of the two factor shares of total income has no impact on inter-personal income inequality, as the Gini coefficient does not depend on any of the two factor shares in this particular distributional configuration (see Ranaldi (2017)). In addition to that, it shall be noticed that when the polarization curve for a given income source lyes below the $\mathscr{L}^e$ curve, then the polarization curve for the other income source lyes above the $\mathscr{L}^e$ curve.

2.2 The Maximum-Polarization Curve

In order to construct a measure of income-source polarization, together with a benchmark of zero inequality in income composition we need a benchmark of maximum inequality in income composition. This justifies the need for the following definition of maximum inequality in income composition.

[3] Such a distribution of income sources represents the long-run distribution of factors across individuals in a neoclassical framework in which heterogeneity of both non-accumulated and accumulated factors are considered (Bertola et all (2005)). The latter is also the underlying distribution of factors in the *New Capitalism 2* society defined by Milanovic (2017).

Definition 2.2. *We say that income composition inequality is maximum when the bottom $p\%$ of the income distribution has an income consisting only of the source κ (i.e. p s.t. $y_p = \mathscr{L}(\boldsymbol{y}, p) = \kappa$), and the top $(1-p)\%$ of the income distribution has an income uniquely determined by the source κ_- (i.e. $1-p$ s.t. $y_{1-p} = 1 - \mathscr{L}(\boldsymbol{y}, p) = \kappa_-$), where $\kappa_- = 1 - \kappa$ and $\kappa = \pi, w$.*

Formally, we define the *Maximum-Polarization Curve*, $\mathscr{L}^{max}(\boldsymbol{\kappa}, p)$, corresponding to the distribution $\boldsymbol{\kappa}$, as follows:

$$\mathscr{L}^{max}(\boldsymbol{\kappa}, p) = \begin{cases} \mathscr{L}^{M}(\boldsymbol{\kappa}, p) = \begin{cases} \mathscr{L}(\mathbf{y}, p) & \text{for } p \leq p^{'} \\ \kappa & \text{for } p > p^{'} \end{cases} \\ \mathscr{L}^{m}(\boldsymbol{\kappa}, p) = \begin{cases} 0 & \text{for } p \leq p^{''} \\ \mathscr{L}(\mathbf{y}, p) - \kappa_- & \text{for } p > p^{''} \end{cases} \end{cases}, \tag{2}$$

with $p^{'}$ s.t. $\mathscr{L}(\mathbf{y}, p^{'}) = \kappa$, $p^{''}$ s.t. $\mathscr{L}(\mathbf{y}, p'') = 1 - \kappa$ and $\boldsymbol{\kappa} = \pi, w$. In addition, we have that:

(i) $\mathscr{L}^{max}(\boldsymbol{\kappa}, p) = \mathscr{L}^{M}(\boldsymbol{\kappa}, p)$ if $\mathscr{L}(\boldsymbol{\kappa}, p) \geq \mathscr{L}^{e}(\boldsymbol{\kappa}, p)$ $\forall p$ and $\exists$ p^* s.t. $\mathscr{L}(\boldsymbol{\kappa}, p^*) > \mathscr{L}^{e}(\boldsymbol{\kappa}, p^*)$

(ii) $\mathscr{L}^{max}(\boldsymbol{\kappa}, p) = \mathscr{L}^{m}(\boldsymbol{\kappa}, p)$ if $\mathscr{L}^{e}(\boldsymbol{\kappa}, p) \leq \mathscr{L}(\boldsymbol{\kappa}, p)$ $\forall p$ and $\exists$ p^{**} s.t. $\mathscr{L}^{e}(\boldsymbol{\kappa}, p^{**}) < \mathscr{L}(\boldsymbol{\kappa}, p^{**})$

The two conditions above are rather strong, implying that the two curves must not intersect alongside the distribution of income. A weaker condition is, instead, to consider the area covered by each curve, which can in turns be approximated by its mean, as follows:

(i) $\mathscr{L}^{max}(\boldsymbol{\kappa}, p) = \mathscr{L}^{M}(\boldsymbol{\kappa}, p)$ if $\sum_{i=1}^{n}\sum_{j=1}^{i} \eta_j^k > \sum_{i=1}^{n}\sum_{j=1}^{i} y_j$

(ii) $\mathscr{L}^{max}(\boldsymbol{\kappa}, p) = \mathscr{L}^{m}(\boldsymbol{\kappa}, p)$ if $\sum_{i=1}^{n}\sum_{j=1}^{i} \eta_j^k < \sum_{i=1}^{n}\sum_{j=1}^{i} y_j$

where $\eta_j^k = \alpha_j$ if $\kappa = \pi$ and $\eta_j^k = \beta_j$ when $\kappa = w$. The two expressions $\mathscr{L}^{M}(\boldsymbol{\kappa}, p)$ and $\mathscr{L}^{m}(\boldsymbol{\kappa}, p)$ are the benchmarks of maximum polarization when, roughly, the polarization curve for income source κ stands above, and below the zero-polarization curve respectively.

2.3 The Income-Factor Polarization Index

In the previous sections, together with the polarization curve for income source, which displays the distribution of an income source across the population, we have discussed its benchmarks of zero, and

of maximum-polarization respectively. The latter allow us to capture the extent to which two income sources are polarized across the population: when the curve $\mathscr{L}(\boldsymbol{\kappa}, p)$ is close to the curve $\mathscr{L}^e(\boldsymbol{\kappa}, p)$, then income composition inequality is low, while when the curve $\mathscr{L}^e(\boldsymbol{\kappa}, p)$ is close to the curve $\mathscr{L}^{max}(\boldsymbol{\kappa}, p)$, then income composition inequality is high. In order to provide with a single number capable of precisely capturing the degree of income-source polarization in the economy, we introduce the *Functional Income Distribution Index*, $\mathscr{I}_f$. If we call $\mathscr{A}$ the area between the zero-polarization curve and the polarization curve for income source κ, formally:

$$\mathscr{A}(\boldsymbol{\kappa}) = \frac{1}{2n}\sum_{i=1}^{n}\left[\left(\mathscr{L}^e\left(\boldsymbol{\kappa}, \frac{i}{n}\right) + \mathscr{L}^e\left(\boldsymbol{\kappa}, \frac{i-1}{n}\right)\right) - \left(\mathscr{L}\left(\boldsymbol{\kappa}, \frac{i}{n}\right) + \mathscr{L}\left(\boldsymbol{\kappa}, \frac{i-1}{n}\right)\right)\right], \tag{3}$$

and $\mathscr{B}$ the area between the zero-polarization curve and the maximum-polarization curve:

$$\mathscr{B}^{max}(\boldsymbol{\kappa}) = \left|\frac{1}{2n}\sum_{i=1}^{n}\left[\left(\mathscr{L}^e\left(\boldsymbol{\kappa}, \frac{i}{n}\right) + \mathscr{L}^e\left(\boldsymbol{\kappa}, \frac{i-1}{n}\right)\right) - \left(\mathscr{L}^{max}\left(\boldsymbol{\kappa}, \frac{i}{n}\right) + \mathscr{L}^{max}\left(\boldsymbol{\kappa}, \frac{i-1}{n}\right)\right)\right]\right|, \tag{4}$$

with $max = m, M$, we define the *Income-Factor Polarization Index*, $\mathscr{I}_f(\boldsymbol{\kappa})$, corresponding to the distribution $\boldsymbol{\kappa}$, as follows:

$$\mathscr{I}_f(\boldsymbol{\kappa}) = \frac{\mathscr{A}(\boldsymbol{\kappa})}{\mathscr{B}^{max}(\boldsymbol{\kappa})}, \tag{5}$$

with $\boldsymbol{\kappa} = \boldsymbol{\pi}, \mathbf{w}$. This measure has considerable intuitive appeal: it is the area between the zero-polarization curve $\mathscr{L}^e(\boldsymbol{\kappa}, p)$ and the polarization curve for income source $\mathscr{L}(\boldsymbol{\kappa}, p)$, divided by the area between the zero-polarization curve $\mathscr{L}^e(\boldsymbol{\kappa}, p)$ and the maximum-polarization curve $\mathscr{L}^{max}(\boldsymbol{\kappa}, p)$. The measure lyes therefore between -1 (the poor own source κ and the rich own source κ_-) and 1 (the poor own source κ_- and the rich own source κ). Note that the area between the curves $\mathscr{L}^M(\boldsymbol{\kappa}, p)$ and $\mathscr{L}^e(\boldsymbol{\kappa}, p)$, and that between the curves $\mathscr{L}^e(\boldsymbol{\kappa}, p)$ and $\mathscr{L}^m(\boldsymbol{\kappa}, p)$ are the same for specific functional form of $\mathscr{L}(\mathbf{y}, p)$, and for given values of π and w (see the appendix for further details).

Although it may seem of little interest to consider negative values of the index, they have a powerful meaning in terms of *income-composition dynamics*, as stated by the following definition.

Definition 2.3. *Let $sign_{t,t+1}$ be the sign of $\mathscr{I}_f^t(\boldsymbol{\kappa}) \cdot \mathscr{I}_f^{t+1}(\boldsymbol{\kappa})$, where $\mathscr{I}_f^t(\boldsymbol{\kappa})$ is the Income-Factor Polarization Index at time t, while $\mathscr{I}_f^{t+1}(\boldsymbol{\kappa})$ the one at time $t+1$. We say that a change in the structure of income composition across the distribution of income occurs at time t if $sign_{t,t+1} < 0$.*

When a change in sign occurs at time $t+1$ (i.e. $sign_{t,t+1} < 0$), those who mainly owned source κ at time t receive principally source κ_- at time $t+1$ and *vice versa*.

Let us now go back to the mathematical structure of the index. The normalization coefficient $\mathscr{B}^m(\boldsymbol{\kappa})$ is a function of $\mathscr{L}(\mathbf{y}, p)$, κ and $p^{''}$, while $\mathscr{B}^M(\boldsymbol{\kappa})$ is a function of $\mathscr{L}(\mathbf{y}, p)$, κ and $p^{'}$. For simplicity, let us generally call $\mathscr{B}(\boldsymbol{\kappa})$ the denominator of the income-factor polarization index. A more compact expression for the latter is, for $\kappa = \pi$, the following:

$$\mathscr{I}_f(\boldsymbol{\pi}) = \mathscr{B}(\boldsymbol{\kappa}) w\pi(\tilde{\mu}_w - \tilde{\mu}_\pi), \tag{6}$$

where $\tilde{\mu}_\pi = \frac{1}{2n}\sum_{i=0}^{n}\left(\sum_{j=0}^{i}\alpha_j + \sum_{j=0}^{i+1}\alpha_j\right)$ and $\tilde{\mu}_w = \frac{1}{2n}\sum_{i=0}^{n}\left(\sum_{j=0}^{i}\beta_j + \sum_{j=0}^{i+1}\beta_j\right)$ are the areas covered by the polarization curve for wages, and the polarization curve for profits respectively. Similarly, for $\kappa = w$, we have:

$$\mathscr{I}_f(\mathbf{w}) = \mathscr{B}(\boldsymbol{\kappa}) w\pi(\tilde{\mu}_\pi - \tilde{\mu}_w). \tag{7}$$

Note that the functions $\tilde{\mu}_\pi$ and $\tilde{\mu}_w$ have a precise dynamics: they increase (decrease) when the source in question moves towards the bottom (top) of the distribution. They are indeed two metrics that provide a rough approximation of income-factor polarization. In a similar manner, the function $\tilde{\mu}_y$ is a measure of income inequality: when it rises so does the surface of the Lorenz curve, by therefore reducing its distance from the bisector (and therefore by reducing the Gini coefficient).

We can also observe that the term $\tilde{\mu}_\pi$ (and therefore $\tilde{\mu}_w$ and $\tilde{\mu}_y$) can be expressed as follows[4]:

$$\tilde{\mu}_\pi = \sum_{i=1}^{n}\alpha_i\left(\frac{2n-2i+1}{2n}\right),$$

2.4 From capital income share to income inequality: a single *transmission* tool

At this point of the analysis, given the way the *Income-Factor Polarization Index* $\mathscr{I}_f$ is designed, its relationship with the Gini coefficient $\mathscr{G}$ deserves to be set up. In order to make this possible, let us start by considering the relationship between $\tilde{\mu}_y$ and $\mathscr{I}_f(\boldsymbol{\kappa})$ as follows:

$$\tilde{\mu}_y = \frac{\mathscr{I}_f(\boldsymbol{\kappa})}{\mathscr{B}(\boldsymbol{\kappa})\kappa} + \tilde{\mu}_\kappa.$$

[4]Indeed, note that $\tilde{\mu}_\pi = \frac{1}{2n}\sum_{i=0}^{n}\left(\sum_{j=0}^{i}\alpha_j + \sum_{j=0}^{i+1}\alpha_j\right) = \frac{1}{2n}\sum_{i=1}^{n}\left(2\sum_{j=1}^{i}\alpha_j + \alpha_i\right) = \frac{1}{n}\sum_{i=0}^{n}\sum_{j=0}^{i}\alpha_j + \frac{1}{2n}\sum_{i=0}^{n}\alpha_i$ from which we obtain the result.

whence we can write:

$$\mathscr{G} = 1 - 2\tilde{\mu}_y = 1 - 2\left(\frac{\mathscr{I}_f(\boldsymbol{\kappa})}{\mathscr{B}(\boldsymbol{\kappa})\kappa} - \tilde{\mu}_\kappa\right),$$

and after some rearrangement it yields:

$$\mathscr{G} = 1 - 2\left(\kappa\tilde{\mu}_\kappa - (1+\kappa)\tilde{\mu}_{\kappa_-}\right). \tag{8}$$

The Gini coefficient can, therefore, be expressed as a function of the two rough metrics of income-factor polarization $\tilde{\mu}_\kappa$ and $\tilde{\mu}_{\kappa_-}$, and of the total income share κ. By now taking the derivative of $\mathscr{G}$ with respect to κ, we find the following result:

$$\frac{\partial\mathscr{G}}{\partial\kappa} = 2(\tilde{\mu}_{\kappa_-} - \tilde{\mu}_\kappa). \tag{9}$$

The elasticity of inter-personal income Gini to changes in the functional income distribution, to use the words of Milanovic (2017), is (two times) the difference between the average of the two polarization curves. Indeed, if we take $\kappa = \pi$, an increase in the value of $\tilde{\mu}_w$ will transfer the wages from the top, to the bottom of the income distribution, by thus boosting the effect that a rise in the capital share of income π will have on income inequality. If we now consider the standard decomposition of total income Gini into inequality contributed by each income source[5]:

$$\mathscr{G} = \kappa\mathscr{R}_\kappa\mathscr{G}_\kappa + \kappa_-\mathscr{R}_{\kappa_-}\mathscr{G}_{\kappa_-}, \tag{10}$$

where $\mathscr{R}_\kappa = \frac{cov(r(y),\kappa)}{cov(r(\kappa),\kappa)}$ is the correlation ratio between the source κ and total income, $r(y)$ and $r(\kappa)$ are the individual's ranks according to total income and source κ respectively, and $\mathscr{G}_\kappa$ is the Gini coefficient of income source κ, we can write:

$$\frac{\partial\mathscr{G}}{\partial\kappa} = \mathscr{R}_\kappa\mathscr{G}_\kappa - \mathscr{R}_{\kappa_-}\mathscr{G}_{\kappa_-}, \tag{11}$$

and by combining both equations 9 and 11 we get:

$$\tilde{\mu}_{\kappa_-} - \tilde{\mu}_\kappa = \frac{\mathscr{R}_\kappa\mathscr{G}_\kappa - \mathscr{R}_{\kappa_-}\mathscr{G}_{\kappa_-}}{2}. \tag{12}$$

According to Milanovic, "for the rising share of capital income to increase overall income Gini, we need therefore to have two "transmission" tools, Gini coefficient of capital income and $\mathscr{R}_\pi$, positive

[5]See Milanovic (2017) for further details.

and high" (Milanovic, 2017, p. 6), or, from a more formal point of view, the following condition must hold: $\mathscr{R}_\pi \mathscr{G}_\pi > \mathscr{R}_w \mathscr{G}_w$. It does appear that such a condition is well captured by the sign of the Income-Factor Polarization Index, which indeed exclusively depends on $\tilde{\mu}_{\kappa_-} - \tilde{\mu}_\kappa$. Therefore, the previous result shows that, for the analysis of the relationship between capital income share and personal income inequality, only *one*, instead of *two* transmission tools (i.e. the Gini coefficient of capital income and the correlation ratio above-mentioned) is required, namely the Income-Factor Polarization index, $\mathscr{I}_f$.

3 Income-Factor Polarization in a *Two-Person* Economy

Let us now focus our attention on the scenario in which the population is divided into two groups (i.e. $n = 2$). Such an exercise is of interest for two main reasons. First of all, due to data availability, it may be difficult sometimes to compute the index previously illustrated, which requires information concerning the composition of individual incomes for the whole population[6]. Second of all, the $n = 2$ version of the index has some interesting mathematical properties which deserve to be exposed.

Let us call y_p the income of the bottom $p\%$ of the income distribution, and y_{1-p} the income of the top $(1-p)\%$, with $y_p \in]0, \frac{1}{2}[$. The income-factor polarization index with $n = 2$ takes the following mathematical form:

$$\mathscr{I}_{f,2}(\boldsymbol{\kappa}, p) = b_{\kappa,p} w\pi (\eta_p^{\kappa_-} - \eta_p^{\kappa}) = t_{\kappa,p}\ \rho\ I_f(\boldsymbol{\kappa}, p) \tag{13}$$

where $\rho = w\pi$, $I_f(\boldsymbol{\kappa}, p) = \eta_p^{\kappa_-} - \eta_p^{\kappa}$, and the normalization coefficient $b_{\kappa,p}$ is defined as follows:

$$b_{\kappa,p} = \begin{cases} \frac{1}{y_p \kappa} & \text{if } \ y_p > \eta_p^{\kappa} \\ \\ \frac{1}{[\min(y_p, \kappa) - y_p \kappa]} & \text{if } \ y_p < \eta_p^{\kappa} \end{cases}$$

The $n = 2$ version of the income-factor polarization index can be thus considered as the product of three elements, namely: $t_{\kappa,p}$, ρ and $I_f(\boldsymbol{\kappa}, p)$. A particular way to grasp their meaning is to rewrite

[6] This information is generally provided by the surveys, which tend to underreport individuals with income at the top of the distribution.

the index as follows:

$$\mathscr{I}_{f,2}(\boldsymbol{\kappa}, p) = b_{\kappa,p} \begin{vmatrix} w & 0 \\ 0 & \pi \end{vmatrix} \begin{vmatrix} \eta_p^{\kappa-} & \eta_p^{\kappa} \\ \eta_{1-p}^{\kappa-} & \eta_{1-p}^{\kappa} \end{vmatrix},$$

where the product of the two determinants ρ and $I_f(\boldsymbol{\kappa}, p)$ is simply the determinant of the following matrix A:

$$A = \begin{pmatrix} \eta_p^{\kappa-} w & \eta_p^{\kappa} \pi \\ \eta_{1-p}^{\kappa-} w & \eta_{1-p}^{\kappa} \pi \end{pmatrix}.$$

The income-factor polarization index can be therefore rewritten as the product between the determinants of two matrixes, and a normalizing coefficient. The first determinant, ρ, adjusts the degree of polarization for the level of income sources: if a negative shock in the share of profit occurred, it would be meaningless to talk about polarization, given that one of the two sources would be completely missing in the economy. The second determinant, $I_f(\boldsymbol{\kappa}, p)$, is, instead, the channel through which the issue of polarization is addressed. Interestingly, we can noticed that the matrix A^*:

$$A^* = \begin{pmatrix} \beta_p & \alpha_p \\ \beta_{1-p} & \alpha_{1-p} \end{pmatrix},$$

such that $\det A^* = I_f(\boldsymbol{\kappa}, p)$, comes from the following relationship:

$$\bar{y} = A^* \bar{x},$$

where $\bar{y} = \begin{pmatrix} y_p \\ y_{1-p} \end{pmatrix}$ and $\bar{x} = \begin{pmatrix} w \\ \pi \end{pmatrix}$, which is equivalent to the following system of equations:

$$\begin{cases} y_p = \beta_p w + \alpha_p \pi \\ y_{1-p} = \beta_{1-p} w + \alpha_{1-p} \pi. \end{cases}$$

When the matrix A^* is nonsingular (i.e. $\det A^* \neq 0$, thus $I_f(\boldsymbol{\kappa}, p) \neq 0$), then we can write: $\bar{x} = (A^*)^{-1} \bar{y}$.

Let us now illustrate several properties of the $I_f(\boldsymbol{\kappa}, p)$. First of all, the profit to wages ratio can be expressed as follows:

$$\frac{\pi}{w} = \frac{\frac{1}{1+\varphi} - \beta_{1-p}}{\frac{\varphi}{1+\varphi} - \alpha_p}, \tag{14}$$

where $\varphi = \frac{y_p}{y_{1-p}}$, from which we simply derive the following result.

Proposition 3.1. *A variation of φ has no effect on $\frac{\pi}{w}$ iff $I_f(\boldsymbol{\kappa}, p) = 0$. Formally*[7]*:*

$$\frac{\partial \frac{\pi}{w}}{\partial \varphi} = 0 \iff I_f(\boldsymbol{\kappa}, p) = 0. \tag{15}$$

This result sheds light on the relationship between income inequalities (φ) and factor shares of income ($\frac{\pi}{w}$). Indeed, it appears that these two aspects are intimately tied by the $I_f(\boldsymbol{\kappa}, p)$ index, i.e. by the way through which income sources are allocated between the two groups. A variation of φ does not affect the ratio $\frac{\pi}{w}$ when the determinant of the matrix A^* equals zero.

Let us now consider the relationship between the determinant $I_f(\boldsymbol{\kappa}, p)$ and the between-group Gini Coefficient, G.[8] Precisely:

$$\frac{\partial G}{\partial \kappa} = I_f(\boldsymbol{\kappa}_-, p)\ p. \tag{16}$$

An increase in the source κ reduces the between-groups inequality G depending on the degree of income sources polarization, and on the share of poor people p. If we let p be equal to $\frac{1}{2}$, thus if we divide the population into two groups of equal size (i.e. 50% each), and if we set $\kappa = \pi$, then we get:

$$\frac{\partial G}{\partial \pi} = \frac{\alpha_{\frac{1}{2}} - \beta_{\frac{1}{2}}}{2}, \tag{17}$$

which looks very much like equation 9. Indeed, in such a case the condition for the rising share of capital income to increase income Gini is that $I_f(\boldsymbol{\kappa}, p) > 0$, or that $\det A^* > 0$.

4 Emirical Application

In the present section we apply the methodology previously illustrated to the case of Italy. The data comes from the *Survey on Household Income and Wealth* conducted by the Bank of Italy, which covers a period between 1989 and 2014 [9]. Following on Ranaldi (2017), capital income is defined as the sum of real-estate income, which includes actual rents and imputed rents, net self-employment income and income from financial assets, which latter is the sum of interest on deposits, interest on government securities, income from other securities and interest payments. Labor income is, instead, the sum of

[7] It is easy to notice that $\frac{\partial \frac{\pi}{w}}{\partial \varphi} > 0$ when $I_f(\boldsymbol{\kappa}, p) > 0$. In particular, when $I_f(\boldsymbol{\kappa}, p) = 1$, an increase of φ raises the ratio $\frac{\pi}{w}$ of the same amount. Indeed, when $I_f(\boldsymbol{\kappa}, p) = 1$ then $y_p = \pi$ and $y_{1-p} = w$, thus $\frac{\pi}{w} = \frac{y_p}{y_{1-p}}$.

[8] See the appendix for further details.

[9] The surveys, which comprise around 8000 families and 20000 individuals, are available for the following years only: 1989, 1991, 1993, 1995, 1998, 2000, 2002, 2004, 2006, 2008, 2010, 2012, 2013, 2014.

payroll income and pensions and net transfers. The former includes net wages, salaries and fringe benefits, while the latter covers pensions, arrears, financial assistance, scholarships, and alimony and gifts.

Figures 3 and 4 illustrate the *Lorenz curve for income* (continuous line), the *polarization curve for profits* (dashed line), the *zero-polarization curve* (dotted line) and the *maximum polarization curve* (ultra-dotted line) for 1989 and 2014 respectively. In both years the polarization curve for profits lyes below the zero-polarization curve, suggesting that profits are mainly concentrated at the high-end of the income distribution. However, in 1989 the distance between the two curves is larger than that in 2014, implying a decrease in income-factor polarization. In fact, the index $\mathscr{I}_f(\boldsymbol{\pi})$ reduced by **0.22** points from 1989 to 2014, by moving precisely from **0.49** to **0.27**, while the Gini coefficient remained almost constant (**0.34** in 1989 and **0.36** in 2014). Figure 5 displays the overall dynamics of the two indicators during the 25 years, and four major considerations can be drawn.

First of all, the two metrics do not seem to show any correlation. Indeed, the dynamics of income composition inequality appears to have an independent path from that of income inequality.

Second of all, the income-factor polarization index fluctuates more than the Gini coefficient, suggesting that income composition across the population has been more subject to changes than income levels.

Third of all, if we compare our results with that of Ricci (2016), which studies the evolution of the middle-income groups in Italy between 2002 and 2012 using the Esteban and Ray (ER) approach, we clearly see that *income polarization* á la ER, and *income-factor polarization* here introduced follow different trends. In fact, according to Ricci, income polarization decreases between 2002 and 2006, and increases between 2006 and 2012, differently from the dynamics of the $\mathscr{I}_f(\boldsymbol{\pi})$ index. Such result strengthen the point according to which the study of income-factor polarization can display different *allures* from that of income polarization.

Fourth of all, three different trends of the $\mathscr{I}_f(\boldsymbol{\pi})$ index can be further identified: two decreasing (1989:1995 and 2004:2014) and one increasing (1995:2004). However, the mild increase of the income-factor polarization index during the middle period (2004:2014) seems to leave relatively unaffected the overall decreasing trend of the index, by rising the question of whether Italy is moving towards a *multiple sources of income society*. Another way of reading this result is that the elasticity of inter-

personal income Gini to changes in capital income share decreased during the period considered. This implies that, today, a boost in the profit share of income would hamper income inequality less than how it would have done in the past. Figure 6 compares the index $\mathscr{I}_f(\boldsymbol{\pi})$ with the correlation ratio between capital and total income $\mathscr{R}_c$ proposed by Milanovic (2017). As expected, the two indices show a significantly high, and positive correlation (**0.82**). However, two main differences can be observed. The first difference is that the ratio suggested by Milanovic fluctuates less than our index, which is unsurprising given the way in which they are designed. In fact, the ratio depends on people's rank according to total income, variable that is less susceptible to variations in the income distribution than the Lorenz curve. The second difference is that the $\mathscr{R}_c$ displays values that are different in scale with those displayed by the Gini, in contrast to what happen with the income-factor polarization index, as we can clearly see from figure 7.

At this point of the analysis, it is of utmost importance to analyze the role played by the two components of the index $\mathscr{I}_f(\boldsymbol{\pi})$, $\tilde{\mu}_w$ and $\tilde{\mu}_\pi$, in shaping its overall dynamics. Firstly, note that the difference between $\tilde{\mu}_w$ and $\tilde{\mu}_\pi$, $(\tilde{\mu}_w - \tilde{\mu}_\pi)$ is, as expected, the major driver of the indicator (see figure 8), even though the two metrics $\tilde{\mu}_w$ and $\tilde{\mu}_\pi$ follow completely different paths (see figure 9). Particularly, if we compare their evolutions with that of the indicator (figures 10 and 11), few considerations can be quickly drawn.

The first consideration is that the decreasing trend of the index between 1989 and 1995 is mainly due to a reduction of $\tilde{\mu}_w$, given that $\tilde{\mu}_\pi$ does not considerably fluctuates. We remind that a reduction of $\tilde{\mu}_w$ does imply that the labor income moves towards the top of the income distribution. Such shift is also associated with a strong increase in overall income inequality, as shown by the almost simultaneous rise of the Gini coefficient in figure 12.

The second consideration is that also during the period between 1995 and 2004, the index seems to be driven by fluctuations of $\tilde{\mu}_w$. However, differently from the previous period, here we see the rise of $\tilde{\mu}_w$, which indeed increases the indicator.

Finally, the third consideration is that the significant fall of the $\mathscr{I}_f(\boldsymbol{\pi})$ during the last period analyzed (2004:2014) seems to be associated with both an increase in $\tilde{\mu}_\pi$, and a decrease in $\tilde{\mu}_w$. We can therefore conclude that the labor income flew towards the rich, while the capital income towards the

poor, entailing that a structural change in the composition of individual's income has occurred in Italy over the last decade.

Having said that, in order to draw a complete picture of the dynamics of income composition inequality in Italy during the lasts three decades, we should also explore the underlying political Italian context, as well as its major historical events. In addition, the data used tends to underreport people with high incomes, implying that our results should be considered with extreme caution. However, given the theoretical nature of the present work, a deeper analysis would go well beyond the scopes of our research, paving therefore the way towards future research on the issue.

5 Conclusion

We here proposed a framework for the analysis of inequality in income composition. For this reason, a metric of income-factor polarization which measures the distance between the polarization curve for income source and the zero-polarization curve (Ranaldi, 2017), named *Income-Factor Polarization Index*, was introduced and discussed. We showed that the sign of the index summarizes the two conditions for the rising share of capital income to increase overall income Gini mentioned by Milanovic (2017) in a single one. Notably, when the index is positive, than an increase in the factor share of income (such as the profit share of income) would rise income inequality; the opposite happens when the index is negative. This result allows us to define such tool of analysis as an elasticity of inter-personal income Gini to changes in the factor share of income, to use the words of Milanovic. Finally, we exposed the method via an empirical application to the case of Italy between 1989 and 2014, showing that a sharp decline of income-factor polarization occurred during the time considered.

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6 List of Figures

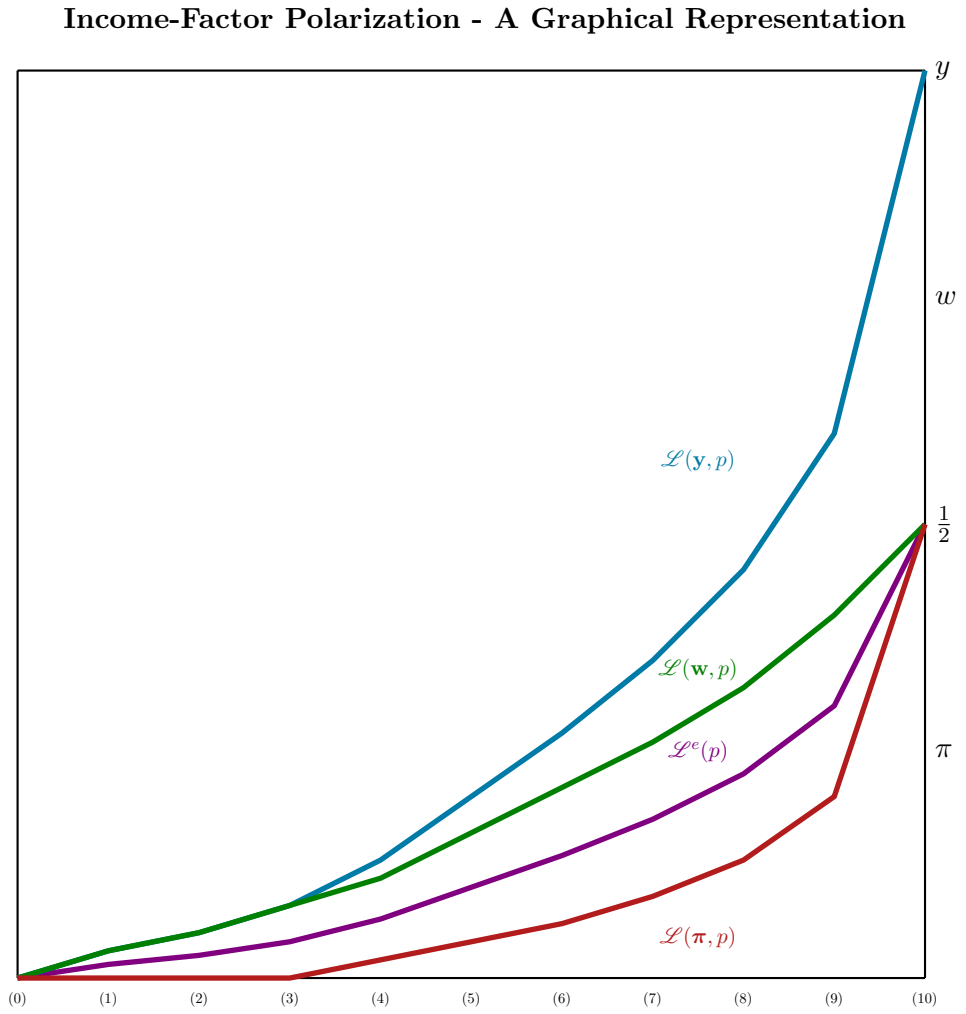

Figure 1: A graphical representation of the *Polarization Curve for Profits* $\mathscr{L}(\boldsymbol{\pi},p)$, the *Lorenz Curve for Income* $\mathscr{L}(\mathbf{y},p)$, the *Polarization Curve for Wages* $\mathscr{L}(\mathbf{w},p)$ and the *Zero-Polarization Curve* $\mathscr{L}^e(p)$ with 10 individuals (or groups) and equal sources of income in the economy ($\pi = w$).

Income-Factor Polarization - A Graphical Representation with $n = 2$

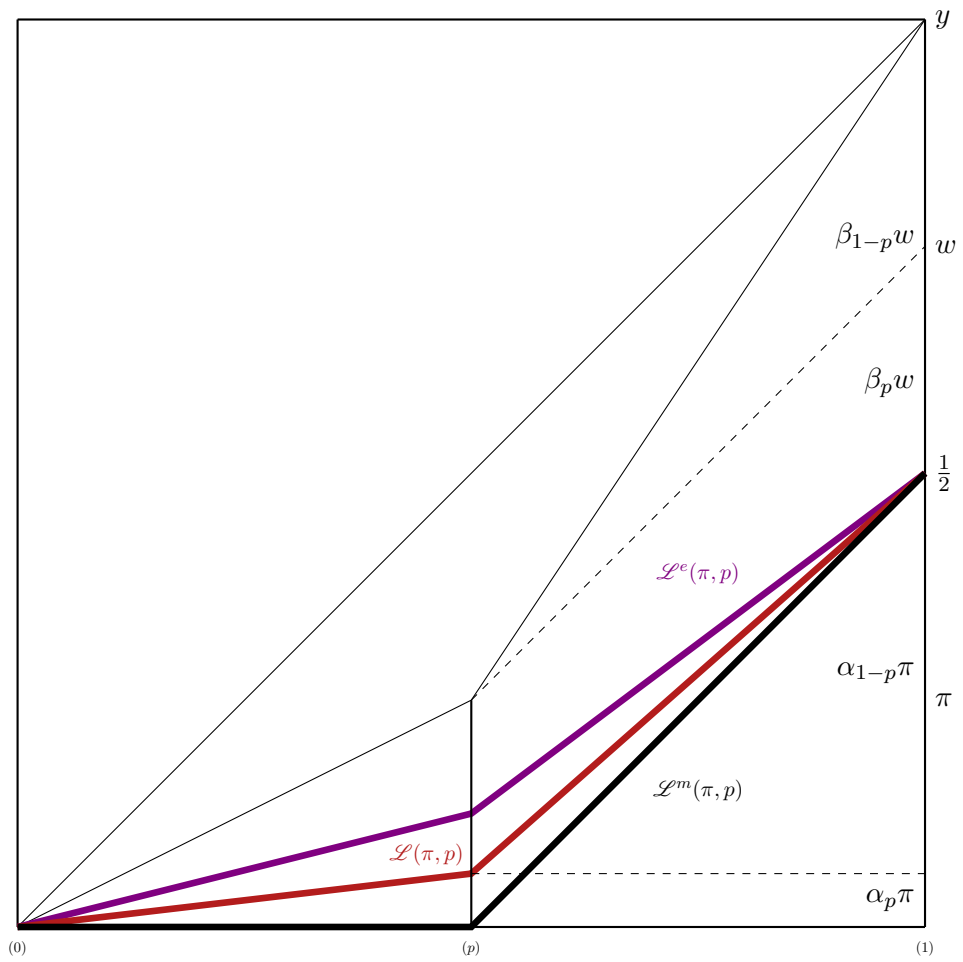

Figure 2: A graphical representation of the methodology in which two people (or groups) with different income ($y_p < y_{1-p}$, with $p = \frac{1}{2}$), and two sources of the same amount ($\pi = w$) are compared. The carnelian line $\mathscr{L}(\boldsymbol{\pi}, p)$ is the *Polarization Curve for Profits*, while the violet line $\mathscr{L}^e(p)$ is the *Zero-Polarization Curve*. The following values have been here assigned: $y_p = 0.25$, $\pi = w = \frac{1}{2}$, $\alpha_p = 0.12$ and $\beta_p = 0.38$.

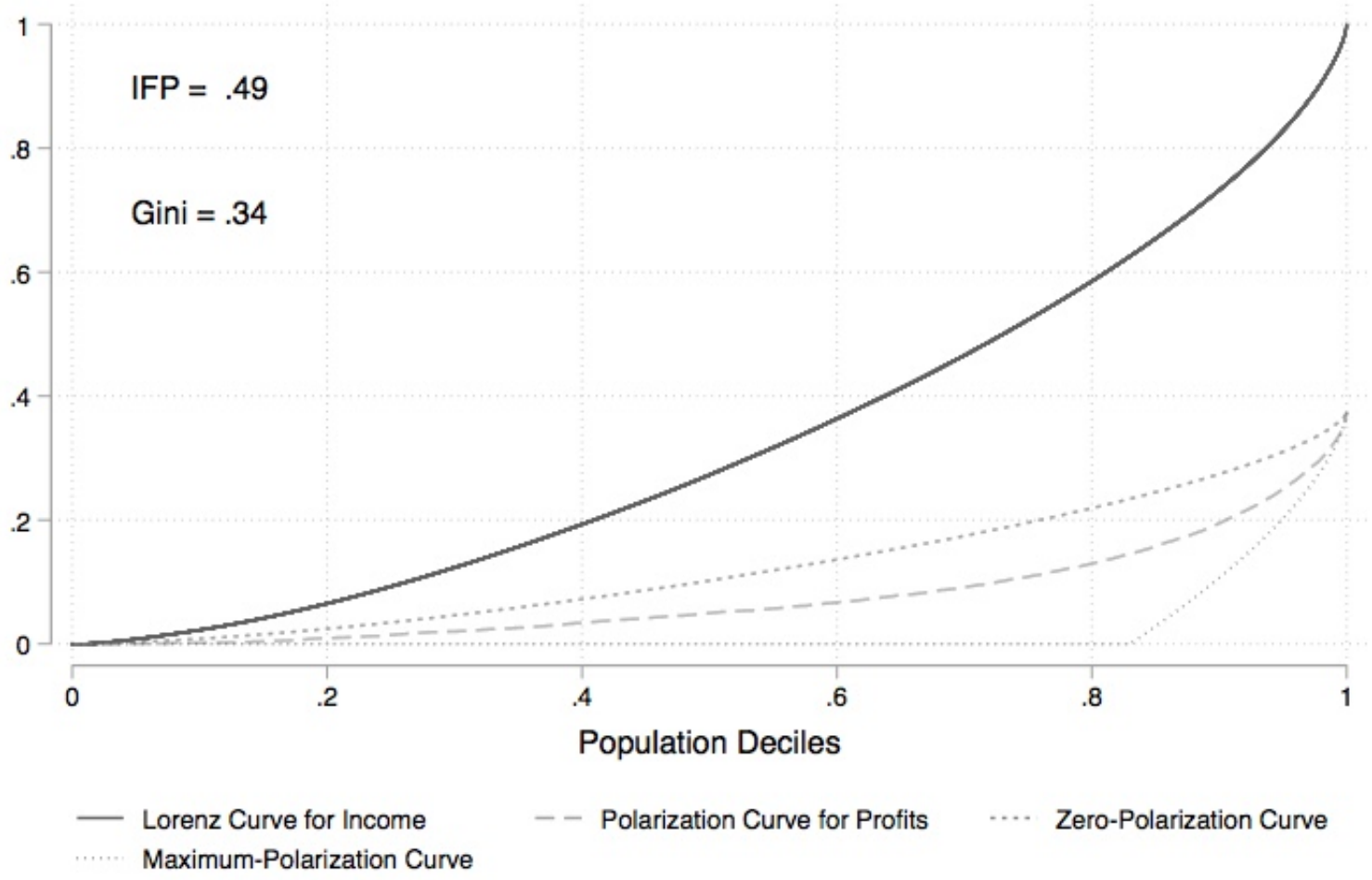

Figure 3: The polarization curve for profits (dashed line), the zero-polarization curve (dotted line), the Lorenz curve for income (continuous line), and the maximum-polarization curve (ultra-dotted line) for Italy in 1989 are presented using data from the 1989 *Survey on Household Income and Wealth* carried out by the Bank of Italy.

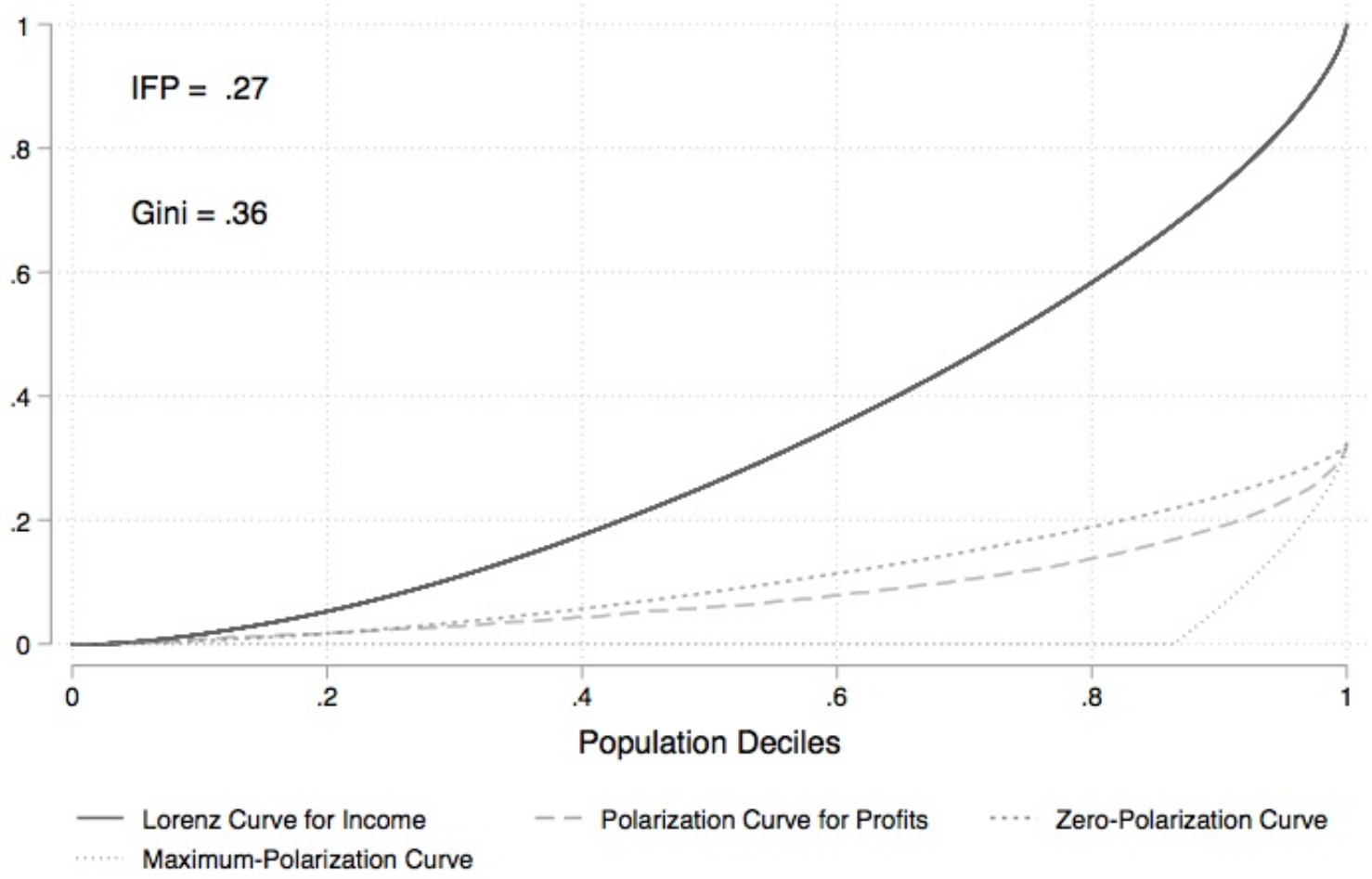

Figure 4: The polarization curve for profits (dashed line), the zero-polarization curve (dotted line), the Lorenz curve for income (continuous line), and the maximum-polarization curve (ultra-dotted line) for Italy in 2014 are presented using data from the 2014 *Survey on Household Income and Wealth* carried out by the Bank of Italy.

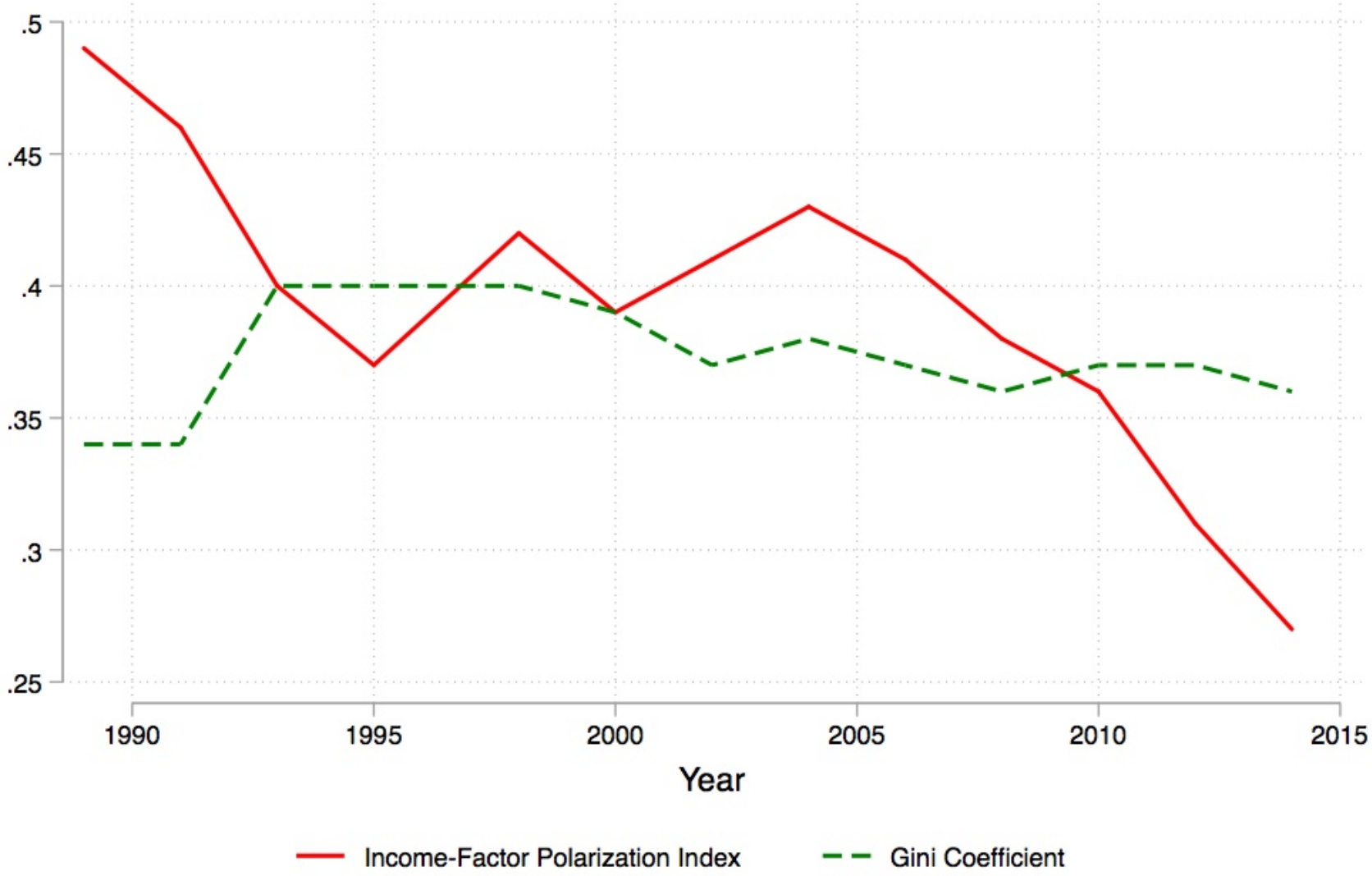

Figure 5: The continuous line displays the evolution of the *Functional Income Distribution Index* $\mathscr{I}_f$ for Italy between 1989 and 2014, while the dashed line shows the dynamics of the Gini index for Italy during the same period. The data are taken from the *Survey on Household Income and Wealth* carried out by the Bank of Italy.

The $\mathscr{I}_f$ Index and the $\mathscr{R}_\pi$ Ratio - Italy 1989:2014

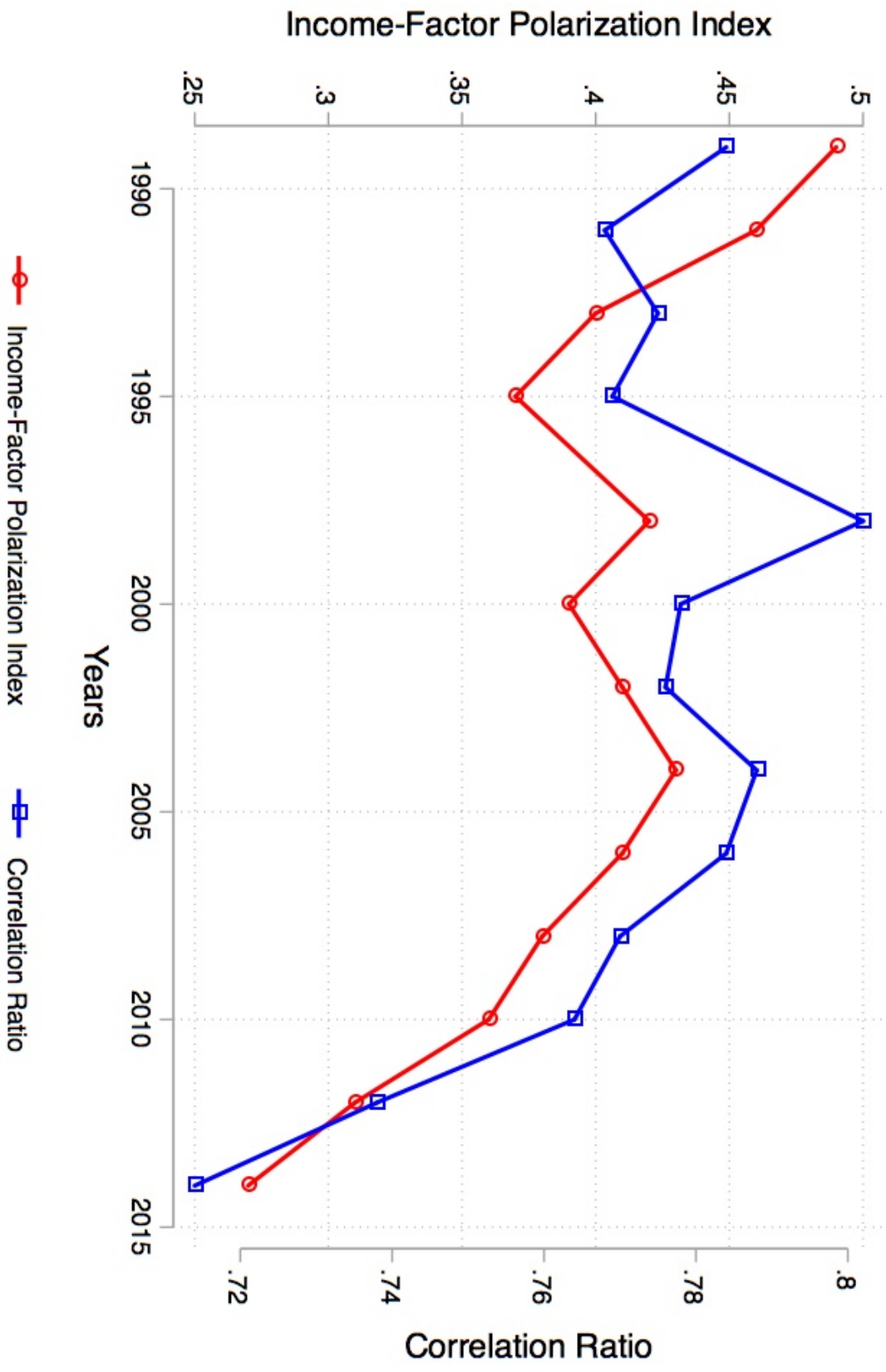

Figure 6: The figure shows the evolution of the index $\mathscr{I}_f$, and of the ratio $\mathscr{R}_\pi$ proposed by Milanovic to study the elasticity of inter-personal income Gini to changes in the functional income distribution (Milanovic, 2017). Their correlation coefficient is **0.82**.

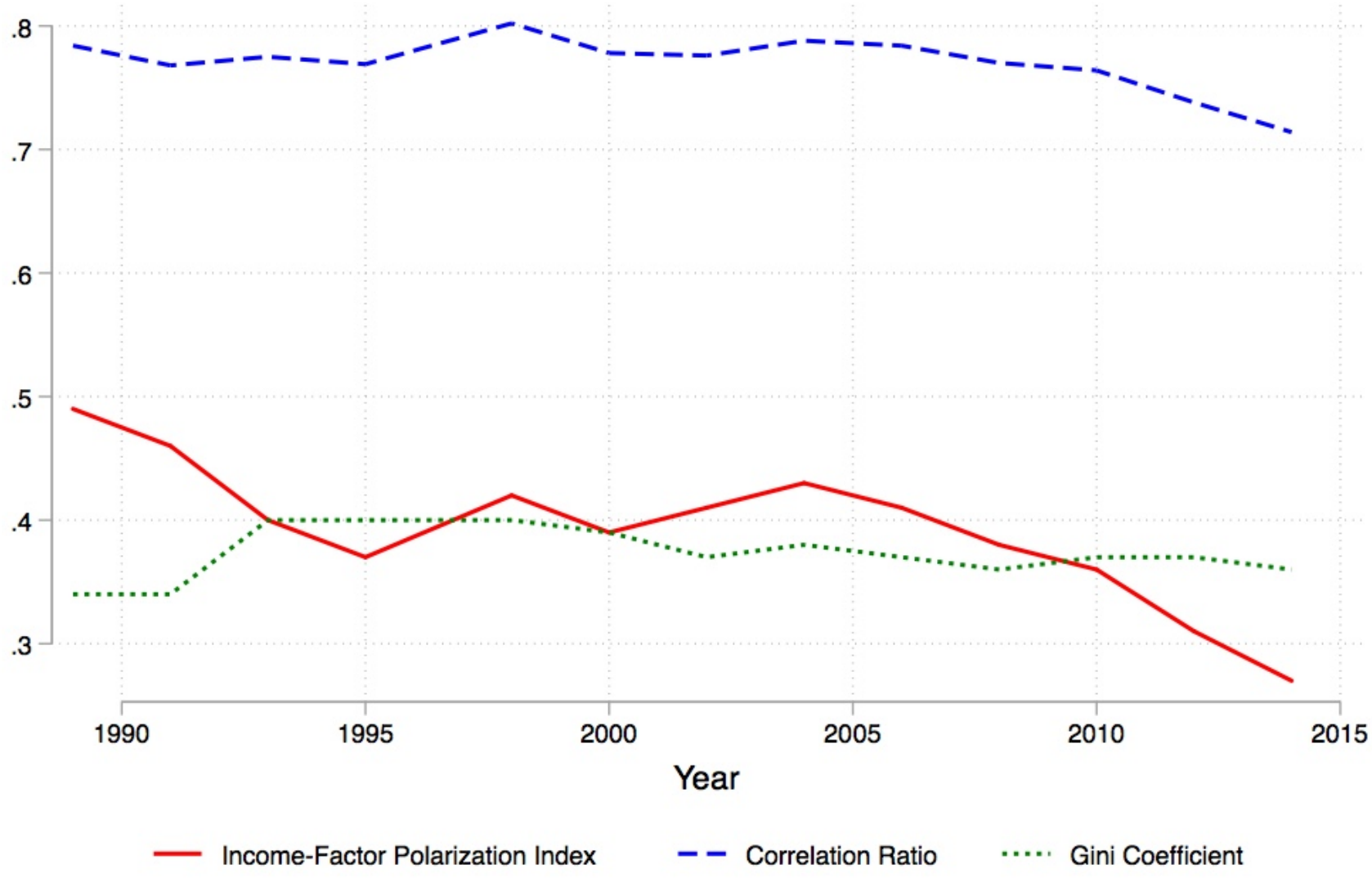

Figure 7: The figure shows the evolution of the index $\mathscr{I}_f$ (continuous line), of the ratio $\mathscr{R}_\pi$ proposed by Milanovic to study the elasticity of inter-personal income Gini to changes in the functional income distribution (dotted line), and of the Gini coefficient (dashed line).

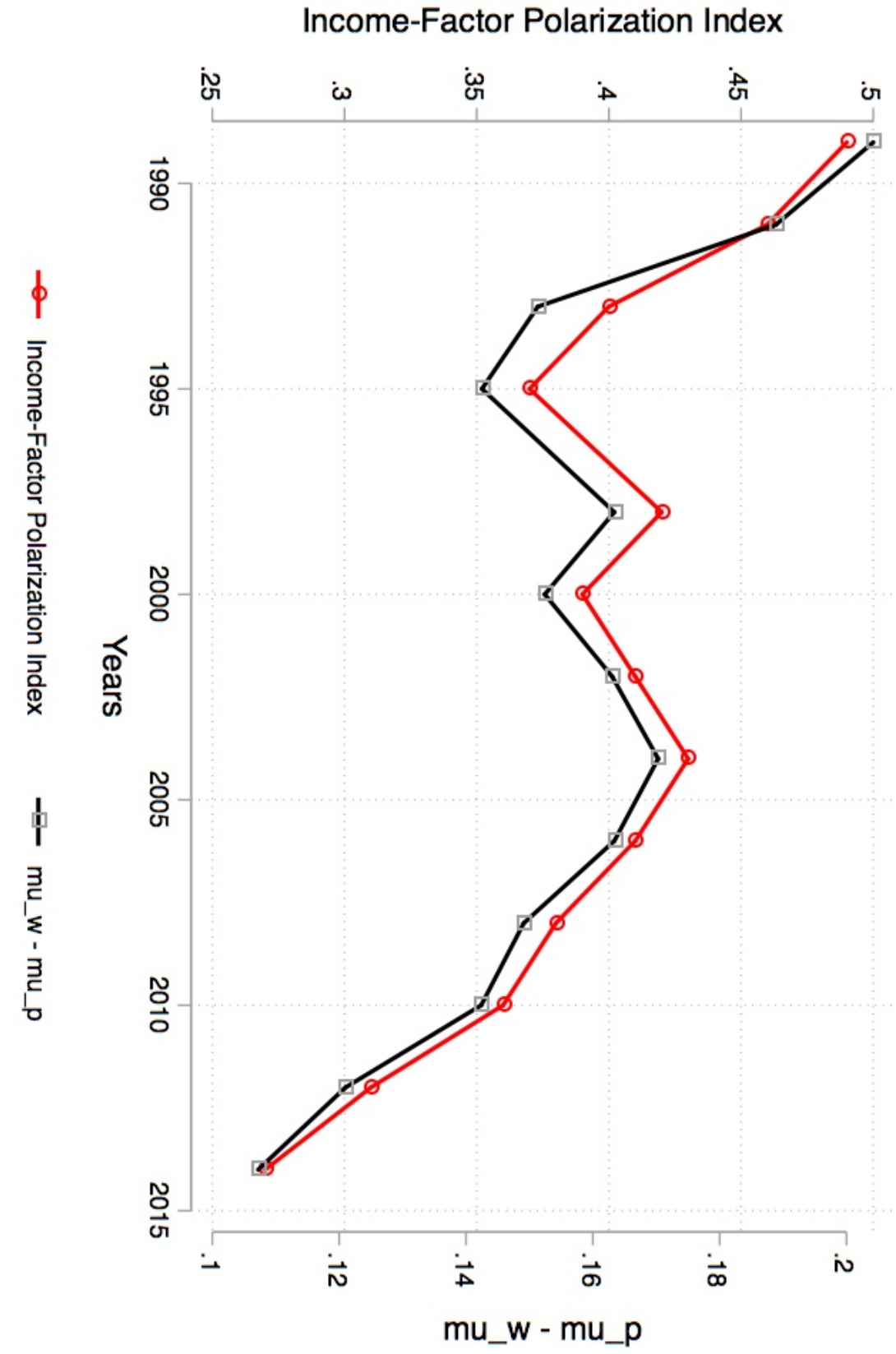

Figure 8: The figure shows the evolution of the index $\mathscr{I}_f$ and of its component $\tilde{\mu}_w - \tilde{\mu}_\pi$.

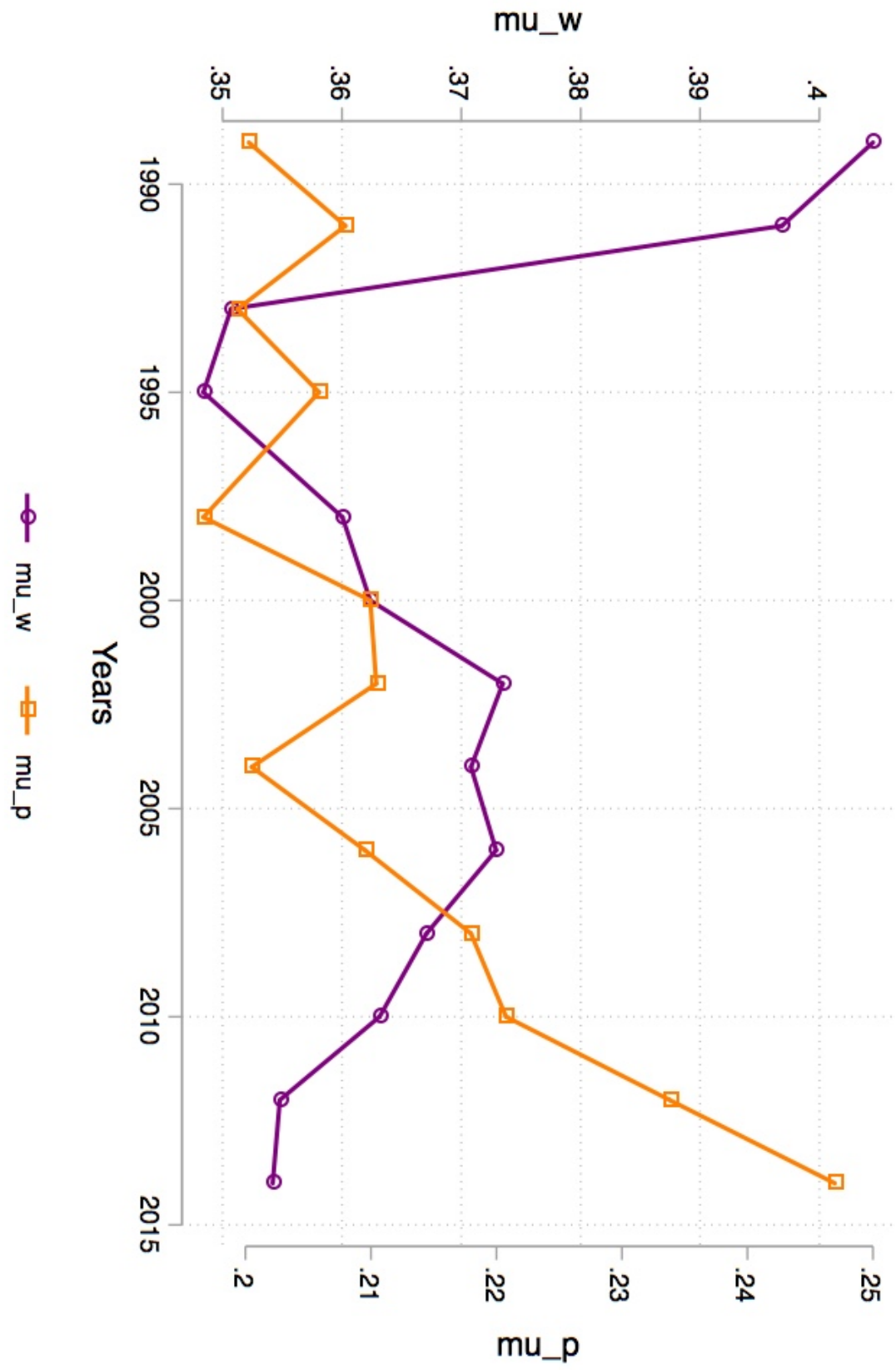

Figure 9: The figure shows the evolution of the two components $\tilde{\mu}_w$ and $\tilde{\mu}_\pi$ of the index.

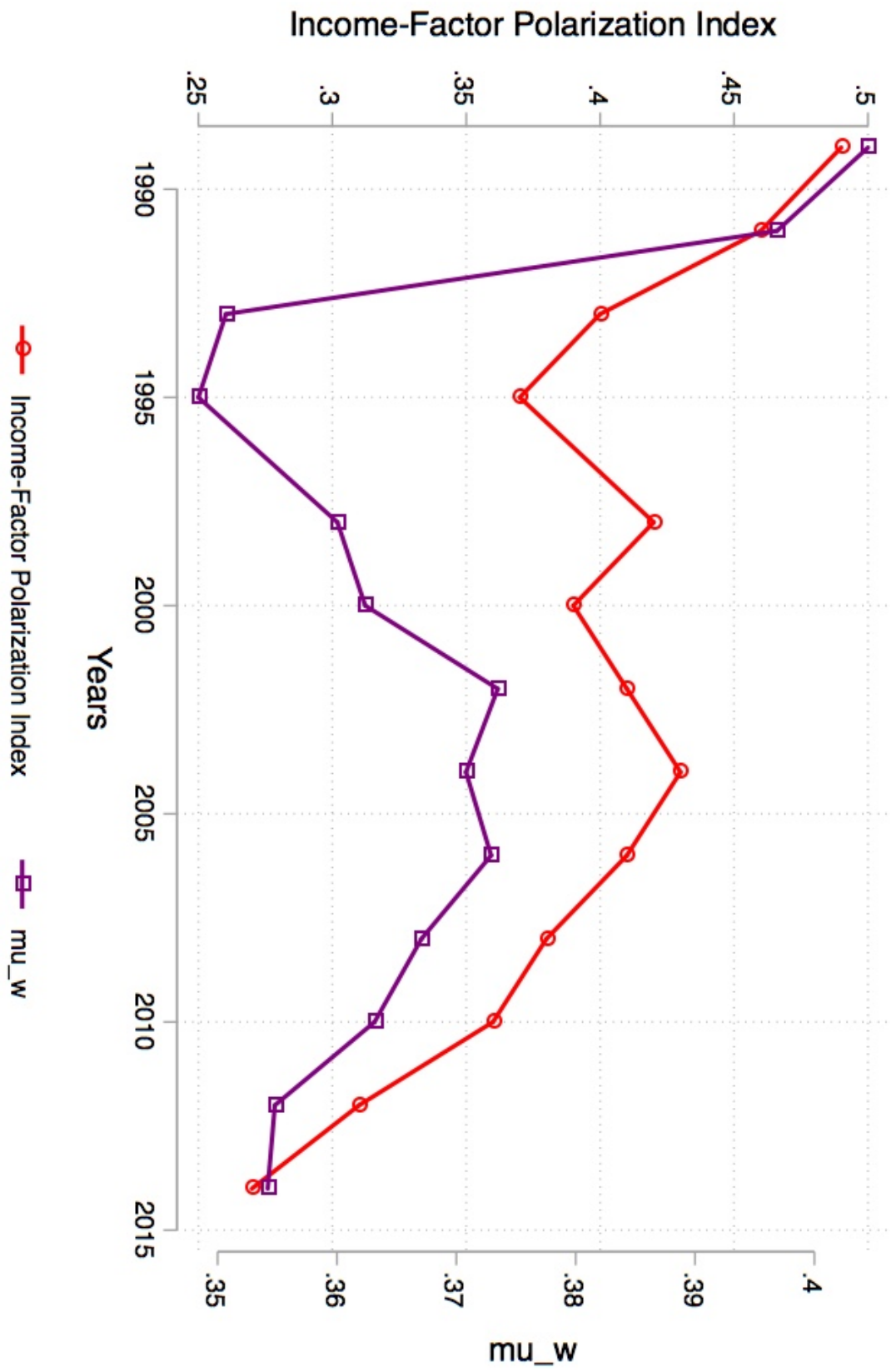

Figure 10: The figure shows the evolution of the index $\mathscr{I}_f$ together with its component $\tilde{\mu}_w$.

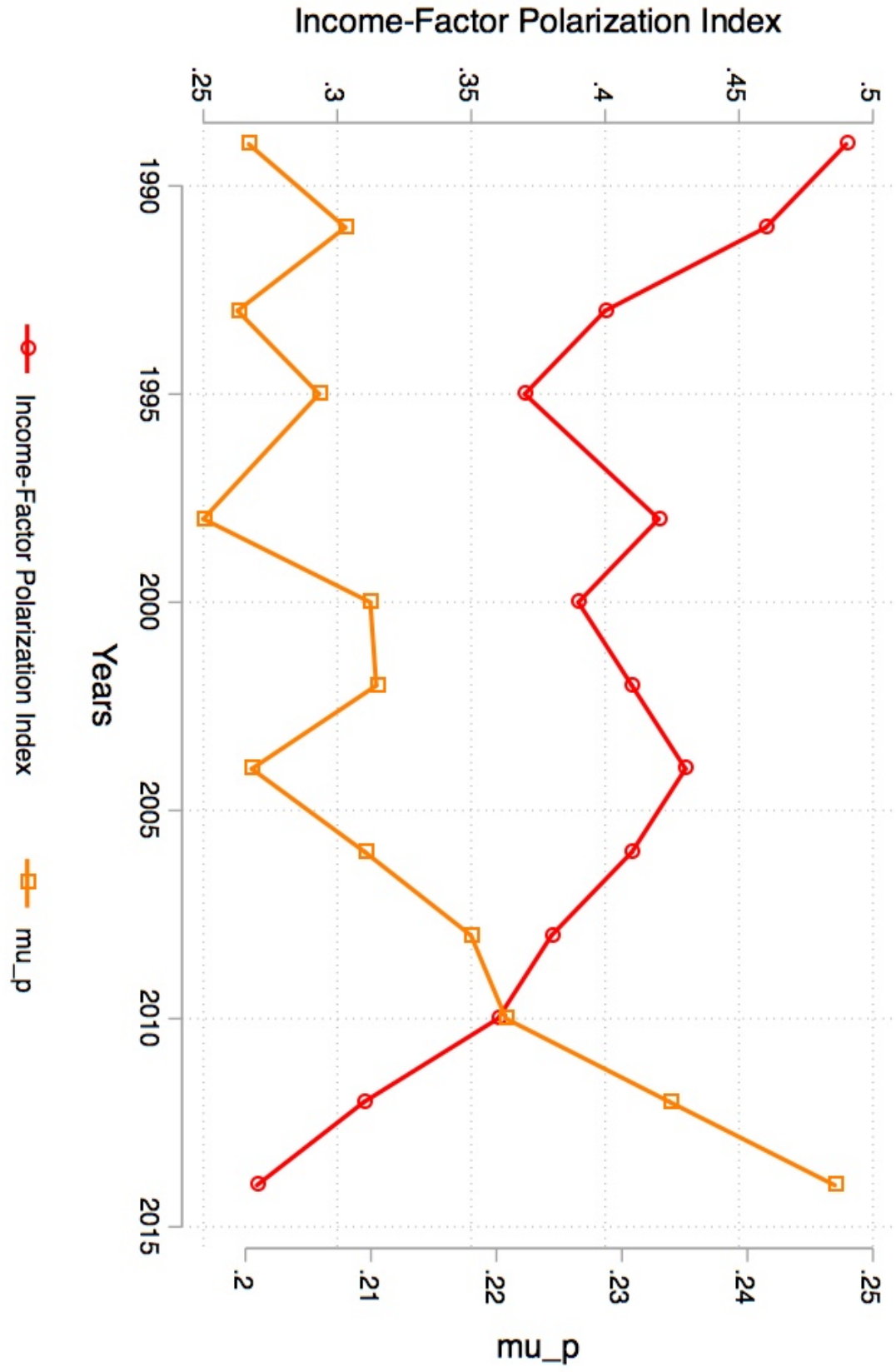

Figure 11: The figure shows the evolution of the index $\mathscr{I}_f$ together with its component $\tilde{\mu}_\pi$.

The Gini Coefficient and the $\tilde{\mu}_w$ Component - Italy 1989:2014

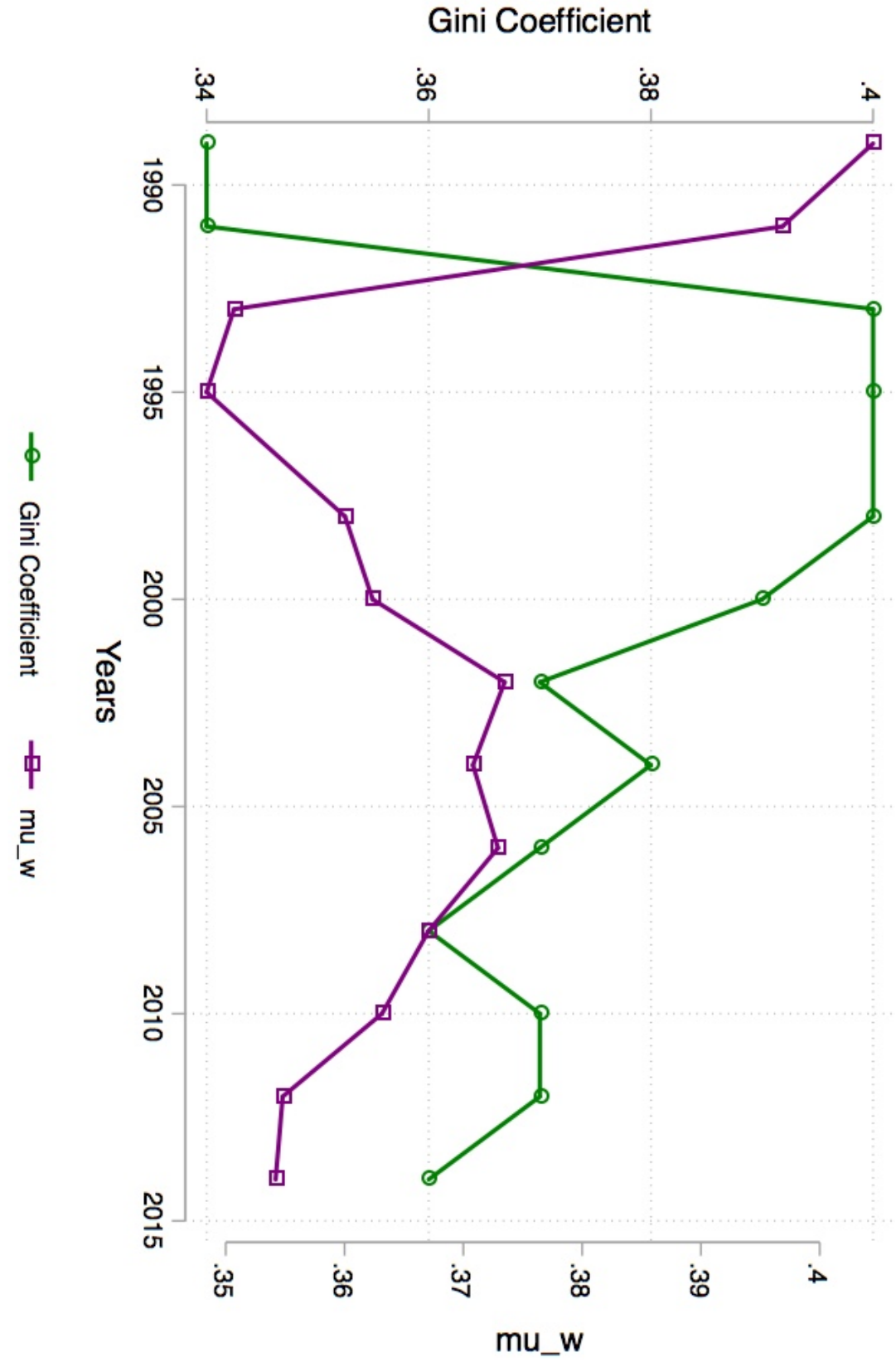

Figure 12: The figure shows the evolution of the Gini coefficient and that of the $\tilde{\mu}_w$ component of the $\mathscr{I}_f$ Index.

The Gini Coefficient and the $\tilde{\mu}_\pi$ Component - Italy 1989:2014

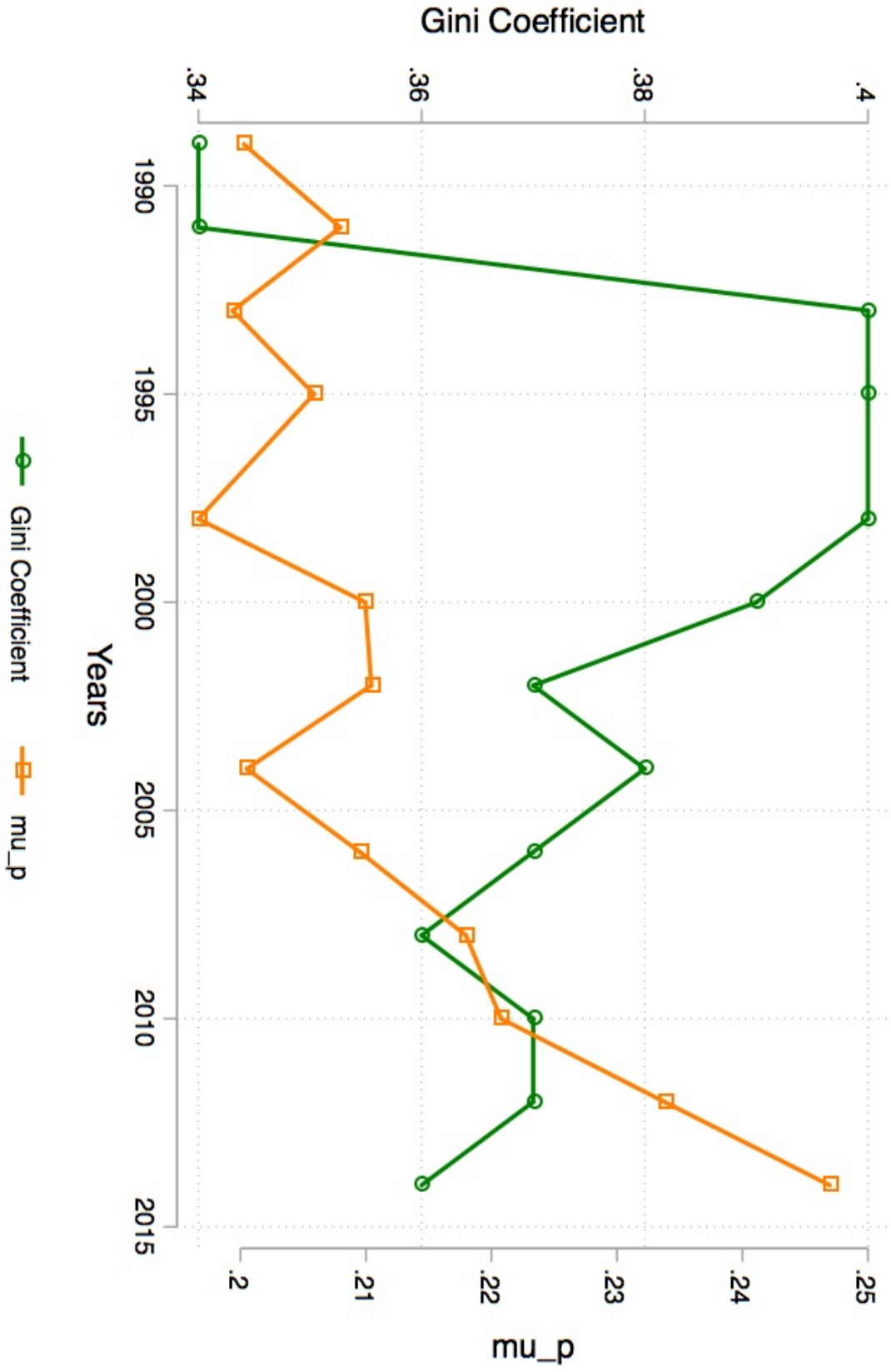

Figure 13: The figure shows the evolution of the Gini coefficient and that of the $\tilde{\mu}_\pi$ component of the $\mathscr{I}_f$ Index.

A Result 3.1

Provided that $y_p = \alpha_p\pi + \beta_p w$, and $y_{1-p} = \alpha_{1-p}\pi + \beta_{1-p}w$, where $y_p + y_{1-p} = y = \pi + w$, we can write:

$$\begin{aligned}
\varphi &= \frac{\beta_p w + \alpha_p \pi}{\beta_{1-p} w + \alpha_{1-p}\pi}, && \iff \\
y_p(\beta_{1-p} w + \alpha_{1-p}\pi) &= y_{1-p}(\beta_p w + \alpha_p \pi), && \iff \\
\frac{\pi}{w} &= \frac{\beta_p y_{1-p} - \beta_{1-p} y_p}{\alpha_{1-p} y_p - \alpha_p y_{1-p}}, && \iff \\
\frac{\pi}{w} &= \frac{\beta_p - \varphi\beta_{1-p}}{-\alpha_p + \varphi\alpha_{1-p}}, && \iff \\
\frac{\pi}{w} &= \frac{1 - (1-\varphi)\beta_{1-p}}{\varphi - (1-\varphi)\alpha_p}, && \iff \\
\frac{\pi}{w} &= \frac{\frac{1}{1-\varphi} - \beta_{1-p}}{\frac{\varphi}{1+\varphi} - \alpha_p}.
\end{aligned}$$

If we now take the first derivative of $\frac{\pi}{w}$ with respect to φ, and we further manipulate, we obtain result 3.1.

B Relationship between the Gini coefficient and the Income-Factor Polarization Index ($n = 2$)

Let us rewrite y_1 (from equation 1) as follows:

$$y_1 = \beta_1 w \pm \alpha_1 w + \alpha_1 \pi.$$

After some algebraic manipulations, we get:

$$y_1 = I_f w + \alpha_1,$$

where I_f is the distribution's component of the index $\mathscr{I}_f$. Let us now recall the expression of the Gini coefficient:

$$G = 1 - \sum_{k=1}^{n} (x_{k+1} - x_k)(y_{k+1} + y_k),$$

where the whole population is divided into n groups, and x_k, y_k represent the bottom $x_k\%$ of the population, and their cumulative income respectively. When $n = 2$ we can write:

$$G = 1 - py_p - (1-p)y_p,$$

where p is the share of the poor group, and $(1-p)$ the share of the rich. The following can be derived:

$$G = 1 - p(I_f w + \alpha_p) - (1-p)$$

whence[10]:

$$G = p(\alpha_{1-p} - I_f w)$$

from which, by taking the derivative with respect to w, we obtain equation 16.

C The normalization coefficient $\mathscr{B}(\boldsymbol{\kappa})$

As stated before in the article, in this section we will show that for specific functional forms of $\mathscr{L}(\mathbf{y}, p)$, and for given values of π and w, the following relationship holds true:

$$\mathscr{L}^M(\boldsymbol{\kappa}) - \mathscr{L}^e(\boldsymbol{\kappa}) = \mathscr{L}^e(\boldsymbol{\kappa}) - \mathscr{L}^m(\boldsymbol{\kappa}). \tag{18}$$

For simplicity, let us move to the continuous space. Suppose, therefore, we have continuous distribution functions $\mathbf{y}$, $\boldsymbol{\pi}$, $\mathbf{w}$. The relationship 18 is, thus, equivalent to the following one:

$$\kappa \int_0^1 \mathscr{L}(\mathbf{y}, p)dp - \int_{p''}^1 (\mathscr{L}(\mathbf{y}, p) - \kappa_-)\, dp = \int_0^{p'} \mathscr{L}(\mathbf{y}, p)dp + \left(1 - p^{'}\right)\kappa - \kappa \int_0^1 \mathscr{L}(\mathbf{y}, p)dp. \tag{19}$$

We remember that $p^{'}$ s.t. $\mathscr{L}(\mathbf{y}, p^{'}) = \kappa$, $p^{''}$ s.t. $\mathscr{L}(\mathbf{y}, p'') = 1 - \kappa$ and $\boldsymbol{\kappa} = \pi, w$. From equation 19 we can write that:

$$2\kappa \int_0^1 \mathscr{L}(\mathbf{y}, p)dp = \int_{p''}^1 (\mathscr{L}(\mathbf{y}, p) - \kappa_-)\, dp + \int_0^{p'} \mathscr{L}(\mathbf{y}, p)dp + \left(1 - p^{'}\right)\kappa.$$

If we call $p^{'} = f(\kappa)$ and $p^{''} = f(\kappa_-)$, where $f(y) = \mathscr{L}^{-1}(\mathbf{y}, p)$, then after further manipulations we get:

$$\int_0^1 \mathscr{L}(\mathbf{y}, p)dp = 1 + \frac{1}{\kappa - \kappa_-} \int_{f(\kappa)}^{f(\kappa_-)} \mathscr{L}(\mathbf{y}, p)dp + \frac{\kappa f(\kappa) - \kappa_- f(\kappa_-)}{\kappa - \kappa_-},$$

which is true only if the following relation is satisfied:

$$(\kappa - \kappa_-) \int_0^1 f(y)dy = \int_\kappa^{\kappa_-} f(y)dy. \tag{20}$$

It can be easily checked that equation 20 is true for $\pi = w$, $\pi = 1$, $w = 1$, regardless of the functional form of $\mathscr{L}$, and for the family of functions of the form $f(x) = x^n$, for $n = 1, +\infty, -\infty$ only.

[10] It can be noticed that $G = p(\alpha_{1-p} - I_f w) = p(1 - y_p) = p y_{1-p}$, which is a different way to express the two-groups Ginin coefficient. Indeed, it clearly appears from the equation that inequality rises when either the share of poor people increases, or when the income share of the rich augments.