Inheritance of Convexity for Partition Restricted Games

Alexandre Skoda

UNIVERSITÉ PARIS 1
PANTHÉON SORBONNE

Documents de Travail du
Centre d'Economie de la Sorbonne

CES Working Papers

Inheritance of Convexity for Partition Restricted Games

Alexandre SKODA

2016.40

Maison des Sciences Économiques, 106-112 boulevard de L'Hôpital, 75647 Paris Cedex 13
http://centredeconomiesorbonne.univ-paris1.fr/
ISSN : 1955-611X

Inheritance of Convexity for Partition Restricted Games

A. Skoda*

Abstract

A correspondence $\mathcal{P}$ associates to every subset $A \subseteq N$ a partition $\mathcal{P}(A)$ of A and to every game (N, v), the $\mathcal{P}$-restricted game $(N, \overline{v})$:

$$\overline{v}(A) := \sum_{F \in \mathcal{P}(A)} v(F), \text{ for all } A \subseteq N. \tag{1}$$

We give necessary and sufficient conditions on $\mathcal{P}$ to have inheritance of convexity from (N, v) to $(N, \overline{v})$. The main condition is a cyclic intersecting sequence free condition. As a consequence, we only need to verify inheritance of convexity for unanimity games and for the small class of extremal convex games (N, v_S) (for any $\emptyset \neq S \subseteq N$) defined for any $A \subseteq N$ by $v_S(A) = |A \cap S| - 1$ if $|A \cap S| \geq 1$, and $v_S(A) = 0$ otherwise. In particular when $(N, \overline{v})$ corresponds to Myerson's network-restricted game inheritance of convexity can be verified by this way. For the $\mathcal{P}_{min}$ correspondence ($\mathcal{P}_{min}(A)$ is built by deleting edges of minimum weight in the subgraph G_A of a weighted communication graph G), we show that inheritance of convexity for unanimity games already implies inheritance of convexity. Assuming only inheritance of superadditivity, we also compute the Shapley value of the restricted game $(N, \overline{v})$ for an arbitrary correspondence $\mathcal{P}$.

Keywords: communication network, cooperative game, restricted game, partitions.

1 Introduction

We consider, on a given finite set N, with $|N| = n$, an arbitrary correspondence $\mathcal{P}$ which associates to every subset $A \subseteq N$ a partition $\mathcal{P}(A)$ of A. Then for every game (N, v) we define the restricted game $(N, \overline{v})$ associated with $\mathcal{P}$ by:

$$\overline{v}(A) = \sum_{F \in \mathcal{P}(A)} v(F), \text{ for all } A \subseteq N. \tag{2}$$

*Corresponding author. Université de Paris I, Centre d'Economie de la Sorbonne, 106-112 Bd de l'Hôpital, 75013 Paris, France. E-mail: alexandre.skoda@univ-paris1.fr

Documents de travail du Centre d'Economie de la Sorbonne - 2016.40

We will more simply refer to this game as the $\mathcal{P}$-restricted game. v is the characteristic function of the game, $v : 2^N \to \mathbb{R}$, $A \mapsto v(A)$ and satisfies $v(\emptyset) = 0$. Through many accurate choices for the correspondence $\mathcal{P}$, the new game $(N, \overline{v})$ can take into account many combinatorial structures and differents aspects of cooperation restrictions.
The first founding example is the Myerson's correspondence $\mathcal{P}_M$ associated with communication games [13]. Communication games are cooperative games (N, v) defined on the set of vertices N of an undirected graph $G = (N, E)$, where E is the set of edges. For every coalition $A \subseteq N$, we consider the induced graph $G_A := (A, E(A))$, where $E(A)$ is the set of edges of E with ends in A. $\mathcal{P}_M(A)$ is the set of connected components of G_A. The $\mathcal{P}_M$-restricted game $(N, \overline{v})$, known as Myerson's game, takes into account how the players of N can communicate according to the graph G. Many other correspondences have been considered to define restricted games (see, e.g., [2], [4] [5], [7], [9], [10]).
Of course, for applications as well as for theoretical reasons, it is of major interest to compare the properties of the games v and $\overline{v}$ and at first to decide if we have inheritance of basic properties as superadditivity and convexity from the underlying game (N, v) to the restricted game $(N, \overline{v})$. In this case, we will say we have inheritance of convexity (resp. superadditivity) for the correspondence $\mathcal{P}$. Inheritance of convexity is a nice property as it implies that good properties are inherited, for instance the non-emptyness of the core, and that the Shapley value is in the core.
Let us observe that inheritance of convexity for a correspondence is a strong property, hence it would be useful to consider weaker properties restricting the inheritance to a smaller class of convex games as, for instance, the class of unanimity games (N, u_S). It is also a first key step to prove inheritance in the general case. In a preceding paper [9] we have established necessary and sufficient conditions on $\mathcal{P}$ to have inheritance of superadditivity on one hand (for all $A \subset B \subseteq N$, $\mathcal{P}(A)$ has to be a refinement of $\mathcal{P}(B)$) and of convexity for unanimity games on the other hand (for $A, B \subseteq N$, $\mathcal{P}(A \cap B) = \{F \cap G \neq \emptyset,\ F \in \mathcal{P}(A),\ G \in \mathcal{P}(B)\}$). Henceforth assuming that these two last conditions are realized, we introduce, in the present paper, for arbitrary subsets $A, B \subseteq N$ and for every $D \in \mathcal{P}(A \cup B)$, the family of intersecting sequences $\{C_1, C_2, \ldots, C_l\}$, such that $C_j \subseteq D$, $C_j \in \mathcal{P}(A)$ or $C_j \in \mathcal{P}(B)$, for all j, $1 \leq j \leq l$, $C_j \cap C_{j+1} \neq \emptyset$, for all j, $1 \leq j \leq l-1$, $C_1 \setminus C_2 \neq \emptyset$, and $C_l \setminus C_{l-1} \neq \emptyset$. (This definition is close to the definitions of intersecting subsets and intersecting family in J. Edmonds and R. Giles [6] or in S. Fujishige [11].) If $C_1 = C_l$, we call it a cyclic intersecting sequence of $\mathcal{P}$. We say that $\mathcal{P}$ is cyclic intersecting sequence free if such cyclic intersecting sequence does not exist in $\mathcal{P}$.

The main result of this paper is that we have inheritance of convexity for $\mathcal{P}$ if and only if $\mathcal{P}$ is a cyclic intersecting sequence free correspondence (Theorem 17, page 17). We also prove in Theorem 18 that it is enough to

verify this condition for subsets $A, B \subseteq N$ such that $|A \setminus B| = |B \setminus A| = 1$. In all proofs we extensively use writing any game as a unique linear combination of unanimity games. To prove that the cyclic intersecting sequence free condition is necessary, we have to use the small class of extremal convex games (N, v_S) ($S \subseteq N$, $|S| \geq 2$) with $v_S(A) = |A \cap S| - 1$ if $|A \cap S| \geq 1$, and $v_S(A) = 0$ otherwise. As a consequence, we only have to verify inheritance of convexity for unanimity games and for this last class of extremal convex games (N, v_S) to obtain inheritance for all convex games. It is a surprising and unexpected fact because the class of convex games is much bigger than these two small classes.
We prove that the cyclic intersecting sequence free condition is sufficient by computing explicitly the link between the convexity of the two games (N, v) and $(N, \overline{v})$. For $A, B \subseteq N$, we set:

$$\Delta v(A, B) := v(A \cup B) + v(A \cap B) - v(A) - v(B).$$

We show that, for an explicitely given family of subsets $A_j, B_j \subseteq A \cup B$, $1 \leq j \leq p$ (for some p depending on A and B) we have:

$$\Delta \overline{v}(A, B) = \sum_{j=1}^{p} \Delta v(A_j, B_j).$$

The notions of intersecting (resp. crossing) submodular functions have been highlighted by S. Fujishige [11]. Such functions have to satisfy the submodular inequality only for specific restricted family of subsets. In the same spirit, we prove a more precise formula for $\Delta \overline{v}(A, B)$ (Proposition 11), using a finite family of maximal intersecting connected subsets (defined in Section 2). Particularly, it provides a more complete information about the contribution of superadditivity on one hand and of convexity on the other hand of the game (N, v) to the convexity of $(N, \overline{v})$.

Convexity is a nice property but may be too strong to be always realized in many practical situations. Therefore we have also investigated other weaker convexity properties. For instance, we have restricted convexity to the family $\mathcal{F}$ of connected subsets of a communication game as in [14]. Then we have a result for inheritance of this restricted convexity similar to the one for convexity by restricting the condition to intersecting sequences of connected subsets.

We prove in Section 4 that for Myerson's correspondence it can be very easily established that this correspondence is cyclic intersecting sequence free. Hence within this framework we get a new proof of A. van den Nouweland and P. Borm's result [16]: inheritance of convexity for Myerson's correspondence holds if and only if the graph of the communication game is cycle-complete[1]. In the case of a partition system defined in [2] we con-

[1] A graph $G = (N, E)$ is *cycle-complete* if for any cycle $C = (v_1, e_1, v_2, e_2, \ldots, e_m, v_1)$ in G the subset $\{v_1, v_2, \ldots, v_m\} \subseteq N$ of vertices of C induces a complete subgraph in G.

sider the correspondence associating to any subset $A \subseteq N$, its partition into maximal subsets and we prove that there is no cyclic intersecting sequence. Therefore inheritance of convexity is satisfied if and only if inheritance of convexity for unanimity games holds. If moreover the partition system is an intersecting family, we prove inheritance of convexity is always satisfied as already shown by U. Faigle [7]. We also consider in Section 4 the correspondence $\mathcal{P}_{\min}$ associated to a weighted graph $G = (V, E)$. For a subset $A \subseteq N$, $\mathcal{P}_{\min}(A)$ corresponds to the set of connected components of the subgraph $(A, E(A) \setminus \Sigma(A))$ where $\Sigma(A)$ is the set of minimum weight edges in the subgraph $G_A = (A, E(A))$. Then we show directly that inheritance of convexity for $\mathcal{P}_{\min}$ is equivalent to the weak property of inheritance of convexity for unanimity games by proving the non existence of any cyclic intersecting sequence. A similar equivalence has already been proved in a forthcoming paper [14] for a weaker condition called $\mathcal{F}$-convexity (corresponding to the restriction of convexity to connected subsets [9]). But this result has been established by a completely different method as a consequence of a characterization of inheritance of $\mathcal{F}$-convexity by four conditions on graph edge-weights. In the forthcoming paper [14], we have also completely classified the weighted communication games for which we have inheritance of convexity for $\mathcal{P}_{\min}$. There are very strong restrictions on these weighted graphs, particularly only three different edge-weights may occur. Hence, we have to restrict convexity to $\mathcal{F}$-convexity (as in [9]) if we want to obtain a wide enough class of weighted graphs for which inheritance of $\mathcal{F}$-convexity holds.

In Section 5, we give examples of correspondences with cyclic intersecting sequences for which there is inheritance of convexity for unanimity games but nevertheless, according to the main result of this paper, no inheritance of convexity for all convex games.

The article is organized as follows. In Section 2, we give preliminary definitions and results. In particular, we recall the definitions of convexity, $\mathcal{F}$-convexity and general conditions on a correspondence to have inheritance of superadditivity, convexity or $\mathcal{F}$-convexity established in [9]. Then we establish all preliminary lemmas we will need for further proofs computing the value $\Delta v(A, B)$ at first for an unanimity game $v = u_S$ and subsequently for any game v. We finally introduce all the background (definitions and preliminary lemmas) about intersecting sequences and intersecting connected subsets, we will later use. Section 3 includes the main results and proofs of the paper. In Section 4, we consider examples of cyclic intersecting sequence free correspondences, in particular Myerson's correspondence and the $\mathcal{P}_{\min}$ correspondence. In Section 5, we construct various examples of correspondences with cyclic intersecting sequences. In Section 6, only assuming inheritance of superadditivity for the correspondence $\mathcal{P}$, we explicitly compute the Shapley value of the restricted game $(N, \overline{v})$ and give a minoration of the Shapley values of both games (N, v) and $(N, \overline{v})$ by another simple value. In

Section 7, we conclude with some remarks and suggestions for generalization of these results to other correspondences even when these correspondences have cyclic intersecting sequences.

2 Preliminary definitions and results

A game (N, v) is *zero-normalized* if $v(i) = 0$ for all $i \in N$. We recall that a game (N, v) is *superadditive* if, for all $A, B \in 2^N$ such that $A \cap B = \emptyset$, $v(A \cup B) \geq v(A) + v(B)$. For any given subset $\emptyset \neq S \subseteq N$, the unanimity game (N, u_S) is defined by:

$$u_S(A) = \begin{cases} 1 & \text{if } A \supseteq S, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

We note that u_S is superadditive for all $S \neq \emptyset$. The following result established in [9] gives general conditions on a correspondence $\mathcal{P}$ to have inheritance of superadditivity.

Theorem 1. *Let N be an arbitrary set and $\mathcal{P}$ a correspondence on N. Then the following claims are equivalent:*

1) For all $\emptyset \neq S \subseteq N$, the $\mathcal{P}$-restricted game $(N, \overline{u_S})$ is superadditive.

2) For all subsets $A \subseteq B \subseteq N$, $\mathcal{P}(A)$ is a refinement of the restriction of $\mathcal{P}(B)$ to A.

3) For all superadditive game (N, v) the $\mathcal{P}$-restricted game $(N, \overline{v})$ is superadditive.

Let us consider a game (N, v). For arbitrary subsets A and B of N, we define the value:

$$\Delta v(A, B) := v(A \cup B) + v(A \cap B) - v(A) - v(B).$$

A game (N, v) is *convex* if its characteristic function v is supermodular, *i.e.*, $\Delta v(A, B) \geq 0$ for all $A, B \in 2^N$. We note that u_S is supermodular for all $S \neq \emptyset$. For an arbitrary element $i \in N$ and an arbitrary subset $A \subseteq N \setminus \{i\}$, the derivative of v at A w.r.t. i is defined by:

$$\Delta_i v(A) := v(A \cup \{i\}) - v(A).$$

$\Delta_i v(A)$ is also known as the marginal contribution of player i w.r.t. coalition A. For arbitrary subsets $A \subseteq B \subseteq N \setminus \{i\}$, we define the value:

$$\Delta_i v(A, B) := \Delta_i v(B) - \Delta_i v(A).$$

Of course we have $\Delta_i v(A, B) = \Delta v(A \cup \{i\}, B)$ and $\Delta_i(\Delta_j v)(A) = \Delta v(A \cup \{i\}, A \cup \{j\})$. Then we have equivalent formulations of supermodularity of v:

1. $\Delta v(A,B) \geq 0$, for all $A, B \subseteq N$.

2. $\Delta_i v(A,B) \geq 0$, for all $i \in N$, for all $A \subseteq B \subseteq N \setminus \{i\}$.

3. $\Delta_i(\Delta_j v)(A) \geq 0$, for all $i, j \in N$ with $i \neq j$, for all $A \subseteq N \setminus \{i,j\}$.

Let $\mathcal{F}$ be a *weakly union-closed family* [2] of subsets of N such that $\emptyset \notin \mathcal{F}$. A game v on 2^N is said to be *$\mathcal{F}$-convex* if $\Delta v(A,B) \geq 0$, for all $A, B \in \mathcal{F}$ such that $A \cap B \in \mathcal{F}$. If $\mathcal{F} = 2^N \setminus \{\emptyset\}$ then $\mathcal{F}$-convexity corresponds to convexity.

For a given graph $G = (N,E)$, we say that a subset $A \subseteq N$ is connected if the induced graph $G_A = (A, E(A))$ is connected. In the case of a communication game (N,v), $\mathcal{F}$ will be the family of connected subsets of N. We recall the following result proved in [9]:

Theorem 2. *Let $G = (N,E)$ be an arbitrary graph and let $\mathcal{F}$ be the family of connected subsets of N. Then the following conditions are equivalent:*

1. *v is $\mathcal{F}$-convex.*

2. *$\Delta_i v(A,B) \geq 0$, for all $i \in N$, for all $A \subseteq B \subseteq N \setminus \{i\}$ with A, B, and $A \cup \{i\} \in \mathcal{F}$.*

3. *$\Delta_i(\Delta_j v)(A) \geq 0$, for all i, j in N with $i \neq j$, for all $A \subseteq N \setminus \{i,j\}$ with $A, A \cup \{i\}$, and $A \cup \{j\} \in \mathcal{F}$.*

The next theorem established in [9] gives general abstract conditions on a correspondence $\mathcal{P}$ to have inheritance of convexity for unanimity games.

Theorem 3. *Let N be an arbitrary set, $\mathcal{P}$ an arbitrary correspondence on N, and $\mathcal{F}$ a weakly-union closed family of subsets of N such that $\emptyset \notin \mathcal{F}$. The following conditions are equivalent.*

a) For all $\emptyset \neq S \subseteq N$, the $\mathcal{P}$-restricted game $(N, \overline{u_S})$ is superadditive and $\mathcal{F}$-convex.

b) For all $A \subseteq B \subseteq N$, $\mathcal{P}(A)$ is a refinement of $\mathcal{P}(B)_{|A}$ and for all $A, B \subseteq N$ such that A, B, and $A \cap B$ are in $\mathcal{F}$, $\mathcal{P}(A \cap B) = \{F \cap G \neq \emptyset,\ F \in \mathcal{P}(A),\ G \in \mathcal{P}(B)\}$.

Moreover if $\mathcal{F} = 2^N \setminus \{\emptyset\}$ or if $\mathcal{F}$ corresponds to the set of all connected subsets of a graph then a) and b) are equivalent to:

c) For all $A \subseteq B \subseteq N$, $\mathcal{P}(A)$ is a refinement of $\mathcal{P}(B)_{|A}$ and for all $i \in N$, for all $A \subseteq B \subseteq N \setminus \{i\}$ such that A, B, and $A \cup \{i\}$ are in $\mathcal{F}$ and for all $A' \in \mathcal{P}(A \cup \{i\})_{|A}$, $\mathcal{P}(A)_{|A'} = \mathcal{P}(B)_{|A'}$.

[2] $\mathcal{F}$ is weakly union-closed if $A \cup B \in \mathcal{F}$ for all A, $B \in \mathcal{F}$ such that $A \cap B \neq \emptyset$ [8]. Weakly union-closed families were introduced and analysed in [1, 3] and called union stable systems.

We first prove some useful lemmas for unanimity games.

Lemma 4. *Let us consider a unanimity game (N, u_S) with $S \neq \emptyset$ and subsets $A, B \subseteq N$. Let us define:*

$$\mathcal{S}(A,B) = \{S \subseteq A \cup B,\ S \nsubseteq A,\ S \nsubseteq B\} \tag{4}$$

Then we have:

$$\Delta u_S(A,B) = \begin{cases} 1 & \text{if } S \in \mathcal{S}(A,B), \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Proof. We have:

$$\Delta u_S(A,B) = u_S(A \cup B) + u_S(A \cap B) - u_S(A) - u_S(B).$$

If $S \nsubseteq A \cup B$, obviously $\Delta u_S(A,B) = 0$.
If $S \subseteq A \cap B$, $\Delta u_S(A,B) = 1 + 1 - 1 - 1 = 0$.
If $S \subseteq A$ and $S \nsubseteq B$, $\Delta u_S(A,B) = 1 + 0 - 1 - 0 = 0$ (by symetry if $S \nsubseteq A$ and $S \subseteq B$, $\Delta u_S(A,B) = 0$).
If $S \subseteq A \cup B$, $S \nsubseteq A$ and $S \nsubseteq B$, $\Delta u_S(A,B) = 1 + 0 - 0 - 0 = 1$. □

Lemma 5. *Let $\mathcal{P}$ be a correspondence on N. Let A and B be subsets of N such that $\mathcal{P}(A)$ and $\mathcal{P}(B)$ are refinement of $\mathcal{P}(A \cup B)$ and $\mathcal{P}(A \cap B) = \{F \cap G \neq \emptyset,\ F \in \mathcal{P}(A), G \in \mathcal{P}(B)\}$. Let us consider a unanimity game (N, u_S) with $S \neq \emptyset$. Let us define for a given $D \in \mathcal{P}(A \cup B)$:*

$$\mathcal{S}(A,B,D) = \{S \subseteq D;\ \text{for all } F \in \mathcal{P}(A) \cup \mathcal{P}(B),\ S \nsubseteq F\} \tag{6}$$

Then we have:

$$\Delta \overline{u_S}(A,B) = \begin{cases} 1 & \text{if, for some } D \in \mathcal{P}(A \cup B),\ S \in \mathcal{S}(A,B,D), \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

Proof. Let us begin with a basic fact we will extensively use in the proof. By definition $\overline{u_S}(A) = \sum_{F \in \mathcal{P}(A)} u_S(F)$ and therefore $\overline{u_S}(A) = 1$ if and only if there exists a block $F \in \mathcal{P}(A)$ such that $S \subseteq F$ and $\overline{u_S}(A) = 0$ otherwise. Let us consider $A, B \subseteq N$. We have:

$$\Delta \overline{u_S}(A,B) = \overline{u_S}(A \cup B) + \overline{u_S}(A \cap B) - \overline{u_S}(A) - \overline{u_S}(B).$$

If $\overline{u_S}(A) = 1$ and $\overline{u_S}(B) = 1$, there exist $F \in \mathcal{P}(A)$ and $G \in \mathcal{P}(B)$ such that $S \subseteq F$ and $S \subseteq G$. Then by assumption $F \cap G \in \mathcal{P}(A \cap B)$ and therefore $\overline{u_S}(A \cap B) = 1$. As $\mathcal{P}(A)$ is a refinement of $\mathcal{P}(A \cup B)$ we also have $\overline{u_S}(A \cup B) = 1$. Hence $\Delta \overline{u_S}(A,B) = 0$.
If $\overline{u_S}(A) = 1$ and $\overline{u_S}(B) = 0$, we have $\overline{u_S}(A \cup B) = 1$ (resp. $\overline{u_S}(A \cap B) = 0$) as $\mathcal{P}(A)$ is a refinement of $\mathcal{P}(A \cup B)$ (resp. as there is no block $G \in \mathcal{P}(B)$ with $S \subseteq G$). Hence $\Delta \overline{u_S}(A,B) = 0$. By symmetry, if $\overline{u_S}(A) = 0$ and $\overline{u_S}(B) = 1$, we also have $\Delta \overline{u_S}(A,B) = 0$.
If $\overline{u_S}(A) = 0$ and $\overline{u_S}(B) = 0$, then $\overline{u_S}(A \cap B) = 0$ and therefore $\Delta \overline{u_S}(A,B) = \overline{u_S}(A \cup B)$. Then $\Delta \overline{u_S}(A,B) = 1$ if and only if there exists $D \in \mathcal{P}(A \cup B)$ such that $S \in \mathcal{S}(A,B,D)$. □

Remark 1. By Theorem 3, if for all $\emptyset \neq S \subseteq N$ the $\mathcal{P}$-restricted game $(N, \overline{u_S})$ is convex, then (7) is satisfied for all $A, B \subseteq N$.

It is well known that every cooperative game (N, v) can be written as a unique linear combination of unanimity games:

$$v = \sum_{S \subseteq N} \lambda_S u_S, \tag{8}$$

where $\lambda_\emptyset = 0$ and for $S \neq \emptyset$ the coefficients $\lambda_S \in \mathbb{R}$ are the Harsanyi dividends [12] of v given by $\lambda_S = \sum_{T \subseteq S} (-1)^{|S|-|T|} v(T)$. These coefficients also correspond to the Möbius transform of v.

Lemma 5 implies the following result.

Corollary 6. *Let $\mathcal{P}$ be a correspondence on N. Let A and B be subsets of N such that $\mathcal{P}(A)$ and $\mathcal{P}(B)$ are refinement of $\mathcal{P}(A \cup B)$ and $\mathcal{P}(A \cap B) = \{F \cap G \neq \emptyset,\ F \in \mathcal{P}(A),\ G \in \mathcal{P}(B)\}$. Let (N, v) be a cooperative game and $v = \sum_{S \subseteq N} \lambda_S u_S$, with $\lambda_S \in \mathbb{R}$, its unique decomposition into unanimity games. Then we have:*

$$\Delta \overline{v}(A, B) = \sum_{D \in \mathcal{P}(A \cup B)} \left(\sum_{S \in \mathcal{S}(A,B,D)} \lambda_S \right). \tag{9}$$

Proof. We have by linearity $\Delta \overline{v}(A, B) = \sum_{S \subseteq N} \lambda_S \Delta \overline{u_S}(A, B)$. Then Lemma 5 implies (9). □

Lemma 7. *Let us consider a unanimity game (N, u_S), $i \in N$, and $A \subseteq N \setminus \{i\}$. Then we have:*

$$\Delta_i u_S(A) = \begin{cases} 1 & \text{if } i \in S \text{ and } S \setminus \{i\} \subseteq A, \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

Proof. If $S \subseteq A$ then $\Delta_i u_S(A) = 0$. If $S \not\subseteq A$ then $\Delta_i u_S(A) = u_S(A \cup \{i\})$. As $S \not\subseteq A$ it is equal to 1 if $i \in S$ and $S \setminus \{i\} \subseteq A$, and 0 otherwise. □

Lemma 8. *Let $\mathcal{P}$ be a correspondence on N. Let us consider $i \in N$ and $A \subseteq N \setminus \{i\}$ such that $\mathcal{P}(A)$ is a refinement of $\mathcal{P}(A \cup \{i\})$. We set $\mathcal{P}(A) = \{A_1, A_2, \ldots, A_p\}$, $\mathcal{P}(A \cup \{i\}) = \{A_1^{'}, A_2^{'}, \ldots, A_{p'}^{'}\}$, and define $\mathcal{S}^{'}(A, i) := \{\emptyset \neq S \subseteq N;\ \exists l,\ S \subseteq A_l^{'},\ \forall A_m \subseteq A_l^{'},\ S \cap (A_l^{'} \setminus A_m) \neq \emptyset\}$. Then we have:*

$$\Delta_i \overline{u_S}(A) = \begin{cases} 1 & \text{if } S \in \mathcal{S}^{'}(A, i), \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

Proof. Let S be a non-empty subset of N. We have $\overline{u_S}(A) = 1$ if and only if $\exists A_l \in \mathcal{P}(A)$ such that $S \subseteq A_l$. Theorem 1 implies that, for all $A_l \in \mathcal{P}(A)$, there exists a unique $A_k^{'} \in \mathcal{P}(A \cup \{i\})$ such that $A_l \subseteq A_k^{'}$. Hence if $\overline{u_S}(A) = 1$

then $\overline{u_S}(A \cup \{i\}) = 1$ and $\Delta_i \overline{u_S}(A) = 0$. Therefore $\Delta_i \overline{u_S}(A) = 1$ if and only if $\overline{u_S}(A) = 0$ and $\overline{u_S}(A \cup \{i\}) = 1$, *i.e.*, for all $A_m \in \mathcal{P}(A)$, $S \not\subseteq A_m$ and there exists $A_l^{'} \in \mathcal{P}(A \cup \{i\})$, $S \subseteq A_l^{'}$. As $\mathcal{P}(A)$ is a refinement of $\mathcal{P}(A \cup \{i\})$, we have $A_m \cap S = \emptyset$ for all $A_m \in \mathcal{P}(A)$ such that $A_m \not\subseteq A_l^{'}$. Therefore $\overline{u_S}(A) = 0$ if and only if, for all $A_m \subseteq A_l^{'}$, we have $S \not\subseteq A_m$. $\square$

Remark 2. By Theorem 1, if for all $\emptyset \neq S \subseteq N$ the $\mathcal{P}$-restricted game $(N, \overline{u_S})$ is superadditive, then (11) is satisfied for all $i \in N$ and all $A \subseteq N \setminus \{i\}$.

We say that two subsets A and B are *intersecting subsets* or that A (resp. B) *intersects* B (resp. A) if and only if $A \setminus B$, $B \setminus A$, and $A \cap B$ are non empty. Let $\mathcal{P}$ be an arbitrary correspondence. For given subsets $A, B \subseteq N$, and $D \in \mathcal{P}(A \cup B)$, we define the following family $\mathcal{C}(A, B, D)$ of subsets of D:

$$\mathcal{C}(A, B, D) := \{C \subseteq D;\ C \in \mathcal{P}(A) \text{ or } C \in \mathcal{P}(B)\}.$$

A finite family of subsets $\mathcal{F} = \{C_1, C_2, \ldots, C_l\}$, with $l \geq 2$, is called an *intersecting sequence* if C_k, C_{k+1} are intersecting subsets for all k, $1 \leq k \leq l-1$. A finite family of subsets $\{C_1, C_2, \ldots, C_l\}$ is called an *intersecting sequence w.r.t.* $\mathcal{C}(A, B, D)$ if it is an intersecting sequence such that $C_k \in \mathcal{C}(A, B, D)$ for all k, $1 \leq k \leq l$.

Remark 3. If $A = B$ or $A \cap B = \emptyset$ there is no intersecting sequence w.r.t. $\mathcal{C}(A, B, D)$. Moreover If $\mathcal{P}$ is a correspondence such that for all $A \subset B \subseteq N$, $\mathcal{P}(A)$ is a refinement of $\mathcal{P}(B)$, then there is no intersecting sequence w.r.t. $\mathcal{C}(A, B, D)$ if $A \subset B$ or $B \subset A$. Hence if there exists an intersecting sequence w.r.t. $\mathcal{C}(A, B, D)$, then A is not a subset of B, B is not a subset of A and $A \cap B \neq \emptyset$, *i.e.*, A and B are intersecting subsets.

Remark 4. Obviously two elements of a partition cannot be intersecting subsets, therefore in the definition of an intersecting sequence w.r.t. $\mathcal{C}(A, B, D)$ if $C_k \in \mathcal{P}(A)$ (resp. $\mathcal{P}(B)$) then $C_{k+1} \in \mathcal{P}(B)$ (resp. $\mathcal{P}(A)$), and $C_k \cap C_{k+2} = \emptyset$, for all k, $1 \leq k \leq l-2$. Therefore $\{C_1, C_2, \ldots, C_l\}$ is an intersecting sequence w.r.t. $\mathcal{C}(A, B, D)$ if and only if $C_k \cap C_{k+1} \neq \emptyset$ for all k, $1 \leq k \leq l-1$, $C_1 \setminus C_2 \neq \emptyset$ and $C_l \setminus C_{l-1} \neq \emptyset$.

A finite family of subsets $\mathcal{F} = \{C_1, C_2, \ldots, C_l\}$, with $l \geq 3$, is called a *cyclic intersecting sequence* if $\mathcal{F}$ is an intersecting sequence such that C_1 and C_l are intersecting. Remark 4 implies that if $\mathcal{F}$ is a cyclic intersecting sequence w.r.t. $C(A, B, D)$ then its number of components in $\mathcal{P}(A)$ corresponds to its number of components in $\mathcal{P}(B)$, and therefore l is necessarily even. Remark 4 also implies that $\{C_1, C_2, \ldots, C_l\}$ is a cyclic intersecting sequence w.r.t $\mathcal{C}(A, B, D)$ if and only if $C_k \cap C_{k+1} \neq \emptyset$ for all k, $1 \leq k \leq l-1$, and $C_1 \cap C_l \neq \emptyset$.

Remark 5. If $\{C_1, C_2, \ldots, C_l\}$ is a cyclic intersecting sequence w.r.t. $C(A, B, D)$ then $l = 2l^{'}$, with $l^{'} \geq 2$, and we have $C_k \cap C_{k+1} \subset A \cap B$ for all k, $1 \leq k \leq l-1$, and $C_1 \cap C_l \subset A \cap B$. Then as $A \neq B$ we have $|N| \geq |A \cap B| + 1 \geq 2l^{'} + 1$. In particular if $|N| \leq 4$, there exists no cyclic intersecting sequence w.r.t. $C(A, B, D)$. Moreover if $\mathcal{P}$ is a correspondence such that for all $A \subset B \subseteq N$, $\mathcal{P}(A)$ is a refinement of $\mathcal{P}(B)$ then A and B are intersecting subsets and we have $|N| \geq |A \cap B| + 2 \geq 2l^{'} + 2$. In particular if $|N| \leq 5$, there exists no cyclic intersecting sequence w.r.t. $C(A, B, D)$.

A finite family of subsets $\mathcal{F} = \{C_1, C_2, \ldots, C_l\}$ is called an *elementary intersecting sequence* if it is an intersecting sequence such that $\mathcal{F}$ does not contain a subfamily corresponding to a cyclic intersecting sequence. An intersecting sequence $\{C_1, C_2, \ldots, C_l\}$ w.r.t. $\mathcal{C}(A, B, D)$ is necessarily elementary if $l \leq 3$. If $l \geq 4$ then $\{C_1, C_2, \ldots, C_l\}$ is an elementary intersecting sequence w.r.t. $\mathcal{C}(A, B, D)$ if and only if $C_j \cap C_k = \emptyset$ for all j, k, with $1 \leq j \leq l-3$, and $k = j + 2p + 1$, with $1 \leq p \leq \lfloor \frac{l-j-1}{2} \rfloor$, $C_k \cap C_{k+1} \neq \emptyset$ for all k, $1 \leq k \leq l-1$, and $C_1 \setminus C_2 \neq \emptyset$ and $C_l \setminus C_{l-1} \neq \emptyset$.

A family $\mathcal{F} = \{C_1, C_2, \ldots, C_l\}$ is called an *intersecting connected family* if for any pair of elements C_j, C_k, $1 \leq j < k \leq l$, there exists an intersecting sequence with all elements in $\mathcal{F}$ and C_j and C_k as end-sets (w.l.o.g. we can always assume that this intersecting sequence is elementary, deleting some subsets if necessary). In particular if $l = 1$, $\mathcal{F} = \{C_1\}$ is also called an intersecting connected family. A subset $C \subseteq N$ is called an *intersecting connected subset* (w.r.t. $\mathcal{C}(A, B, D)$) if it corresponds to the union of elements of an intersecting connected family $\mathcal{F}$ (of subsets of $\mathcal{C}(A, B, D)$). In this case we also say that $\mathcal{F}$ induces C. In particular a subset $C \in \mathcal{C}(A, B, D)$ is also called an intersecting connected subset w.r.t. $\mathcal{C}(A, B, D)$.

We will sometimes refer to the graph $\Gamma_{\mathcal{F}}$ associated to a family of subsets $\mathcal{F} = \{C_1, C_2, \ldots, C_l\}$. The vertices of $\Gamma_{\mathcal{F}}$ correspond to the subsets $C_1, C_2, \ldots, C_l$. Each pair $\{C_j, C_m\}$, with $1 \leq j < m \leq l$, is an edge of $\Gamma_{\mathcal{F}}$ if and only if C_j intersects C_m. It immediately results from the definitions that an intersecting connected sequence built with elements of $\mathcal{F}$ corresponds to a path in $\Gamma_{\mathcal{F}}$, and a cyclic intersecting connected sequence corresponds to a cycle in $\Gamma_{\mathcal{F}}$. Moreover we have that $\mathcal{F}$ is an intersecting connected family if and only if $\Gamma_{\mathcal{F}}$ is connected.

Remark 6. If C is an intersecting connected subset w.r.t. $\mathcal{C}(A, B, D)$ and if C intersects a subset $\tilde{C} \in \mathcal{C}(A, B, D)$, then $C \cup \tilde{C}$ is still an intersecting connected subset w.r.t. $\mathcal{C}(A, B, D)$.

Proof of Remark 6. Let $\mathcal{F} = \{C_1, C_2, \ldots, C_l\}$ be an intersecting connected family w.r.t. $\mathcal{C}(A, B, D)$ inducing C. Let us consider the family of subsets $\mathcal{F}^{'} = \mathcal{F} \cup \{\tilde{C}\}$. Then $\Gamma_{\mathcal{F}^{'}}$ is a connected graph and therefore $\mathcal{F}^{'}$ is an intersecting connected family w.r.t. $\mathcal{C}(A, B, D)$ inducing $C \cup \tilde{C}$. $\square$

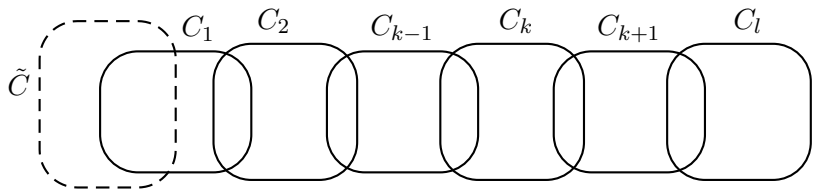

Figure 1: Elementary intersecting sequence $\{C_1, C_2, \ldots, C_l\}$ and $\tilde{C}$.

An intersecting connected subset is maximal if it is maximal for inclusion. Hence if C is a maximal intersecting connected subset w.r.t. $\mathcal{C}(A, B, D)$ and if $\tilde{C} \in \mathcal{C}(A, B, D)$ then $\tilde{C} \subseteq C$ or $\tilde{C} \cap C = \emptyset$. Every intersecting connected subset is contained in a maximal intersecting connected subset.

For a given intersecting connected family $\{C_1, C_2, \ldots, C_l\}$, we define for all k, $1 \leq k \leq l$, the subsets:

$$\tilde{C}_k = \cup_{j=1}^{k} C_j,$$

and we set:

$$\tilde{C}_0 = \emptyset.$$

Lemma 9. *For any intersecting connected subset C, there exists an intersecting connected family $\{C_1, C_2, \ldots, C_l\}$ inducing C such that C_k and $\tilde{C}_{k-1}$ are intersecting for all k, $2 \leq k \leq l$.*

Proof. Let C be an intersecting connected subset. By definition there exists an intersecting connected family $\{C_1, C_2, \ldots, C_l\}$ inducing C. Then we can build a family satisfying the condition of the lemma by induction on k. For $k = 2$ we can assume that C_1 and C_2 are intersecting subsets after renumbering if necessary. Let us assume $k > 2$ and that we have built $\{C_1, C_2, \ldots, C_{k-1}\}$. If $C = \tilde{C}_{k-1}$, the construction ends. Otherwise there exists C_j, with $k \leq j \leq l$, such that $C_j \not\subseteq \tilde{C}_{k-1}$. We can assume $j = k$ after renumbering if necessary. As $C_1 \neq C_k$ there exists an elementary intersecting sequence of subsets of $\mathcal{F}$ with C_1 and C_k as end-sets. At least one subset C_j of this sequence is not a subset of $\tilde{C}_{k-1}$ (as $C_k \not\subseteq \tilde{C}_{k-1}$) and intersects one subset of $\tilde{C}_{k-1}$. Then C_j and $\tilde{C}_{k-1}$ are intersecting. We can assume $j = k$ after renumbering if necessary. □

3 Inheritance of convexity

We will prove that to have inheritance of convexity from the underlying games to the $\mathcal{P}$-restricted games the following condition has to be satisfied.

Cyclic Intersecting Sequence Free Condition. *For all pairs of intersecting subsets $A, B \subseteq N$, and for all $D \in \mathcal{P}(A \cup B)$, there is no cyclic intersecting sequence w.r.t. $\mathcal{C}(A, B, D)$.*

Lemma 10. *Let us consider $A, B \subseteq N$. Let $\mathcal{P}$ be a correspondence on N such that:*

1) $\mathcal{P}(A \cap B) = \{F \cap G \neq \emptyset,\ F \in \mathcal{P}(A),\ G \in \mathcal{P}(B)\}$.

2) For a given $D \in \mathcal{P}(A \cup B)$, there is no cyclic intersecting sequence w.r.t. $C(A, B, D)$.

Then for any intersecting connected family $\{C_1, C_2, \ldots, C_l\}$ w.r.t. $C(A, B, D)$, we can always assume that C_k intersects only one subset C_m, with $1 \leq m \leq k-1$, after renumbering if necessary. Then $C_k \cap \tilde{C}_{k-1} \in \mathcal{P}(A \cap B)$ for all k, $2 \leq k \leq l$.

Proof. Let us consider an intersecting connected family $\mathcal{F} = \{C_1, C_2, \ldots, C_l\}$ w.r.t. $C(A, B, D)$, and the associated graph $\Gamma_{\mathcal{F}}$. By assumption $\Gamma_{\mathcal{F}}$ is connected and cycle-free and therefore a tree. Then $\Gamma_{\mathcal{F}}$ has at least two leaf vertices and after renumbering if necessary we can suppose that C_l corresponds to a leaf vertex of Γ. Therefore C_l intersects only one subset C_m with $1 \leq m \leq l-1$. By the same reasoning we can successively consider the restriction Γ_k of the graph $\Gamma_{\mathcal{F}}$ to $\{C_1, C_2, \ldots, C_k\}$. Γ_k is still a tree and we can suppose after renumbering that C_k is a leaf vertex of Γ_k. Therefore C_k intersects only one subset C_m, with $1 \leq m \leq k-1$. Then, by definition of $C(A, B, D)$, if C_k intersects C_m with $1 \leq m \leq k-1$, we can assume w.l.o.g. $C_k \in \mathcal{P}(A)$ and $C_m \in \mathcal{P}(B)$. Then $C_k \cap C_m$ corresponds to a block of $\mathcal{P}(A \cap B)$. As $C_k \cap C_m = C_k \cap \tilde{C}_{k-1}$, we get $C_k \cap \tilde{C}_{k-1} \in \mathcal{P}(A \cap B)$. $\square$

The next proposition gives sufficient conditions to have inheritance of convexity.

Proposition 11. *Let $\mathcal{P}$ be a correspondence on N. Let A and B be subsets of N such that:*

1) $\mathcal{P}(A)$ and $\mathcal{P}(B)$ are refinement of $\mathcal{P}(A \cup B)$ and $\mathcal{P}(A \cap B) = \{F \cap G \neq \emptyset,\ F \in \mathcal{P}(A),\ G \in \mathcal{P}(B)\}$.

2) For all $D \in \mathcal{P}(A \cup B)$, there is no cyclic intersecting sequence w.r.t. $C(A, B, D)$.

If for all $D \in \mathcal{P}(A \cup B)$, we denote by $\mathcal{C}(D)$ the partition of D into maximal intersecting connected subsets w.r.t. $C(A, B, D)$, then we have:

$$\Delta\overline{v}(A, B) = \sum_{D \in \mathcal{P}(A \cup B)} \left[\sum_{C \in \mathcal{C}(D)} \hat{\Delta} v(C) + v(D) - \sum_{C \in \mathcal{C}(D)} v(C) \right]. \tag{12}$$

where if $\{C_1, C_2, \ldots, C_l\}$ is an intersecting connected family w.r.t. $C(A, B, D)$ inducing C such that C_{k+1} intersects $\tilde{C}_k$ for all k, $1 \leq k \leq l-1$, then:

$$\hat{\Delta} v(C) = \sum_{k=1}^{l} \Delta v(\tilde{C}_{k-1}, C_k). \tag{13}$$

To prove Proposition 11 we need the following lemmas.

Lemma 12. *Let (N, v) be a cooperative game and $v = \sum_{S \subseteq N} \lambda_S u_S$, with $\lambda_S \in I\!R$, its unique decomposition into unanimity games. Let $\mathcal{P}$ be a correspondence on N. Let us consider a pair of intersecting subsets $A, B \subseteq N$ and $D \in \mathcal{P}(A \cup B)$. Let C be a intersecting connected subset w.r.t. $\mathcal{C}(A, B, D)$ and $\tilde{C}$ be a subset in $\mathcal{C}(A, B, D)$ intersecting C. If there is no cyclic intersecting sequence w.r.t. $\mathcal{C}(A, B, D)$, then we have:*

$$\sum_{S \in \mathcal{S}(A,B,D),\ S \subseteq C \cup \tilde{C},\ S \not\subseteq C} \lambda_S = \Delta v(C, \tilde{C}). \tag{14}$$

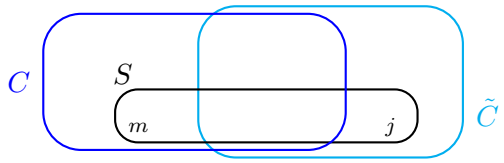

Figure 2: $m \in S \cap (C \setminus \tilde{C})$ and $j \in S \cap (\tilde{C} \setminus C)$.

Proof. By linearity $\Delta v(C, \tilde{C}) = \sum_{S \subseteq N} \lambda_S \Delta u_S(C, \tilde{C})$. Then by Lemma 4 we have:

$$\Delta v(C, \tilde{C}) = \sum_{S \in \mathcal{S}(C, \tilde{C})} \lambda_S \tag{15}$$

where $\mathcal{S}(C, \tilde{C}) = \{S \subseteq C \cup \tilde{C}, S \not\subseteq C, S \not\subseteq \tilde{C}\}$. Let us define the following family of subsets of N:

$$\mathcal{G} := \{S \in \mathcal{S}(A, B, D); S \subseteq C \cup \tilde{C}, S \not\subseteq C\}.$$

We will prove that $\mathcal{G} = \mathcal{S}(C, \tilde{C})$.
Let us consider a subset $S \in \mathcal{G}$. As $S \in \mathcal{S}(A, B, D)$ and as $\tilde{C} \in \mathcal{C}(A, B, D)$, we have $S \not\subseteq \tilde{C}$. Then $\mathcal{G} \subseteq \mathcal{S}(C, \tilde{C})$.
To prove that $\mathcal{S}(C, \tilde{C}) \subseteq \mathcal{G}$, we only have to prove that any $S \in \mathcal{S}(C, \tilde{C})$ is in $\mathcal{S}(A, B, D)$. Let m be an element of $S \cap (C \setminus \tilde{C})$ and j be an element of $S \cap (\tilde{C} \setminus C)$ as represented in Figure 2. By assumption $C \cap \tilde{C} \neq \emptyset$ and C is an intersecting connected subset w.r.t. $\mathcal{C}(A, B, D)$. Therefore there exists an elementary intersecting sequence $\{C_1, C_2, \ldots, C_l\}$ w.r.t. $\mathcal{C}(A, B, D)$ such that $m \in C_1$, $j \in C_l = \tilde{C}$ and $C_k \subseteq C$ for all k, $1 \leq k \leq l - 1$.
By contradiction, let us assume that $S \subseteq C_{l+1}$ for some $C_{l+1} \in \mathcal{C}(A, B, D)$. Note that it implies $l \geq 3$ (as C_{l+1} intersects C_1 and C_l, C_1 and C_l are blocks of the same partition). Then we obtain a cyclic intersecting sequence $(C_1, C_2, \ldots, C_l = \tilde{C}, C_{l+1})$ w.r.t. $\mathcal{C}(A, B, D)$ ($C_k \cap C_{k+1} \neq \emptyset$ for all k, $1 \leq k \leq l-1$, and $m \in C_1 \cap C_{l+1}$ and $j \in C_l \cap C_{l+1}$), a contradiction. Hence $S \in \mathcal{S}(A, B, D)$ and $\mathcal{G} = \mathcal{S}(C, \tilde{C})$. Then (15) implies $\Delta v(C, \tilde{C}) = \sum_{S \in \mathcal{G}} \lambda_S$. $\square$

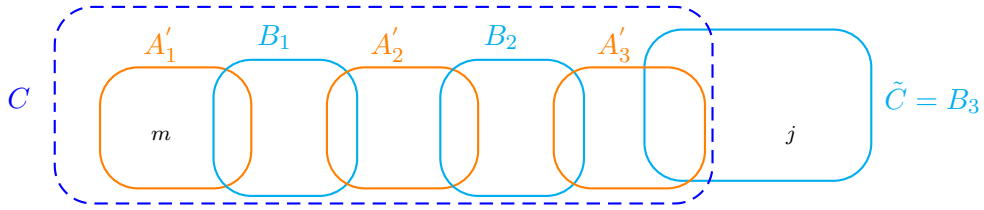

Figure 3: Elementary intersecting sequence $\{A_1', B_1, A_2', B_2, A_3', \tilde{C}\}$.

Lemma 13. *Let (N, v) be a cooperative game and $v = \sum_{S \subseteq N} \lambda_S u_S$, with $\lambda_S \in I\!R$, its unique decomposition into unanimity games. Let $\mathcal{P}$ be a correspondence on N. Let us consider $A, B \subseteq N$ and $D \in \mathcal{P}(A \cup B)$. Let C be an intersecting connected subset induced by an intersecting connected family $\{C_1, C_2, \ldots, C_l\}$ w.r.t. $\mathcal{C}(A, B, D)$, such that C_k intersects $\tilde{C}_{k-1}$, for all k, $2 \leq k \leq l$. If there is no cyclic intersecting sequence w.r.t. $\mathcal{C}(A, B, D)$, then we have:*

$$\sum_{S \in \mathcal{S}(A,B,D),\ S \subseteq C} \lambda_S = \sum_{k=1}^{l} \Delta v(\tilde{C}_{k-1}, C_k) = \hat{\Delta} v(C). \tag{16}$$

Proof. We apply Lemma 12 with $C = \tilde{C}_{k-1}$ and $\tilde{C} = C_k$ so that $C \cup \tilde{C} = \tilde{C}_k$, we get:

$$\sum_{S \in \mathcal{S}(A,B,D),\ S \subseteq \tilde{C}_k,\ S \not\subseteq \tilde{C}_{k-1}} \lambda_S = \Delta v(\tilde{C}_{k-1}, C_k). \tag{17}$$

Adding all equations (17) for all k, $1 \leq k \leq l$, we obtain (16) (note that a subset $S \in \mathcal{S}(A, B, D)$ cannot verify $S \subseteq C_1$, hence there is no term for $k = 1$ in the sum (17), and as $\tilde{C}_0 = \emptyset$ we have $\Delta v(\tilde{C}_0, C_1) = 0$). □

Proof of Proposition 11. Let (N, v) be a given game and $v = \sum_{S \subseteq N} \lambda_S u_S$ with $\lambda_S \in \mathbb{R}$, its unique decomposition into unanimity games. By Corollary 6, Claim 1 implies:

$$\Delta \overline{v}(A, B) = \sum_{D \in \mathcal{P}(A \cup B)} \left(\sum_{S \in \mathcal{S}(A,B,D)} \lambda_S \right). \tag{18}$$

Let us consider a given block $D \in \mathcal{P}(A \cup B)$ and let $\{C_1, C_2, \ldots, C_L\}$ be the finite family of all maximal intersecting connected subsets w.r.t. $\mathcal{C}(A, B, D)$ in D (with an obvious change of notations as $C_1, C_2, \ldots, C_l$ were before subsets inducing a given intersecting connected subset C). Note that by Claim 1 $\{C_1, C_2, \ldots, C_L\}$ is a partition of D.
Let us define the two following families of subsets of N:

$$\mathcal{G}(D) := \{S \in \mathcal{S}(A, B, D);\ S \not\subseteq C_k,\ \forall k,\ 1 \leq k \leq L\},$$

$$\mathcal{G}'(D) := \{S \subseteq D;\, S \not\subseteq C_k,\, \forall k,\, 1 \leq k \leq L\}.$$

We obviously have $\mathcal{G}(D) \subseteq \mathcal{G}'(D)$. To prove that $\mathcal{G}'(D) = \mathcal{G}(D)$, we only have to prove that any $S \in \mathcal{G}'(D)$ is in $\mathcal{S}(A, B, D)$. Such an S meet at least two subsets C_{k_1} and C_{k_2} of the family $\{C_k, 1 \leq k \leq L\}$. S cannot be a subset of any $C \in \mathcal{C}(A, B, D)$, otherwise such a C will meet C_{k_1} and C_{k_2}, contradicting their maximality. Therefore $S \in \mathcal{S}(A, B, D)$ and $\mathcal{G}(D) = \mathcal{G}'(D)$. As $\{C_1, C_2, \ldots, C_L\}$ is a partition of D, we have:

$$\sum_{S \in \mathcal{S}(A,B,D)} \lambda_S = \sum_{S \in \mathcal{G}(D)} \lambda_S + \sum_{k=1}^{L} \left(\sum_{S \in \mathcal{S}(A,B,D),\, S \subseteq C_k} \lambda_S \right). \tag{19}$$

and:

$$\sum_{S \in \mathcal{G}'(D)} \lambda_S = \sum_{S \subseteq D} \lambda_S - \sum_{k=1}^{L} \left(\sum_{S \subseteq C_k} \lambda_S \right) = v(D) - \sum_{k=1}^{L} v(C_k). \tag{20}$$

By Lemma 9, for any maximal intersecting connected subset C_i there exists an intersecting connected family $\{C_{i,1}, C_{i,2}, \ldots, C_{i,l}\}$, inducing C_i, such that $C_{i,k}$ intersects $\tilde{C}_{i,k-1}$ for all k, $2 \leq k \leq l$. Then Lemma 13 applied to each maximal intersecting connected subset C_k gives:

$$\sum_{S \in \mathcal{S}(A,B,D),\, S \subseteq C_k} \lambda_S = \hat{\Delta} v(C_k), \tag{21}$$

for all k, $1 \leq k \leq L$. As $\mathcal{G}(D) = \mathcal{G}'(D)$, using (19), (20), (21), we obtain:

$$\sum_{S \in \mathcal{S}(A,B,D)} \lambda_S = v(D) - \sum_{k=1}^{L} v(C_k) + \sum_{k=1}^{L} \hat{\Delta} v(C_k). \tag{22}$$

As $\mathcal{C}(D)$ is the family of all maximal intersecting connected subset C w.r.t. $C(A, B, D)$, (22) can be re-written:

$$\sum_{S \in \mathcal{S}(A,B,D)} \lambda_S = \sum_{C \in \mathcal{C}(D)} \hat{\Delta} v(C) + v(D) - \sum_{C \in \mathcal{C}(D)} v(C). \tag{23}$$

Finally using (18), we obtain:

$$\Delta \overline{v}(A, B) = \sum_{D \in \mathcal{P}(A \cup B)} \left[\sum_{C \in \mathcal{C}(D)} \hat{\Delta} v(C) + v(D) - \sum_{C \in \mathcal{C}(D)} v(C) \right]. \tag{24}$$

□

Remark 7. If (N, v) is superadditive, then (24) implies:

$$\Delta \overline{v}(A, B) \geq \sum_{D \in \mathcal{P}(A \cup B)} \sum_{C \in \mathcal{C}(D)} \hat{\Delta} v(C). \tag{25}$$

If (N, v) is convex, each term $\Delta v(C_k, \tilde{C}_{k-1})$ in (13) is non-negative and then $\hat{\Delta} v(C) \geq 0$ for all $C \in \mathcal{C}(D)$ and therefore $\Delta \overline{v}(A, B) \geq 0$. Hence if the assumptions of Proposition 11 are satisfied for all $A, B \subseteq N$ then $(N, \overline{v})$ is convex. Therefore we have sufficient conditions for inheritance of convexity.

Corollary 14. *Let $\mathcal{P}$ be a correspondence on N satisfying the following conditions:*

1) For all $A, B \subseteq N$, $\mathcal{P}(A)$ and $\mathcal{P}(B)$ are refinement of $\mathcal{P}(A \cup B)$ and $\mathcal{P}(A \cap B) = \{F \cap G \neq \emptyset,\ F \in \mathcal{P}(A), G \in \mathcal{P}(B)\}$.

2) $\mathcal{P}$ satisfies the Cyclic Intersecting Sequence Free Condition.

Then there is inheritance of convexity for $\mathcal{P}$.

For some specific correspondences and for superadditive games, Proposition 11 also gives sufficient conditions for inheritance of $\mathcal{F}$-convexity.

Corollary 15. *Let $\mathcal{F}$ be a weakly union closed family of subsets of N and let $\mathcal{P}$ be a correspondence on N such that, for every $A \in \mathcal{F}$, $\mathcal{P}(A)$ is a partition of A into subsets of $\mathcal{F}$. Moreover let us assume that this correspondence $\mathcal{P}$ satisfies the Cyclic Intersecting Sequence Free condition for every $A, B \in \mathcal{F}$ such that $A \cap B \in \mathcal{F}$, and the following conditions:*

1) For all $A \subset B \subseteq N$, $\mathcal{P}(A)$ is a refinement of $\mathcal{P}(B)_{|A}$.

2) For all $A, B \subseteq N$ such that A, B and $A \cap B$ are in $\mathcal{F}$, $\mathcal{P}(A \cap B) = \{F \cap G \neq \emptyset,\ F \in \mathcal{P}(A), G \in \mathcal{P}(B)\}$.

Let (N, v) be a superadditive and $\mathcal{F}$-convex game. Then for all $A, B \subseteq N$ such that A, B and $A \cap B$ are in $\mathcal{F}$, we have:

$$\Delta \overline{v}(A, B) \geq 0,$$

i.e., *$(N, \overline{v})$ is $\mathcal{F}$-convex.*
In particular, let us consider a communication game (N, v) which is superadditive and $\mathcal{F}$-convex for the family $\mathcal{F}$ of connected subsets of N, then the $\mathcal{P}$-restricted game $(N, \overline{v})$ is $\mathcal{F}$-convex.

Proof. Let us consider $A, B \subseteq N$ such that A, B, and $A \cap B$ are in $\mathcal{F}$. Note that Claims 1 and 2 imply Claim 1 in Proposition 11. Therefore Proposition 11 implies:

$$\Delta \overline{v}(A, B) = \sum_{D \in \mathcal{P}(A \cup B)} \left[\sum_{C \in \mathcal{C}(D)} \hat{\Delta} v(C) + v(D) - \sum_{C \in \mathcal{C}(D)} v(C) \right]. \tag{26}$$

As (N, v) is superadditive we have $v(D) - \sum_{C \in \mathcal{C}(D)} v(C) \geq 0$. Each block C in $\mathcal{C}(D)$ is induced by an intersecting connected family $\{C_1, C_2, \ldots, C_l\}$

w.r.t. $\mathcal{C}(A,B,D)$. Lemma 10 implies that, after renumbering if necessary, C_k intersects $\tilde{C}_{k-1}$ and $C_k \cap \tilde{C}_{k-1} \in \mathcal{P}(A \cap B)$ for all k, $2 \leq k \leq l$. Then $\tilde{C}_k$ is in $\mathcal{F}$ for all k, $1 \leq k \leq l$ and $C_k \cap \tilde{C}_{k-1}$ is also in $\mathcal{F}$ for all k, $2 \leq k \leq l$, as by assumption all $F \in \mathcal{P}(A \cap B)$ are in $\mathcal{F}$. As (N,v) is $\mathcal{F}$-convex, we have $\Delta v(\tilde{C}_{k-1}, C_k) \geq 0$, for all k, $2 \leq k \leq l$, and then $\hat{\Delta} v(C) \geq 0$. Therefore $\Delta \overline{v}(A,B) \geq 0$. Hence $(N,\overline{v})$ is $\mathcal{F}$-convex. □

Remark 8. $\mathcal{P}_M$ (resp. $\mathcal{P}_{\min}$) is such that for all $A \subseteq N$, $\mathcal{P}_M(A)$ (resp. $\mathcal{P}_{\min}(A)$) is a partition of A into connected subsets of G_A.

We will now prove that the Cyclic Intersecting Sequence Free Condition is necessary to have inheritance of convexity. We have to consider specific supermodular functions (close to modular functions).

Lemma 16. *Let $S \subseteq N$, with $|S| \geq 2$, be a finite subset of elements in N. The function v_S defined, for every $A \subseteq N$, by:*

$$v_S(A) = \begin{cases} |A \cap S| - 1 & \text{if } |A \cap S| \geq 2, \\ 0 & \text{otherwise.} \end{cases} \tag{27}$$

is supermodular.

Proof. Let us consider $A, B \in 2^N$. We set $A' = A \cap S$ and $B' = B \cap S$. If $|A' \cup B'| \leq 1$ we obviously have $\Delta v_S(A,B) = 0$. Let us assume $|A' \cup B'| \geq 2$. If $|A' \cap B'| = 0$, then $v_S(A \cup B) + v_S(A \cap B) = v_S(A \cup B) = |A' \cup B'| - 1 = |A'| + |B'| - 1 = 1 + (|A'| - 1) + |B'| - 1$. Then in any subcase we have $\Delta v_S(A,B) \geq 0$ and moreover $\Delta v_S(A,B) = 1$ if $|A'| \geq 1$ and $|B'| \geq 1$.
If $|A' \cap B'| = 1$, then $v_S(A \cup B) + v_S(A \cap B) = v_S(A \cup B) = |A' \cup B'| - 1 = |A'| + |B'| - 2 = (|A'| - 1) + (|B'| - 1) = v_S(A) + v_S(B)$. Therefore $\Delta v_S(A,B) = 0$. If $|A' \cap B'| \geq 2$, we obviously have $\Delta v_S(A,B) = |A' \cup B'| - 1 + |A' \cap B'| - 1 - (|A'| - 1) - (|B'| - 1) = 0$. □

Remark 9. v_S is supermodular and close to a modular function, but not modular. Indeed $v_S(i) = 0$ for all $i \in N$ but $v_S(\{j,k\}) = 1$ if $\{j,k\} \subseteq S$ and $v_S(\{j,k\}) = 0$ otherwise. In fact, we will only need to consider these functions v_S for even values of $|S|$ (and for $|S| \geq 4$).

Theorem 17. *Let $\mathcal{P}$ be a correspondence on N satisfying one of the two following equivalent conditions:*

1) For all $\emptyset \neq S \subseteq N$, the $\mathcal{P}$-restricted game $(N, \overline{u_S})$ is convex.

2) a) For all $A \subset B \subseteq N$, $\mathcal{P}(A)$ is a refinement of $\mathcal{P}(B)_{|A}$.
b) For all $A, B \subseteq N$, $\mathcal{P}(A \cap B) = \{F \cap G \neq \emptyset,\ F \in \mathcal{P}(A), G \in \mathcal{P}(B)\}$.

Then the following statements are equivalent:

A) $\mathcal{P}$ satisfies the Cyclic Intersecting Sequence Free Condition.

B) There is inheritance of convexity for $\mathcal{P}$.

Moreover, if $\mathcal{F}$ is a weakly union closed family of subsets of N and if $\mathcal{P}$ is such that for every $A \in \mathcal{F}$, $\mathcal{P}(A)$ is a partition of A into subsets of $\mathcal{F}$, and if we assume in 1) that the game $(N, \overline{u_S})$ is only superadditive and $\mathcal{F}$-convex (or in 2b that we only consider subsets $A, B \subseteq N$ such that A, B and $A \cap B$ are in $\mathcal{F}$) then, we have inheritance of $\mathcal{F}$-convexity if and only if the Cyclic Intersecting Sequence Free Condition in A) is satisfied for all subsets $A, B \subseteq N$ such that A, B and $A \cap B$ are in $\mathcal{F}$.
Particularly, if N is the set of vertices of a graph $G = (N, E)$ we can choose for $\mathcal{F}$ the family of connected subsets of N.

Remark 10. 1) and 2) are equivalent by Theorem 3.

Proof of Theorem 17. Let us assume that 1) and 2) are satisfied. Then by Corollary 14 (resp. Corollary 15) the Cyclic Intersecting Sequence Free Condition is a sufficient condition to have inheritance of convexity (resp. $\mathcal{F}$-convexity).
We finally prove that it is also a necessary condition. Let us consider $A, B \subseteq N$, and $D \in \mathcal{P}(A \cup B)$. We set $\mathcal{P}(A) = \{A_1, A_2, \ldots, A_p\}$, and $\mathcal{P}(B) = \{B_1, B_2, \ldots, B_q\}$. Let us consider a cyclic intersecting sequence $\{C_1, C_2, \ldots, C_l\}$ w.r.t. $\mathcal{C}(A, B, D)$. We can assume $C_{2i-1} = A_i$ for all i, $1 \leq i \leq \frac{l}{2}$, and $C_{2i} = B_i$ for all i, $1 \leq i \leq \frac{l}{2}$, after renumbering if necessary. We select an element $j_{2k-1} \in A_k \cap B_k$ for all k, $1 \leq k \leq \frac{l}{2}$, an element j_{2k} in $B_k \cap A_{k+1}$ for all k, $1 \leq k \leq \frac{l}{2} - 1$, and an element $j_l \in B_{\frac{l}{2}} \cap A_1$ as represented in Figure 4 with $l = 4$. Let us consider $S := \{j_1, j_2, \ldots, j_l\}$ (l is

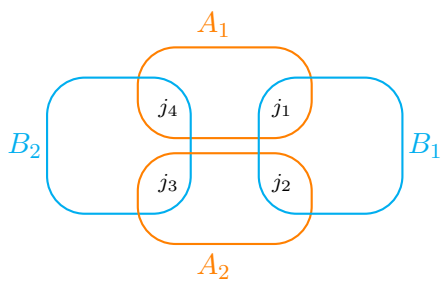

Figure 4: Cyclic intersecting sequence $\{A_1, B_1, A_2, B_2\}$.

even).
For any subset $C \subseteq N$, we set $\mathcal{P}(C) = \{C_1, C_2, \ldots, C_r, \ldots, C_s\}$ and we assume w.l.o.g. that $C_j \cap S \neq \emptyset$ for $j \leq r$ and $C_j \cap S = \emptyset$ for $r + 1 \leq j \leq s$. We consider the function v_S defined by:

$$v_S(C) = \begin{cases} |C \cap S| - 1 & \text{if } |C \cap S| \geq 2, \\ 0 & \text{otherwise.} \end{cases} \tag{28}$$

We have already seen (Lemma 16) that v_S is a supermodular function. By definition, we have:

$$(29)\quad \overline{v_S}(C) = \sum_{j=1}^{s} v_S(C_j) = \sum_{j=1}^{r}(|C_j \cap S| - 1) \;=\; |C \cap S| - r$$
$$= \; |C \cap S| - |\mathcal{P}(C)_{|S}|.$$

Using (29), we have (by construction of S) $\overline{v_S}(A) = \overline{v_S}(B) = l - \frac{l}{2} = \frac{l}{2}$. By Claim 2b, $\mathcal{P}(A \cap B)_{|S} = (\mathcal{P}(A) \cap \mathcal{P}(B))_{|S} = \{\{j_1\}, \{j_2\}, \ldots, \{j_l\}\}$, and therefore $\overline{v_S}(A \cap B) = l - l = 0$. Finally, as $S \subseteq D$, $\mathcal{P}(A \cup B)_{|S} = \{D \cap S\} = \{S\}$, $\overline{v_S}(A \cup B) = l - 1$. Therefore we obtain:

$$(30)\qquad \Delta\overline{v_S}(A, B) = (l - 1) + 0 - \frac{l}{2} - \frac{l}{2} = -1.$$

Hence $\overline{v_S}$ is not supermodular and there is no inheritance of convexity from (N, v_S) to $(N, \overline{v_S})$. Note that v_S is superadditive and $\mathcal{F}$-convex and we can establish the same contradiction to $\mathcal{F}$-convexity of $(N, \overline{v_S})$ with A and B such that A, B, and $A \cap B$ are in $\mathcal{F}$. □

Using Proposition 11, we will now show it is enough to verify the Cyclic Intersecting Sequence Free Condition for smaller families of intersecting pairs of subsets (A, B) for which $|A \backslash B| = 1$ or for which $|A \backslash B| = 1$ and $|B \backslash A| = 1$. For $i \in N$, for $A \subset B \subseteq N \setminus \{i\}$ and $B' \in \mathcal{P}(B \cup \{i\})$, we set:

$$\mathcal{C}_i(A, B, B') := \mathcal{C}(A \cup \{i\}, B, B').$$

For $i, j \in N$ with $i \neq j$, for $A \subseteq N \setminus \{i, j\}$ and $A' \in \mathcal{P}(A \cup \{i, j\})$, we set:

$$\mathcal{C}_{i,j}(A, A') := \mathcal{C}(A \cup \{i\}, A \cup \{j\}, A').$$

Theorem 18. *Let $\mathcal{P}$ be a correspondence on N satisfying one of the two following equivalent conditions:*

1) For all $\emptyset \neq S \subseteq N$, the $\mathcal{P}$-restricted game $(N, \overline{u_S})$ is convex.

2) a) For all $A \subset B \subseteq N$, $\mathcal{P}(A)$ is a refinement of $\mathcal{P}(B)_{|A}$.

b) For all $A, B \subseteq N$, $\mathcal{P}(A \cap B) = \{F \cap G \neq \emptyset,\ F \in \mathcal{P}(A),\ G \in \mathcal{P}(B)\}$

Then the following statements are equivalent:

A) $\mathcal{P}$ satisfies the Cyclic Intersecting Sequence Free Condition.

B) For all $i \in N$, for all $A \subset B \subseteq N \setminus \{i\}$ and for all $B' \in \mathcal{P}(B \cup \{i\})$, there is no cyclic intersecting sequence w.r.t. $\mathcal{C}_i(A, B, B')$.

C) For all $i, j \in N$ with $i \neq j$, for all $A \subseteq N \setminus \{i, j\}$ and for all $A' \in \mathcal{P}(A \cup \{i, j\})$, there is no cyclic intersecting sequence w.r.t. $C_{i,j}(A, A')$.

D) There is inheritance of convexity for $\mathcal{P}$

Proof. Obviously A) implies B) which implies C).
Let us assume C) satisfied. Let us consider a game (N, v), $i, j \in N$ with $i \neq j$ and $A \subseteq N \setminus \{i, j\}$. By definition $\Delta_i(\Delta_j \overline{v})(A) = \Delta \overline{v}(A \cup \{i\}, A \cup \{j\})$. Applying Proposition 11 to $A \cup \{i\}$ and $A \cup \{j\}$, we get:

$$\Delta_i(\Delta_j \overline{v})(A) = \sum_{A' \in \mathcal{P}(A \cup \{i,j\})} \left[\sum_{C \in \mathcal{C}(A')} \hat{\Delta} v(C) + v(A') - \sum_{C \in \mathcal{C}(A')} v(C) \right].$$

If (N, v) is convex, we get $\Delta_i(\Delta_j \overline{v}(A)) \geq 0$ for all i, j in N with $i \neq j$ and for all $A \subset N \setminus \{i, j\}$, *i.e.*, $(N, \overline{v})$ is convex. Hence there is inheritance of convexity and by Theorem 17, A) is satisfied. □

Remark 11. If $\mathcal{F}$ is the family of connected subsets of a communication graph $G = (N, E)$, then under the same assumptions as in Theorem 17, and using Theorem 2, we have inheritance of $\mathcal{F}$-convexity if and only if B) (resp. C)) is satisfied for all $i \in N$ and for all $A \subset B \subseteq N \setminus \{i\}$ such that A, B, and $A \cup \{i\}$ are in $\mathcal{F}$ (resp. for all i, j in N with $i \neq j$ and for all $A \subseteq N \setminus \{i, j\}$ such that that A, $A \cup \{i\}$, and $A \cup \{j\}$ are in $\mathcal{F}$).

4 Examples of cyclic intersecting sequence free correspondences

We now give examples of correspondences satisfying the Cyclic Intersecting Sequence Free Condition. Then by Theorem 17, if these correspondences satisfy inheritance of convexity for unanimity games, they also satisfy inheritance of convexity.

The first example is given by a correspondence $\mathcal{P}$ verifying for all $A \subset B \subseteq N$, $\mathcal{P}(A) = \mathcal{P}(B)_{|A}$. Let us consider $A, B \subseteq N$, and $D \in \mathcal{P}(A \cup B)$. As $\mathcal{P}(A) = \mathcal{P}(A \cup B)_{|A}$ and $\mathcal{P}(B) = \mathcal{P}(A \cup B)_{|B}$ there exist unique $F \in \mathcal{P}(A)$ and $G \in \mathcal{P}(B)$ such that $D = F \cup G$. Hence there exists an intersecting sequence w.r.t. $\mathcal{C}(A, B, D)$ if and only if F and G are intersecting subsets and this intersecting sequence $\{F, G\}$ has length 2. As a cyclic intersecting sequence has length at least 4, the Cyclic Intersecting Sequence Free Condition is necessarily satisfied. Moreover for all $A, B \subseteq N$, $\mathcal{P}(A \cap B) = \mathcal{P}(A)_{|A \cap B} = \mathcal{P}(B)_{|A \cap B}$. Hence $\mathcal{P}$ satisfies the conditions of Theorem 3, and therefore there is inheritance of convexity for unanimity games. Then we have inheritance of convexity for $\mathcal{P}$.

Let us now consider a weakly union-closed family $\mathcal{F}$ on N. Let us moreover assume that $\emptyset \in \mathcal{F}$ and $\{i\} \in \mathcal{F}$ for all $i \in N$. Such a family is called a partition system in [2]. For a given subset $A \subseteq N$, we set

$\mathcal{F}(A) = \{F \in \mathcal{F}, F \subseteq A\}$. It immediately results from the definitions that the subfamily of maximal subsets of $\mathcal{F}(A)$ is a partition of A. We denote by $\mathcal{P}_{\mathcal{F}}$ the correspondence associating to any subset $A \subseteq N$ the partition $\mathcal{P}_{\mathcal{F}}(A)$ into maximal subsets of $\mathcal{F}(A)$.

1) Let us consider $A \subset B \subseteq N$ and $A_j \in \mathcal{P}_{\mathcal{F}}(A)$. Then $A_j \in \mathcal{F}(B)$ and there exists $B_m \in \mathcal{P}_{\mathcal{F}}(B)$ such that $A_j \subseteq B_m$. Hence $\mathcal{P}_{\mathcal{F}}(A)$ is a refinement of $\mathcal{P}_{\mathcal{F}}(B)_{|A}$ (then by Theorem 1 there is inheritance of superadditivity for $\mathcal{P}_{\mathcal{F}}$).

2) For $i \in N$, $A \subset B \subseteq N \setminus \{i\}$, and $B^{'} \in \mathcal{P}(B \cup \{i\}$, we now prove that there is no cyclic intersecting sequence w.r.t. $\mathcal{C}_i(A, B, B^{'})$ for $\mathcal{P}_{\mathcal{F}}$. Let us consider $A_l^{'} \in \mathcal{P}_{\mathcal{F}}(A \cup \{i\})$ and $B_m \in \mathcal{P}_{\mathcal{F}}(B)$ such that $A_l^{'} \cap B_m \neq \emptyset$. Let us assume $i \notin A_l^{'}$. Then we have $A_l^{'} \subseteq A$ and therefore $A_l^{'} \in \mathcal{F}(A)$. As $A_l^{'}$ is maximal in $\mathcal{F}(A \cup \{i\})$, it is also maximal in $\mathcal{F}(A)$ and therefore $A_l^{'} \in \mathcal{P}_{\mathcal{F}}(A)$. By 1), $\mathcal{P}_{\mathcal{F}}(A)$ is a refinement of $\mathcal{P}_{\mathcal{F}}(B)$ and therefore $A_l^{'} \cap B_m = A_l^{'}$. Hence $A_l^{'} \setminus B_m = \emptyset$ and $A_l^{'}$ and B_m are not intersecting subsets. Thus a block $B_m \in \mathcal{P}_{\mathcal{F}}(B)$ can only cross the unique block $A_1^{'}$ of $\mathcal{P}(A \cup \{i\})$ containing i. Therefore there is no cyclic intersecting sequence w.r.t. $\mathcal{C}_i(A, B, B')$. Moreover any intersecting sequence w.r.t. $\mathcal{C}_i(A, B, B')$ has length at most 3.

Let us now consider that $\mathcal{F}$ satisfies the previous conditions and is moreover an intersecting family[3]. Let us consider $i \in N$, $A \subset B \subseteq N \setminus \{i\}$, and $A^{'} \in \mathcal{P}_{\mathcal{F}}(A \cup \{i\})$. We want to prove that $\mathcal{P}_{\mathcal{F}}(A)_{|A'} = \mathcal{P}_{\mathcal{F}}(B)_{|A'}$. Let us consider B_m in $\mathcal{P}_{\mathcal{F}}(B)$ such that $A^{'} \cap B_m \neq \emptyset$. Then $A^{'} \cap B_m \in \mathcal{F}(A)$ and there exists A_j in $\mathcal{P}_{\mathcal{F}}(A)$ such that $A^{'} \cap B_m \subseteq A_j$. Now as $A_j \in \mathcal{P}_{\mathcal{F}}(A)$, we have $A_j \in \mathcal{F}(A)$. As $A \subset A \cup \{i\}$ (resp. $A \subset B$) we also have $A_j \in \mathcal{F}(A \cup \{i\})$ (resp. $A_j \in \mathcal{F}(B)$) and therefore there exists $A^* \in \mathcal{P}_{\mathcal{F}}(A \cup \{i\})$ (resp. $B^* \in \mathcal{P}_{\mathcal{F}}(B)$) such that $A_j \subseteq A^*$ (resp. $A_j \subseteq B^*$). As $A^{'} \cap A_j \neq \emptyset$ (resp. $B_m \cap A_j \neq \emptyset$), we also have $A^{'} \cap A^* \neq \emptyset$ (resp. $B_m \cap B^* \neq \emptyset$) and therefore, by definition of $\mathcal{P}_{\mathcal{F}}$, $A^{'} = A^*$ (resp. $B_m = B^*$). Hence we have $A_j \subseteq A^{'} \cap B_m$ and therefore $A_j = A^{'} \cap B_m$. We get $\mathcal{P}_{\mathcal{F}}(A)_{|A'} = \mathcal{P}_{\mathcal{F}}(B)_{|A'}$. Then by Theorem 3 there is inheritance of convexity for unanimity games. Finally by Theorem 18 there is inheritance of convexity for $\mathcal{P}_{\mathcal{F}}$ and $\mathcal{P}_{\mathcal{F}}$ satisfies the Cyclic Intersecting Sequence Free Condition.

Moreover for $i \in N$, and $A \subset B \subseteq N \setminus \{i\}$, 2) and Theorem 3, imply that $A \cup \{i\}$ and B satisfy the conditions of Proposition 11, and therefore:

$$\Delta_i \overline{v}(A, B) = \sum_{B' \in \mathcal{P}(B \cup \{i\})} \left[\sum_{C \in \mathcal{C}(B')} \hat{\Delta} v(C) + v(B') - \sum_{C \in \mathcal{C}(B')} v(C) \right].$$

[3]A family $\mathcal{F} \subseteq 2^N$ is an intersecting family if for all $A, B \in \mathcal{F}$ such that $A \cap B \neq \emptyset$ we have $A \cap B$ and $A \cup B$ in $\mathcal{F}$.

Then we only need to assume the superadditivity and the $\mathcal{F}$-convexity of the game (N, v) to obtain the convexity of $(N, \overline{v})$. This last result was already proved by Faigle [7] for intersecting convex games (in [7] a game $\tilde{v}$ different from $\overline{v}$ is considered but these games coincide when v is superadditive).

We now consider the $\mathcal{P}_{\min}$ correspondence. $\mathcal{P}_{\min}$ does not always satisfy the Cyclic Intersecting Sequence Free Condition but we prove that the condition is satisfied if there is inheritance of convexity for unanimity games. The following results are also valid for the local $\tilde{\mathcal{P}}_{\min}$-restricted game where for any subset A of N, $\tilde{\mathcal{P}}_{\min}(A)$ is defined by $\tilde{\mathcal{P}}_{\min}(A) = \{\mathcal{P}_{\min}(A_1), \mathcal{P}_{\min}(A_2), \ldots, \mathcal{P}_{\min}(A_l)\}$ where A_1, A_2, ..., A_l are the connected components of G_A.

Proposition 19. *Let us consider a weighted graph $G = (N, E, w)$ and the family $\mathcal{F}$ of connected subsets of N. Let us assume that for all $\emptyset \neq S \subseteq N$, the $\mathcal{P}_{\min}$-restricted game $(N, \overline{u_S})$ is convex (resp. $\mathcal{F}$-convex). Then $\mathcal{P}_{\min}$ satisfies the Cyclic Intersecting Sequence Free Condition. Moreover for all $\{i, j\} \subset N$, $A \subseteq N \setminus \{i, j\}$ and $A' \in \mathcal{P}_{\min}(A \cup \{i, j\})$, an intersecting sequence w.r.t. $\mathcal{C}_{i,j}(A, A')$ has length at most 3.*

Proof. For any subset $A \subseteq N$, we define $\sigma(A) := \min_{e \in E(A)} w(e)$. Let us consider $\{i, j\} \subset N$, $A \subseteq N \setminus \{i, j\}$ and $A' \in \mathcal{P}_{\min}(A \cup \{i, j\})$. By contradiction, let us consider an intersecting sequence of length 4, $\{C_1, C_2, C_3, C_4\}$, w.r.t. $\mathcal{C}_{i,j}(A, A')$. We set $\mathcal{P}_{\min}(A \cup \{i\}) = \{A_1^{'}, A_2^{'}, \ldots, A_{p'}^{'}\}$ and $\mathcal{P}_{\min}(A \cup \{j\}) =$

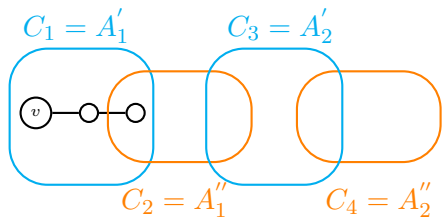

Figure 5: Intersecting sequence $\{C_1, C_2, C_3, C_4\}$ and path γ.

$\{A_1^{''}, A_2^{''}, \ldots, A_{p''}^{''}\}$. Interchanging i and j and renumbering if necessary, we can assume $C_1 = A_1^{'}$, $C_2 = A_1^{''}$, $C_3 = A_2^{'}$, $C_4 = A_2^{''}$, as represented in Figure 5.

Let us first assume $i \notin A_1^{'}$. Then we have $A_1^{'} \subseteq A \subset A \cup \{j\}$ and therefore any edge $e_1^{'}$ in $E(A_1^{'})$ satisfies $w(e_1^{'}) \geq \sigma(A \cup \{j\})$. As $A_1^{'}$ and $A_1^{''}$ are intersecting subsets, there exists a vertex $v \in A_1^{'} \setminus A_1^{''}$. By definition of $A_1^{'}$, we can find a path γ in $A_1^{'}$ linking v to $A_1^{''}$ so that for all edge $e_1^{'}$ in γ, $w(e_1^{'}) > \sigma(A \cup \{i\})$. If for all edge $e_1^{'}$ in γ we have $w(e_1^{'}) > \sigma(A \cup \{j\})$, then γ is also a path in $A_1^{''}$ and therefore $v \in A_1^{''}$, a contradiction. Hence, there exists an edge $e_1^{'} \in \gamma$ such that:

$$\sigma(A \cup \{j\}) = w(e_1^{'}) > \sigma(A \cup \{i\}). \tag{31}$$

If $j \notin A_1^{''}$, by the same reasoning as before, interchanging i and j, $A_1^{'}$ and $A_1^{''}$, we get $\sigma(A \cup \{i\}) > \sigma(A \cup \{j\})$, contradicting (31). Hence $j \in A_1^{''}$. Then $j \notin A_2^{''}$, and we can apply again the same reasoning as before to the pair of subsets $(A_2^{'}, A_2^{''})$, interchanging respectively i and j, $A_1^{'}$ and $A_2^{''}$, $A_1^{''}$ and $A_2^{'}$ so that $\sigma(A \cup \{i\}) > \sigma(A \cup \{j\})$, still contradicting (31).
Let us now assume $i \in A_1^{'}$. Hence we have $i \notin A_2^{'}$ and we can follow the same reasoning as in the preceding case with $i \notin A_1^{'}$. Hence an intersecting sequence w.r.t. $\mathcal{C}_{i,j}(A, A')$ has length at most 3 and therefore there is no cyclic intersecting sequence w.r.t. $\mathcal{C}_{i,j}(A, A')$. Then Theorem 18 implies that $\mathcal{P}_{\min}$ satisfies the Cyclic Intersecting Sequence Free Condition. □

Corollary 20. *Let us consider a weighted graph $G = (N, E, w)$ and the family $\mathcal{F}$ of connected subsets of N. Then the following properties are equivalent:*

1) For all $\emptyset \neq S \subseteq N$, the $\mathcal{P}_{\min}$-restricted game $(N, \overline{u_S})$ is convex (resp. $\mathcal{F}$-convex).

2) For each convex (resp. superadditive and $\mathcal{F}$-convex) game (N, v), the $\mathcal{P}_{\min}$-restricted game $(N, \overline{v})$ is convex (resp. $\mathcal{F}$-convex).

Proof. By Proposition 19 and Theorem 17, we have that 1) implies 2). And obviously 2) implies 1). □

Let us consider a graph $G = (N, E)$ and a game (N, v). We now consider the correspondence $\mathcal{P}_M$ associating to any subset $A \subseteq N$ its partition into connected components. Then the $\mathcal{P}_M$-restricted game $(N, \overline{v})$ corresponds to Myerson's restricted game. $\mathcal{P}_M$ is a particular case of the correspondence $\mathcal{P}_{\mathcal{F}}$ defined page 21 taking for $\mathcal{F}$ the family of connected subsets of N. Hence $\mathcal{P}_M$ satisfies the following result.

Proposition 21. *Let $G = (N, E)$ be a graph and let us consider Myerson's correspondence $\mathcal{P}_M$. Then for all $i \in N$, for all $A \subset B \subseteq N \setminus \{i\}$ and for all $B^{'} \in \mathcal{P}(B \cup \{i\}$, there is no cyclic intersecting sequence w.r.t. $\mathcal{C}_i(A, B, B^{'})$.*

Hence by Theorem 18, to verify inheritance of convexity from an underlying game to Myerson's restricted game we only need to check the inheritance for unanimity games as in van den Nouweland and Borm [16]. We will use Theorem 3 to study inheritance of convexity for unanimity games and give a new proof of the following result.

Theorem 22 (van den Nouweland and Borm [16])**.** *Let $G = (N, E)$ be a graph and let us consider Myerson's correspondence $\mathcal{P}_M$. There is inheritance of convexity for $\mathcal{P}_M$ if and only if G is cycle-complete.*

Proof. Let us assume that for all $\emptyset \neq S \subseteq N$, the $\mathcal{P}_M$-restricted game $(N, \overline{u_S})$ is convex. We will prove that every cycle C of G is complete by induction on $|V(C)|$. Let us assume that it is true for any cycle C with

$|V(C)| \leq m-1$, and let us consider a cycle $C = \{1, e_1, 2, e_2, \ldots, m, e_m, 1\}$ with $m \geq 4$. Let us assume that C is not complete. We can suppose w.l.o.g. that the edge $\{1, l\}$ is not a chord of C for some l, $3 \leq l \leq m-1$, after renumbering if necessary. If C has no chord, we choose arbitrarily a vertex i

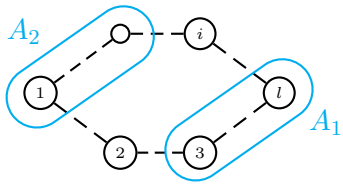

Figure 6: Chordless cycle.

with $l < i \leq m$. We define $A_1 := \{3, \ldots, l, \ldots, i-1\}$, $A_2 := \{i+1, \ldots, m, 1\}$, as represented in Figure 6, $A := A_1 \cup A_2$, and $B := A \cup \{2\} = V(C) \setminus \{i\}$. Then we have $A \subseteq B \subseteq N \setminus \{i\}$ and $\mathcal{P}_M(A) = \{A_1, A_2\}$, $\mathcal{P}_M(A \cup \{i\}) = \{A \cup \{i\}\}$, and $\mathcal{P}_M(B) = \{B\}$. Taking $A' = A \cup \{i\}$, we get $\mathcal{P}_M(B)_{|A'} = \{A\} \neq \{A_1, A_2\} = \mathcal{P}_M(A)_{|A'}$, and it contradicts Theorem 3.

Let us now assume that C has at least one chord $\{i, j\}$, with $1 \leq i < j \leq m$. If $m = 4$ then we have $l = 3$, and $i = 2$, $j = 4$ after renumbering if necessary. We consider $A_1 = \{l\}$ and $A_2 = \{1\}$ as represented in Figure 7, $A = A_1 \cup A_2$, and $B = A \cup \{j\}$. We get the same contradiction as before.

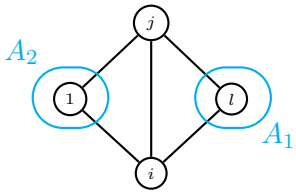

Figure 7: Cycle of size 4 with chord $\{i, j\}$.

Let us now assume $m \geq 5$. If $1 \leq i < j \leq l$ or if $l \leq i < j \leq m$ as represented in Figure 8, we can consider a smaller cycle C' (using chord $\{i, j\}$) which

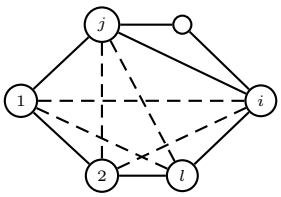

Figure 8: Cycle with chord $\{i, j\}$ and $l \leq i < j \leq m$.

contains the vertices $1, i, j, l$ with $|V(C')| \leq m-1$. C' is complete by induction. Therefore $e := \{1, l\}$ is a chord of C' and also a chord of C, a contradiction. If now $1 \leq i < l < j \leq m$, as $m \geq 5$ at least one of the cycles $C' := \{1, 2, \ldots, i, j, j+1, \ldots, m, 1\}$, $C'' := \{i, i+1, \ldots, l, \ldots, j, i\}$, has

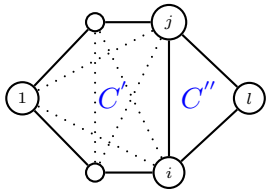

Figure 9: Cycle with chord $\{i, j\}$ and $1 < i < l < j \leq m$.

a size larger than 4. Let us assume w.l.o.g. $|V(C^{'})| \geq 4$, as represented in Figure 9. By induction $C^{'}$ and $C^{''}$ are complete. Hence $\{1, i\}$ or $\{1, j\}$ is a chord of $C^{'}$ and therefore of C. Let us assume w.l.o.g. $\{1, i\} \in E(C)$. Then we can consider the cycle $\tilde{C} := \{1, i, i+1, \ldots, m, 1\}$. As $|V(\tilde{C})| < |V(C)|$, $\tilde{C}$ is complete and $\{1, l\}$ is a chord of $\tilde{C}$ and therefore of C, a contradiction.

Let us now assume that the graph G is cycle-complete. Let us consider $A \subseteq B \subseteq N \setminus \{i\}$ such that $\mathcal{P}_M(A) = \{A_1, A_2, \ldots, A_p\}$, $\mathcal{P}_M(B) = \{B_1, B_2, \ldots, B_q\}$. We can assume $\mathcal{P}_M(A \cup \{i\}) = \{A_1 \cup A_2 \cup \ldots \cup A_t \cup \{i\}, A_{t+1}, \ldots, A_p\}$ for some t, $1 \leq t \leq p$, after renumbering if necessary. Then there exist edges $\{i, j_1\}, \{i, j_2\}, \ldots, \{i, j_t\}$ with $j_k \in A_k$ for all k, $1 \leq k \leq t$, and there is no edge linking i to subsets $A_{t+1}, \ldots, A_p$. By connectivity we have:

$$A_j \cap B_k = \emptyset \text{ or } A_j \subseteq B_k, \forall A_j \in \mathcal{P}_M(A),\ B_k \in \mathcal{P}_M(B). \tag{32}$$

Let us consider a component $A^{'} \in \mathcal{P}_M(A \cup \{i\})$. If $A^{'} = A_k$ for some k, $t+1 \leq k \leq p$, we have $\mathcal{P}_M(B)_{|A^{'}} = \mathcal{P}_M(B)_{|A_k} = \{A_k\} = \mathcal{P}_M(A)_{|A^{'}}$. Now if $A^{'} = A_1 \cup A_2 \cup \ldots \cup A_t \cup \{i\}$, let us assume $\mathcal{P}_M(B)_{|A^{'}} \neq \mathcal{P}_M(A)_{|A^{'}}$, *i.e.*, $\mathcal{P}_M(B)_{|A^{'}} \neq \{A_1, A_2, \ldots, A_t\}$. Then (32) implies that $A_1 \cup A_2 \subseteq B_1$, after renumbering if necessary. As A_1 and A_2 are in the connected component B_1, there exists an elementary path γ in B_1 linking j_1 and j_2. As $i \notin B_1$, $\{i, j_1\}$,

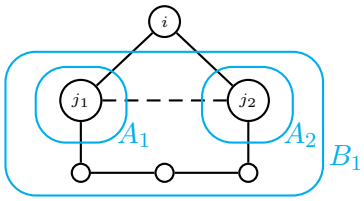

Figure 10: Cycle C induced by $\{i, j_1\}, \{i, j_2\}$, and γ.

$\{i, j_2\}$, and γ form a cycle C as represented in Figure 10. By assumption C is complete, therefore $\{j_1, j_2\}$ is an edge in G_A connecting A_1 to A_2, a contradiction. Therefore $\mathcal{P}_M(B)_{|A^{'}} = \mathcal{P}_M(A)_{|A^{'}}$ for all $A^{'} \in \mathcal{P}_M(A \cup \{i\})$. By Theorems 3 and 17 we have inheritance of convexity for $\mathcal{P}_M$. $\square$

5 Examples of correspondences with cyclic intersecting sequences

We consider a finite set $N = \{1, 2, \ldots, 2n, 2n+1, 2n+2\}$. Thus $|N| = 2n+2$. We set $i = 2n+1$, $j = 2n+2$ and $A = \{1, 2, \ldots, 2n\}$. We denote by $\mathcal{P}_{\text{sing}}(A)$ the partition of A into singletons. For two given partitions $\mathcal{P}_1(A)$ and $\mathcal{P}_2(A)$ of A we define the partition:

$$\mathcal{P}(A) := \{A_1 \cap A_2;\ A_1 \in \mathcal{P}_1(A), A_2 \in \mathcal{P}_2(A), \text{with } A_1 \cap A_2 \neq \emptyset\}.$$

We will consider partitions $\mathcal{P}_1(A)$ and $\mathcal{P}_2(A)$ such that:

$$\mathcal{P}(A) = \mathcal{P}_{\text{sing}}(A). \tag{33}$$

For instance, we can take $\mathcal{P}_1(A) = \{\{1, 3, \ldots, 2n-1\}, \{2, 4, \ldots, 2n\}\}$ the partition in odd (resp. even) integers and $\mathcal{P}_2(A) = \{\{1, 2\}, \{3, 4\}, \ldots, \{2k-1, 2k\}, \ldots, \{2n-1, 2n\}\}$.
We now define $\mathcal{P}(A \cup \{i\}) := \{\mathcal{P}_1(A), \{i\}\}$ and $\mathcal{P}(A \cup \{j\}) := \{\mathcal{P}_2(A), \{j\}\}$. We set $B := A \cup \{j\}$ so that $A \subset B \subseteq N \setminus \{i\}$. According to the definition of $\mathcal{P}(A \cup \{i\})$ and $\mathcal{P}(B)$, if $A^{'} \in \mathcal{P}(A \cup \{i\})$, then $A^{'} \in \mathcal{P}_1(A)$ or is the singleton $\{i\}$ and if $\tilde{B} \in \mathcal{P}(B)$, $\tilde{B} \in \mathcal{P}_2(A)$ or is the singleton $\{j\}$. Hence, by assumption (33) on $\mathcal{P}_1(A)$ and $\mathcal{P}_2(A)$, $\mathcal{P}(B)_{|A'}$ is the singleton partition of $A^{'}$. Therefore we have $\mathcal{P}(B)_{|A'} = \mathcal{P}_{\text{sing}}(A^{'}) = \mathcal{P}(A)_{|A'}$. Of course, we can interchange the roles of i and j. We complete the definition of $\mathcal{P}$ by setting $\mathcal{P}(N) := \{A \cup \{j\}, \{i\}\} = \{B, \{i\}\}$ or $\mathcal{P}(N) := \{N\}$. Hence for $B^{'} \in \mathcal{P}(B \cup \{i\}) = \mathcal{P}(N)$, $B^{'} = B$ or $\{i\}$ or $B^{'} = N$. Finally, for $C \neq A, A \cup \{i\}, B = A \cup \{j\}, N$, we set $\mathcal{P}(C) := \mathcal{P}_{\text{sing}}(C)$. Then obviously, for all $\tilde{A} \subset \tilde{B} \subseteq N$, $\mathcal{P}(\tilde{A})$ is a refinement of $\mathcal{P}(\tilde{B})$ and for all $k \in N$, for all $\tilde{A} \subset \tilde{B} \subseteq N \setminus \{k\}$ and all $A^{'} \in \mathcal{P}(\tilde{A} \cup \{k\})$, we have $\mathcal{P}(\tilde{A})_{|A'} = \mathcal{P}(\tilde{B})_{|A'}$ (by construction if $k = i$ or j and otherwise for trivial reasons $A^{'}$ being a singleton). Hence by Theorem 3, for such a correspondence $\mathcal{P}$, we always have inheritance of convexity for unanimity games.

In the following examples, we only need to make an accurate choice for $\mathcal{P}_1(A)$ and $\mathcal{P}_2(A)$ to obtain many cyclic intersecting sequences in $\mathcal{P}$.

Example 1
We take $\mathcal{P}_1(A) = \{A_1^{'}, A_2^{'}\}$ with $A_1^{'} = \{1, 3, \ldots, 2n-1\}$ and $A_2^{'} = \{2, 4, \ldots, 2n\}$, and $\mathcal{P}_2(A) = \{B_1, B_2, \ldots, B_n\}$ where $B_k = \{2k-1, 2k\}$, for all k, $1 \leq k \leq n$. Hence we have $A_1^{'} \cap B_k = \{2k-1\}$ and $A_2^{'} \cap B_k = \{2k\}$, for all k, $1 \leq k \leq n$. Then we have $\mathcal{P}(A \cup \{i\}) = \{A_1^{'}, A_2^{'}, \{i\}\}$ and $\mathcal{P}(B) = \mathcal{P}(A \cup \{j\}) = \{B_1, B_2, \ldots, B_n, \{j\}\}$. Each sequence $\{A_1^{'}, B_k, A_2^{'}, B_l, A_1^{'}\}$, with $1 \leq k < l \leq n$, corresponds to a cyclic intersecting sequence w.r.t. $\mathcal{C}_i(A, B, B^{'})$ with $B^{'} = B$ or N. Thus we have $\frac{n(n-1)}{2}$ cyclic intersecting sequences of length 4 as represented in Figure 11.

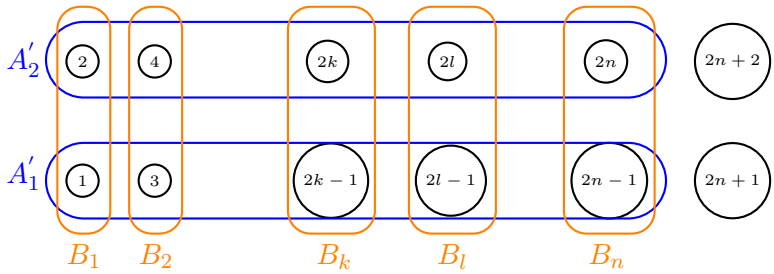

Figure 11: Cyclic intersecting sequences.

Example 2
We consider the partitions $\mathcal{P}_1(A) = \{\{1,2\},\{3,4\},\ldots,\{2k-1,2k\},\ldots,\{2n-1,2n\}\}$, $\mathcal{P}_2(A) = \{\{2,3\},\{4,5\},\ldots,\{2k,2k+1\},\ldots,\{2n-2,2n-1\},\{2n,1\}\}$, and $\mathcal{P}(A)$ which is by construction the singleton partition $\{\{1\},\{2\},\ldots,\{2n\}\}$. We set $A_k^{'} := \{2k-1,2k\}$, for all k, $1 \leq k \leq n$, $B_k := \{2k,2k+1\}$, for all k, $1 \leq k \leq n-1$, and $B_n := \{2n,1\}$. By construction, we obtain a cyclic intersecting sequence $C := \{A_1^{'}, B_1, A_2^{'}, B_2, \ldots, A_k^{'}, B_k, \ldots, A_n^{'}, B_n, A_1^{'}\}$ w.r.t. $\mathcal{C}_i(A,B,B^{'})$ with $B^{'} = B$ or N, of length $2n$ as represented in Figure 12 with $n = 2$.

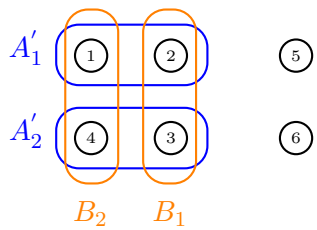

Figure 12: Cyclic intersecting sequence $\{A_1^{'}, B_1, A_2^{'}, B_2, A_1^{'}\}$.

Example 3
We now build a similar example with several given cyclic intersecting sequences. For that, we consider a first partition of A into p subsets $\{A_1, A_2, \ldots, A_p\}$ such that $|A_k| = 2l_k$ with $l_k \geq 2$, for all k, $1 \leq k \leq p$. Hence we have $n = l_1 + l_2 + \ldots + l_p$. On each A_k, we consider the numbering induced by the numbering of A (we could also consider a specific numbering depending on k) and then we can construct two partitions $\mathcal{P}_1(A_k)$ and $\mathcal{P}_2(A_k)$ by the same procedure as before for A in Example 2 (replacing A by A_k). Then the intersection of a subset of $\mathcal{P}_1(A_k)$ with a subset of $\mathcal{P}_2(A_k)$ is either a singleton or the empty set. We now define the partitions $\mathcal{P}_1(A)$ and $\mathcal{P}_2(A)$ by:

$$\mathcal{P}_r(A) = \{\mathcal{P}_r(A_1), \mathcal{P}_r(A_2), \ldots, \mathcal{P}_r(A_k), \ldots, \mathcal{P}_r(A_p)\},\ r = 1,2. \tag{34}$$

Then $\mathcal{P}(A) = \mathcal{P}_{\text{sing}}(A)$. As before we define $\mathcal{P}(A \cup \{i\}) = \{\mathcal{P}_1(A), \{i\}\}$, $\mathcal{P}(A \cup \{j\}) = \{\mathcal{P}_2(A), \{j\}\}$, $\mathcal{P}(N) = \{A \cup \{j\}, \{i\}\}$ or $\mathcal{P}(N) = \{N\}$, and

$B = A \cup \{j\}$. As in the preceding examples, we have inheritance of convexity for unanimity games (for all $\tilde{A} \subset \tilde{B} \subseteq N$, $\mathcal{P}(\tilde{A})$ is a refinement of $\mathcal{P}(\tilde{B})$ and for all $k \in N$, for all $\tilde{A} \subset \tilde{B} \subseteq N \setminus \{k\}$, for all $A' \in \mathcal{P}(\tilde{A} \cup \{k\})$, $\mathcal{P}(\tilde{A})_{|A'} = \mathcal{P}(\tilde{B})_{|A'}$ is the singleton partition).
But now, by construction, for each subset A_k of A, $1 \leq k \leq p$, we obtain a cyclic intersecting sequence w.r.t. $\mathcal{C}_i(A, B, B^{'})$ with $B^{'} = B$ or N, of length $2l_k$, with blocks of $\mathcal{P}(B)$ and $\mathcal{P}(A \cup \{i\})$ in A_k. For instance, if $|N| = 12$ taking $A_1 = \{1, 2, 3, 4, 5, 6\}$ and $A_2 = \{7, 8, 9, 10\}$, we get two cyclic intersecting sequences as represented in Figure 13.

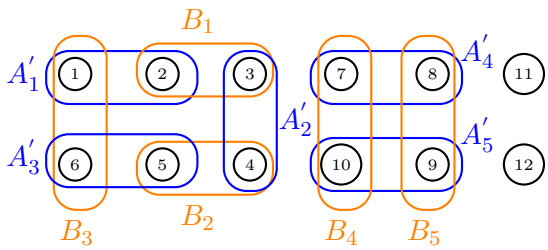

Figure 13: Two cyclic intersecting sequences of length 6 and 4.

6 The Shapley value of the reduced game

In this section, we compute the Shapley value of the $\mathcal{P}$-restricted game $(N, \overline{v})$, assuming that there is inheritance of superadditivity for $\mathcal{P}$. Let us recall that the Shapley value of the game (N, v) is a vector $\Phi(v) \in R^n$ defined by:

$$\Phi_i(v) = \sum_{i \notin A \subset N} C_{|A|} \; \Delta_i v(A)$$

where the constant $C_{|A|}$ is defined by $C_{|A|} = \frac{|A|!(n-1-|A|)!}{n!}$ and $i \in N$.
Hence the Shapley value of the game $(N, \overline{v})$ is given by:

$$\Phi_i(\overline{v}) = \sum_{i \notin A \subset N} C_{|A|} \; \Delta_i \overline{v}(A).$$

We use the decomposition of v as a linear combination of unanimity games $v = \sum_{S \subseteq N} \lambda_S u_S$. Then we have by linearity $\overline{v} = \sum_{S \subseteq N} \lambda_S \overline{u_S}$ and $\Delta_i \overline{v}(A) = \sum_{S \subseteq N} \lambda_S \Delta_i \overline{u_S}(A)$. Let us recall that for $A \subseteq N \setminus \{i\}$ we set $\mathcal{P}(A) = \{A_1, A_2, \ldots, A_p\}$ and $\mathcal{P}(A \cup \{i\}) = \{A_1^{'}, A_2^{'}, \ldots, A_{p'}^{'}\}$. As there is inheritance of superadditivity for $\mathcal{P}$, Theorem 1 and Lemma 8 imply:

$$\Delta_i \overline{v}(A) = \sum_{S \in \mathcal{S}'(A,i)} \lambda_S = \sum_{l=1}^{p'} \sum_{S \in \mathcal{S}_l'(A,i)} \lambda_S \tag{35}$$

where $\mathcal{S}_l^{'}(A,i) := \{\emptyset \neq S \subseteq A_l^{'},\ \forall A_m \subseteq A_l^{'},\ S \cap (A_l^{'} \setminus A_m) \neq \emptyset\}$ and $\mathcal{S}^{'}(A,i) := \bigcup_l \mathcal{S}_l^{'}(A,i)$. Henceforward we will assume that $i \in A_1^{'}$ (after renumbering if necessary).

We slightly change the notation of $A_1^{'}$ by setting $\mathcal{P}(A \cup \{i\}) = \{A_1^{'} \cup \{i\}, A_2^{'}, \ldots, A_{p'}^{'}\}$ so that now we have $A_1^{'} \subseteq A$ and (35) can be written:

$$\Delta_i \overline{v}(A) = \sum_{S \subseteq A_1^{'} \cup \{i\}, \forall A_m \subseteq A_1^{'},\, S \not\subseteq A_m} \lambda_S + \sum_{l=2}^{p^{'}} \sum_{S \in \mathcal{S}_l^{'}(A,i)} \lambda_S.$$

We now separate in the first sum, terms with $i \in S$ from the others:

$$\Delta_i \overline{v}(A) = \sum_{i \in S,\, S \setminus \{i\} \subseteq A_1^{'}, \forall A_m \subseteq A_1^{'},\, S \not\subseteq A_m} \lambda_S + \sum_{S \subseteq A_1^{'}, \forall A_m \subseteq A_1^{'},\, S \not\subseteq A_m} \lambda_S + \sum_{l=2}^{p^{'}} \sum_{S \in \mathcal{S}_l^{'}(A,i)} \lambda_S. \tag{36}$$

Let us observe that, if $i \in S$, then $S \not\subseteq A_m$ (as $i \notin A$) and (36) becomes:

$$\Delta_i \overline{v}(A) = \sum_{i \in S,\, S \setminus \{i\} \subseteq A_1^{'}} \lambda_S + \sum_{l=1}^{p^{'}} \sum_{S \in \mathcal{S}_l^{'}(A,i)} \lambda_S. \tag{37}$$

By Lemma 7 applied to $A_1^{'}$, we have $\Delta_i v(A_1^{'}) = \sum_{i \in S,\, S \setminus \{i\} \subseteq A_1^{'}} \lambda_S$ and then (37) becomes:

$$\Delta_i \overline{v}(A) = \Delta_i v(A_1^{'}) + \sum_{l=1}^{p^{'}} \sum_{S \in \mathcal{S}_l^{'}(A,i)} \lambda_S. \tag{38}$$

But (as the A_m's are disjoints subsets), we obviously have:

$$\sum_{S \in \mathcal{S}_l^{'}(A,i)} \lambda_S = \sum_{S \subseteq A_l^{'}} \lambda_S - \sum_{A_m \subseteq A_l^{'}} \left[\sum_{S \subseteq A_m} \lambda_S \right] \tag{39}$$

which is equivalent to:

$$\sum_{S \in \mathcal{S}_l^{'}(A,i)} \lambda_S = v(A_l^{'}) - \sum_{A_m \subseteq A_l^{'}} v(A_m). \tag{40}$$

Then (38) and (40) imply:

$$\Delta_i \overline{v}(A) = \Delta_i v(A_1^{'}) + \sum_{l=1}^{p^{'}} \left[v(A_l^{'}) - \sum_{A_m \subseteq A_l^{'}} v(A_m) \right]. \tag{41}$$

If the game (N, v) is superadditive, (41) implies the following inequality:

$$\Delta_i \overline{v}(A) \geq \Delta_i v(A_1^{'}). \tag{42}$$

If the game (N, v) is convex, as $A_1^{'} \subseteq A \subseteq N \setminus \{i\}$, we have the inequality:

$$\Delta_i v(A) \geq \Delta_i v(A_1^{'}). \tag{43}$$

To give a more intrinsic formulation of relations (41) to (43), we now define, for all $A \subseteq N \setminus \{i\}$ the subset $A_{(i)}^{'} \subseteq A$ by:

$$A_{(i)}^{'} \cup \{i\} \in \mathcal{P}(A \cup \{i\}). \tag{44}$$

$A_{(i)}^{'} \cup \{i\}$ is the unique block in $\mathcal{P}(A \cup \{i\})$ containing i. Then (41) can be rewritten as follows:

$$\Delta_i \overline{v}(A) = \Delta_i v(A_{(i)}^{'}) + \sum_{A' \in \mathcal{P}(A \cup \{i\})_{|A}} \left[v(A^{'}) - \sum_{\tilde{A} \in \mathcal{P}(A),\, \tilde{A} \subseteq A'} v(\tilde{A}) \right]. \tag{45}$$

To compute and estimate the Shapley value of $(N, \overline{v})$, we define two new values $\Phi_i^{'}(\overline{v})$ and $\Phi_i^{''}(\overline{v})$ (this last value is directly related to the superadditivity of (N, v)):

$$\Phi_i^{'}(\overline{v}) = \sum_{i \notin A \subset N} C_{|A|}\ \Delta_i v(A_{(i)}^{'}) \tag{46}$$

$$\Phi_i^{''}(\overline{v}) = \sum_{i \notin A \subset N} C_{|A|} \left[\sum_{A' \in \mathcal{P}(A \cup \{i\})_{|A}} \left[v(A^{'}) - \sum_{\tilde{A} \in \mathcal{P}(A),\, \tilde{A} \subseteq A'} v(\tilde{A}) \right] \right]. \tag{47}$$

Then (45) implies:

$$\Phi_i(\overline{v}) = \Phi_i^{'}(\overline{v}) + \Phi_i^{''}(\overline{v}). \tag{48}$$

If the game (N, v) is superadditive, we have $\Phi_i^{''}(\overline{v}) \geq 0$ and if (N, v) is convex we moreover have (as $A_{(i)}^{'} \subseteq A \subseteq N \setminus \{i\}$) $\Phi_i(v) \geq \Phi_i^{'}(\overline{v})$. Hence we have the following result.

Theorem 23. *Let $\mathcal{P}$ be a correspondence on N. Let us assume that there is inheritance of superadditivity for $\mathcal{P}$. Then the Shapley value of the $\mathcal{P}$-restricted game $(N, \overline{v})$ is given by:*

$$\Phi_i(\overline{v}) = \Phi_i^{'}(\overline{v}) + \Phi_i^{''}(\overline{v}). \tag{49}$$

If the game (N, v) is superadditive, we have:

$$\Phi_i(\overline{v}) \geq \Phi_i^{'}(\overline{v}). \tag{50}$$

If the game (N, v) is convex, we have:

$$\min(\Phi_i(v), \Phi_i(\overline{v})) \geq \Phi_i^{'}(\overline{v}). \tag{51}$$

7 Conclusion

Our main result gives necessary and sufficient conditions on a correspondence $\mathcal{P}$ to have inheritance of convexity or $\mathcal{F}$-convexity. Moreover we have proved that for the Myerson's correspondence and for the $\mathcal{P}_{\min}$ correspondence, we only need to verify inheritance of convexity for unanimity games because of the non existence of a cyclic intersecting sequence.
Does a similar result hold for the correspondence $\mathcal{P}_G$ associated with the strength of a graph presented in [9], which gives natural partitions and coincides with $\mathcal{P}_{\min}$ on cycle-free graphs? Does there exist cyclic intersecting sequences in the case of $\mathcal{P}_G$? As the inheritance of superadditivity for $\mathcal{P}_G$ is not always satisfied and as its characterization is not obvious, the answer seems not easy at all.

If there is inheritance of convexity for unanimity games for a given correspondence $\mathcal{P}$ on N and if there exist cyclic intersecting sequences, another interesting question is to study the class of convex games (N, v) such that the $\mathcal{P}$-restricted game $(N, \overline{v})$ is also convex. We may call such games $\mathcal{P}$-convex games.

It would be also of great interest to study the complexity of the problem. We can hope that in many specific situations, we do not need to consider all cyclic intersecting sequences but only a few of them. For instance, for the $\mathcal{P}_{\min}$ correspondence and the inheritance of $\mathcal{F}$-convexity, we proved in a forthcoming paper [15] that we only have to consider a polynomial number of paths and cycles associated with a minimum weight spanning tree. Then we are able to construct a polynomial time algorithm to decide of the inheritance of $\mathcal{F}$-convexity though the problem looks *a priori* highly non polynomial.

References

[1] E. Algaba. Extensión de juegos definidos en sistemas de conjuntos. PHD thesis, Univ. of Seville, Spain, 1998.

[2] E. Algaba, J.M. Bilbao, and J.J. Lopez. A Unified Approach to Restricted Games. *Theory and Decision*, 50(4):333-345, 2001.

[3] E. Algaba, J.M. Bilbao, P. Borm and J.J. Lopez. The position value for union stable systems. *Mathematical Methods of Operations Research*, 52:221-236, 2000.

[4] J. M. Bilbao. Cooperative games on combinatorial structures. *Kluwer Academic Publishers*, Boston, 2000.

[5] J.M. Bilbao. Cooperative Games under Augmenting Systems. *SIAM Journal on Discrete Mathematics*, 17(1):122-133, 2003.

[6] J. Edmonds and R. Giles. A min-max relation for submodular functions on graphs. *Annals of Discrete Mathematics*, 1:185-204, 1977.

[7] U. Faigle. Cores of Games with Restricted Cooperation. ZOR - *Methods and Models of Operations Research*, 33(6):405–422, 1989.

[8] U. Faigle, M. Grabisch, and M. Heyne. Monge extensions of cooperation and communication structures. *European Journal of Operational Research*, 206:104-110, 2010.

[9] M. Grabisch and A. Skoda. Games induced by the partitioning of a graph. *Annals of Operations Research*. 201(1):229-249, Springer, 2012.

[10] M. Grabisch. The core of games on ordered structures and graphs. *Annals of Operations Research*. 204(1):33-64, Springer, 2013.

[11] S. Fujishige, Submodular functions and optimization, *Annals of Discrete Mathematics*, 58, Second edition, Elsevier, 2005.

[12] J.C. Harsanyi. A simplified bargaining model for the n-person cooperative game. *International Economic Review*, 4:194-220, 1963.

[13] R.B. Myerson. Graphs and Cooperation in Games. *Mathematics of Operations Research*, 2(3):225–229, 1977.

[14] A.Skoda. Convexity of network restricted games induced by minimum partitions. *Documents de travail du Centre d'Economie de la Sorbonne*, 2016.19, 2016.

[15] A. Skoda. Complexity of the inheritance of $\mathcal{F}$-convexity for restricted games induced by minimum partitions, 2016.

[16] A. van den Nouweland and P. Borm. On the Convexity of Communication Games. *International Journal of Game Theory*, 19(4): 421-30, 1991.