# Mutual rankings

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**Abstract**

This paper analyzes some ranking methods in two-sided settings through their axiomatization. In these settings, there are two sets (the sides) and each member of one side evaluates each member of the other side. Such settings with mutual evaluations abound, for instance between buyers and sellers, students and teachers, or individuals and clubs.

**Keywords** ranking, scores, two-sided, overlapping groups, mutual evaluation systems, bipartite graph.

**JEL** D71, D85.

## 1 Introduction

The enormous increase in interest in ranking methods based on data is clear in academic life, where students, researchers, universities are graded, or on Internet, where Web pages are ranked, movies and books are rated. Rankings may rely on a simple count, as the Science Citation Index, which adds up citations, or on more sophisticated computations, as in the recursive methods on which Google PageRank is based. A large number of situations involve two sides, each providing data on the other side. For example, buyers rate sellers and vice versa on eBay, or students evaluate their teachers and teachers grade them. The objective of this paper is to analyze some ranking methods in these *two-sided* settings through their axiomatization.

In a two-sided setting, there are two sets (the sides), $M$ and $N$, and the members of one side provide *evaluations* on the other side's members. A ranking method assigns *scores* to the members of each side based on these mutual evaluations. The vector of scores on a side, say $M$, is called the *ranking* of $M$. The ranking is defined up to a multiplicative factor, meaning that it assigns relative strengths to the elements in $M$.[2]

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[2]The ranking thus not only orders the elements of $M$ but also assigns (relative) values to them, as the

The ranking on one side naturally depends on the evaluations on its members, say the ranking on $M$ depends on the evaluations provided by $N$. It may also depend on their own evaluations on $N$ if these convey information on the relevance of the evaluations provided by $N$'s members. Consider students and teachers for example. The teachers' scores depend on the students' evaluations on them. One may also think that the evaluations provided by a highly-graded student are more relevant than those by a low-graded one. In that case, the teachers' scores should depend not only on how students evaluate them but also on how teachers grade the students. This introduces a feedback between the evaluations of each side. Whether this feedback is desirable is an important feature in the design of a ranking method.

I characterize a method, called *mutual centrality method*, that takes this feedback into account. The method relies on the dominant eigenvector of the square matrix that stacks the evaluations. The characterization involves two axioms. The first one, called *additivity under unanimity*, pertains to the situation where all members of one side are unanimous on their evaluations on the other side. The second one, called *consistency across sides*, requires that replacing the evaluations on the agents of one side by their assigned scores does not affect the rankings. The two axioms are shown to be independent.

In some contexts, one may want the rankings to be insensitive to the total of the evaluations provided by each agent, a property called *intensity-invariance*. Considering the students-teachers example, a teacher may be much more indulgent than another one. To avoid the degree of indulgence to influence the rankings of the students or of the teachers, a solution is to scale the grades of each teacher so that they sum (say) to one before computing the method. This results in an intensity-invariant 'version' of the mutual centrality method. The resulting method is characterized by intensity-invariance together with the two previous axioms properly restricted to scaled evaluations.

The feedback between the two sides' evaluations may be thought as non desirable in some situations. A method that excludes this feedback is said to be *impartial across sides*. The axiom requires the ranking on the members of one side to be independent of the evaluations they provide on the other side, say the ranking of the students does not depend on how they evaluate the teachers. Impartiality can be justified by strategic issues: if $i$'s evaluations on the other side do not impact $i$'s score, then $i$ has no incentive to lie.[3] Impartial methods decompose into two methods, each one specifying the ranking on one side as a function

impact factors or PageRank. The term 'ranking' might not be appropriate though this is a rather standard terminology.

[3]Though, this does not count for reputation effects. On eBay for example, there is evidence that buyers refrain from posting bad ratings in fear of retaliation.

of the sole evaluations provided by the other side on them. The counting method, which assigns scores equal to the totals of the received evaluations, is a prominent impartial method. I characterize it through two axioms, uniformity and homogeneity, in addition to impartiality. Another impartial method, called *congruence method*, accounts not only on the received evaluations' totals but also on the joint pattern of these evaluations. The method reflects the idea that a concentration of good evaluations by a group of evaluators is a good signal, hence reinforce each other. The ranking on one side is defined by a dominant eigenvector of the evaluation matrix multiplied by its transpose. The congruence ranking can be shown to minimize the distance in the joint evaluations between the observed matrix and that associated to a unanimous ranking.

This paper is related to recent studies that have characterized ranking methods based on evaluations or citations. These studies consider *one-sided* or *peers'* settings. In a one-sided setting, a set of 'experts' provide evaluations on the items to be ranked. The peers' setting obtains when the experts coincide with the items, that is, when everybody evaluates every other agent as, for example, journals citing each other or Web pages linking toward other pages.

In the peers' setting, the eigenvalue centrality method defines the ranking to be the dominant eigenvector of the matrix of evaluations. Several axiomatizations of eigencentrality have been provided (Palacios-Huerta and Volij 2004, Slutzki and Volij 2006 for its intensity-invariant version, Altman and Tennenholtz 2005 for its ordinal version, and Kitti 2016). The mutual centrality method is related to eigenvalue centrality since it is built on the dominant eigenvector of a matrix (it does not coincide with it due to the independence of the rankings between the two sides). The two-sided setting allows for a different characterization.

Several axioms used in this paper appeared in the literature in various forms and contexts. The consistency 'principle' underlies studies in fair division, bankruptcy problems (Young 1987), apportionment problems (Balinski and Demange 1989 where it is called uniformity) or cooperative games (Hart and Mas-Colell 1989) for example. As explained in Thomson (1990), the principle requires that changing part of the data of a problem by the solution a method gives to them does not lead to a contradiction, i.e., does not lead to reassess the solution for the others. Consistency across sides requires the principle to apply when the data on a side is changed by its solution. Uniformity and homogeneity are used to characterize the counting method in the one-sided setting (Demange 2014), and here in the two-sided setting under impartiality across sides. The intensity-invariance property is widely used in peers' settings. In the context of ranking Websites for example, in which the numbers of links from various sites vary widely, PageRank deflates a link from a site by the total number of links from that site, thereby preventing sites to increase their score by inflating the number of links to sites that point to them. In the context of economic journals, in which citations intensities substantially differ across fields, Palacios-Huerta and Volij (2004) show that factoring out intensity has a non-negligible impact on the ranking. In the one-sided setting, the handicap-based method is shown to be the counterpart of the counting method when intensity-invariance is required or when evaluations represent shares (Demange 2014). Finally, Clippel, Moulin, and Tideman (2008) introduce the impartial property in the context of 'division of a dollar', in which everybody recommends the (relative) share that every other agent should receive (thus a peers' setting). Impartiality requires that the share of any agent is determined exclusively by the reports of other agents. The property is extended to a two-sided setting by requiring that the score of any agent is determined exclusively by the evaluations provided by the other side.

The paper is also related to the studies on affiliation networks, sometimes referred to as two-modes networks, conducted in sociology (see e.g. Borgatti and Everett (1997) and Faust (1997) for discussions on these networks). An affiliation network describes the membership of individuals to groups, clubs, or boards. Analyzing the network may help to uncover underlying friendship or power relationships from the pattern of common membership (overlap between two individuals' participation) and common attendance (overlap between two groups). For example, studies on 'interlocking directorates' consider the participation of business leaders to the boards of different companies, aiming at detecting possible alliance strategies from the overlap of the boards, starting with Mintz and Schwartz (1981). An affiliation network is represented by a bipartite graph in which one side is composed of the individuals, the other side is composed of the groups and an edge connects an individual $i$ and a group $\lambda$ in this graph if $i$ is a member of $\lambda$. Bonacich in (1972) and (1991) proposes to simultaneously assign group and individual centralities to the adjacency matrix associated to the bipartite graph. This matrix is symmetric and has elements either equal to 1 to represent membership of $i$ to $\lambda$ or 0 otherwise. This paper extends this approach to more general weighted and possibly directed relations between two sides by defining and characterizing the mutual centrality method. Salonen (2015) and (2016) also proposes to extend Bonacich approach and relates the ranking to the equilibria in a noncooperative games or to minimum norm solutions.[4]

The paper is organized as follows. Section 2 presents the framework, Section 3 introduces and characterizes the mutual centrality method and its intensity-invariant version. It also introduces the 'competitive' counting method and shows the independence of the used

[4]I thank a referee to point out these works to me.

axioms. Section 4 defines impartial methods and characterizes the counting and congruence methods.

## 2 Ranking methods in two-sided settings

### 2.1 The framework

In a two-sided setting, two distinct sets of entities (or 'sides'), denoted by $M$ and $N$, interact and provide data on each other, called *evaluations*. An entity may represent an individual, a firm, a school or group, and an evaluation can be interpreted in different ways depending on the context as, say, an application, a recommendation, a grade, or an affiliation, as illustrated in the examples in Section 2.2.

In the following, the members of $M$ and $N$ are called agents, keeping in mind the different interpretations. Members of $M$ are indexed by roman letters, generic $i$, and those of $N$ by greek ones, generic $\lambda$. Let $m$ denote the cardinality of $M$ and $n$ the cardinality of $N$. Each member of one side provides evaluations on each member of the opposite side. For example each $i$ in $M$ provides evaluation $v_{\lambda i}$ on each $\lambda$ in $N$. All the evaluations by $M$ are represented by the $n \times m$ matrix $\boldsymbol{v} = (v_{\lambda i})$, in which $i$'s column represents $i$'s evaluations on the members of $N$. Row $\lambda$ thus represents all the evaluations on $\lambda$. Similarly, the evaluations by the members of $N$ on those of $M$ are described by the $m \times n$ matrix $\boldsymbol{u} = (u_{i\lambda})$, in which $\lambda$'s column represents $\lambda$'s evaluations on the members of $M$.

The data of a problem is thus summarized by the pair of evaluation matrices $(\boldsymbol{u}, \boldsymbol{v})$. The pair satisfies the constraints associated with the setting. The first example in Section 2.2 considers affiliation networks, in which $\boldsymbol{v}$ is the transpose of $\boldsymbol{u}$, denoted by $\widetilde{\boldsymbol{u}}$, whereas in the second and third examples, $\boldsymbol{v}$ is not a priori related to $\boldsymbol{u}$ so that any non-negative pair $(\boldsymbol{u}, \boldsymbol{v})$ is feasible. Let $\mathcal{F}$ denote the feasible set.

Given evaluations $(\boldsymbol{u}, \boldsymbol{v})$ in $\mathcal{F}$, one seeks to evaluate the members of $M$ and $N$ by assigning them non-negative *scores*. Let $r_i$ denote $i$'s score and $s_\lambda$ denote $\lambda$'s score. The scores provide the relative strength of the members within each side, $M$ or $N$. This reflects two properties. First, the members of $M$ and $N$ are not comparable between each other, so comparing their scores, i.e., the $r_i$ with the $s_\lambda$, has no meaning. Second, within a side, not only the order of the scores matters but also their values, up to a multiplicative constant. Therefore, calling $\mathbf{r} = (r_i)$ the *ranking* on $M$, all the vectors proportional to $\mathbf{r}$, $\mathbf{r}' = k\mathbf{r}$ for $k > 0$, written as $\mathbf{r}' \propto \mathbf{r}$, provide the same ranking on $M$. It is often convenient to choose the (unique) *unit* vector whose components sum to one. Similarly $\mathbf{s} = (s_\lambda)$ is a ranking on $N$ and all $\mathbf{s}' \propto \mathbf{s}$ provide the same ranking. A ranking method assigns a ranking on each side to each feasible problem $(\boldsymbol{u}, \boldsymbol{v})$. Formally,

**Definition 1** *Given $M$, $N$ and the set $\mathcal{F}$ of feasible pairs of evaluation matrices, a ranking method $F$ assigns to each feasible pair $(\boldsymbol{u}, \boldsymbol{v})$ a ranking $\mathbf{r} = F_M(\boldsymbol{u}, \boldsymbol{v})$ on $M$ and a ranking $\mathbf{s} = F_N(\boldsymbol{u}, \boldsymbol{v})$ on $N$.*

As an example, the **counting** method assigns scores proportional to the received evaluations' totals. Given a matrix $\boldsymbol{u}$, $u_{i+}$ denotes the total in row $i$, $u_{\lambda+} = \sum_{\lambda \in M} u_{i\lambda}$. (Similar notation is used for any matrix). The counting rankings are

$$\mathbf{r} \propto (\sum_{\lambda \in N} u_{i\lambda})_{i \in M} \text{ and } \mathbf{s} \propto (\sum_{i \in M} v_{\lambda i})_{\lambda \in N}. \tag{1}$$

Other methods are introduced in the next sections.

Finally, it is sometimes convenient to view a problem $(\boldsymbol{u}, \boldsymbol{v})$ as a weighted, possibly directed, bipartite graph. Recall that a bipartite graph is a graph whose vertices can be divided into two disjoint sets $M$ and $N$ such that every edge connects a vertex in $M$ to one in $N$ or a vertex in $N$ to one in $M$. Define the bipartite graph with edge $(\lambda, i)$ if $u_{i\lambda} > 0$ and weight $u_{i\lambda}$ and edge $(i, \lambda)$ if $v_{\lambda i} > 0$ and weight $v_{\lambda i}$; when $\boldsymbol{v}$ is equal to $\widetilde{\boldsymbol{u}}$, the transpose of $\boldsymbol{u}$, the graph is 'undirected'. Conversely any weighted directed bipartite graph is associated to a problem.

## 2.2 Examples

Let us illustrate the setting with three examples.

1. **Affiliation networks**. These networks represent the participation of individuals to events or their affiliations to clubs. $M$ describes the set of individuals and $N$ the set of events or clubs. For example, the studies on interlocking directorates referred to in the introduction consider the participation of business leaders to the boards of different companies, in which $M$ is the set of business leaders and $N$ is the set of boards of various companies.

   An affiliation network is represented by an undirected bipartite graph. The evaluations $\boldsymbol{u}$ and $\boldsymbol{v}$ are defined by setting $u_{\lambda i}$ and $v_{i\lambda}$ both equal to 1 if $i$ is a member of $\lambda$ and both to 0 otherwise. These evaluations reflect the fact that the affiliation of $i$ to $\lambda$ represents a mutual agreement: to be affiliated with $\lambda$, $i$ must be accepted by $\lambda$ and be willing to be a member of $\lambda$. More generally, allowing for a 'weighted' intensity for the affiliation between $i$ and $\lambda$, $v_{\lambda i} = u_{i\lambda}$ any nonnegative number, an affiliation network has symmetric or dual evaluations: $\boldsymbol{v}$ is equal to the transpose $\widetilde{\boldsymbol{u}}$ of $\boldsymbol{u}$.

2. **Mutual applications.** Consider the situation in which the members on one side express their preferences on the other side before an assignment. In the academic world for example, the students are asked to weigh the universities and each university provides grades on the students. The evaluations here are respectively the students weights and the universities grades. The students' evaluations are likely to differ because their distance to universities or their preferred specializations differ. Similarly, the universities' evaluations are likely to differ: even if universities have access to the same information on the students, say the results of a general final exam, they may weigh differently the fields. Moreover the evaluations are likely not to be dual. Consider for example the case where the evaluations are 1 or 0, where a student's evaluation is 1 on a university to which she applies and a university's evaluation is 1 on a student it finds acceptable. A student who applies to a university may not be found acceptable by it and a student who is acceptable to a university may not apply to it: $\boldsymbol{v}$ typically differs from $\widetilde{\boldsymbol{u}}$. Here, a student's score is meant to capture how much she is demanded relative to other students, and similarly for universities.[5]

3. **Mutual recommendations/grades.** Recommendations on exchanges are more and more widespread. For example, buyers and sellers rate their transactions on eBay. In what follows, it is assumed that there is a recommendation of each $i$ on each $\lambda$ and vice versa. In practice, this assumption is not necessarily met at the unit level. One can assign the average of the recommendations or use techniques to group similar users and fill the missing data, as used by Netflix recommendation system. The mutual evaluations of students and teachers fit this example as well. This case illustrates why one may want to consider intensity-invariant methods (see Section 3.2). A teacher may be severe while another one is indulgent, meaning that the average of the former is much lower than the average of the latter. An intensity-invariant method 'normalizes' the evaluations, hence is insensitive to the degree of indulgence.

# 3 Mutual centrality

This section introduces and characterizes the mutual centrality method through two axioms. It also introduces the 'competitive' counting method and shows that the axioms are independent. Finally, Section 3.2 defines and characterizes an intensity-invariant version of the mutual centrality method.

[5]Note that the problem of determining the scores differs from that of the final assignment of students to universities.

### 3.1 Defining and characterizing the mutual centrality method

In a peers' setting, eigenvector centrality considers that a node is important if it is pointed to by other important nodes. This produces the ranking given by the dominant eigenvector of the evaluation matrix. In the two-sided setting, mutual centrality applies a similar principle. Though, as the sides are not comparable, i.e., as each ranking on $M$ and $N$ is defined up to a scalar, eigenvector centrality has to be adapted.

Let us first consider the square $(m+n)\times(m+n)$ matrix gathering the evaluations: $\boldsymbol{\pi} = \begin{pmatrix} 0 & \boldsymbol{u} \\ \boldsymbol{v} & 0 \end{pmatrix}$. Assume $\boldsymbol{\pi}$ irreducible (the condition is interpreted below). Due to Perron-Frobenius theorem, $\boldsymbol{\pi}$ has a positive eigenvector, unique up to a scalar, associated to its largest eigenvalue, which is positive. Such an eigenvector is called dominant. Denoting by $\rho$ the largest eigenvalue of $\boldsymbol{\pi}$, a dominant eigenvector $(\mathbf{r}, \mathbf{s})$ satisfies

$$r_i = \frac{1}{\rho}\sum_{\lambda\in N} u_{i\lambda}s_\lambda \text{ for each } i \in M \text{ and } s_\lambda = \frac{1}{\rho}\sum_{i\in M} v_{\lambda i}r_i \text{ for each } \lambda \in N. \tag{2}$$

The scores given by $\mathbf{r}$ and $\mathbf{s}$ exhibit a *mutually reinforcing relationship.* Considering mutual recommendations between sellers and buyers for example, these scores say that a 'good' buyer -one with a high score- is one who is highly evaluated by many good sellers and a good seller is one who is highly evaluated by many good buyers. (This type of justification appears in affiliation networks, as described at the end of this section.)

As the rankings on $M$ and $N$ are each defined up to a scalar, mutual centrality assigns to $M$ the rankings proportional to $\mathbf{r}$ and to $N$ those proportional to $\mathbf{s}$. To define and interpret the method, the next Lemma is useful. Let $\mathcal{I}$ denote the set of evaluations $(\boldsymbol{u}, \boldsymbol{v})$ such that $\boldsymbol{\pi}$ is irreducible. As shown below, the irreducibility of $\boldsymbol{\pi}$ is equivalent to that of both $\boldsymbol{uv}$ and $\boldsymbol{vu}$.

**Lemma 1** *Given $(\boldsymbol{u}, \boldsymbol{v})$ in $\mathcal{I}$ the following properties on positive vectors $\mathbf{r}$ and $\mathbf{s}$ are equivalent:*

*(i) $\mathbf{r}$ and $\mathbf{s}$ are respectively dominant eigenvectors of $\boldsymbol{uv}$ and $\boldsymbol{vu}$*

*(ii) $\mathbf{r}$ is a dominant eigenvector of $\boldsymbol{uv}$ and $\mathbf{s} \propto \boldsymbol{v}\mathbf{r}$*

*(iii) There is positive $k$ for which $(\mathbf{r}, k\mathbf{s})$ is a dominant eigenvector of $\boldsymbol{\pi}$.*

*Furthermore the dominant eigenvalues of $\boldsymbol{uv}$ and $\boldsymbol{vu}$ are identical, equal to $\rho^2$.*

**Proof** In what follows let us denote by $\sigma$ the dominant eigenvalue of $\boldsymbol{uv}$.

(i) is equivalent to (ii). Take $\mathbf{r}$ a positive dominant eigenvector of $\boldsymbol{uv}$: $\boldsymbol{uv}\mathbf{r} = \sigma\mathbf{r}$. Multiplying by $\boldsymbol{v}$, we obtain $\boldsymbol{vuv}\mathbf{r} = \sigma\boldsymbol{v}\mathbf{r}$. Thus $\boldsymbol{v}\mathbf{r}$ is a positive eigenvector of $\boldsymbol{vu}$ associated to $\sigma$. This implies that $\sigma$ is the dominant eigenvalue of $\boldsymbol{vu}$ and, by irreducibility of $\boldsymbol{vu}$, that the dominant eigenvectors of $\boldsymbol{vu}$ are all proportional to $\boldsymbol{v}\mathbf{r}$. This proves the equivalence between (i) and (ii).

(iii) implies (i). Assume (iii). Take $(\mathbf{r}, k\mathbf{s})$ a dominant eigenvector of $\boldsymbol{\pi}$. This writes: $k\boldsymbol{u}\mathbf{s} = \rho\mathbf{r}$ and $\boldsymbol{v}\mathbf{r} = k\rho\mathbf{s}$. Multiply the second equation by $\boldsymbol{u}$: $\boldsymbol{uv}\mathbf{r} = k\rho\boldsymbol{u}\mathbf{s}$; using the first equation, we obtain $\boldsymbol{uv}\mathbf{r} = \rho^2\mathbf{r}$. Hence $\mathbf{r}$ is a dominant eigenvector of $\boldsymbol{uv}$ (since $\mathbf{r}$ is a positive vector) and $\rho^2$ is its dominant eigenvalue. Similarly, multiplying the first equation $k\boldsymbol{u}\mathbf{s} = \rho\mathbf{r}$ by $\boldsymbol{v}$ yields $k\boldsymbol{vu}\mathbf{s} = \rho\boldsymbol{v}\mathbf{r} = k\rho^2\mathbf{s}$. Hence $\mathbf{s}$ is a dominant eigenvector of $\boldsymbol{vu}$ and $\rho^2$ its dominant eigenvalue. This proves (i) and $\sigma = \rho^2$.

(i) implies (iii). Take $\mathbf{r}$ a dominant eigenvector of $\boldsymbol{uv}$ and set $\mathbf{s}' = \frac{1}{\sqrt{\sigma}}\boldsymbol{v}\mathbf{r}$. We have $\boldsymbol{u}\mathbf{s}' = \frac{1}{\sqrt{\sigma}}\boldsymbol{uv}\mathbf{r}$; since $\boldsymbol{uv}\mathbf{r} = \sigma\mathbf{r}$, we obtain $\boldsymbol{u}\mathbf{s}' = \sqrt{\sigma}\mathbf{r}$. Thus both equations $\boldsymbol{u}\mathbf{s}' = \sqrt{\sigma}\mathbf{r}$ and $\boldsymbol{v}\mathbf{r} = \sqrt{\sigma}\mathbf{s}'$ hold: this proves that $(\mathbf{r}, \mathbf{s}')$ is a dominant eigenvector of $\boldsymbol{\pi}$, that is (iii), and $\sqrt{\sigma}$ is the dominant eigenvalue of $\boldsymbol{\pi}$. ■

**Definition 2** *The mutual centrality method assigns to each* $(\boldsymbol{u}, \boldsymbol{v})$ *in* $\mathcal{I}$ *the rankings* $\mathbf{r}$ *and* $\mathbf{s}$ *given respectively by the dominant eigenvectors of* $\boldsymbol{uv}$ *and* $\boldsymbol{vu}$*, or satisfying the equivalent conditions (ii) or (iii) in Lemma 1.*

The mutual ranking on $M$ thus satisfies

$$r_i = \frac{1}{\rho^2} \sum_{j \in M} (uv)_{ij} r_j \text{ for each } i \in M. \tag{3}$$

The matrix $\boldsymbol{uv}$ reflects some form of competition between the agents in $M$, as will be studied below. I provide now an axiomatization of the mutual centrality method.

**Additivity under unanimity.** The members of $M$ are said to be *unanimous* when their evaluations on $N$ are equal: $v_{\lambda i} = y_\lambda$ for each $i$ and each $\lambda$. The matrix $\boldsymbol{v}$ takes the form $dg(\boldsymbol{y})\mathbb{1}_{n,m}$ where $dg(\boldsymbol{y})$ is the $n \times n$ diagonal matrix with the vector $\boldsymbol{y} = (y_\lambda)$ on the diagonal and $\mathbb{1}_{n,m}$ is the $n \times m$ matrix made of 1. Unanimity of $N$ is defined similarly.

**Definition 3** *The method* $F$ *is additive under unanimity on* $\mathcal{F}$ *if for all problems* $(\boldsymbol{u}, \boldsymbol{v})$ *for which* $M$ *is unanimous,* $\boldsymbol{v} = dg(\boldsymbol{y})\mathbb{1}_{n,m}$ *for some* $\boldsymbol{y}$*, then*

$$\mathbf{r} \propto (\sum_{\lambda} u_{i\lambda} y_\lambda)_{i \in M} \textit{ and } \mathbf{s} \propto \boldsymbol{y} \tag{4}$$

*and for all problems* $(\boldsymbol{u}, \boldsymbol{v})$ *for which* $N$ *is unanimous,* $\boldsymbol{u} = dg(\boldsymbol{x})\mathbb{1}_{m,n}$ *for some* $\boldsymbol{x}$*, then*

$$\mathbf{r} \propto \boldsymbol{x} \textit{ and } \mathbf{s} \propto (\sum_{i} v_{\lambda i} x_i)_{\lambda \in N} \tag{5}$$

To illustrate the property, consider the example of ratings between buyers and sellers. Assume that buyers all rate identically each seller. According to (4), the sellers' scores coincide with these common ratings and a buyer's score is obtained as the weighted sum of the ratings that sellers assign to her, where a seller's weight is his score. Additivity under unanimity encompasses two notions. First, the evaluations of one side on the other, say of $M$ on $N$, are relevant to judge the evaluations provided by $N$ on $M$. Second, the (unanimous) evaluation obtained by a member of $N$ acts as a 'price' on his own evaluations on $M$. If the rating of seller $\lambda$ is twice that of $\mu$, $\lambda$'s evaluations on the buyers are doubled relative to those of $\mu$.

**Consistency across sides.** Consistency across sides applies the 'consistency principle' (see the introduction) to be satisfied when the data on one side is changed by its solution. Specifically, replacing the evaluations on the members of one side by their assigned scores does not affect the rankings.

**Definition 4** *The method $F$ is consistent across sides on $\mathcal{F}$ if, letting $(\mathbf{r}, \mathbf{s}) = F(\boldsymbol{u}, \boldsymbol{v})$:*

$$F(\boldsymbol{u}, \boldsymbol{v}) = F(\boldsymbol{u}, dg(\mathbf{s})\mathbb{1}_{n,m}) \text{ and } F(\boldsymbol{u}, \boldsymbol{v}) = F(dg(\mathbf{r})\mathbb{1}_{m,n}, \boldsymbol{v}).$$

To understand the first equation, let us replace the evaluations by each member of $M$ by the assigned ranking $\mathbf{s}$ on $N$: the matrix $\boldsymbol{v}$ is replaced by $dg(\mathbf{s})\mathbb{1}_{n,m}$. The first equation thus says that the rankings are not affected by this change. Similarly for the second equation when $\boldsymbol{u}$ is replaced by the unanimous ranking $\mathbf{r}$.

**Proposition 1** *The mutual centrality method is the only ranking method on $\mathcal{I}$ that is additive under unanimity and consistent across sides.*

**Proof** The mutual centrality method is additive under unanimity. Assume the members of $M$ all provide the same evaluation $\boldsymbol{y}$ on $N$. The equation on the right hand side in (2) writes $s_\lambda = \frac{1}{\rho}\sum_{i \in M} y_\lambda r_i$ for each $\lambda$ in $N$ hence $\mathbf{s}$ is proportional to $\boldsymbol{y}$. Plugging these values into the equation on the left hand side of (2) yields that $\mathbf{r}$ is proportional to $(\sum_\lambda u_{i\lambda} y_\lambda)_{i \in M}$: this proves additivity under unanimity. Consistency across sides straightforwardly follows from (2).

Conversely, let $F$ satisfy the axioms. Let $(\mathbf{r}, \mathbf{s}) = F(\boldsymbol{u}, \boldsymbol{v})$. By consistency across sides, $\mathbf{r} \propto F_M(\boldsymbol{u}, dg(\mathbf{s})\mathbb{1}_{n,m})$. By additivity under unanimity, $F_M(\boldsymbol{u}, dg(\mathbf{s})\mathbb{1}_{n,m}) \propto (\sum_\lambda u_{i\lambda} s_\lambda)_{i \in M}$. Hence there is $k > 0$ such that $r_i = \alpha \sum_\lambda u_{i\lambda} s_\lambda$ for each $i$. Exchanging the roles of $M$ and $N$, there is $\beta > 0$ such that $s_\lambda = \beta \sum_i v_{\lambda i} r_i$ for each $\lambda$. Combining these relations, we obtain that $\mathbf{r}$ and $\mathbf{s}$ are respectively dominant eigenvectors of $\boldsymbol{uv}$ and $\boldsymbol{vu}$: this proves that $F$ is the mutual centrality method . ■

Next, let us study the matrix $\boldsymbol{uv}$ and introduce the competitive counting method. This method will be useful to show the independence between the two axioms.

**The competitive counting method** The matrix $\boldsymbol{uv}$ reflects some form of competition between the agents in $M$. $(\boldsymbol{uv})_{ij}$ is equal to $\sum_\lambda u_{i\lambda}v_{\lambda j}$. In the example of mutual applications with 0-1 values, it is equal to the number of universities to which $j$ applies that find $i$ acceptable. Thus, $(\boldsymbol{uv})_{ij}$ reflects the competition posed by $i$ on $j$ for $i \neq j$, and $(\boldsymbol{uv})_{ii}$ reflects the adequacy in the demand and acceptance of $i$. More generally, $(\boldsymbol{uv})_{ij}$ is high when the members in $N$ who are highly valued by $j$ highly value $i$. Let us call $\boldsymbol{uv}$ the *competitive* matrix for $M$ and similarly $\boldsymbol{vu}$ the competitive matrix for $N$.

The *competitive counting* method assigns to each side the counting scores computed on its competitive matrix, i.e., the counting scores of $\boldsymbol{uv}$ to $M$ and those of $\boldsymbol{vu}$ to $N$. Rearranging the sums:

$$\mathbf{r} \propto (\sum_\lambda u_{i\lambda}v_{\lambda+})_{i\in M} \text{ and } \mathbf{s} \propto (\sum_i v_{\lambda i}u_{i+})_{\lambda\in N}. \tag{6}$$

The scores of agents in $M$ are thus given by the sums of the received evaluations weighted by the evaluators' counting score, and similarly for the scores of $N$.

The three methods introduced so far, the counting, competitive counting and mutual centrality ones, generally produce different results. Consider the example

| $\boldsymbol{u}$ | $\alpha$ | $\beta$ |
|---|---|---|
| $a$ | 0 | $k$ |
| $b$ | 2 | 1 |

| $\boldsymbol{v}$ | $a$ | $b$ |
|---|---|---|
| $\alpha$ | 1 | 0 |
| $\beta$ | 0 | 2 |

| $\boldsymbol{uv}$ | $a$ | $b$ |
|---|---|---|
| $a$ | 0 | $2k$ |
| $b$ | 2 | 2 |

| $\boldsymbol{vu}$ | $\alpha$ | $\beta$ |
|---|---|---|
| $\alpha$ | 0 | $k$ |
| $\beta$ | 4 | 2 |

where $k$ is a positive parameter. $k$ has a positive impact on the competitiveness of $a$ over $b$ and $\alpha$ over $\beta$. The competitiveness of $a$ over $b$ is equal to $2k$ due to the fact that $a$ is valued $k$ by $\beta$ who is valued 2 by $b$. The competitiveness of $\alpha$ over $\beta$ is equal to $k$ due to the fact $\alpha$ is valued 1 by $a$ who is valued $k$ by $\beta$.

On $M$, the counting scores are proportional to $(k/3, 1)$, the competitive counting ones to $(k/2, 1)$ and the mutual centrality ones to $(\sqrt{k+\frac{1}{4}} - \frac{1}{2}, 1)$. They differ for generic $k$.

$a$'s competitive counting score is always larger than his counting score: since $\beta$'s counting score is twice that of $\alpha$, the competitive counting method, according to (6), values more the evaluations from $\beta$ than the counting one. The result follows because $a$ is valued by $\beta$ only whereas $b$ is more valued by $\alpha$ than $\beta$.

Comparing $a$'s competitive counting and mutual centrality scores, the former is smaller for $k<2$. It can be understood by considering (2) and the double effect of $k$: $k$ increases $a$'s grade but $a$ is valued only by $\beta$ whose score decreases with $k$: we have $\frac{r_a}{r_b}=k\frac{s_\beta}{s_\alpha+s_\beta}$; since the competitiveness of $\beta$ relative to $\alpha$ decreases with $k$, $\beta$'s score decreases relative to $\alpha$'s one, as can be checked by computing the mutual centrality scores on $N$: $(\frac{1}{2}(\sqrt{k+\frac{1}{4}}-\frac{1}{2}),1)$.

The relationships between the three rankings are better understood by considering a recursive approach to the mutual centrality scores.[6] The competitive counting method can be seen as intermediate between the counting and mutual centrality methods. Let us start with the counting scores. The idea is to recompute the scores by weighting each evaluation by the evaluators' scores. This provides new scores. Iterating, the algorithm can be shown to converge to a dominant eigenvector of the matrix $\boldsymbol{\pi}$, under the additional assumption of ergodicity.Formally, start with the counting rankings $\mathbf{r}^1$ and $\mathbf{s}^1$. The recursive equations at $t+1$ are

$$\mathbf{r}^{t+1}\propto(\sum_\lambda u_{i\lambda}s^t_\lambda)_{i\in M} \text{ and } \mathbf{s}^{t+1}\propto(\sum_i v_{\lambda i}r^t_i)_{\lambda\in N} \tag{7}$$

that is, at step $t+1$, the scores in $M$ are proportional to the sum of the received evaluations weighted by the evaluators' scores at $t$, i.e. $\lambda$'s evaluations are weighted by $s^t_\lambda$, and similarly the scores in $N$ are proportional to the sum of the received evaluations weighted by the evaluators' scores at $t$, i.e. $i$'s evaluations are weighted by $r^t_i$.

Assume the matrix $\boldsymbol{\pi}$ to be ergodic, i.e., a power of $\boldsymbol{\pi}$ has all its elements positive. Then, choosing unit vectors, the sequence $(\mathbf{r}^t,\mathbf{s}^t)$ converges to an eigenvector of $\boldsymbol{\pi}$. Along the algorithm, the rankings are the counting one at step 1, the competitive counting one at step 2 as follows from (7), and the mutual centrality one asymptotically.

**Independence between additivity under unanimity and consistency across sides**
We show that the two axioms are independent. Clearly, the counting method is consistent across sides but is not additive under unanimity.

Let us show that the competitive counting method is additive under unanimity but not consistent across sides. Additivity under unanimity straightforwardly follows[7] from (6). The method is not consistent across sides: Replace the evaluations $\boldsymbol{v}$ by unanimous evaluations equal to the competitive counting score $\mathbf{s}$ of $N$. Thanks to additivity under unanimity, the scores of $M$ become proportional to the sums of the received evaluations

[6] This approach has been used by various authors in the peers' setting. Liebowitz and Palmer (1984) use the first steps of the algorithm to rank economic journals.

[7] If $M$'s members unanimously evaluate $N$ by $\boldsymbol{y}$, then $\sum_i v_{\lambda,i}u_{i+}=y_\lambda\sum_i u_{i+}$, hence $\mathbf{s}\propto\boldsymbol{y}$. Furthermore, since $v_{\lambda+}=my_\lambda$, $\mathbf{r}\propto(\sum_\lambda u_{i\lambda}y_\lambda)_{i\in M}$ by (6).

weighted by $\mathbf{s}$, which differ in general from the sums weighted by $N$'s counting scores. This is illustrated by the previous example. Let us replace the evaluations $\boldsymbol{v}$ on $N$ by unanimous ones equal to $(k, 6)$ the competitive counting scores on $N$. By additivity under unanimity, the competitive counting scores in $M$ become proportional to $(3k, k+3)$ (obtained as $(0 \times 1 + k \times 6, 2 \times k + 1 \times 6)$). These are not proportional to the competitive counting scores of $N$, $(k, 2)$, for $k$ different from 3: the method is not consistent.

**Irreducibility** The irreducibility of the competitive matrices $\boldsymbol{uv}$ and $\boldsymbol{vu}$ can be interpreted as saying that any two agents in the same side are in competition, directly or indirectly. Consider for example $\boldsymbol{uv}$. $\boldsymbol{uv}$ is irreducible if for each pair $(i, j)$ there is an integer $p$ for which $\boldsymbol{uv}^{(p)}(i, j) > 0$. If $p = 1$, $j$ poses competition on $j$ as we have seen. For $p > 1$, $\boldsymbol{uv}^{(p)}(i, j) > 0$ means that there is a succession of $p$ individuals in $N$ starting from $i$ and ending to $j$ such that each one poses competition on the follower. Alternatively, in the underlying directed bipartite graph, there is a (directed) path linking any two agents in $M$.

The irreducibility of $\boldsymbol{\pi}$ is equivalent to that of both $\boldsymbol{uv}$ and $\boldsymbol{vu}$. Since $\boldsymbol{\pi}$ is irreducible if there is a (directed) path linking any two agents in $M \cup N$, it straightforwardly implies the irreducibility of both $\boldsymbol{uv}$ and $\boldsymbol{vu}$. Conversely, when both $\boldsymbol{uv}$ and $\boldsymbol{vu}$ are irreducible, no row nor column of $\boldsymbol{\pi}$ can be null, so that any path from an $i$ to a $\lambda$ can be obtained through a $j$ connected to $\lambda$: $\boldsymbol{\pi}$ is irreducible.

**Affiliation networks** Recall that an affiliation network represents the affiliation of individuals to groups (see Section 2.2). Bonacich (1972) proposes to *simultaneously* assign individual and group centralities in an affiliation network by considering the scores that satisfy for some $\alpha$

$$r_i = \alpha \sum_{\lambda / i \text{ is in board } \lambda} s_\lambda \text{ for each } i \text{ and } s_\lambda = \alpha \sum_{i / \text{ is in board } \lambda} r_i \text{ for each } \lambda \qquad (8)$$

These equations correspond to (2) where $u_{\lambda i}$ and $v_{i\lambda}$ are both equal to 1 if $i$ is a member of board $\lambda$ and both to 0 otherwise. The mutual centrality method thus extends this idea to any two-sided problem. Quoting Bonacich (1991) in the context of the membership to boards, (8) says that *A central firm gets its central position from the board membership patterns of its members. They belong to the variety of boards that make that firm central. If its members belong to a constricted set of other boards, that firm would not be central. Dually a central individual should be one who belongs to a variety of important firms. One kind of centrality cannot be defined without reference to the other.*

In the Web context, Kleinberg (1999) uses a similar idea to define the Hits method, based on good 'hubs and authorities'.[8] The method distinguishes two weights for each Web page, one associated with the relevance or authority of a page, which should help users to find the relevant pages, the other with the adequacy of a page to point toward the relevant pages, which identifies the good hubs (not necessarily useful to Internet users). Specifically, the method assigns the (authorities) ranking $\mathbf{r} = (r_i)$ and the (hubs) ranking $\mathbf{s} = (s_i)$ that satisfy

$$r_i = \sum_{j/j \text{ points to } i} s_j \text{ for each } i \text{ and } s_j = \sigma \sum_{i/i \text{ points to } j} r_i \text{ for each } j \tag{9}$$

for some positive $\sigma$. Quoting Kleinberg *Hubs and authorities exhibit what could be called a mutually reinforcing relationship: a good hub is a page that points to many good authorities; a good authority is a page that is pointed to by many good hubs.*

Comparing (9) with (2), the Hits method coincides with the mutual centrality method applied to the bipartite graph in which each side coincides with the set of Websites, and the evaluations are dual, given by $u_{ij} = v_{ji}$ equal to 1 if $j$ points to $i$ or to 0 otherwise, i.e. this corresponds to an affiliation network.

Proposition 1 does not characterize the mutual centrality method on the set of affiliation networks because the axioms pertain to changes in the evaluations of one side that leave unchanged those of the other side. Applying such a change to an affiliation network, the obtained evaluations are no longer the transpose of each other, hence do not represent an affiliation network.

### 3.2 The invariant mutual centrality method

In some contexts, as explained in the introduction, one may want the rankings to be insensitive to the total of the evaluations provided by each agent, a property called intensity-invariance. In the example of students-teachers, this means that the degree of indulgence or severity of a teacher should not influence the rankings of the students or of the teachers. Formally, $\boldsymbol{u}'$ is a scaled version of $\boldsymbol{u}$ if the columns of $\boldsymbol{u}'$ and $\boldsymbol{u}$ are proportional: for each $\lambda$ in $N$, $\lambda$'s evaluations in $\boldsymbol{u}$ and $\boldsymbol{u}'$ are proportional, i.e., there is a positive number $k_\lambda$ such that $u'_{i\lambda} = k_\lambda u'_{i\lambda}$ for each $i$. A scaled version $\boldsymbol{v}'$ of $\boldsymbol{v}$ is defined similarly by requiring $i$'s evaluations in $\boldsymbol{v}$ and $\boldsymbol{v}'$ to be proportional for each $i$ in $M$. The axiom of intensity-invariance is defined on any feasible set $\mathcal{F}$ that is closed by scaling the evaluations, i.e., starting with

[8] Relationships between the invariant and Hits methods are analyzed in Kitti (2016).

$(\boldsymbol{u},\boldsymbol{v})$ in $\mathcal{F}$, $(\boldsymbol{u}',\boldsymbol{v}')$ belongs to $\mathcal{F}$ for any scaled versions $\boldsymbol{u}'$ of $\boldsymbol{u}$ and $\boldsymbol{v}'$ of $\boldsymbol{v}$. Call such a set scaled-invariant.

**Definition 5** *A method $F$ defined on the scaled-invariant $\mathcal{F}$ is* **intensity-invariant** *if $F(\boldsymbol{u},\boldsymbol{v}) = F(\boldsymbol{u}',\boldsymbol{v}')$ for any $(\boldsymbol{u},\boldsymbol{v})$ in $\mathcal{F}$ and for any scaled versions $\boldsymbol{u}'$ of $\boldsymbol{u}$ and $\boldsymbol{v}'$ of $\boldsymbol{v}$.*

Basically, an intensity-invariant method is characterized by its values on the evaluations that are each normalized say to sum to 1. Denote by $[\mathcal{F}]$ the set of irreducible matrices $\boldsymbol{\pi}$ whose columns sum to 1. Given $F$, one builds an intensity-invariant method by scaling the evaluations of each agent before applying $F$. Formally let $[\boldsymbol{\pi}]$ be the matrix whose columns are proportional to those of $\boldsymbol{\pi}$ and sum to 1. Define $[F]$ by: for each $\boldsymbol{\pi}$, $[F](\boldsymbol{\pi}) = F([\boldsymbol{\pi}])$. $[F]$ is intensity-invariant and coincides with $F$ on $[\mathcal{F}]$.

The invariant mutual centrality method is defined to be $[F]$ where $F$ is the mutual centrality method. The dominant eigenvalue of $[\boldsymbol{\pi}]$ (as well as that of $[\boldsymbol{u}][\boldsymbol{v}]$ and $[\boldsymbol{v}][\boldsymbol{u}]$) is equal to 1. Thus, applying (2), the unit rankings $(\mathbf{r},\mathbf{s}) = [F](\boldsymbol{u},\boldsymbol{v})$ satisfy

$$\mathbf{r} = (\sum_{\lambda\in N}[u_{i\lambda}]s_\lambda)_{i\in M} \text{ and } \mathbf{s} = (\sum_{i\in M}[v_{\lambda i}]r_i)_{\lambda\in N}. \tag{10}$$

The invariant mutual centrality method is characterized in a similar way as the mutual centrality one, by restricting the two axioms on $[\mathcal{I}]$ and adding intensity-invariance. This is stated in the following corollary to Proposition 1.

**Corollary 1** *The invariant mutual centrality method is the only ranking method on $\mathcal{I}$ that is intensity-invariant on $\mathcal{I}$, additive under unanimity and consistent across sides on $[\mathcal{I}]$.*

**Proof** Let $F$ be the mutual centrality method. By construction $[F]$ is invariant. To show additivity under unanimity on $[\mathcal{I}]$, let $\boldsymbol{u}$ be in $[\mathcal{I}]$, $\boldsymbol{u} = [\boldsymbol{u}]$, and $M$ be unanimous on $N$ with unit evaluations $\boldsymbol{y}$. We have $[F](\boldsymbol{u},\boldsymbol{v}) = F(\boldsymbol{u}, dg(\boldsymbol{y})\mathbb{1}_{n,m})$. Since $F$ is additive under unanimity, it follows that $[F]_M(\boldsymbol{u},\boldsymbol{v})$ is proportional to $(\sum_\lambda u_{i\lambda}y_\lambda)_{i\in M}$ and $[F]_N(\boldsymbol{u},\boldsymbol{v})$ to $\boldsymbol{y}$; this proves that $[F]$ is additive under unanimity on $[\mathcal{I}]$. Similarly $[F]$ is consistent across sides on $[\mathcal{I}]$ (and even on $\mathcal{I}$) because $F$ is.

To show the converse, let $G$ be intensity-invariant on $\mathcal{I}$, additive under unanimity and consistent across sides on $[\mathcal{I}]$. Since $G$ is intensity-invariant, $(\mathbf{r},\mathbf{s}) = G([\boldsymbol{u}],[\boldsymbol{v}])$ for any $(\boldsymbol{u},\boldsymbol{v})$. Using additivity under unanimity and consistency across sides on $[\mathcal{I}]$, the proof of Proposition 1 applies to the pair $([\boldsymbol{u}],[\boldsymbol{v}])$. This proves that $\mathbf{r}$ and $\mathbf{s}$ are respectively dominant eigenvectors of $[\boldsymbol{u}][\boldsymbol{v}]$ and $[\boldsymbol{v}][\boldsymbol{u}]$, i.e., $G = [F]$. ∎

The same approach straightforwardly applies to define methods that are intensity-invariant on one side only. Normalizing the evaluations of one side only, say $M$, the obtained method is intensity-invariant over $M$. A similar characterization as in Corollary 1 is obtained, requiring intensity-invariance on $M$, additivity under unanimity and consistency across sides for all irreducible matrices $\boldsymbol{u}$ and all the normalized ones $\boldsymbol{v}$.

# 4 Impartial methods

The property of impartiality across sides requires that the score of any agent is determined exclusively by the evaluations of the agents of the other side: The ranking on $M$ only depends on $\boldsymbol{u}$ and that on $N$ only on $\boldsymbol{v}$. Formally

**Definition 6** *The method $F$ is impartial (across sides) if $F_M(\boldsymbol{u}, \boldsymbol{v})$ is independent of $\boldsymbol{v}$ and $F_N(\boldsymbol{v})$ is independent of $\boldsymbol{u}$, so we can write $\mathbf{r} = F_M(\boldsymbol{u})$ and $\mathbf{s} = F_N(\boldsymbol{v})$.*

Impartiality across sides implies consistency across sides; actually it is a much stronger axiom. Impartiality is incompatible with additivity under unanimity because the latter precisely requires the ranking on $M$ to vary with their evaluations on $N$ when these are unanimous: For a fixed $\boldsymbol{u}$, $\mathbf{r}$ varies with $\boldsymbol{y}$ as can be seen from (4). There are many impartial methods, each one obtained by choosing a ranking method. An impartial method is thus characterized by characterizing each one-sided method, $F_M$ and $F_N$. In what follows, I consider methods $F_M$ that are defined on the set $\mathcal{P}_M$ of non-negative $m \times n$ evaluation matrices $\boldsymbol{u}$ and similarly for $F_N$.

The counting method defined by (1) is a benchmark. It has been characterized in various settings. The ordinal ranking associated to the counting method has been axiomatized by Rubinstein (1980) for tournaments, in which $i$ wins over $j$ or $j$ wins over $i$, and by Brink and Gilles (2009) for weighted directed graphs. I use here the characterization in Demange (2014), which applies to our framework in which the (relative) values taken by the scores, and not only the order, matter. The characterization relies on two properties, either uniformity and homogeneity, or exactness and homogeneity.

Uniformity and exactness bear on some specific evaluations, hereafter called row-balanced. A matrix is said to be *row-balanced* if each row receives the same total. Equivalently, rows obtain equal scores under the counting method. Uniformity views row-balanced matrices as 'neutral': there is no rationale for distinguishing between the agents on one side if they are assigned identical totals by the other side. Exactness asks the converse property that agents obtain equal scores only if they receive identical totals. Formally let $\mathcal{P}_M$ be the set of evaluation on $M$, i.e. the set of $m \times n$ non-negative matrices.

**Definition 7** *Let $F_M$ a method be defined on $\mathcal{P}_M$. $F_M$ is* **uniform** *if $F_M(\boldsymbol{u}) = \mathbb{1}_M$ for* *each row-balanced $\boldsymbol{u}$ in $\mathcal{P}_M$. $F_M$ is* **exact** *if $F_M(\boldsymbol{u}) = \mathbb{1}_M$ implies that $\boldsymbol{u}$ is row-balanced. Uniformity and exactness of $F_N$ are defined similarly.*

The next property is called *homogeneity*. The property is very natural when evaluations are cardinal.[9] Let us start with evaluations on $M$ and multiply each evaluation on $i$ by the same scalar $\rho_i$. Homogeneity requires $i$'s score to be multiplied by the same factor relative to other scores. Formally, starting with $\boldsymbol{u}$ in $\mathcal{P}_M$, multiplying each evaluation on $i$ by $\rho_i$ multiplies row $i$ by $\rho_i$; the modified evaluation matrix writes $dg(\boldsymbol{\rho})\boldsymbol{u}$ where $\boldsymbol{\rho}$ is the vector with $i$'s component equal to $\rho_i$ and all others equal to 1. Homogeneity requires $F_M(dg(\boldsymbol{\rho})\boldsymbol{u})$ to be (proportional to) the ranking $dg(\boldsymbol{\rho})F_M(\boldsymbol{u})$. The property is required for each row $i$, so iteration yields the following equivalent definition.

**Definition 8** *A method $F_M$ is* **homogeneous** *on $M$ if for each $\boldsymbol{u}$ in $\mathcal{P}_M$ and each positive $m$-vector $\boldsymbol{\rho} = (\rho_i)$, $F_M(dg(\boldsymbol{\rho})\boldsymbol{u})$ is the ranking $dg(\boldsymbol{\rho})F_M(\boldsymbol{u})$. Homogeneity of $F_N$ is defined similarly.*

The following proposition follows straightforwardly from the impartiality axiom and Proposition 3 in Demange (2014).

**Proposition 2** *The counting method is the only method that is impartial, homogeneous and uniform on $N$ and $M$. The counting method is the only method that is impartial, homogeneous and exact on $N$ and $M$.*

**The congruence method** The congruence method assigns to $(\boldsymbol{u}, \boldsymbol{v})$ the rankings $\mathbf{r}$ and $\mathbf{s}$ respectively dominant eigenvectors of $\boldsymbol{u}\widetilde{\boldsymbol{u}}$ and $\boldsymbol{v}\widetilde{\boldsymbol{v}}$, assuming these matrices irreducible. The word *congruence* refers to the fact that the method departs from the counting method by considering how evaluations jointly depart from the counting scores, as explained below.

Under unanimous evaluations $\mathbf{r}$, the term $(\boldsymbol{u}\widetilde{\boldsymbol{u}})_{ij}$ is equal to $nr_ir_j$. The following proposition states that the method seeks for the unanimous ranking $\mathbf{r}$ that minimizes some errors between the unanimous evaluation matrix $\widetilde{\mathbf{r}}\mathbf{r}$ and $\boldsymbol{u}\widetilde{\boldsymbol{u}}$, and similarly for $\mathbf{s}$. The proof extends the argument used in Bonacich (1972) for affiliation networks to the general two-sided setting. The argument bears on the eigenvectors of a symmetric matrix, as is applied in principal components analysis though on centered variables.

Let $\mathcal{V}$ be the set of matrices $(\boldsymbol{u}, \boldsymbol{v})$ such that both $\boldsymbol{u}\widetilde{\boldsymbol{u}}$ and $\boldsymbol{v}\widetilde{\boldsymbol{v}}$ are irreducible.

[9] The homogeneity axiom introduced in Palacios-Huerta and Volij (2004) differs since it bears on a given matrix that has two proportional rows.

**Proposition 3** *The congruence method is the only method on $\mathcal{V}$ that is impartial and where* $\mathbf{r}$ *minimizes the sum of the squared errors* $\sum_{ij}[(\boldsymbol{u}\widetilde{\boldsymbol{u}})_{ij} - nr_ir_j)]^2$ *and* $\mathbf{s}$ *minimizes* $\sum_{\lambda\mu}[(\boldsymbol{v}\widetilde{\boldsymbol{v}})_{\lambda\mu} - ms_\lambda s_\mu)^2]$.

**Proof of Proposition 3** Consider the function $\sum_{ij}[(\boldsymbol{u}\widetilde{\boldsymbol{u}})_{ij} - nr_ir_j)]^2$. The first order conditions for minimization write $\sum_j[(\boldsymbol{u}\widetilde{\boldsymbol{u}})_{ij} - mr_ir_j]r_j = 0$ for each $i$, or $\sum_j(\boldsymbol{u}\widetilde{\boldsymbol{u}})_{ij}r_j = (n\sum_j r_j^2)r_i$ for each $i$. Hence $\mathbf{r}$ is an eigenvector of $\boldsymbol{u}\widetilde{\boldsymbol{u}}$. Diagonalizing the symmetric matrix $\boldsymbol{u}\widetilde{\boldsymbol{u}}$ in an orthogonal basis, one checks that the global minimum is obtained for the largest eigenvalue. ∎

The congruence method is not uniform, as shown by the following example.

**Example** Let $M$ and $N$ have each three elements and the evaluation matrix $\boldsymbol{u}$ be

| $\boldsymbol{u}$ | $\alpha$ | $\beta$ | $\gamma$ |
|---|---|---|---|
| $a$ | 1 | 1 | 0 |
| $b$ | 1 | 1 | 0 |
| $c$ | 1/2 | 1/2 | 1 |

which gives

| $\boldsymbol{u}\widetilde{\boldsymbol{u}}$ | $a$ | $b$ | $c$ |
|---|---|---|---|
| $a$ | 2 | 2 | 1 |
| $b$ | 2 | 2 | 1 |
| $c$ | 1 | 1 | 1.5 |

.

The counting scores are all equal to 2. The congruent scores satisfy $r_a = r_b \approx 1.57r_c$. This proves that the method is not uniform. In the above example, $a$ and $b$'s scores are higher than $c$'s due to the good evaluations of $\alpha$ and $\beta$ on them, which reinforce each other. More generally, the departure from the counting scores is due to the pattern of joint evaluations. This can be seen from the following decomposition of the $ij$-th term of matrix $\boldsymbol{u}\widetilde{\boldsymbol{u}}$. $(\boldsymbol{u}\widetilde{\boldsymbol{u}})_{ij}$ is equal to $\sum_\lambda u_{i\lambda}u_{j\lambda}$ or

$$(\boldsymbol{u}\widetilde{\boldsymbol{u}})_{ij} = \sum_\lambda (u_{i\lambda} - mu_{i+})(u_{j\lambda} - mu_{j+}) + mu_{i+}u_{j+}.$$

The term $(\boldsymbol{u}\widetilde{\boldsymbol{u}})_{ij}$ is increasing in the counting scores of each $i$ and $j$ and in the congruence in the evaluations of both: If those who value $i$ more than the average also value $j$ highly, and have dispersed preferences on others agents, then $(\boldsymbol{u}\widetilde{\boldsymbol{u}})_{ij}$ is higher than $nu_{i+}u_{j+}$. Thus, a group of agents tend to have higher scores than their counting scores if those who value them have concentrated high evaluations on them. Roughly speaking, a group of individuals with similar evaluations weigh more than their aggregate.

Finally, note that the mutual centrality and the congruence methods coincide for affiliation networks, since $\boldsymbol{v} = \widetilde{\boldsymbol{u}}$. The characterization in Proposition 3 applies to any pair of evaluation matrices, in particular it applies to affiliation networks (because the minimization of errors does not consider variations in the evaluation matrices).

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