Power-Law Distribution in the Debt-to-Fiscal Revenue Ratio: Empirical Evidence and a Theoretical Model

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Abstract

This paper provides evidence that the external debt-to-fiscal revenue ratio in the emerging countries has a power-law distribution. Such a distribution reflects the fact that debt distress or debt crises are extreme events that have been found to happen fairly often. We formally test the hypothesis of a power-law, going further than the usual visual inspection of the distribution of the variable of interest on a doubly logarithmic scale. We further show that such a distribution can be derived from a theoretical model in which the debt dynamics is explained by tax evasion and corruption. Using the framework of an optimal stochastic growth model, we model the debt-to-fiscal revenue ratio as a diffusion process for which the stochastic steady state distribution is derived using the properties of Itô diffusion processes.

Keywords: Power-law, stochastic growth, external debt, emerging countries.
JEL Classification: C14, C51, C61

1. Introduction

External debt distress and episodes of foreign debt crises have been recurrent in the emerging countries history, at least since the beginning 1980s. They have taken different forms: tightened restrictions on access to international capital markets, accumulation of large stocks of financial liabilities,

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debt restructuring, bailouts packages provided by the international financial organizations, debt rescheduling, delays in paying debt service, strong increases in bond spreads, arrears of principal or interest on external obligations. The question regarding the policies designed to tackle sovereign debt crises has given rise to a voluminous literature. The issue is not only the ways out of debt distress episodes (cleaning up after the crises occur), but more importantly how to lean against the wind (understand the chain of events leading to external debt crises and try to act in a preventive way). In the empirical literature, two approaches are usually explored.

A first branch of the literature on sovereign debt distress in the emerging economies attempts to identify those factors that are triggers of debt crises and define threshold values on the driving variables that permit early alert of debt stress outcomes. This literature builds upon a signal extraction approach by looking at the behavior of different macroeconomic variables that have been historically associated with debt crises: inflation, trade imbalances, fiscal and monetary policies, financial stress, extreme episodes of capital flows, etc. Debt crises have also often been associated with currency and banking crises. This literature therefore explores the triggers of sovereign debt crises and debt stress episodes using early warning models. The literature is massive and we refer the reader to some seminal papers by Berg and Patillo (1999), Herrera and Garcia (1999), Kamin and Babson (1999), Kaminsky and Reinhart (1999), Ciarlone and Trebeschi (2005) and to recent contributions by Arellano (2008), Boonman and Kuper (2014), Manasse and Roubini (2009), Savona and Vezzoli (2015).

A second branch of the literature examines over-indebtedness through indicators of debt sustainability. The frequency of situations of debt distress in the emerging countries has been explained by the snowball effect (high interest rates above the economic growth rate), the original sin (countries borrow in foreign currency), currency mismatch (the currency denomination of the countries assets differ from that of their liabilities), the fiscal sin (fiscal policy pro-cyclical bias), the initial stock of debt. Debt sustainability analysis (DSA) has been developed as a way of getting insight into the probability that external debt becomes unsustainable in the face of plausible shocks. Probabilistic frameworks are therefore used to provide estimation of upper and lower intervals of the future evolution of debt. The idea is to assess, prospectively, which changes in policies should be undertaken today to prevent the future occurrence of debt crises if adverse shocks were to happen.

Here also the literature is massive. For recent contributions, we refer the reader to the papers by Adler and Sosa (2013), Aguiar and Amador (2014), Tanner and Samake (2008).

This paper proposes a new approach of debt distress or debt crises in the emerging economies, based on the study of the distribution of external debts. We investigate debt distress as changes in the ratio of external debt over fiscal revenue that are extreme or large (relative to normal situations), but not rare. A situation of debt distress or debt crisis is defined by the probability that the debt ratio jumps above a given threshold. We consider distributions in the family of the so-called extreme values distribution (GEV) of Jenkinson (1955), with a specific attention to distributions with a decay like a power function rather than those with an exponential rate of decay. The reasons are twofold. First, debt crises in the emerging countries have been found to happen fairly often (they are not rare). Secondly, changes from normal to abnormal debt situations do not necessarily occur abruptly. Indeed, unlike in the developed countries, extreme debt events in the emerging countries are not events occurring only under extraordinary economic events (shocks hitting the economies or the financial systems implying a huge variance in the data that disappear when the shocks are switched off). Conversely, the sources of uncertainty causing changes in the ratio of external debts are endogenous to the economic system, since the factors determining the dynamics of debt in normal time are the same factors triggering debt stress episodes. For both reasons, we need distributions whose tails decay more slowly than those of Weibull or Gumpel distributions. We therefore consider distributions in the basin of attraction of Fréchet distributions. The candidate distributions being potentially large (Pareto, Cauchy, Student-t,) we focus on a class of distributions that we motivate theoretically : power-law functions.

After providing empirical evidence that the distributions of their external debt-to-fiscal revenues ratio is represented by a power-law function, we propose a theoretical explanation of the occurrence of such a power-law. More specifically, we develop a continuous time stochastic optimal growth model where the dynamics of debt is described by a Itô diffusion process. The sources of uncertainty are intrinsic to the economic system and come from both tax evasion (households only report a fraction of their income to the fiscal administration) and corruption (a fraction of fiscal revenues is stolen by corrupted bureaucrats). Fraud activity is risky because the households can be caught with a given probability. Similarly, the fiscal administration faces a situation of uncertainty because the corrupted bureaucrats are not known with certainty. It only has a suspicion that some of its civil servants are corrupted. We examine the asymptotic behavior of our variables using the concept of steady state in a stochastic sense. The steady state is defined in terms of a probability density function which is shown to be a power-law.

The remainder of the paper is organized as follows. Section 1 establishes empirical evidence that the external debt-to-fiscal revenue ratio has a power-law distribution in the emerging countries. We go beyond the visual inspection of a log-log graph of the distribution of the series, by estimating the scale coefficient and by testing formerly the hypothesis of a power-law distribution. Section 2 contains our theoretical model. Finally, Section 3 concludes.

2. Empirical power-laws in foreign debt-to-public revenue ratio

Let us look at the data in Figures 1 to 3. We plot the ratio of external debt over public revenues for some major emerging countries. The data are quarterly from 1980 (1990 for some countries) to 2014 and are taken from Oxford Economics and national sources (central banks and statistical institutes). Foreign debt data do not only comprise the general government's external debt but also include external debt issued by the private sector. Public revenues data correspond to central and local government revenues in local currency, including different types of taxes, interest receipts and dividends from state owned enterprises. To compare them with the ratio of external debt on GDP, these public revenues are then divided by nominal GDP (at market prices, in local currency)

We show the non-parametric distribution of the ratio using both a Gaussian and Epachenikov Kernel. The countries are shared over three regional areas : Latin America, Asia and Europe. We see that many of the distributions have fat tails in their right side and do not seem to fit with a Normal law. The probability of finding extreme values for the external debt ratio is higher than for a Gaussian distribution. These extreme values reflect situations of external debt vulnerability, which may cause financial tensions (rising interest rates, currency depreciation, capital outflows). To prevent such situations, policymakers may seek to calculate the probability that the debt ratio exceeds a certain value which is believed to correspond to episodes of debt stress or crises. A simpler way to proceed is to estimate a parametric law that approximate the observed distributions. In what follows, we show that the extreme values observed in Figures 1 to 3 follow power-law distributions, more specifically Pareto type 1 distributions.

Formally, a continuous variable x obeys a power-law with parameter α if its probability density function is written as

$$p(x) = \begin{cases} Cx^{-\alpha}, & x \geq x_{min} \\ 0, & x < x_{min} \end{cases}, \tag{1}$$

with the normalizing constant $C = \frac{\alpha-1}{(x_{min})^{-\alpha+1}}$ and x_{min} being the lower bound of the power-law behavior.

The m-th order moment is given by

$$\langle x^m \rangle = \int_{x_{min}}^{\infty} x^m p(x)dx = \frac{\alpha-1}{\alpha-1-m} x_{min}^m, \quad m \geq \alpha - 1. \tag{2}$$

For finite size samples, the central moments like the mean and the variance are not finite. Typical values for Pareto laws in economics are found for α in the interval $(2,3)$ for which the variance and higher-order moments are infinite.

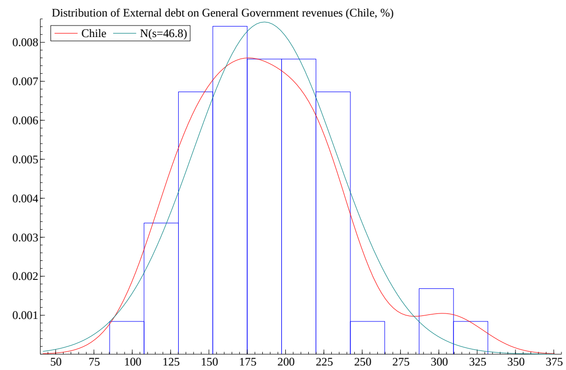
2.1. Estimates of power-law based on bootstrap methods

We first estimate the lower bound x_{min} from which the data can be considered to follow a power-law. We follow Clauset and Newman (2009). For a given x_{min} (assumed to be known), we compute the maximum likelihood estimator of α :

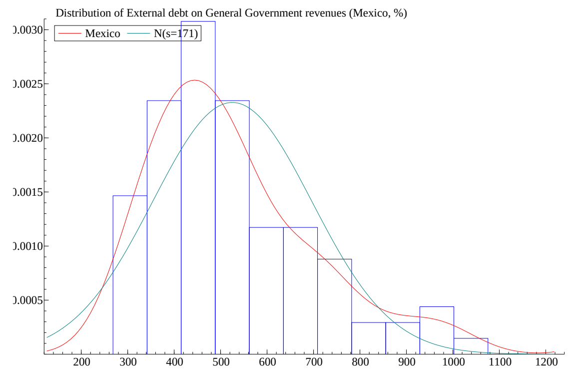
$$\hat{\alpha} = 1 + n\big[\sum_{i=1}^{n} ln(\frac{x_i}{x_{min}})\big]^{-1}, \tag{3}$$

whose standard error is equal to $\frac{\hat{\alpha}-1}{\sqrt{n}} + O(1/n)$, with n being the number of observed values being higher than x_{min}. $\hat{\alpha}$ corresponds to the shape parameter of the best fitting power-law, for this given value of x_{min}. Then, we compute the distance between the CDFs of the empirical data and the best fitting model (weighted Kolmogorov-Smirnov statistic) :

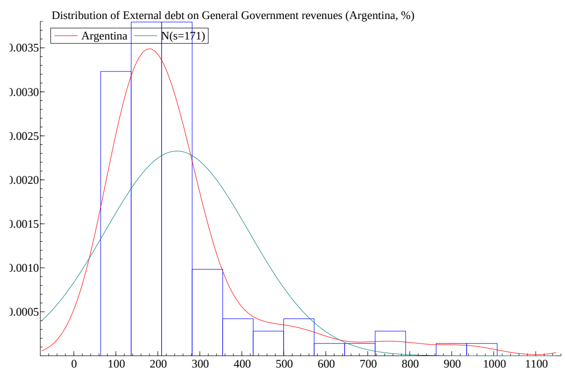
$$D^*(x_{min}) = \max_{x \geq x_{min}} \frac{|S(x) - P(x)|}{\sqrt{P(x)(1-P(x))}}, \tag{4}$$



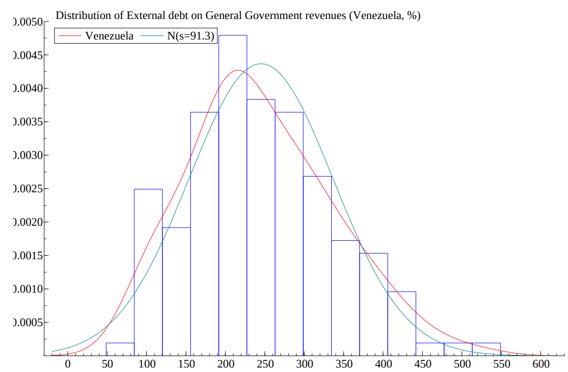
1. Chile



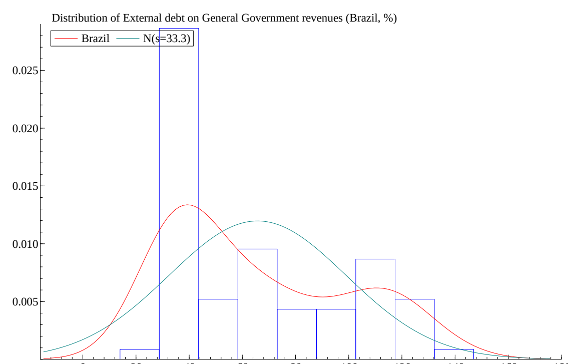
2. Mexico



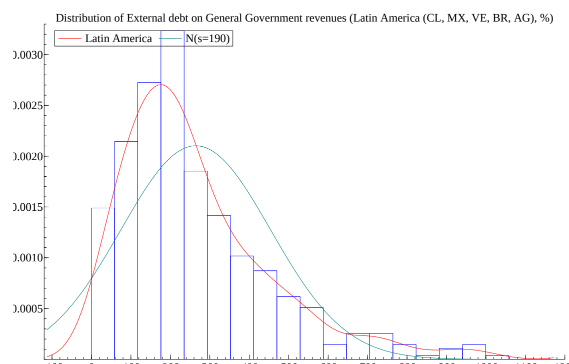
3. Argentina



4. Venezuela

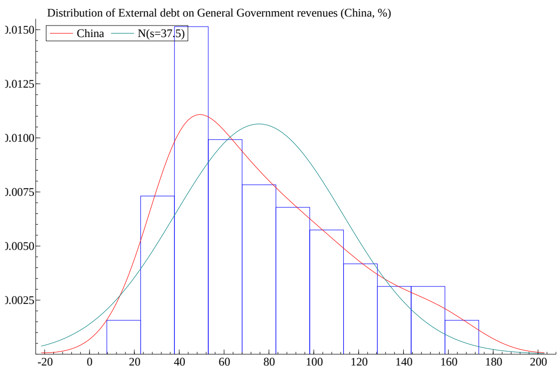


5. Brazil

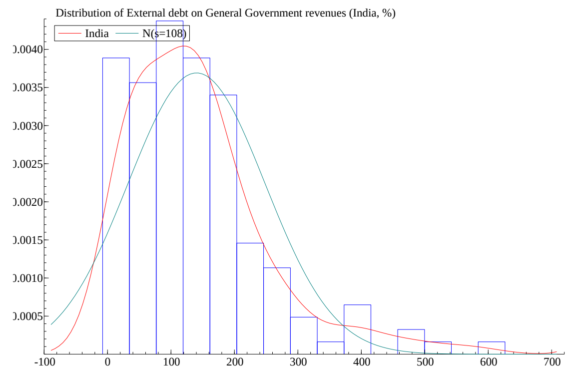


6. Latin America

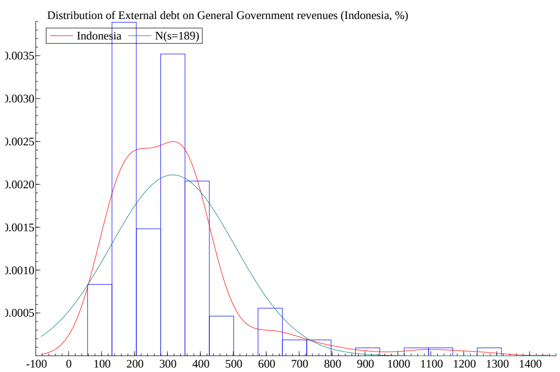
Figure 1: Distribution of External debt as share of public revenues (%), Latin America



1. China

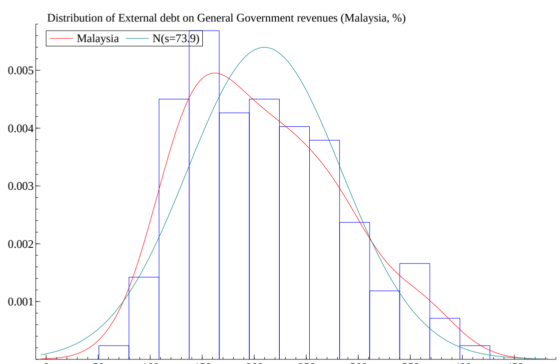


2. India

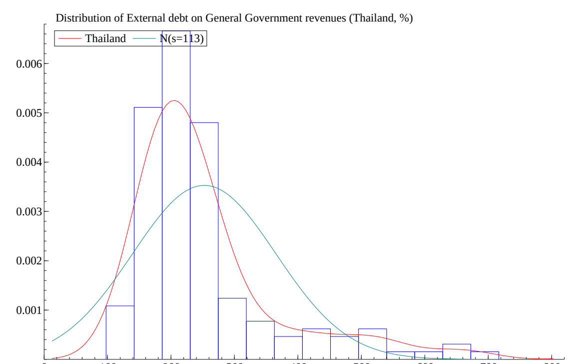


3. Indonesia

4. South Korea



5. Malaysia



6.Thailand

Figure 2: Distribution of External debt as share of public revenues (%), Asia

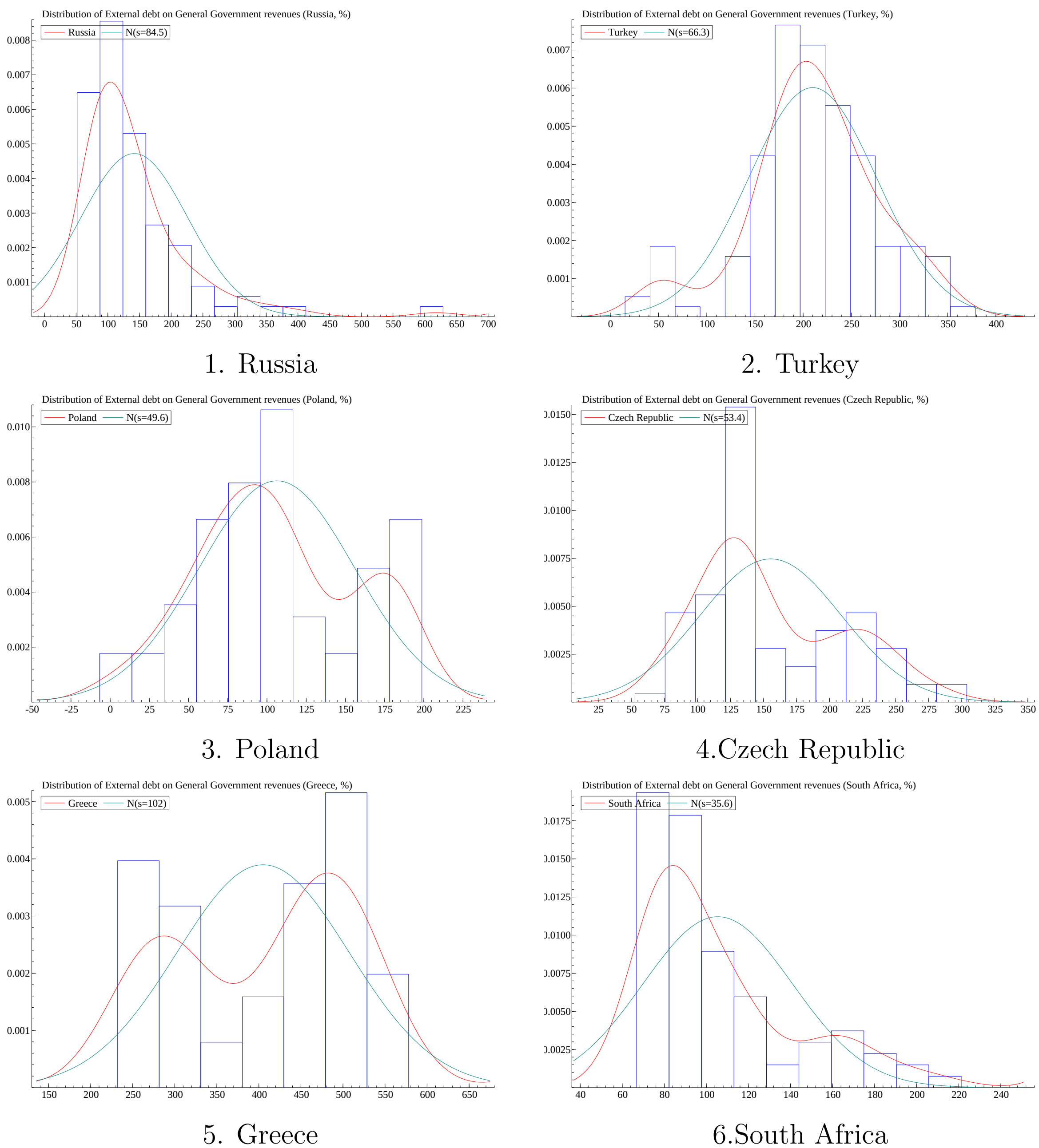


Figure 3: Distribution of External debt as share of public revenues (%), Europe, Russia and South Africa

where $S(x)$ is the CDF of the data for the values higher than x_{min}, and $P(x)$ is the CDF of the power-law model that best fits the data for $x \geq x_{min}$. Using a grid of possible values for x_{min}, the "optimal" x_{min} is finally computed so as to make the distributions of the empirical data and of the best fitting power-law as close as possible:

$$\hat{x}_{min} = \underset{x_{min}}{\operatorname{argmin}} D^*(x_{min}). \tag{5}$$

It minimizes D^* on a range of possible values for x_{min}. The mean and standard errors of the estimated parameters $\hat{x}_{min}$ and $\hat{\alpha}(\hat{x}_{min})$ are obtained by a bootstrap procedure.

2.2. Regression-based approaches

The previous method is based on the assumption that a power-law can fit the data well, which is not necessarily true. To evaluate whether a Pareto law really fits the data, we proceed by using a rank-regression approach proposed by Gabaix and Ibragimov (2011).

By OLS, we run the log-log rank-size regression (for $\gamma = 0.5$):

$$log(t - \gamma) = a - b\, log(Z(t)) \;\Leftrightarrow log(Rank - \gamma) = a - b\; log(Size), \tag{6}$$

where t is the rank of the observations Z(t) (ranked in decreasing order). The standard error of the shape coefficient b id $\sqrt{\frac{2}{n}}\hat{b}$, with n the "total" rank, i.e. the number of observations higher than x_{min} (i.e. following a power-law).

Another way to compute this standard error is to estimate the opposite regression:

$$log(Z(t)) = a_2 - b_2\, log(t - \gamma), \tag{7}$$

where the shape coefficient is obtained by $\frac{1}{\hat{b_2}}$ whose standard error is $\sqrt{\frac{2}{n}}\frac{1}{\hat{b_2}}$. One can plot $log(t - 0.5)$ against $log(Z(t))$, as well as the straight line whose equation we have estimated, such that we see if the data fits well with the power-law on this log-log plot. Gabaix and Ibragimov (2011) show that the bias is optimally reduced for $\gamma = 0.5$.

2.3. Evidence of Pareto laws in foreign debt data

Tables 1 and 2 show the estimates for our sample of countries. Graphs of log-log plots of the estimates are not reported to save place but are available upon request to authors. The estimates suggest that many of the distributions of the ratio of the external debt over public revenues can be approximated by power-laws. The scale exponent lies between 2 and 4 which are typical values for which power-law behaviors are usually observed. This has several implications for the analysis of the emerging countries' external debt.

	x_{min}	s.e. (bootstrap)	α	s.e. (bootstrap)
Chile	132,3	19,9	3,99	1,52
Mexico	336,7	45,7	3,48	0,63
Argentina	111,1	48,02	2,52	0,41
Venezuela	135,6	45,7	2,92	0,80
Brazil	31,2	12,3	2,43	0,42
China	34,2	17,7	2,32	0,45
India	20,04	17,62	1,59	0,07
Indonesia	114,4	43,1	2,09	0,19
South Korea	78,5	11,9	2,75	0,29
Malaysia	119,4	18,04	3,05	0,50
Thailand	157,2	17,4	3,56	0,47
Russia	73,8	17,4	2,82	0,38
Turkey	122,4	43,4	3,49	1,34
Poland	51,4	33,9	2,49	0,89
Czech Republic	103,6	20,7	3,68	0,79
Greece	260,5	63,4	2,81	1,49
South Africa	71,7	7,1	3,98	0,61

Table 1: Estimated lower bound x_{min} and associated α (bootstrap)

Firstly, the standard analysis of debt sustainability, that consists of examining how some economic shocks can make the debt ratio deviate from their "normal" levels, can be misleading. Indeed, with power-laws each new event changes the central and higher moments of the distribution, so that debt does not necessarily stabilize around an equilibrium value. It is better to focus on finding critical levels around which the debt oscillates and from which debt crises can occur.

	b (Reg 1)	standard error	1/b (Reg 2)	standard error
Chile	4,16	0,85	4,79	0,98
Mexico	3,33	0,51	3,7	0,57
Argentina	1,92	0,29	2,03	0,31
Venezuela	2,95	0,37	3,55	0,44
Brazil	1,81	0,31	2,37	0,40
China	1,97	0,26	2,43	0,32
India	1,04	0,12	1,43	0,17
Indonesia	1,84	0,22	2,14	0,25
South Korea	2,21	0,27	2,67	0,33
Malaysia	2,87	0,35	3,55	0,44
Thailand	2,77	0,34	2,89	0,35
Russia	2,22	0,34	2,28	0,35
Turkey	3,87	0,47	4,74	0,58
Poland	2,33	0,34	3,2	0,46
Czech Republic	3,18	0,50	3,8	0,60
Greece	3,13	0,65	5,32	1,11
South Africa	3,19	0,50	3,48	0,55

Table 2: Estimated α using rank-1/2 regression

Secondly, in powers-law distributions, extreme events are not rare. This accords with the finding that some emerging countries have been found to be "serial defaulters" on their foreign debt : if they have defaulted in the past, they are very likely to default again in the future (the literature on serial sovereign default is abundant. For a recent literature review, see Asonuma (2016)).

3. Tax evasion and corruption as causes of power-laws in foreign debt ratios: a theoretical model

What causes high external debt in the emerging countries? The literature suggests that episodes of foreign debt distress have been systematically associated with other macroeconomic and financial imbalances, among which hyper-inflation, banking crises, asset price bubbles, credit booms, capital flows, the cyclical nature of fiscal and monetary policies (see Kaminsky and Vegh (2005), Reinhart and Rogoff (2008), Reinhart and Rogoff (2011), Reinhart and Rogoff (2013)).

What is distinctive in our paper relative to previous works is that we explain theoretically how outcomes of high external debt ratio depicted by a power-law distribution can arise from loose civic capital or governance, captured by tax evasion and bureaucratic corruption in the fiscal administration. Both these factors weaken a country's fiscal position by contracting government's revenues. Public spending is therefore financed through a forcible foreign borrowing. Since tax revenues accounts for the greatest part of public revenues and given that interest receipts and dividends from state owned enterprises are exogenous (in the sense that they do not directly depend upon tax evasion and corruption behaviors), we focus on tax revenues as the denominator of the external debt ratio. We propose a simple continuous time stochastic growth model from which the exact asymptotic distribution of foreign debt-to-fiscal revenue ratio is obtained using results from the theory of stochastic differential equations.

Two types of uncertainties are introduced into the model. First, we consider lotteries on tax evasion and corruption behaviors which give us the expected return to cheating for both households and bureaucrats, and the risks of being a fraudulent and a corrupt government. Secondly, we consider diffusion processes as follows.

Consider a random variable $X(t)$. Its change between time t and $t + dt$ is defined by the Itô stochastic differential equation:

$$dX(t) = f(X(t))dt + g(X(t)dZ. \tag{8}$$

f and g are two continuous functions and Z is a Brownian motion : $dZ \approx N(0, dt)$ and dt is of infinitesimal order. Moreover, we assume that Z is a Markovian process in the sense that dZ depends neither on t nor on its past values. This assumption will allow us to use the general properties of diffusion processes to derive a probability distribution for our main variable of interest.

3.1. Production

We consider an open economy specialized in the production of a single good. This good is the numeraire and its price is normalized to 1. The country comprises N inhabitants (consumer/taxpayers and workers). All the variables in the model are expressed in per-capita terms. The individuals differ in their behaviors as taxpayers (some are fraudulent, others are not), but their consumption and production decisions are the same.

The single domestic good of the economy is produced with capital, labor (fixed amount) and public spending

$$y(t) = k(t)^{\beta} g(t)^{1-\beta}, \tag{9}$$

where $y(t)$ is per-capita income, $k(t)$ is private capital-labor ratio, $g(t)$ is per-capita productive public spending (infrastructure, public goods and services). We assume constant returns to scale ($0 < \beta < 1$).

Domestic capital yields a stochastic return. The flow of private rate of return on capital over the period $(t, t + dt)$ is subject to a stochastic disturbance:

$$dR_k(t) = r_k dt + du_k, \tag{10}$$

where the drift component $r_k = \frac{\partial F}{\partial k}$ is a constant return to private capital and the diffusion component du_k is a Brownian motion.

3.2. Tax evasion

A consumer is faced with the decision of whether or not to pay taxes on his actual income $y(t)$. The income tax rate τ is constant. He hides a fraction $e(t)$ of the income $(0 < e(t) < 1)$. Therefore $[1 - e(t)]y(t)$ is the amount of reported income. The probability of being detected as a fraudulent by the administration is p $(0 < p < 1)$ and, if caught, the taxpayer pays the due tax plus a penalty. b is the penalty rate on the amount of hidden income. Therefore, the consumer pays $\tau[1 - e(t)]y(t)$ if he is not detected and $\tau y(t) + be(t)y(t)$ if he is caught. The return of each unit of concealed income is described by the following lottery:

$$x = \begin{cases} 1, & \text{w.p. } 1-p \\ -b, & \text{w.p. } p \end{cases} . \tag{11}$$

The expected gain and risk of the lottery are defined by

$$\overline{x} = (1-p) - pb, \ \sigma^2 = \mathbb{E}(x^2) - \mathbb{E}(x)^2 = \mathbb{E}(x^2) - \bar{x}^2. \tag{12}$$

$\overline{x}$ and σ^2 are function of p and b.

3.3. Corruption and government's choice

The tax administration consists of bureaucrats whose number is the same as the consumer and a fraction p_1 of whom are corrupted and a fraction $1-p_1$ who are non-corrupted $(0 < p_1 < 1)$[3]. Bureaucratic corruption arises when civil servants charged with the responsibility for collecting taxes choose to extort a fraction $\theta(t)$ of tax revenues. They steal a fraction of tax resources which normally serves to finance public goods that enter as inputs in the production activity.

A fraction $p_1\theta(t)$ of tax revenues is stolen and converted to unproductive public spending (or public consumption). Per-capita public spending is therefore described by

$$g(t) = \begin{cases} (1 - p_1\theta(t))\big[\tau y(t) + be(t)y(t)\big], & \text{w.p. } p \\ (1 - p_1\theta(t))\tau(1 - e(t))y(t), & \text{w.p. } (1-p) \end{cases} , \tag{13}$$

and the dynamics of productive public spending is given by the following stochastic differential equation (SDE):

$$dg(t) = \bar{g}(t)dt + \tilde{g}(t)dW^g, \tag{14}$$

[3]The assumption that the number of bureaucrats equals those of consumers is made for technical simplicity and does not change our arguments.

where W^g is a Brownian motion, $\bar{g}(t)$ is the expected productive public expenditure from the above lottery and $\tilde{g}^2(t) = \mathbb{V}(g(t)) = \mathbb{E}(g(t)^2) - \mathbb{E}(g(t))^2$ is the variance of the lottery. Unproductive public spending (the share of diverted tax revenues which we consider to be public consumption) is given by

$$C^G(t) = \begin{cases} 0, & \text{w.p. } (1-p_1) \\ \theta(t)\big[\tau y(t) + be(t)y(t)\big], & \text{w.p. } pp_1 \\ \theta(t)\tau(1-e(t))y(t), & \text{w.p. } p_1(1-p) \end{cases}. \tag{15}$$

The dynamics of public consumption is described by the following SDE

$$dC^G(t) = \bar{C^G}(t)dt + \tilde{C^G}(t)dW^{C^G}, \tag{16}$$

where $W^{C^G}(t)$ is a Brownian motion and

$$\bar{C^G}(t) = \mathbb{E}(C^G(t)) = p_1\theta(t)y(t)\Big[pbe(t) + p\tau e(t) + \tau(1-e(t))\Big], \tag{17}$$

$$\tilde{C^G}^2 = \mathbb{V}(C^G(t)) = \mathbb{E}(C^G(t)^2) - \mathbb{E}(C^G(t))^2. \tag{18}$$

In this stochastic framework, the balanced budget constraint describes how the flow of tax revenues is assigned to the flows of productive and unproductive public spending:

$$dT(t) = \big[\bar{g}(t) + \bar{C^G}(t)\big]dt + \tilde{g}dW^g(t) + \tilde{C^G}dW^{C^G}, \tag{19}$$

where

$$\bar{C^G}(t) = p_1\theta(t)y(t)\Big[pbe(t) + p\tau e(t) + \tau(1-e(t))\Big] \tag{20}$$

$p_1\theta(t)$ can be interpreted as a corruption index ranging from 0 (low corruption) to 1 (high corruption). It depends upon the proportion of corrupted bureaucrats and the degree of prevarication in the use of public resources. We assume that the bureaucrats' punishment for being corrupt is nil. This is a distinguishing feature of countries with poor governance (lack of transparency and accountability, inefficient court systems). Moreover, it is assumed that $\theta(t) > 0$ in the sense that corruption is a rent seeking activity that is privately beneficial, and that $\theta(t) < 1$, because corruption inhibits growth (by reducing the productivity of private capital) and the amount of the rent. Therefore, the bureaucrats' aim is to define an optimal degree of prevarication (how much to steal) that maximizes public consumption.

The government maximizes its intertemporal expected utility function defined on public consumption subject to the constraint (19):

$$\max_{n^G,n^C} \mathbb{E}\int_0^{+\infty} \frac{1}{\gamma}(T(t)n_C(t))^{\gamma}e^{-\rho t}dt,\ \rho > 0,\ -\infty < \gamma < 1, \tag{21}$$

subject to

$$\frac{dT(t)}{T(t)} = \psi^T(t)dt + \sigma_{w^T}(t)dw^T,\ T \in (0,\infty), \tag{22}$$

and

$$n_g(t) + n_C(t) = 1, \tag{23}$$

where

$$n_g(t) = \frac{\bar{g}(t)}{T(t)},\ n_C(t) = \frac{\bar{C}^G(t)}{T(t)},\ \psi^T(t) = \frac{\bar{g}(t)}{T(t)} + \frac{\bar{C}^G(t)}{T(t)}, \tag{24}$$

$$\sigma^2_{w^T}(t) = \tilde{n}_g(t)^2\sigma^2_{W^g}(t) + \tilde{n}^2_C(t)\sigma^2_{W^{C^G}}(t), \tag{25}$$

$$\tilde{n}_g(t) = \frac{\tilde{g}(t)}{T(t)},\ \tilde{n}_C(t) = \frac{\tilde{C}^G(t)}{T(t)}. \tag{26}$$

The solution to the maximization problem of the government gives an optimal path of the share of public consumption out of total fiscal revenues defined by (we omit the index t for purpose of simplification) [4]

$$\hat{n_C} = \frac{\sigma^2_{W^g} + \sqrt{\sigma^4_{W^g} - 4(\sigma^2_{W^{C^G}} + \sigma^2_{W^g})(\frac{\rho-\gamma}{1-\gamma} + \frac{1}{2}\sigma^2_{w^T}\gamma)}}{2(\sigma^2_{W^{C^G}} + \sigma^2_{W^g})} \tag{27}$$

This solution illustrates the role of the relative aversion risk coefficient $(1-\gamma)$. When corruption takes place, there is a chance that lean times could occur in the future. A bureaucrat could suffer from loss in his future consumption because the outcome of diverting tax revenues is lower productive public spending, lower production and thus lower tax base.

An interesting question is whether high risk averse bureaucrats (when γ becomes more and more negative) are inclined to be less dishonest (which could explain that high risk aversion reduces the critical level of corruption, because being "honest" today leaves more room for maneuver to be able to

[4] See Appendices A and C.

steal more resources tomorrow), or whether they adopt a behavior suggesting that "a half a loaf is better than no loaf at all" (in this case, starting from an initial value, we could expect the path of $\hat{n_C}$ to decrease in time, because the more they steal today, the less the possibility of diverting resources tomorrow since the tax base decreases). It is easily seen that the sign of the derivative of refnc with respect to γ is indeterminate so that both outcomes can happen.

3.4. The open economy and external debt

Insofar as tax evasion and corruption do not allow the domestic country to produce the amount of the single domestic good that households would like to consume, the country need to buy a foreign good from abroad. We adopt the convention that imports of goods is equivalent to external borrowing, so that external debt is defined as the domestic country's net international current account position. We make the following assumptions:

- P^M is the price of imports and therefore $1/P^M$ is a proxy of the terms of trade. Import price changes are described by the following SDE:

$$\frac{dP^M(t)}{P^M(t)} = \pi dt + dP, \tag{28}$$

  where π defines changes in the terms of trade and P is a Brownian motion.

- Debt is denominated in terms of the foreign output and its price is expressed in terms of the price of the numeraire $P^M D$. The interest rate paid on foreign debt is stochastic and described by

$$dR_f(t) = r_f dt + du_f,\ r_f = i^* + \pi \text{ and } du_f = dP. \tag{29}$$

  where r_f is the real world interest rate assumed to be constant.

3.5. Consumer's choice

The domestic country comprises agents who consume the domestic and the imported goods. The agent bears domestic assets (he owns the firm) and foreign liabilities (he pays back the foreign debt). His wealth constraint is given by

$$W(t) = K(t) - P^M(t)D(t),\ W \in (0, \infty). \tag{30}$$

This implies

$$\frac{K(t)}{W(t)} - \frac{P^M(t)D(t)}{W(t)} = n_K(t) - n_f(t) = 1, \tag{31}$$

where $n_K(t)$ and $n_f(t)$ are the shares of capital and foreign debt in total wealth with $0 < n_K(t) < 1$.

The stochastic wealth accumulation equation is given by

$$\begin{array}{rcl} dW(t) & = & \left[1 - \tau + \bar{x}\tau e(t)\right] k(t)^{\beta} g(t)^{1-\beta} dt \\ & & -(C^P(t) + P^M(t)C^M(t))dt \\ & & +\sigma\left[\tau e(t)k(t)^{\beta} g(t)^{1-\beta}\right] dW^W \\ & & +k(t)dR_k(t) - P^M(t)D(t)dR_f(t) \end{array} . \tag{32}$$

This equation has several drift and diffusion components:

- Line 1 describes the drift component of the disposable income. The household can inflate his disposable thanks to tax evasion activity. Given (9), we have $(1-\tau)k(t)^{\beta}g(t)^{1-\beta} = (1-\tau)y(t)$. This is the disposable income when there is no tax evasion. $e(t)\tau(t)y(t)$ is the amount of tax evaded. Since tax evasion is a risky activity, we need to consider the expected return of tax evasion that is $\bar{x}e(t)\tau y(t)$ where $\bar{x}$ is defined by (12).

- Line 2 defined the drift component of consumption, that is the household's consumption of domestic and foreign goods that are assumed to proceed at non-stochastic rates over the interval $(t, t+dt)$.

- Line 3 corresponds to the diffusion component of disposable income or the risk associated to fiscal evasion. The standard error of the Brownian motion W^W is $\sigma\tau e(t)y(t)$ where σ is defined by (12).

- Line 4 (first expression) $k(t)dR_k(t)$ describes both the deterministic and the stochastic shocks to the productivity of private capital.

- Line 4 (second expression) $P^M(t)D(t)dR_f(t)$ describes the influence of the international financial markets and of the terms of trade shocks which impact the interest rate paid on external debt.

The consumer's objective is to choose his domestic and foreign consumption, the amount of income to hide and his portfolio of net assets (assets minus liabilities) in order to maximize the expected value of discounted utility subject to the constraint given by (C.2):

$$\max_{C^P, C^M, e, n_K, n_f} = \mathbb{E}\int_0^{+\infty} \frac{1}{\mu}\big[C^P(t)^{\eta} C^M(t)^{1-\eta}\big]^{\mu} e^{-\rho t} dt,\ \rho > 0,\ - < \mu < 1, \tag{33}$$

subject to

$$\frac{dW(t)}{W(t)} = \psi^W(t)dt + \sigma_{z^W}(t)dz^W,\ n_K(t) - n_f(t) = 1, \tag{34}$$

and $W(0) = w_0$, where

$$\begin{aligned}\psi^W(t) \quad = \quad & \big[1 - \tau + \bar{x}\tau e(t)\big] n_k^{\beta}(t) n_g^{1-\beta}(t)(\tfrac{T(t)}{W(t)})^{1-\beta} - \tfrac{CPI(t)C(t)}{W} + \\ & n_k(t) r_k - (i^* + \pi) n_f(t),\end{aligned} \tag{35}$$

$$\sigma_{z^W} dz^W = \sigma\tau e(t) n_k^{\beta}(t) n_g^{1-\beta}(t)(\frac{T(t)}{W(t)})^{1-\beta} dW^W + n_k(t)du_k - n_f(t)du_f. \tag{36}$$

$n_k(t) = \frac{k(t)}{W(t)}$ and $n_f(t) = \frac{P(t)D(t)}{W(t)}$ are respectively the share of capital and foreign debt in total wealth[5].

The optimal path is described by the following system of recursive equations[6]:

$$\begin{cases} \tilde{C}(t) = (\delta\mu)^{\frac{1}{\mu-1}} (P^M(t))^{\frac{1-\eta}{\mu-1}} W(t) \\ \tilde{e}(t) = \frac{\bar{x}\tau\tilde{n}_k(t)^{\beta} n_g(t)^{1-\beta}(\frac{T(t)}{W(t)})^{1-\beta}}{(1-\mu)\left[\sigma\tau\tilde{n}_k(t)^{\beta} n_g(t)^{1-\beta}(\frac{T(t)}{W(t)})^{1-\beta}\sigma_{W^W}(t)\right]^2} \\ \tilde{n}_k(t) = \frac{r_k - (i^*+\pi) + (1-\mu)\sigma_{W^f}(t)^2 + \sqrt{\triangle}}{2(1-\mu)(\sigma_{W^k}(t)^2 + \sigma_{W^f}(t)^2)} \\ \tilde{n}_f(t) = \tilde{n}_k(t) - 1 \end{cases}$$

[5]Assuming the same rate of time preference for the government and the household is made for simplicity and does not have any consequence on our results.

[6]See Appendices B and C.

$$\triangle = \Big[r_k - i^* - \pi + (1-\mu)\sigma^2_{W^f}\Big]^2$$
$$- 4(1-\mu)(\sigma^2_{W^k} + \sigma^2_{W^f})\big[(1-\mu)\beta\sigma^2\tau^2\tilde{e}^2\sigma^2_{W^W}n_y^2 - \beta\big[1-\tau+\bar{x}\tau\tilde{e}\big]n_y\big]. \quad (37)$$

From the first equation, we see that total private consumption (private domestic goods plus imports) increases with wealth. Moreover, an increase in import prices (deterioration of the terms of trade) raises the nominal debt and accordingly reduces the nominal net wealth (since $W(t) = K(t) - P^M(t)D(t)$). In this case the households reduces his consumption (balance sheet effect).

From the second equation of the system, it is seen that the incentive for tax avoidance increases when the return on each unit of hidden income increases relative to the risk of being caught ($\tilde{e}(t)$ is an increasing function of $\bar{x}/\sigma^2$). Moreover the taxpayer is more likely to cheat when per-capita income $y(t)$ decreases. This happens, either when private capital decreases (the households holds less assets in his portfolio) or when corruption by bureaucrats increases (in this case n_g diminishes or, equivalently, n_c increases). Finally, T/W measures the implicit corruption and fraud tax on the household's net wealth. Indeed, these illegal activities implies that the consumer must borrow from abroad (in the form of imports) and therefore bears the costs of the interest rate charged on external debt. This cost is internalized by the consumer and has an inhibiting effect on tax fraud (we see that $\tilde{e}$ is a decreasing function of the ratio T/W).

From the third equation, we see that the capital share in total wealth depends upon the following factors:

- $r_k - (i^* + \pi)$ is the expected net rate of return of capital (the capital owned as share of the domestic firm minus the capital borrowed from abroad). It is proportional to the risks of both types of capital $(\sigma_{W^k}(t)^2 + \sigma_{W^f}(t)^2)$.
- the ratio $\sigma_{W^f}(t)^2/(\sigma_{W^k}(t)^2 + \sigma_{W^f}(t)^2)$ captures the household's dehedging behavior. When the risk of debt holding increases, the household want to own more domestic capital (which is possible by reducing tax evasion).
- We have an additional term $\Delta^{1/2}$ that depends on $\tilde{e}$, W/T, $\hat{n}_c$.

4. Power-law as steady state distribution for the external debt ratio

From Equation (30), the ratio of nominal external debt over nominal fiscal revenues can be defined by

$$\frac{P^M(t)D(t)}{T(t)} = \frac{K(t)}{T(t)} - \frac{W(t)}{T(t)}. \tag{38}$$

For a given value of the capital/tax income, changes in the external debt/tax revenues ratio are negatively correlated to those in $W(t)/T()$. Given the invariance properties of power-law distributions, for a given value of the ratio $K(t)/T(t)$, the existence of a Pareto law for $W(t)/T(t)$ implies that the external debt ratio will also follow a Pareto law. Thus, we define the macroeconomic equilibrium in terms of the ratio of net wealth-to-tax revenues.

A power-law probability density function is derived by proceeding in several steps. First, we specify the dynamics of the ratio $W(t)/T(t)$ by using the Itô's lemma. We thus obtain a Itô stochastic differential equation formulation. Then, a theorem of the existence of a stochastic steady state is proved by using some properties of Markov chain diffusion processes. After that, we show that the stochastic steady state can be defined by a power-law with shape parameter depending upon our key variables of interest in the model related to tax evasion and corruption. Finally, we propose some simulations of the model.

4.1. Macroeconomic equilibrium as a diffusion process

Definition 4.1. Stochastic macroeconomic equilibrium

A stochastic macroeconomic equilibrium is defined as a path of wealth-to-fiscal revenues ratio $\{\lambda(t)\}$, where

$$\lambda(t) = \left\{ \frac{W(t)}{T(t)} \right\}_{t=0}^{\infty}, \tag{39}$$

along which the agents' choice (bureaucrats and households) are optimal and the current account is balanced (imports are financed by foreign borrowing).

Theorem 4.1. *The stochastic dynamics for the wealth-fiscal revenue ratio is a diffusion process defined by the SDE*

$$d\lambda(t) = \Omega_1(\lambda)dt + \Omega_2^{\frac{1}{2}}(\lambda)dB, \quad \lambda(t) = \frac{W(t)}{T(t)} \in \left[0, \infty\right], \tag{40}$$

where

$$\Omega_1(\lambda) = \lambda\Big[\tilde{n}_g^2(\lambda) + \tilde{n}_C^2(\lambda) - \tilde{n}_g(\lambda) - \tilde{n}_C(\lambda) +$$
$$\big[1 - \tau + \bar{x}\tau\tilde{e}(\lambda)\big]\tilde{n}_y(\lambda) - CPI\frac{\tilde{C}}{W} + n_k(\lambda)r_k - (i^* + \pi)n_f(\lambda)\Big],$$
$$\Omega_2(\lambda) = \lambda^2\Big[\tilde{n_g}(\lambda)^2 + \tilde{n_C}(\lambda)^2 + \sigma^2\tau^2 e(\lambda)^2 n_y(\lambda)^2 + n_k(\lambda)^2 + n_f(\lambda)^2\Big].$$

Proof. We first write the bureaucrats' and consumers' constraints (Equations (22) and(34) which give us the separate dynamics of $dT(t)$ and $dW(t)$):

$$\mathrm{dW(t)} = \mathrm{W(t)}\big[\text{ 1-}\tau + \bar{x}\tau e(\lambda)\big]n_y(\lambda) - CPI\frac{C}{W} + n_k(\lambda)r_k - (i^* + \pi)n_f(\lambda)\Big]dt$$
$$+W(t)\Big[\sigma\tau e(\lambda)n_y(\lambda)dW^W + n_k(\lambda)du_k - n_f(\lambda)du_f\Big],$$
$$\mathrm{dT(t)} = \mathrm{T(t)}\ \Big[\ \mathrm{n}_g(\lambda) + n_C(\lambda)\Big]dt + T(t)\Big[\tilde{n_g}(\lambda)dW^g(t) + \tilde{n_C}(\lambda)dW^{C^G}\Big].$$

To obtain the dynamics of the ratio $\lambda(t) = W(t)/T(t)$, we use the Itô's lemma.

Itô's lemma. Let X(t) in $\mathbb{R}^2$ be a diffusion process and F(X) a $\mathbb{C}^2$ map from $\mathbb{R}^2$ to $\mathbb{R}$, then

$$dF(X) = F_x dX + \frac{1}{2}dX'F_{xx}dX, \tag{41}$$

with F_x and F_{xx} representing, respectively, the matrix of partial derivatives of F and the Hessian matrix.

We define $X = (T, W)'$, $dX = (dT, dW)'$, $F(X) = \frac{W}{T}$,

$$F_x = \begin{pmatrix} \frac{\partial F}{\partial T} \\ \frac{\partial F}{\partial W} \end{pmatrix} = \begin{pmatrix} \frac{-W}{T^2} \\ \frac{1}{T} \end{pmatrix},$$

$$F_{xx} = \begin{pmatrix} \frac{\partial^2 F}{\partial T^2} & \frac{\partial^2 F}{\partial T \partial W} \\ \frac{\partial^2 F}{\partial W \partial T} & \frac{\partial^2 F}{\partial W^2} \end{pmatrix} = \begin{pmatrix} \frac{2W}{T^3} & \frac{-1}{T^2} \\ \frac{-1}{T^2} & 0 \end{pmatrix}.$$

From (41) we obtain

$$d(\frac{W}{T}) = \frac{-W}{T^2}dT + \frac{1}{T}dW + \frac{W}{T^3}dT^2 - \frac{1}{T^2}dTdW. \tag{42}$$

We get the final form of Equation (40) by using the Levy characterization of diffusion processes and by considering the following properties of Wiener processes. Consider two Wiener processes w_i and w_j. We have:

$$(dw_i)^2 = dt,\ < dt.dw_i >= 0\ \forall i \neq j,\ dt^2 = 0.$$

For purpose of simplicity, we assume the following correlation structure of two Wiener processes: $d < w_i, w_j >= 0$, where $< w_i, w_j >$ is the quadratic variation process for the components of the Wiener processes. ■

4.2. Existence of a steady state distribution

The concept of steady state for diffusion processes is defined in a stochastic sense. Instead of a point, $\lambda(t)$ converges to a set of values in a basin of attraction. Define λ^* as the attractor. For any variable $Z(\lambda)$ in the bassin of attraction, there exists two real numbers $\epsilon > 0$ and $\delta > 0$ such that:

$$d\big(z(\lambda), z(\lambda^*)\big) = inf\left\{\epsilon > 0 : Pr\big(\,|Z(\lambda) - Z(\lambda^*)\,| > \epsilon \leq \delta\right\}.$$

The probability that $Z(\lambda)$ is outside the ball of radius ϵ centered at $Z(\lambda^*)$ is very small. This means that, once λ has reached its long-term attractor, all the variables in the model which depend upon λ will also reach their own basin of attraction. For Itô diffusion processes, the properties of the values in the basin of attraction of the variable of interest can be studied by constructing their distribution called the steady state distribution of the diffusion process. The issue here is therefore to study the convergence in distribution of the variable $\lambda(t)$. To do this, we use the mathematical tools of the theory of Markov chains to prove the existence and derive stationary probability measures. The techniques are similar to those used in a few papers dealing with continuous time stochastic growth models (see Bourguignon (1974), Merton (1975), Chang and Maliaris (1987)).

4.2.1. Asymptotic stochastic solutions of Itô diffusion processes

We first recall some mathematical properties of steady state distributions of Itô diffusion processes (see Feller (1952), Feller (1954), Ito and McKean (1996)).

Let us consider the following SDE:

$$dx = a(x)dt + b^{1/2}(x)dz,\ dz \approx N(0, dt),\ x \in \big[0, \infty\big], \tag{43}$$

with $a(\cdot)$ and $b(\cdot)$ being continuous and differentiable functions of x.

Consider $X(t)$ the solution of the SDE and define the transition probability as

$$P(x,t,x_0,t_0) = Pr\big[X(t) \leq x \,\big|X(0) = x_0\big].(44)$$

The probability density $\pi(x,t,x_0)$ satisfies the Kolmogorov-Fokker-Planck equation:

$$\frac{1}{2}\frac{\partial^2}{\partial x^2}\big[b(x)\pi(x,t,x_0)\big] - \frac{\partial}{\partial x}\big[a(x)\pi(x,t,x_0)\big] = \frac{\partial \pi(x,t,x_0)}{\partial t}. \tag{45}$$

The steady state density function, obtained by integrating (45), must satisfy

$$p(x) = c_1 m(x) + c_2 S(x),\ p(x) = \lim_{t\to\infty} \pi(x,t,x_0), \tag{46}$$

where

$$M(x) \equiv \int_{x_0}^{x} m(u)du,\ S(x) \equiv \int_{x_0}^{x} s(u)du, \tag{47}$$

with

$$m(x) \equiv \frac{exp\big[2J(x)\big]}{b(x)},\ s(x) \equiv exp\big[-2J(x)\big],\ J(x) \equiv \int_{x_0}^{x} \frac{a(u)}{b(u)}du. \tag{48}$$

c_1 and c_2 are constants of integration ensuring that $p(x)$ is a true probability density. $s(x)$, $S(x)$ and $m(x)$ are called, respectively, the scale density function, the scale function and the speed density function of the stochastic process.

Existence of steady state distribution. A time-invariant distribution function $P(x)$ exists if and only if

$$\lim_{\substack{x\to 0\\ x\to\infty}} S(x) = \mp\infty \ and\ |M(x)| = \int_0^{\infty} m(u)du < \infty. \tag{49}$$

The existence of steady state distribution implies that the boundaries of the process are inaccessible. A corollary is that, if the boundaries are inaccessible, then $\pi(x,t,x_0)$ converges towards a probability density function defined by (46) with $c_2 = 0$ (for rigorous proofs, see Feller (1952), Feller (1954), Ito and McKean (1996)).

4.2.2. A theorem of the existence of a steady state distribution for the wealth-to-tax revenues ratio $\lambda(t)$

Now, we must prove that the boundaries 0 and $+\infty$ are inaccessible for the wealth-to-tax revenues ratio $\lambda(t)$. To do that, several preliminary remarks are in order.

First, $\lambda(t)$ is the ratio of two variables $W(t)$ and $T(t)$. We assume that $T \in (0, +\infty)$ and $W \in (0, +\infty)$. By assumption 0 is thus an inaccessible boundary for W and T. This means that we do not consider the extreme situation of balance of payment crisis with high indebtedness. $T = W = 0$ can figure out a situation in which corruption and tax evasion are so important that this yields a depletion of tax revenues ($T \to 0$). In this case, the country reaches a high level of debt which reduces the household's net wealth to a low level ($W \to 0$).

Second, the fact that by assumption, T and W do not reach the zero boundary, does not mean that 0 is inaccessible for λ. Indeed,

$$\lim_{W \to +\infty} \lambda = +\infty, \ \lim_{T \to \infty} \lambda = 0. \tag{50}$$

Therefore to prove that 0 and $+\infty$ inaccessible boundaries for λ, we must prove that $+\infty$ is an inaccessible boundary for both W and T.

Theorem 4.2. *Let us consider the SDE of T and W in a compact form using the Levy characterization*

$$dT(t) = a_1(T, \lambda^*)dt + b_1^{\frac{1}{2}}((T, \lambda^*))dw^T,$$

where

$$a_1(T, \lambda^*) = T(t)\Big[n_g(\lambda^*) + n_C(\lambda^*)\Big] = T(t)a(\lambda^*),$$

$$b_1(T, \lambda^*) = T^2(t)\Big[\tilde{n}_g^2(\lambda^*) + \tilde{n}_C^2(\lambda^*)\Big] = T^2(t)b(\lambda^*),$$

$$dW(t) = d_1(W, \lambda^*)dt + h_1^{\frac{1}{2}}((W, \lambda^*))dz^W,$$

where

$$d_1(W, \lambda^*) = W(t)\Big[\big[1 - \tau + \bar{x}\tau\tilde{e}(\lambda^*)\big]\tilde{n}_y(\lambda^*) - CPI\tfrac{\tilde{C}}{W} +$$

$$\tilde{n}_k(\lambda^*)r_k - (i^* + \pi)\tilde{n}_f(\lambda^*)\Big] = W(t)d(\lambda^*),$$

$$h_1(W, \lambda^*) = W^2(t)\Big[\sigma^2\tau^2\tilde{e}(\lambda^*)^2\tilde{n}_y^2(\lambda^*) + \tilde{n}_k^2(\lambda^*) + \tilde{n}_f^2(\lambda^*)\Big] = W^2(t)h(\lambda^*).$$

Sufficient conditions for the existence of a steady state distribution for the ratio of wealth-to-fiscal revenue are:

$$a)\ 2a(\lambda^*) - b(\lambda^*) < 0,\ and\ b)\ 2d(\lambda^*) - h(\lambda^*) < 0 \tag{51}$$

Proof. *i)*

We first prove that $\lim_{T\to+\infty} S(T, \lambda^*) = +\infty$

Using the Levy representation of the SDE of T, as given in the theorem, we compute the scale density function of $T(t)$ as

$$s(T, \lambda^*) = exp\left\{-2\int_{T_0}^{T} \frac{a_1(u, \lambda^*)}{b_1(u, \lambda^*)}du\right\},\ T_0 = T(0), \tag{52}$$

or

$$s(T, \lambda^*) = exp\left\{-2\frac{a(u, \lambda^*)}{b(u, \lambda^*)}\int_{T_0}^{T} \frac{1}{u}du\right\} = \Big[\frac{T}{T_0}\Big]^{-2\frac{a(\lambda*)}{b(\lambda^*)}}, \tag{53}$$

Then, we calculate the scale function

$$S(T, \lambda^*) = \int_{T_0}^{T} s(u)du = \frac{b(\lambda^*)(T_0)^{\frac{2a(\lambda^*)}{b(\lambda^*)}}}{-2a(\lambda^*) + b(\lambda^*)}\left\{[T]^{\overline{-2a(\lambda^*)+b(\lambda^*)}} - [T_0]^{-2a(\lambda^*)+b(\lambda^*)}\right\}, \tag{54}$$

We see that

$$\lim_{T\to+\infty} S(T, \lambda^*) = +\infty,\ if\ 2a(\lambda^*) - b(\lambda^*) < 0. \tag{55}$$

ii)

Using a similar approach by considering the SDE of W(t), we get

$$\lim_{W\to+\infty} S(W, \lambda^*) = +\infty, if\ 2d(\lambda^*) - h(\lambda^*) < 0 \tag{56}$$

■

Remark 1. *Condition b) in(51) means that to avoid an infinite increase of wealth, the risk-adjusted return of net wealth must be capped, which implies that the risk-adjusted return of tax evasion and the marginal productivity of capital should not exceed a threshold value, and that the cost of borrowing abroad cannot be too low. Condition a) implies that the share of fiscal revenues that is diverted is not infinite, if the risk-adjusted return of corruption is capped.*

4.3. Invariant density function for the external debt-to-fiscal revenues ratio

Theorem 4.3. *Let $f(\lambda)$ be the invariant steady-state density function of $\lambda(t)$. Assume that $T > 0, W > 0$ and that the conditions a) and b) of Theorem 4.2 hold. Then, $f(\lambda)$ for the SDE of the wealth-to-fiscal revenues ratio*

$$d\lambda(t) = \Omega_1(\lambda)dt + \Omega_2^{\frac{1}{2}}(\lambda)dB(t),$$
$$\Omega_1 = K_2^*\lambda = \lambda\Big[\tilde{n}_g^2(\lambda) + \tilde{n}_C^2(\lambda) - \tilde{n}_g(\lambda) - \tilde{n}_C(\lambda) + \big[1 - \tau + \bar{x}\tau\tilde{e}(\lambda)\big]\tilde{n}_y(\lambda) - CPI\tfrac{\tilde{C}}{W} + \tilde{n}_k(\lambda)r_k - (i^* + \pi)\tilde{n}_f(\lambda)\Big],$$
$$\Omega_2 = K_3^*\lambda^2 = \lambda^2\Big[\tilde{n}_g^2(\lambda) + \tilde{n}_C^2(\lambda) + \sigma^2\tau^2\tilde{e}^2(\lambda)\tilde{n}_y^2(\lambda) + \tilde{n}_k^2(\lambda) + \tilde{n}_f^2(\lambda)\Big]$$

is

$$\begin{array}{rcl} f(\lambda) & = & C_0\, m(\lambda) = \frac{1-2\frac{K_2^*}{K_3^*}}{\tilde{\lambda}}\big(\frac{\lambda}{\tilde{\lambda}}\big)^{-2(1-\frac{K_2^*}{K_3^*})} \\ & = & \frac{\alpha-1}{\tilde{\lambda}}\big(\frac{\lambda}{\tilde{\lambda}}\big)^{-\alpha}, \end{array}$$

$\alpha = 2(1 - K_2^*/K_3^*)$, $\tilde{\lambda}$ is an arbitrary λ defined as the lower bound from which the power-law holds.

Proof.

We compute the speed density function as

$$m(\lambda) = \frac{exp(2J(\lambda))}{\Omega_2(\lambda)}, \quad J(\lambda) = \int_{\tilde{\lambda}}^{\lambda} \frac{\Omega_1(u)}{\Omega_2(u)}du, \tag{57}$$

We therefore get

$$\begin{array}{rcl} C_0 m(\lambda) & = & C_0\frac{1}{K_3^*\lambda^2}exp\big[2\int_{\tilde{\lambda}}^{\lambda}\frac{K_2^*u}{K_3^*u^2}du\big] \\ & = & C_0\frac{1}{K_3^*\lambda^2}exp\big[2\int_{\tilde{\lambda}}^{\lambda}\frac{K_2^*}{K_3^*u}du\big] \\ & = & C_0\frac{1}{K_3^*\lambda^2}exp\big[2\frac{K_2^*}{K_3^*}ln(\frac{\lambda}{\tilde{\lambda}})\big] \\ & = & C_0\frac{1}{K_3^*\lambda^2}\big[\frac{\lambda}{\tilde{\lambda}}\big]^{2\frac{K_2^*}{K_3^*}} \\ & = & \frac{C_0\tilde{\lambda}^{-2}}{K_3^*}\big(\frac{\lambda}{\tilde{\lambda}}\big)^{-2(1-\frac{K_2^*}{K_3^*})} \end{array} \tag{58}$$

This corresponds to a power-law with $\alpha = 2(1 - K_2^*/K_3^*)$.

We then calculate the normalizing constant C_0, such that $\int_{\tilde{\lambda}}^{+\infty} f(u)du = 1$. This yields $C_0 = -\tilde{\lambda}(2K_2^* - K_3^*)$. By replacing in the expression above, we get $f(\lambda)$.

■

The μ^{th}-order moment is given by

$$\mathbb{E}(\lambda^\mu) = \int_{\tilde{\lambda}}^{+\infty} u^\mu f(u)du = \frac{\alpha - 1}{\alpha - 1 - \mu}\tilde{\lambda}^\mu. \tag{59}$$

All moments, except the mean, diverge when $2 < \alpha < 3$. This means that they do not converge as more observations are added to the sample and that the distribution has time-varying tails.

In the expression of α, K_2^* and K_3^* refer respectively to the mean and volatility of net wealth. The effect of a change in a parameter of the model on the shape parameter thus depends upon a trade-off between the mean level of wealth and its variability. One cannot exclude to have nonlinear effects depending upon the respective response of the mean and variance component of wealth to changes in the parameters. Rather than perform analytical exercises, we do some simulations of the model.

4.4. Some simulations of the model

We now simulate the model to examine the impact of tax evasion and corruption on the scale parameter α. Since the shape parameter is positive (by definition of a power-law function), the ratio (K_2^*/K_3^*) varies in the interval $(-\infty, 1)$. When (K_2^*/K_3^*) varies from $-\infty$ to 1, α decreases, thereby implying that the pdf of λ will tend to Pareto laws as the ratio diminishes. This ratio can be interpreted as the combined net returns of illegal activities (fraud and corruption) relative to their risk. Since, we have seen that net wealth and debt are negatively correlated, one can understand that, an increase in the returns to illegal activity relative to their risk implies heavier tails in the pdf of the external debt ratio and therefore a higher likelihood of the occurrence of extreme debt events.

We first calibrate the model to have a benchmark situation (baseline scenario). The macroeconomic variables are chosen in such a way to simulate an economy with some plausible macroeconomic characteristics of emerging economies. The aggregate income is assumed to be 1000 billion dollars with aggregate tax revenues of 400 billions. The share of tax revenues over total wealth is 50%. The marginal productivity of capital is 5% and the interest rate on external debt is 7%. Corrupt bureaucrats steal 30% of tax revenues, tax evaders hide 10% of their income. The probability to be caught as a tax fraudulent is one-half, the probability of being a corrupt bureaucrat is 0.5. If they are caught, the consumer/tax evader must pay a penalty of 20% of the hidden income. The values of the other parameters (variances, tax rate, and the parameters computed from the calibrated variables) are given below (see Table 3).

	Shape parameter $\alpha = 2(1 - K_2/K_3)$	**2,80**			
K_2		-9,69		**Characteristics of the Economy**	
K_3		24,12	y	aggregate revenue	1000
	Fiscal evasion		T	aggregate tax revenue	400
e	Share of the household's concealed revenue	0,1	T/W	share of tax revenue on aggregate wealth	0,5
$\bar{x}$	$\mathbb{E}$ of the yield of a concealed unit of taxed revenue	0,4	C/W	$= n_c * T/W$	0,10
	Administration's fight against fiscal evasion		β	coeff. of the production function (on k)	0,5
τ	Tax rate	0,1	μ	Arrow Prat coefficient	**-20**
p	Probability of being caught by the administration	0,5	CPI	CPI index $= P^{1-\eta}$	100
b	Fine HH has to pay if caught by the administration	0,2	r_k	interest rate on capital	0,05
$\bar{g}$	$\mathbb{E}$ of taxes (legally collected)	249,0	r_f	$= i * + \pi$ (interest rate on external debt)	0,07
	Corruption				
theta θ	Share of tax ceased by the corrupted government	0,3	n_g	$g_{bar(t)}/T$	0,8
p1	Probability of the government to be corrupted	0,5	n_c	$C^{G_b ar}/T = 1 - n_g$	0,2
	Volatilities		n_K	K/W given by $n_k^{\beta} = n_y/(n_g T/W)^{1-\beta}$	3,9
σ	variance of the consumer's lottery	0,6	n_f	$PD/W = n_K - 1$	2,9
σ_{Ww}	variance on concealed revenue not ceased by the gvt	1	n_y	$= y/W = yT/W/T$	1,25
σ_{Wk}	variance of the growth of return to capital dR_k	1	n_g^{tilde}		0,63
σ_{Wf}	variance of the growth of i_t on external debt dR_f	1	n_c^{tilde}		0,16

Table 3: Calibration of the parameters for the computation of the shape parameter α
Note: in blue: coefficients or parameters fixed for the calibration
in orange: coefficients or parameters computed

Impact of fighting fiscal evasion on the shape parameter

Fighting against fiscal evasion reduces the frequency of extreme events of external debt (see Figure 4). This happens for a higher probability p of catching those consumers convinced of fraud, or a higher penalty rate b. A corollary result is that tax avoidance have more negative repercussions on foreign debt (with more heavy tails indicating more frequent extreme events), if the evaders take advantage of favorable conditions: deficiencies of administrative surveillance, opaque fiscal environment.

Meanwhile, as the figures suggest, a highly punitive system can create a harmful situation in terms of debt. Indeed, the inverted bell curves (especially for b) suggest that, above a given threshold, there may be a fiscal cost for the government: the higher the penalty rate, the more consumers will try to figure out ways to stop paying their income tax. Such a nonlinear effect is in line with the literature suggesting that there is no compelling evidence of how tax compliance is affected by punishment through more efficient tax audits, or tax penalties (see Alm (2012), Murphy (2008), Slemrod (2007)).

The fact that a severe punishment may increase the way people behave in not fulfilling their tax obligations (thereby implying here a higher foreign debt level) has been motivated by different arguments. First, people do not necessarily report their tax liabilities on the basis of legal obligations, but in good faith according to what they believe to be correct. High penalty rate can be though of as reflecting abusive transactions between the tax administration and tax payers, especially if government officials are known to be corrupted (they respond to tax fraud by ratcheting up penalties). A second explanation is more in line with our model. The penalty regime does affect the taxpayer's behavior. Here the tax evaders pays twice: a fixed penalty of the evaded income plus an interest rate on debt (they internalize the effect of fraud on debt, because this reduces their net wealth). The fact that they bear this additional interest rate cost should persuade the consumers to limit their tax noncompliance below a certain level (if the amount of debt service paid from foreign borrowing increases, their compliance rate is likely to rise). In this context, attempts by the government to coerce and threaten taxpayers into compliance through high penalty rates could undermine the legitimacy of the tax administration's authority.

Impact of corruption on the shape parameter

Figure 5 suggests that more corrupt bureaucrats (increase in p_1) and a higher tax diversion (increase in θ) imply a thiner tail on the pdf of λ (α increases) and therefore more frequent extreme events for the external debt ratio.

The negative harmful effects are, however, reversed when the share of resources stolen becomes higher above a certain threshold. The humped curve representing a nonlinear relationship between α and θ suggests that tax diversion of tax resources can have de-stabilizing or stabilizing effects on foreign indebtedness (stabilizing meaning a lower likelihood of adverse high debt scenarios in the right-hand side of the pdf of the external debt ratio, that is a decrease in α). An increase in θ biases the composition of spending by

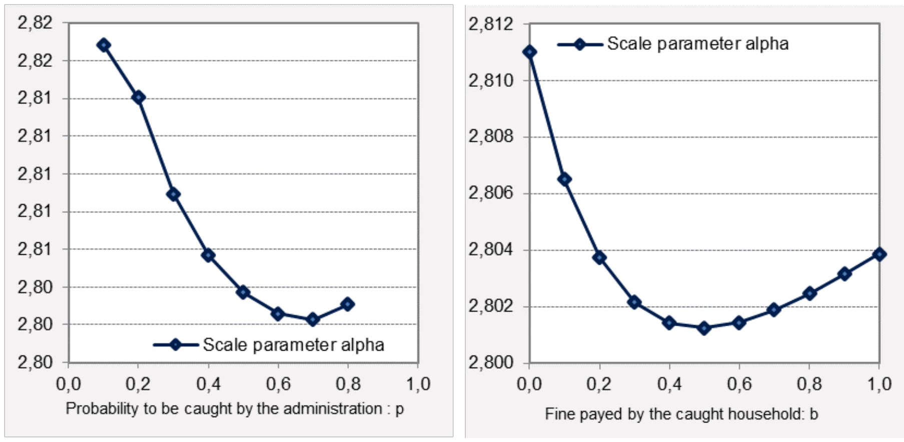


Figure 4: Simulations: impact of changes in p, b on α

raising the share of unproductive spending relative to productive spending. The country needs to borrow more from abroad and the mean net wealth (expressed as share of total taxes) diminishes (portfolio-adjustment effect). But, as θ takes higher values, the expected lifetime path of returns from corruption becomes more decreasing (the bureaucrats exhaust rapidly their corruption opportunity). This contributes to stabilize foreign borrowing.

5. Conclusion

Our intention in this paper has been to suggest that the emerging countries' external debt ratio can well be described by power-law distributions, and to close the gap between the empirical evidence and the theoretical framework within which such power-law functions can be obtained analytically. We consider the role of corruption and tax evasion about which there is still much to learn concerning their implications on the emerging countries' debt crises or debt stress episodes.

Our claim that foreign debt ratio can be the outcome of Pareto laws is obtained rigorously, by using econometric tests rather than simply plotting log-log graphs of the presumed distributions of the data. This is important, since a non-negligible amount of empirical papers estimates shape parameters by assuming that the distribution is a Pareto law, while this assumption needs to be tested formally.

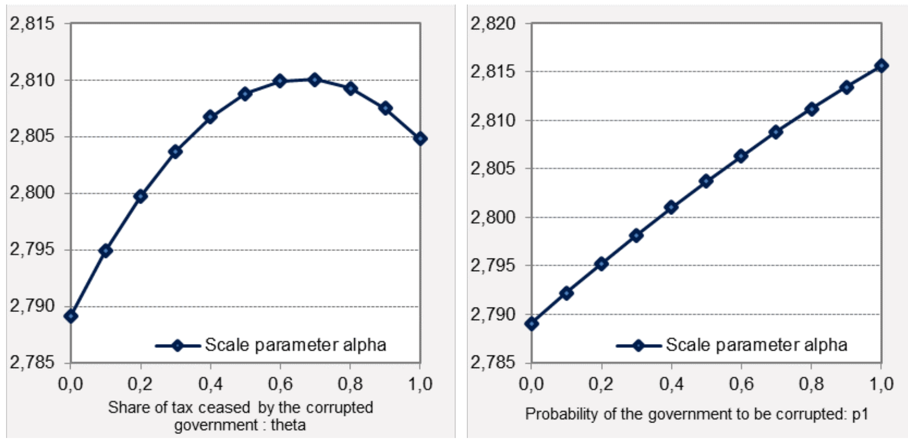


Figure 5: Simulations: impact of changes in θ, p_1 on α

Our purpose was then to provide a simple illustration of how a corrupt economy with tax evasion might explain this empirical observation. Our model incorporates the essential features of emerging economies: foreign aid and borrowing is often a substitute to domestic taxation, bureaucrats officially appointed to make tax audits are sometimes inclined to seek private ways of using the collected tax rather than leaving them available for productive investment, the detection of illegal taxation activities by the governments is not necessarily efficient. Though they are present in the model, for sake of completeness, our analysis is not meant to consider the role of macroeconomic shocks such as terms of trade, supply shocks, interest rates shock on foreign debt. We prefer to focus on the role of endogenous uncertainty, meaning that the tails of the Pareto law do not only depend upon the stochastic components of the diffusion processes, but also on the probability of detecting fraudulent people and their risk-taking behavior, on the probability of facing a corrupt bureaucrat and on the risk-taking behavior of bureaucrats.

This paper could be extended by investigating other distributions that are likely to characterize foreign debt, notably the family of upper incomplete Gamma distributions. Enriching the class of distributions for external debt would help in the detection of early warning stress debt episodes in the emerging countries.

Appendix A. Government's optimal choice

This appendix and the next ones present how the government's and household's optimal choices are calculated. The methodology is based on techniques used in continuous time optimization models (see for instance Chang (2004), Turnovsky (2000)).

The government's choice is as follows:

$$\max_{\bar{C^G}(t)} = \mathbb{E}\int_0^{+\infty} \frac{1}{\gamma}\bar{C^G}(t)^{\gamma} e^{-\rho t} dt \tag{A.1}$$

subject to

$$dT(t) = [\frac{\bar{g}(t)}{T(t)} + \frac{\bar{C^G}(t)}{T(t)}]T(t)dt + [\frac{\tilde{g}dW^g(t)}{T(t)} + \frac{\tilde{C^G}dW^{C^G}(t)}{T(t)}]T(t). \tag{A.2}$$

This constraint can be expressed as:

$$\frac{dT(t)}{T(t)} = \psi^T dt + \sigma_{w^T} dw^T, \tag{A.3}$$

with

$$\begin{aligned} \psi^T &= \tfrac{\bar{g}(t)}{T(t)} + \tfrac{\bar{C^G}(t)}{T(t)}, \\ \sigma_{w^T} dw^T &= \tfrac{\tilde{g}dW^g(t)}{T(t)} + \tfrac{\tilde{C^G}dW^{C^G}(t)}{T(t)}. \end{aligned} \tag{A.4}$$

Let us define $n_g = \frac{\bar{g}(t)}{T(t)}$, $n_C = \frac{\bar{C^G}(t)}{T(t)}$, $\tilde{n_g} = \frac{\tilde{g}}{T(t)}$ and $\tilde{n_C} = \frac{\tilde{C^G}}{T(t)}$. We then have:

$$\begin{aligned} \psi^T &= n_g + n_C, \\ \sigma_{w^T} dw^T &= \tilde{n_g} dW^g(t) + \tilde{n_C} dW^{C^G}(t), \end{aligned} \tag{A.5}$$

such that

$$\sigma^2_{w^T} = \tilde{n_g}^2 \sigma^2_{W^g} + \tilde{n_C}^2 \sigma^2_{W^{C^G}}. \tag{A.6}$$

The program becomes

$$\max_{n^G, n^C} = \mathbb{E}\int_0^{+\infty} \frac{1}{\gamma}(Tn_C)^{\gamma} e^{-\rho t} dt \tag{A.7}$$

subject to

$$\begin{aligned} \tfrac{dT(t)}{T(t)} &= \psi^T dt + \sigma_{w^T} dw^T, \\ 1 &= n_g + n_C. \end{aligned} \tag{A.8}$$

The differential generator of the value function V(T,t) is defined by:

$$L\big[V(T,t)\big] \equiv \frac{\partial V}{\partial t} + \psi^T T \frac{\partial V}{\partial T} + \frac{1}{2}\sigma^2_{w^T} T^2 \frac{\partial^2 V}{\partial T^2}. \tag{A.9}$$

We assume V to be of the following time separable form:

$$V(T,t) = e^{-\rho t} X(T). \tag{A.10}$$

And government choses n_C and n_g maximising the following Lagrangian:

$$\text{Lagrangian} \quad = \quad e^{-\rho t}\tfrac{1}{\gamma}(Tn_C)^\gamma + L\big[e^{-\rho t}X(T)\big] + e^{-\rho t}\lambda(1 - n_g - n_C). \tag{A.11}$$

The partial derivative with respect to n_C is:

$$T^\gamma n_C^{\gamma-1} + TX_T - T^2 X_{TT}\sigma^2_{W^C G} n_C = \lambda. \tag{A.12}$$

The partial derivative with respect to n_g is:

$$TX_T - T^2 X_{TT}\sigma^2_{W^g} n_g = \lambda. \tag{A.13}$$

Putting these equations together with $1 = n_g + n_C$ leads to

$$T^\gamma n_C^{\gamma-1} = T^2 X_{TT}\big[\sigma^2_{W^C G} n_C - \sigma^2_{W^g} n_g\big], \tag{A.14}$$

and

$$T^\gamma n_C^{\gamma-1} = T^2 X_{TT}\big[(\sigma^2_{W^C G} + \sigma^2_{W^g})n_C - \sigma^2_{W^g}\big]. \tag{A.15}$$

Besides, the value function must satisfy the Bellman equation

$$\max_{n_C, n_g}\left\{\frac{1}{\gamma}e^{-\rho t}(Tn_C)^\gamma + L\big[e^{-\rho t}X(T)\big]\right\} = 0. \tag{A.16}$$

To solve it, we substitute the optimized value of n_C and n_g to solve the resulting equation in X(T):

$$\tfrac{1}{\gamma}T^\gamma \hat{n_C}^\gamma - \rho X(T) + TX_T + \tfrac{1}{2}\sigma^2_{w^T} T^2 X_{TT} = 0. \tag{A.17}$$

We postulate X(T) of the form

$$X(T) = \delta T^\gamma, \tag{A.18}$$

with δ to be determined. This yields to

$$\begin{array}{rcl} TX_T & = & \gamma X(T), \\ T^2 X_{TT} & = & \gamma(\gamma-1)X(T). \end{array} \tag{A.19}$$

Using this, the Bellman equation becomes:

$$\frac{1}{\gamma}T^{\gamma}\hat{n_C}^{\gamma} - \rho X(T) + \gamma X(T) + \frac{1}{2}\sigma^2_{w^T}\gamma(\gamma-1)X(T) = 0. \tag{A.20}$$

According to (A.15), $(T\hat{n_C})^{\gamma}$ is given by:

$$\begin{array}{rcl} T^{\gamma}n_C^{\gamma-1} & = & T^2 X_{TT}\big[(\sigma^2_{W^{C^G}} + \sigma^2_{W^g})n_C - \sigma^2_{W^g}\big] \\ & = & \gamma(\gamma-1)\big[(\sigma^2_{W^{C^G}} + \sigma^2_{W^g})n_C - \sigma^2_{W^g}\big]X(T). \end{array} \tag{A.21}$$

We can substitute it in (A.20) and divide by $X(T)$:

$$(\sigma^2_{W^{C^G}} + \sigma^2_{W^g})\hat{n_C}^2 - \sigma^2_{W^g}\hat{n_C} + \frac{\rho-\gamma}{1-\gamma} + \frac{1}{2}\sigma^2_{w^T}\gamma = 0, \tag{A.22}$$

which leads to the second-order differential equation with:

$$\triangle = \sigma^4_{W^g} - 4(\sigma^2_{W^{C^G}} + \sigma^2_{W^g})(\frac{\rho-\gamma}{1-\gamma} + \frac{1}{2}\sigma^2_{w^T}\gamma). \tag{A.23}$$

Solutions (if $\triangle$ is positive) are of the form

$$\hat{n_C} = \frac{\sigma^2_{W^g} \pm \sqrt{\triangle}}{2(\sigma^2_{W^{C^G}} + \sigma^2_{W^g})}. \tag{A.24}$$

With $\hat{n_C}$ positive we have

$$\hat{n_C} = \frac{\sigma^2_{W^g} + \sqrt{\sigma^4_{W^g} - 4(\sigma^2_{W^{C^G}} + \sigma^2_{W^g})(\frac{\rho-\gamma}{1-\gamma} + \frac{1}{2}\sigma^2_{w^T}\gamma)}}{2(\sigma^2_{W^{C^G}} + \sigma^2_{W^g})}. \tag{A.25}$$

Appendix B. The consumer's optimal choice

Government and households are assumed to have the same time preference coefficient ρ). The objective function is

$$\max_{C^P, C^M, e, n_K, n_f} = \mathbb{E}\int_0^{+\infty} \frac{1}{\mu}\big[C^P(t)^{\eta}C^M(t)^{1-\eta}\big]^{\mu} e^{-\rho t}dt. \tag{B.1}$$

The household maximizes the intertemporal utility function subject to the constraints given by Equation (34), and with $W(0) = w_0$.

We define the aggregate consumption $C = C^P(t)^\eta C^M(t)^{1-\eta}$.
The consumer price index can be defined as $CPI(t) = P^P(t)^\eta P(t)^{1-\eta}$ which yields to $CPI(t) = P(t)^{1-\eta}$, as the domestic good is the numeraire
We thus have:

$$\frac{dW(t)}{W(t)} = \psi^W dt + \sigma_{z^W} dz^W, \tag{B.2}$$

with

$$\begin{aligned} \psi^W &= \left[1 - \tau + \bar{x}\tau e(t)\right] n_k^\beta n_g^{1-\beta} (\tfrac{T}{W})^{1-\beta} - \tfrac{CPI(t)C(t)}{W} + n_k r_k - (i^* + \pi) n_f, \\ \sigma_{z^W} dz^W &= \sigma\tau e(t) n_k^\beta n_g^{1-\beta} (\tfrac{T}{W})^{1-\beta} dW^W(t) + n_k du_k - n_f du_f, \end{aligned} \tag{B.3}$$

where $n_k = \frac{k}{W}$,$n_g = \frac{g}{T}$ and $n_f = \frac{PD}{W}$ ($0 < n_i < 1$ for $i = k, g, f$).
And we get

$$\sigma_{z^W}^2 = \sigma^2 \tau^2 e(t)^2 n_k^{2\beta} n_g^{2(1-\beta)} (\frac{T}{W})^{2(1-\beta)} \sigma_{W^W}^2 + n_k^2 \sigma_{W^k}^2 + n_f^2 \sigma_{W^f}^2. \tag{B.4}$$

The households' decision is as follows:

$$\max_{C,e,n_K,n_f} \mathbb{E} \int_0^{+\infty} \frac{1}{\mu} C^\mu e^{-\rho t} dt, 0 < \mu < 1 \tag{B.5}$$

subject to

$$\frac{dW(t)}{W(t)} = \psi^W dt + \sigma_{z^W} dz^W, \tag{B.6}$$

$$n_K - n_f = 1, \tag{B.7}$$

$$W(0) = w_0. \tag{B.8}$$

Define V as the value function

$$V(W) = \max_{C,e,n_K,n_f} \mathbb{E} \int_0^{+\infty} \frac{1}{\mu} C^\mu e^{-\rho t} dt. \tag{B.9}$$

Then, the optimal program satisfies the Hamilton Jacobi Bellman equation

$$\rho V(W) = \max_{C,e,n_K,n_f} \tilde{F}(C, e, n_k, n_f) = \max_{C,e,n_K} F(C, e, n_k), \tag{B.10}$$

where

$$F(C,e,n_k) = \frac{1}{\mu}C^{\mu} + V'(W)W\psi^W + \frac{1}{2}V''(W)W^2\sigma^2_{z^W}. \tag{B.11}$$

Using the fact that B.7 which implies $n_f = n_K - 1$, we get the following necessary conditions:

$$\frac{\partial F(.)}{\partial C} = C^{\mu-1} - CPIV'(W) = 0, \tag{B.12}$$

$$\begin{aligned}\frac{\partial F(.)}{\partial e} &= \bar{x}\tau n_k^{\beta} n_g^{1-\beta}(\tfrac{T}{W})^{1-\beta} WV'(W) \\ &+\sigma^2\tau^2 e(t) n_k^{2\beta} n_g^{2(1-\beta)}(\tfrac{T}{W})^{2(1-\beta)}\sigma^2_{W^W}W^2V''(W) = 0,\end{aligned} \tag{B.13}$$

$$\begin{aligned}\frac{\partial F(.)}{\partial n_k} &= \Bigg[\beta\big[1-\tau+\bar{x}\tau e(t)\big]n_k^{\beta-1}n_g^{1-\beta}(\tfrac{T}{W})^{1-\beta} + r_k - (i^*+\pi)\Bigg]WV'(W) \\ &+\Bigg[\beta\sigma^2\tau^2 e(t)^2 n_k^{2\beta-1}n_g^{2(1-\beta)}(\tfrac{T}{W})^{2(1-\beta)}\sigma^2_{W^W} + n_k\sigma^2_{W^k} + (n_k-1)\sigma^2_{W^f}\Bigg]W^2V''(W) = 0.\end{aligned} \tag{B.14}$$

F has an extremum $(\tilde{C}, \tilde{e}, \tilde{n_k})$ defined such as to verify the last three equations. From the first two variables we obtain:

$$\tilde{C} = (CPIV'(W))^{\frac{1}{\mu-1}}, \tag{B.15}$$

$$\tilde{e} = \frac{\bar{x}\tau\tilde{n_k}^{\beta}n_g^{1-\beta}(\frac{T}{W})^{1-\beta}}{AR(W)(\sigma\tau\tilde{n_k}^{\beta}n_g^{1-\beta}(\frac{T}{W})^{1-\beta}\sigma_{W^W})^2}, \tag{B.16}$$

with $AR(W)$ being the Arrow Prat relative risk coefficient defined by $AR(W) = \frac{-WV''(W)}{V'(W)}$.
Assuming V has the form $V(W) = \delta W^{\mu}$, where δ is a constant, we get $AR(W) = (1-\mu)$ constant.
The third equation leads to

$$\begin{aligned}&\beta\big[1-\tau+\bar{x}\tau\tilde{e}\big](\tfrac{T}{W})^{1-\beta}WV'(W)n_g^{1-\beta}n_k^{\beta} \\ &+\beta\sigma^2\tau^2\tilde{e}^2(\tfrac{T}{W})^{2(1-\beta)}\sigma^2_{W^W}W^2V''(W)n_g^{2(1-\beta)}n_k^{2\beta} \\ &+\big[n_k^2(\sigma^2_{W^k}+\sigma^2_{W^f}) - n_k\sigma^2_{W^f}\big]W^2V''(W) \\ &+\big[r_k-(i^*+\pi)\big]WV'(W)n_k = 0.\end{aligned} \tag{B.17}$$

Define n_y as the share of production over total wealth, that is $n_y = \frac{Y}{W}$. We then have $n_g^{1-\beta} n_k^{\beta} (\frac{T}{W})^{1-\beta} = n_y$, which leads to rewrite the condition as follows

$$\begin{aligned} &\beta\big[1-\tau+\bar{x}\tau\tilde{e}\big]WV'(W)n_y \\ &+\beta\sigma^2\tau^2\tilde{e}^2\sigma^2_{W^W}W^2V''(W)n_y^2 \\ &+n_k^2(\sigma^2_{W^k}+\sigma^2_{W^f})W^2V''(W) \\ &+n_k\Big[\big[r_k-(i^*+\pi)\big]WV'(W)-\sigma^2_{W^f}W^2V''(W)\Big]=0, \end{aligned} \tag{B.18}$$

or

$$\begin{aligned} &(1-\mu)(\sigma^2_{W^k}+\sigma^2_{W^f})n_k^2 \\ &+\Big[-\big[r_k-(i^*+\pi)\big]-(1-\mu)\sigma^2_{W^f}\Big]n_k \\ &-\beta\big[1-\tau+\bar{x}\tau\tilde{e}\big]n_y+(1-\mu)\beta\sigma^2\tau^2\tilde{e}^2\sigma^2_{W^W}n_y^2=0. \end{aligned} \tag{B.19}$$

This is a second-order differential equation in n_k. The discriminant is

$$\begin{aligned} \triangle \;=\; &\Big[r_k-i^*-\pi+(1-\mu)\sigma^2_{W^f}\Big]^2 \\ &-4(1-\mu)(\sigma^2_{W^k}+\sigma^2_{W^f})\big[(1-\mu)\beta\sigma^2\tau^2\tilde{e}^2\sigma^2_{W^W}n_y^2-\beta\big[1-\tau+\bar{x}\tau\tilde{e}\big]n_y\big]. \end{aligned} \tag{B.20}$$

Considering the solutions for $\triangle > 0$, we obtain:

$$n_k^{1,2} = \frac{r_k-(i^*+\pi)+(1-\mu)\sigma^2_{W^f}\pm\sqrt{\triangle}}{2(1-\mu)(\sigma^2_{W^k}+\sigma^2_{W^f})}. \tag{B.21}$$

It is easy to see that among both solutions, the following one satisfies the condition $n_k > 0$:

$$n_k = \frac{r_k-(i^*+\pi)+(1-\mu)\sigma^2_{W^f}+\sqrt{\triangle}}{2(1-\mu)(\sigma^2_{W^k}+\sigma^2_{W^f})}. \tag{B.22}$$

The household's optimal choice is therefore described by the following system :

$$\begin{cases} \tilde{C} = (\delta\mu)^{\frac{1}{\mu-1}}P^{\frac{1-\eta}{\mu-1}}W \\ \tilde{e} = \frac{\bar{x}\tau\tilde{n_k}^{\beta}n_g^{1-\beta}(\frac{T}{W})^{1-\beta}}{(1-\mu)(\sigma\tau\tilde{n_k}^{\beta}n_g^{1-\beta}(\frac{T}{W})^{1-\beta}\sigma_{W^W})^2} \\ n_k = \frac{r_k-(i^*+\pi)+(1-\mu)\sigma^2_{W^f}+\sqrt{\triangle}}{2(1-\mu)(\sigma^2_{W^k}+\sigma^2_{W^f})} \\ n_f = n_k - 1 \end{cases}. \tag{B.23}$$

Appendix C. Transversality conditions

The agents' choices must also satisfy the transversality condition. We consider this condition for the consumer (the proof is similar for the government).

For the constant elasticity utility function, the transversality condition is given by:

$$\lim_{T\to\infty} \mathbb{E}\big[W(T)^{\mu}e^{-\rho T}\big] = 0. \tag{C.1}$$

The stochastic differential equation in W is

$$dW(t) = \psi^{W} W(t)dt + \sigma_{z^W} W(t)dz^{W}(t). \tag{C.2}$$

$\psi^{W(t)}$ and $\sigma_{z^{W(t)}}$ (defined by Equations (35) and (36)) converge to constant terms when when $t \to \infty$, so we omit t. We first compute the solution of C.2 for $W(0) = w_0$ (initial condition of wealth), given.

We rewrite C.2 as follows

$$\frac{dW(t)}{W(t)} = \psi^{W} dt + \sigma_{z^W} dz^{W}(t). \tag{C.3}$$

Integrating this equation between 0 and t gives

$$\int_0^t \frac{dW(u)}{W(u)} = \int_0^t \psi^{W}\, dt + \int_0^t \sigma_{z^W}\, dz^{W}(t) = \psi^{W} t + \sigma_{z^W} z^{W}(t). \tag{C.4}$$

We use Itô's formula:

$$df(t, W(t)) = \frac{\partial f(t, W(t))}{\partial t}dt + \frac{\partial f(t, W(t))}{\partial W}dW(t) + \frac{1}{2}\frac{\partial^2 f(t, W(t))}{\partial W^2}(dW(t))^2. \tag{C.5}$$

Taking $f(t, W(t)) = f(W) = ln(W)$, we obtain

$$d\, ln(W(t)) = \frac{dW(t)}{W(t)} - \frac{1}{2}\Big[\frac{dW(t)}{W(t)}\Big]^2. \tag{C.6}$$

Since $\big[dz^{W}(t)\big]^2 = dt$, $dt^2 = 0$ and $dz^{W}(t).dt = 0$, we have

$$\Big[\frac{dW(t)}{W(t)}\Big]^2 = \big[\psi^{W} dt + \sigma_{z^W} dz^{W}(t)\big]^2 = \sigma_{z^W}^2\, dt. \tag{C.7}$$

Thus,

$$d\,ln(W(t)) = \frac{dW(t)}{W(t)} - \frac{\sigma^2_{z^W}}{2}dt. \tag{C.8}$$

And integrating between 0 and t, we have

$$ln(W(t)) - w_0 = \int_0^t \frac{dW(u)}{W(u)} - \frac{\sigma^2_{z^W}}{2}t. \tag{C.9}$$

Replacing by the expression in C.4, we get

$$\frac{ln(W(t))}{w_0} + \frac{\sigma^2_{z^W}}{2}t = \int_0^t \frac{dW(u)}{W(u)} = \psi^W t + \sigma_{z^W} z^W(t), \tag{C.10}$$

and thus

$$W(t) = w_0\, exp\big[(\psi^W - \frac{\sigma^2_{z^W}}{2})t + \sigma_{z^W} z^W(t)\big], \tag{C.11}$$

which yields to

$$W(t)^\mu\, exp(-\rho t) = w_0^\mu\, exp\big[(\mu\psi^W - \frac{\mu\sigma^2_{z^W}}{2} - \rho)t + \mu\sigma_{z^W} z^W(t)\big]. \tag{C.12}$$

This is a geometric brownian motion. Assuming $z^W(t)$ is independent of w_0, one of the properties of such a motion is that,

$$\mathbb{E}\big[W(t)^\mu\, exp(-\rho t)\big] = \mathbb{E}\big[w_0^\mu\big]\, exp\big[(\mu\psi^W - \rho)t\big]. \tag{C.13}$$

In the end, the transversality condition can be rewritten as

$$\lim_{T\to\infty} \mathbb{E}\big[w_0^\mu\big]\, exp\big[-(\rho - \mu\psi^W)T\big] = 0. \tag{C.14}$$

This puts a lower bound on the rate of impatience ρ, since the above condition is satisfied for $\rho > \mu\psi^W$.

For the government, the transversality condition is obtained in a similar way and implies that $\rho > \gamma\psi^T$ with psi^T is defined by Equation(24).

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