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**Determining influential models**

Michel GRABISCH, Agnieszka RUSINOWSKA

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# Determining influential models<sup>\*</sup>

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**Abstract.** We consider a model of opinion formation based on aggregation functions. Each player modifies his opinion by arbitrarily aggregating the current opinion of all players. A player is influential for another player if the opinion of the first one matters for the latter. A generalization of influential player to a coalition whose opinion matters for a player is called influential coalition. Influential players (coalitions) can be graphically represented by the graph (hypergraph) of influence, and the convergence analysis is based on properties of the hypergraphs of influence. In the paper, we focus on the practical issues of applicability of the model w.r.t. the standard opinion formation framework driven by the Markov chain theory. For the qualitative analysis of convergence, knowing the aggregation functions of the players is not required, but one only needs to know the influential coalitions for every player. We propose simple algorithms that permit to fully determine the influential coalitions. We distinguish three cases: the symmetric decomposable model, the anonymous model, and the general model.

**JEL Classification:** C7, D7, D85

**Keywords:** social network, opinion formation, aggregation function, influential coalition, algorithm

## 1 Introduction - dynamic models of opinion formation

Opinion formation models are widely studied in psychology, sociology, economics, mathematics, computer sciences, among others; for overviews, see, e.g., Jackson (2008), Acemoglu and Ozdaglar (2011). A seminal model of opinion formation and imitation has been introduced in DeGroot (1974). In that model, individuals in a society start with initial opinions on a subject. The interaction patterns are described by a stochastic matrix whose entry on row  $j$  and column  $k$  represents the weight ‘that player  $j$  places on the current belief of player  $k$  in forming  $j$ 's belief for the next period’. The beliefs are updated over time. A number of works in the network literature deal with the DeGroot model and its variations. In particular, Jackson (2008) and Golub and Jackson (2010) investigate a model, in which players update their beliefs by repeatedly taking weighted averages of their neighbors’ opinions. One of the issues in the DeGroot framework that these authors deal with concerns necessary and sufficient conditions for convergence of the social influence matrix and reaching a consensus; see additionally Berger (1981). Jackson (2008) also examines the speed of convergence of beliefs, and Golub and Jackson (2010) analyze in the context of the DeGroot model whether consensus beliefs are “correct”,

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i.e., whether the beliefs converge to the right probability, expectation, etc. The authors consider a sequence of societies, where each society is strongly connected and convergent, and described by its updating matrix. In each social network of the sequence, the belief of each player converges to the consensus limit belief. There is a true state of nature, and the sequence of networks is wise if the consensus limit belief converges in probability to the true state as the number of societies grows.

Several other generalizations of the DeGroot model can be found in the literature, e.g., models in which the updating of beliefs can vary in time and circumstances; see e.g., DeMarzo et al. (2003), Krause (2000), Lorenz (2005), Friedkin and Johnsen (1990, 1997). In the model by DeMarzo et al. (2003), players in a network try to estimate some unknown parameter which allows updating to vary over time, i.e., a player may place more or less weight on his own belief over time. The authors study the case of multidimensional opinions, in which each player has a vector of beliefs. They show that, in fact, the individuals' opinions can often be well approximated by a one-dimensional line, where a player's position on the line determines his position on all issues. Friedkin and Johnsen (1990, 1997) study a similar framework, in which social attitudes depend on the attitudes of neighbors and evolve over time. In their model, players start with initial attitudes and then mix in some of their neighbors' recent attitudes with their starting attitudes.

Also other works in sociology related to influence are worth mentioning, e.g., the eigenvector-like notions of centrality and prestige (Katz (1953), Bonacich (1987), Bonacich and Lloyd (2001)), and models of social influence and persuasion by French (1956) and Harary (1959); see also Wasserman and Faust (1994). A sociological model of interactions on networks is also presented in Conlisk (1976); see also Conlisk (1978, 1992) and Lehoczky (1980). Conlisk introduces the interactive Markov chain, in which every entry in a state vector at each time represents the fraction of the population with some attribute. The matrix depends on the current state vector, i.e., the current social structure is taken into account for evolution in sociological dynamics. In Granovetter (1978) threshold models of collective behavior are discussed. These are models in which agents have two alternatives and the costs and benefits of each depend on how many other agents choose which alternative. The author focuses on the effect of the individual thresholds (i.e., the proportion or number of others that make their decision before a given agent) on the collective behavior, discusses an equilibrium in a process occurring over time and the stability of equilibrium outcomes. A certain model of influence is studied in Asavathiratham (2000) and Asavathiratham et al. (2001). The model consists of a network of nodes, each with a status evolving over time. The evolution of the status is according to an internal Markov chain, but transition probabilities depend not only on the current status of the node, but also on the statuses of the neighboring nodes.

Another work on interaction is presented in Hu and Shapley (2003b,a), where authors apply the command structure of Shapley (1994) to model players' interaction relations by simple games. For each player, boss sets and approval sets are introduced, and based on these sets, a simple game called the command game for a player is built. In Hu and Shapley (2003a) the authors introduce an authority distribution over an organization and the (stochastic) power transition matrix, in which an entry in row  $j$  and column  $k$  is interpreted as agent  $j$ 's "power" transferred to  $k$ . The authority equilibrium equa-

tion is defined. In Hu and Shapley (2003a) multi-step commands are considered, where commands can be implemented through command channels.

There is also a vast literature on learning in the context of social networks; see e.g. Banerjee (1992), Ellison (1993), Ellison and Fudenberg (1993, 1995), Bala and Goyal (1998, 2001), Gale and Kariv (2003), Celen and Kariv (2004), Banerjee and Fudenberg (2004). In general, in social learning models agents observe choices over time and update their beliefs accordingly, which is different from the models where the choices depend on the influence of others.

While DeGroot (1974) assumes that players update their opinion by taking weighted averages of the opinions of all players, Grabisch and Rusinowska (2013) investigate a model of opinion formation in which players update their beliefs according to arbitrary aggregation functions. Förster et al. (2013) study the model of opinion formation in which ordered weighted averages are used in the opinion updating process. In this paper we come back to the model of influence based on aggregation functions introduced in Grabisch and Rusinowska (2013) and discuss the practical issues of applicability of this model w.r.t. the “standard” opinion formation framework driven by the Markov chain theory.

The paper is organized as follows. Section 2 gives the basic material on models of influence based on aggregation functions, establishes the notation and terminology, and recalls the essential result which is the basis for the determination of the qualitative part of the influence model. Section 3 addresses the problem of the practical determination of an influence model and focuses on the determination of its qualitative part, which is sufficient for a qualitative analysis of convergence. This determination amounts to identifying influential coalitions, which can be easily obtained by interview of the agents. We distinguish the case of the symmetric decomposable model (influential coalitions reduce to individuals), the anonymous model (only the number of agents matters, not their identity), and the general model. We show how clues on convergence can be obtained in a simple way, even without determining the reduced transition matrix. Section 4 gives some concluding remarks.

## 2 Influence model based on aggregation functions

In this section, we recapitulate the model of influence based on aggregation functions (Grabisch and Rusinowska (2013)). Consider a set  $N := \{1, \dots, n\}$  of players that have to make a yes-no decision on a certain issue. Every player has an initial opinion which may change due to mutual interaction (influence) among players. By  $b_{S,T}$  we denote the probability that the set  $S$  of ‘yes’-voters becomes  $T$  after one step of influence. We assume that the process of influence may iterate, and therefore obtain a stochastic influence process, depicting the evolution of the coalition of ‘yes’-players in time. We assume that the process is Markovian ( $b_{S,T}$  depends on  $S$  and  $T$  but not on the whole history) and stationary ( $b_{S,T}$  is constant over time). States of this finite Markovian process are all subsets  $S \subseteq N$  representing the set of ‘yes’-players, and we have also the transition matrix  $\mathbf{B} := [b_{S,T}]_{S,T \subseteq N}$  which is a  $2^n \times 2^n$  row-stochastic matrix.

For the qualitative description of the convergence of the process, it is sufficient to know the reduced matrix  $\tilde{\mathbf{B}}$  given by

$$\tilde{b}_{S,T} = \begin{cases} 1, & \text{if } b_{S,T} > 0 \\ 0, & \text{otherwise.} \end{cases}$$

and equivalently represented by the transition graph  $\Gamma = (2^N, E)$ , where  $E$  is the set of arcs, its vertices are all possible coalitions, and an arc  $(S, T)$  from state  $S$  to state  $T$  exists if and only if  $\tilde{b}_{S,T} = 1$ .

**Definition 1.** An  $n$ -place aggregation function is a mapping  $A : [0, 1]^n \rightarrow [0, 1]$  satisfying

- (i)  $A(0, \dots, 0) = 0$ ,  $A(1, \dots, 1) = 1$  (boundary conditions)
- (ii) If  $\mathbf{x} \leq \mathbf{x}'$  then  $A(\mathbf{x}) \leq A(\mathbf{x}')$  (nondecreasingness).

To every player  $i \in N$  we associate an aggregation function  $A_i$ , which specifies the way player  $i$  modifies his opinion from the opinions of the players. Let  $\mathbf{A} := (A_1, \dots, A_n)$  denote the vector of aggregation functions. We compute  $\mathbf{A}(1_S) = (A_1(1_S), \dots, A_n(1_S))$ , where  $1_S$  is the characteristic vector of  $S$ , and  $A_i(1_S)$  indicates the probability of player  $i$  to say ‘yes’ at next step when the set of agents saying ‘yes’ is  $S$ . We assume that these probabilities are independent among agents, hence the probability of transition from the yes-coalition  $S$  to the yes-coalition  $T$  is given by

$$b_{S,T} = \prod_{i \in T} A_i(1_S) \prod_{i \notin T} (1 - A_i(1_S)). \quad (1)$$

A detailed study of the convergence of the model is provided in Grabisch and Rusinowska (2013). It is shown in particular that three types of terminal class<sup>1</sup> can exist: singletons, cycles, and regular terminal classes. The first case is when the class is reduced to a single coalition (called *terminal state*), the second one is the case where no convergence occurs because the process endlessly cycles on a sequence of coalitions, and the last case is when the class is a Boolean lattice of the form  $\{S \in 2^N \mid K \subseteq S \subseteq L\}$  for some sets  $K, L$ . In any case,  $N$  and  $\emptyset$  are terminal states (called *trivial terminal states*).

We emphasize two particular aggregation functions. The first one is the well-known *weighted arithmetic mean (WAM)*, defined by

$$\text{WAM}_w(x_1, \dots, x_n) = \sum_{i=1}^n w_i x_i$$

where  $w = (w_1, \dots, w_n)$  is a *weight vector*, i.e.,  $w \in [0, 1]^n$  with the property  $\sum_{i=1}^n w_i = 1$ . The weighted arithmetic mean is used in most models of opinion formation, e.g., the DeGroot model. Another remarkable aggregation function is the *ordered weighted arithmetic mean (OWA)* (Yager (1988)), defined by

$$\text{OWA}_w(x_1, \dots, x_n) = \sum_{i=1}^n w_i x_{(i)}$$

<sup>1</sup> A *class* is a maximal collection of coalitions such that for any two distinct coalitions  $S, T$  in the class, there exists a sequence of transitions inside the class leading from  $S$  to  $T$ . The class is *terminal* if no transition to go outside the class is possible.

where  $w$  is a weight vector, and the inputs have been reordered decreasingly:  $x_{(1)} \geq \dots \geq x_{(n)}$ . Note that unlike the case of WAM, weights do not bear on inputs but on the rank of the inputs, so that the minimum and the maximum are particular cases, by taking respectively  $w = (0, \dots, 0, 1)$  and  $w = (1, 0, \dots, 0)$ . Applied to our context of influence where the input vectors are binary, if every agent aggregates the opinions by an OWA, we obtain an *anonymous influence model*, because each agent updates his opinion according to the *number* of agents saying 'yes', not to which agent says 'yes'. Anonymous influence models have been studied in detail in Förster et al. (2013).

**Definition 2.** Let  $A_i$  be the aggregation function of agent  $i$ . A nonempty coalition  $S \subseteq N$  is yes-influential for  $i$  if

- (i)  $A_i(1_S) > 0$
- (ii) For all  $S' \subset S$ ,  $A_i(1_{S'}) = 0$ .

Similarly, a coalition  $S$  is no-influential for  $i$  if

- (i)  $A_i(1_{N \setminus S}) < 1$
- (ii) For all  $S' \subset S$ ,  $A_i(1_{N \setminus S'}) = 1$ .

We denote by  $\mathcal{C}_i^{\text{yes}}$  and  $\mathcal{C}_i^{\text{no}}$  the collections of yes- and no-influential coalitions for  $i$ . Coalition  $S$  is yes-influential for player  $i$  if, when players in  $S$  say 'yes' and every other player says 'no',  $i$  has a positive probability to say 'yes' (and similarly for no-influential coalitions). Moreover,  $S$  has no superfluous player. If an influential coalition is formed by only one player, then we call it influential player. Note that these collections are never empty, since in case no proper subcoalition of  $N$  is yes- or no-influential,  $N$  would be both yes- and no-influential, by Definition 1. More importantly, each such collection is an *antichain* in  $2^N$ , that is, for any distinct  $S, S'$  members of the collection,  $S \not\subseteq S'$  and  $S' \not\subseteq S$ .

Influential players can easily be represented in a directed graph. Define  $G_{\mathbf{A}}^{\text{yes}}$ , the *graph of yes-influence*, as follows: the set of nodes is the set of agents  $N$ , and there is an arc  $(j, i)$  from  $j$  to  $i$  if  $j$  is yes-influential for  $i$ . The graph of no-influence  $G_{\mathbf{A}}^{\text{no}}$  is defined similarly. The representation of influential coalitions require the more complex notion of hypergraph.

**Definition 3.** We define the following concepts:

- (i) A hypergraph (Berge, 1976)  $H$  is a pair  $(N, \mathcal{E})$  where  $N$  is the set of nodes and  $\mathcal{E}$  the set of hyperedges, where an hyperedge  $S \in \mathcal{E}$  is a nonempty subset of  $N$ . If  $|S| = 2$  for all  $S \in \mathcal{E}$ , then we have a classical graph.
- (ii) A directed hypergraph on  $N$  is a hypergraph on  $N$  where each hyperedge  $S$  is an ordered pair  $(S', S'')$  (called an hyperarc from  $S'$  to  $S''$ ), with  $S', S''$  being nonempty and  $S' \cup S'' = S$ .
- (iii) A directed hyperpath from  $i$  to  $j$  is a sequence  $i_0 S_1 i_1 S_2 i_2 \dots i_{q-1} S_q i_q$ , where  $i_0 := i, i_1, \dots, i_{q-1}, j := i_q$  are nodes,  $S_1 = (S'_1, S''_1), \dots, S_q = (S'_q, S''_q)$  are hyperarcs such that  $S'_k \ni i_{k-1}$  and  $S''_k \ni i_k$  for all  $k = 1, \dots, q$ .

We define the hypergraphs  $H_{\mathbf{A}}^{\text{yes}}, H_{\mathbf{A}}^{\text{no}}$  of yes-influence and no-influence as follows. For  $H_{\mathbf{A}}^{\text{yes}}$ , the set of nodes is  $N$ , and there is an hyperarc  $(C, \{i\})$  for each  $C \in \mathcal{C}_i^{\text{yes}}$  (similarly for  $H_{\mathbf{A}}^{\text{no}}$ ).

Grabisch and Rusinowska (2013) prove that the hypergraphs  $H_{\mathbf{A}}^{\text{yes}}, H_{\mathbf{A}}^{\text{no}}$  (equivalently, the collections  $\mathcal{C}_i^{\text{yes}}$  and  $\mathcal{C}_i^{\text{no}}$  for all  $i \in N$ ) are equivalent to the reduced matrix  $\tilde{\mathbf{B}}$ , and therefore contain the entire qualitative description of the convergence.

**Theorem 1.** *Consider an influence process  $\mathbf{B}$  based on aggregation functions  $\mathbf{A}$ . Then  $\tilde{\mathbf{B}}$  can be reconstructed from the hypergraphs  $H_{\mathbf{A}}^{\text{yes}}$  and  $H_{\mathbf{A}}^{\text{no}}$  as follows: for any  $S, T \in 2^N$ ,  $\tilde{b}_{S,T} = 1$  if and only if*

- (i) *For each  $i \in T$ , there exists a nonempty  $S'_i \subseteq S$  such that  $S'_i$  is yes-influential on  $i$ , i.e.,  $S'_i \in \mathcal{C}_i^{\text{yes}}$ ; and*
- (ii) *For each  $i \notin T$ , there exists a nonempty  $S''_i$  such that  $S''_i \cap S = \emptyset$  and  $S''_i$  is no-influential on  $i$ , i.e.,  $S''_i \in \mathcal{C}_i^{\text{no}}$ .*

*In particular,  $\tilde{b}_{\emptyset,T} = 0$  for all  $T \neq \emptyset$ ,  $b_{\emptyset,\emptyset} = 1$ , and  $\tilde{b}_{N,T} = 0$  for all  $T \neq N$ ,  $b_{N,N} = 1$ .*

Recall that (1) is valid only if the probabilities of saying 'yes' are independent among the agents. Therefore, the presence of correlation among the agents makes the determination of the transition matrix difficult. However,  $\tilde{\mathbf{B}}$  is insensitive to possible correlation among agents, because  $\tilde{b}_{S,T} = 1$  if and only if  $A_i(1_S) > 0$  for every  $i \in T$  and  $A_i(1_S) < 1$  for every  $i \notin T$ , regardless of the correlation among agents.

### 3 Determination of the model

An important issue concerns the determination of an influence model of the above type in a practical situation. This implies that we are making essentially two assumptions:

- 1) Every agent aggregates the opinion of all agents to form his opinion in the next step.
- 2) The aggregation function is monotonically increasing.

The latter assumption implies that anti-conformist behaviors (i.e., the more individuals say 'yes', the more I am inclined to say 'no') cannot be modeled in this framework.

#### 3.1 General considerations

A complete determination of the model amounts either to identifying the transition matrix  $\mathbf{B}$  or all aggregation functions  $A_1, \dots, A_n$  (supposing absence of correlation). Considering the size of the matrix  $\mathbf{B}$  ( $2^n \times 2^n$ ), a statistical determination of  $\mathbf{B}$  seems to be nearly impossible, unless making a huge amount of observations. As for the determination of the aggregation functions, the situation is even worse, since questioning the agents about the aggregation functions (type, parameters) appears to be quite unrealistic. We know from Section 2 that the knowledge of the reduced matrix  $\tilde{\mathbf{B}}$  is enough to obtain a qualitative description of the convergence of the model, which is insensitive to possible correlations among agents. Moreover, the knowledge of  $\tilde{\mathbf{B}}$  (size  $2^{2n}$ ) is equivalent by Theorem 1 to the knowledge of the collections of all yes- and no-influential coalitions (size at most  $2n \binom{n}{\lfloor \frac{n}{2} \rfloor}$ ), which is in turn equivalent to the knowledge of the hypergraphs of yes- and no-influence. In some favorable cases (e.g., the WAM model), the hypergraphs reduce to ordinary graphs. This immediately indicates two ways of identifying the (qualitative part of the) model: either by observation of the transitions, i.e., the evolution of the coalition of the 'yes' agents, or by interview of the agents. In the first case, observing a transition



from  $S$  to  $T$  yields  $\tilde{b}_{ST} = 1$ . In the second case, the interview permits to determine the influential coalitions or the graphs of influence.

In the rest of this section, we mainly focus on the second approach. Concerning the first one, we only mention an important fact. The underlying assumptions of the model make that the reduced matrix  $\tilde{\mathbf{B}}$  is not arbitrary and has specific properties. Recall that  $\tilde{b}_{ST} = 1$  if and only if for all  $i \in T$ ,  $A_i(1_S) > 0$  and for all  $i \notin T$ ,  $A_i(1_S) < 1$ . This implies the following fact:

**Fact 1** *For a given  $S \subseteq N$ ,  $S \neq \emptyset, N$ , the candidates transitions are all sets of the form  $T = K \cup L$ , where*

$$K = \{i \in N \mid A_i(1_S) = 1\}$$

$$L \subseteq \{i \in N \mid 0 < A_i(1_S) < 1\}.$$

Put otherwise,  $\bigcap T$ , the intersection of all possible transitions from  $S$  yields the set  $K = \{i \in N \mid A_i(1_S) = 1\}$ , while  $N \setminus \bigcup T$  yields  $K' = \{i \in N \mid A_i(1_S) = 0\}$ . When  $S$  increases,  $K$  increases while  $K'$  decreases. This fact permits to detect, when  $\tilde{\mathbf{B}}$  is constructed from observations, possible deviations from the model (e.g., presence of anti-conformists).

### 3.2 Determination of influential coalitions

We may distinguish three cases, according to the type of underlying model:

- (i) WAM model (symmetric decomposable model): all aggregation functions are weighted arithmetic means;
- (ii) OWA model (anonymous model): all aggregation functions are ordered weighted averages;
- (iii) general model (no special assumption).

*The symmetric decomposable model.* The case of the WAM model is particularly simple and has been studied in depth in Grabisch and Rusinowska (2013). It has been proved to be equivalent to a symmetric decomposable model. An aggregation model is *decomposable* if for every agent  $i \in N$ , every yes- and no-influential coalition for agent  $i$  is a singleton. Now, an aggregation model is *symmetric* if a yes-influential coalition for  $i$  is also no-influential for  $i$  and vice versa, for every  $i \in N$ . Note that the first property implies that the hypergraphs of yes- and no-influence reduce to ordinary graphs, while the second property implies that the two graphs are identical, and therefore the whole (qualitative) model is represented by a single graph of influence. This makes the interview of the agents particularly simple: it suffices to ask to every agent to whom he asks advice. Then,  $i$  asks advice to  $j$  is translated in the graph of influence by an arc from  $j$  to  $i$ .

In Grabisch and Rusinowska (2013), we have applied this technique to a real case, namely the manager network of Krackhardt (Krackhardt (1987)). The agents are the 21 managers of a small manufacturing firm in the USA, and the network is obtained as follows: each agent  $k$  is asked if he/she thinks that agent  $i$  asks advice to agent  $j$ . Then an arc from  $j$  to  $i$  is put in the graph of influence if a majority of agents thinks that  $i$  asks advice to  $j$ . From the graph, and due to the properties of symmetric decomposable models, many conclusions can be easily drawn on the convergence of the model. In particular,

it is possible to detect the presence of regular terminal classes (Theorem 8 in Grabisch and Rusinowska (2013)). There is also a simple criterion to know if there is no regular terminal class: it suffices that for each agent  $i$ , every agent outside  $\text{cl}(i)$  receives an arc in the influence graph from  $\text{cl}(i)$ , where  $\text{cl}(i)$ , the *closure* of  $i$ , is the set of agents who can reach  $i$  by a path in the influence graph.

*The anonymous model.* In the OWA model, agents do not change their opinion due to particular individuals, but due to the number of individuals saying 'yes'. Therefore, in general these are not decomposable models, and one needs to determine influential coalitions as in the general case. However, because these models are anonymous, a collection  $\mathcal{C}_i^{\text{yes}}$  or  $\mathcal{C}_i^{\text{no}}$  is composed of all sets of a given size  $s$ ,  $1 \leq s \leq n$ , and this is characteristic of an anonymous model. Therefore, under the assumption of anonymity, it suffices to ask to every agent  $i$ :

Q1: Suppose that your opinion on some question is 'yes'. What is the minimal number of agents saying 'no' that may make you change your opinion?

Q2: Suppose that your opinion on some question is 'no'. What is the minimal number of agents saying 'yes' that may make you change your opinion?

Assuming that the answers are respectively  $s$  and  $s'$ , it follows that

$$\mathcal{C}_i^{\text{no}} = \{S \in 2^N \mid |S| = s\}, \quad \mathcal{C}_i^{\text{yes}} = \{S \in 2^N \mid |S| = s'\}.$$

Now, it is easy to see that given  $s, s'$  for agent  $i$ , one can get the form of the weight vector  $w$  in the aggregation function  $\text{OWA}_w$  of agent  $i$  (Proposition 2 in Förster et al. (2013)):

$$w = (\underbrace{00 \cdots 0}_{s'-1 \text{ zeros}}, \bullet \cdots \bullet, \underbrace{00 \cdots 0}_{s-1 \text{ zeros}}),$$

where “ $\bullet$ ” indicates any nonzero weight. In particular, all agents are yes-influential (respectively, no-influential) for  $i$  if and only if  $w_1 > 0$  (respectively,  $w_n > 0$ ).

As for the convergence of the model, it is shown in Förster et al. (2013) (Proposition 3) that no cycle can occur, but the two other types of terminal classes may occur. Terminal states are easily detected as follows:  $S$  of size  $s$  is a terminal state if and only if for every  $i \in S$ , the size of a no-influential coalition is at least  $n - s + 1$ , and for every  $i \notin S$ , the size of a yes-influential coalition is at least  $s + 1$ . The absence of regular terminal classes can also be characterized only through influential coalitions, but the condition is more complex (see Corollary 3 in the aforementioned paper).

*The general model.* We address now the general case, where no special assumption is made on the model, except the following: we assume that each agent is yes- and no-influential on himself, which means that  $A_i(1_i) > 0$ ,  $A_i(1_{N \setminus i}) < 1$  (in other words, the agent trusts his opinion to a nonnull extent). This induces some simplification in the algorithm, but it would not be difficult to generalize it in order to overcome this limitation.

INTERVIEW FOR AGENT  $i$

0. Set  $\mathcal{C}_i^{\text{yes}} = \{\{i\}\}$ ,  $\mathcal{C}_i^{\text{no}} = \{\{i\}\}$ ,  $\mathcal{N}_i^{\text{yes}} = \mathcal{N}_i^{\text{no}} = 2^{N \setminus i}$ .

%  $\mathcal{N}_i^{\text{yes}}, \mathcal{N}_i^{\text{no}}$  are the sets of candidate coalitions.

1. For each agent  $j \in N$ ,  $j \neq i$ , do:

1.1. If  $\{j\} \notin \mathcal{N}_i^{\text{no}}$ , GO TO Step 1.2, otherwise ask:

Q: *Suppose that your opinion on some question is 'yes'. Would you be inclined to change your opinion if Agent  $j$  would say 'no'?*

If the answer is positive:

- add  $\{j\}$  in  $\mathcal{C}_i^{\text{no}}$ , and discard from  $\mathcal{N}_i^{\text{no}}$  all sets containing  $j$ .
- If  $\mathcal{N}_i^{\text{no}} = \emptyset$  or if  $|\mathcal{C}_i^{\text{no}}| = \binom{n}{\lfloor \frac{n}{2} \rfloor}$ , STOP (GO TO STEP 2).

Otherwise, discard  $\{j\}$  from  $\mathcal{N}_i^{\text{no}}$ , and ask:

1.2 Q: *In the list  $\{S \in \mathcal{N}_i^{\text{no}} \mid S \ni j\}$ <sup>2</sup>, could you tell me the first coalition that may make change your opinion if the agents in that coalition would say 'no'?*

If some set  $S$  is answered:

- add  $S$  in  $\mathcal{C}_i^{\text{no}}$ , discard  $S$ , all sets listed before  $S$  and all supersets of  $S$  in  $\mathcal{N}_i^{\text{no}}$ .
- If  $\mathcal{N}_i^{\text{no}} = \emptyset$  or if  $|\mathcal{C}_i^{\text{no}}| = \binom{n}{\lfloor \frac{n}{2} \rfloor}$ , STOP (GO TO STEP 2).
- Go to Step 1.2 again.

Otherwise, discard all sets containing  $j$  in  $\mathcal{N}_i^{\text{no}}$ . If  $\mathcal{N}_i^{\text{no}} = \emptyset$ , STOP (GO TO STEP 2), otherwise proceed to next agent (Step 1).

2. Exactly like Step 1 for  $\mathcal{C}_i^{\text{yes}}$ . Question 1.1. becomes: *Suppose that your opinion on some question is 'no'. Would you be inclined to change your opinion if Agent  $j$  would say 'yes'?*, etc.

We give some examples.

*Example 1.* (braces are omitted for coalitions) Consider  $N = \{1, 2, 3, 4, 5\}$ . We detail the interview for Agent 1.

(i) We have  $\mathcal{C}_1^{\text{no}} = \{1\}$ . We take Agent 2.

*Suppose that your opinion on some question is 'yes'. Would you be inclined to change your opinion if Agent 2 would say 'no'?*

Yes. Hence,  $\mathcal{C}_1^{\text{no}} = \{1, 2\}$ , and  $\mathcal{N}_1^{\text{no}} = \{3, 4, 5, 34, 35, 45, 345\}$ .

(ii) Agent 3.

*Suppose that your opinion on some question is 'yes'. Would you be inclined to change your opinion if Agent 3 would say 'no'?*

Answer: No.  $\mathcal{N}_1^{\text{no}} = \{4, 5, 34, 35, 45, 345\}$ . Next question:

*In the list  $\{34, 35, 345\}$ , could you tell me the first coalition that may make change your opinion if they would say 'no'?*

Answer: 34. Then  $\mathcal{C}_i^{\text{no}} = \{1, 2, 34\}$ , and  $\mathcal{N}_1^{\text{no}} = \{5, 35, 45\}$ .

*In the list  $\{35\}$ , could you tell me the first coalition that may make change your opinion if they would say 'no'?*

Answer: None.  $\mathcal{N}_1^{\text{no}} = \{5, 45\}$ .

(iii) Agent 4.

*In the list  $\{45\}$ , could you tell me the first coalition that may make change your opinion if they would say 'no'?*

Answer: None. Then  $\mathcal{N}_1^{\text{no}} = \{5\}$ .

(iv) Agent 5.

*Suppose that your opinion on some question is 'yes'. Would you be inclined to change your opinion if Agent 5 would say 'no'?*

Answer: No.  $\mathcal{N}_1^{\text{no}} = \emptyset$ . STOP.

<sup>2</sup> The list must be ordered by inclusion.

Finally,  $C_i^{\text{no}} = \{1, 2, 34\}$ . We do the same for  $C_1^{\text{yes}}$ .

(i) Agent 2:

*Suppose that your opinion on some question is 'no'. Would you be inclined to change your opinion if Agent 2 would say 'yes' ?*

Answer: No.  $\mathcal{N}_1^{\text{yes}} = \{3, 4, 5, 23, 24, 25, 34, 35, 45, 234, 235, 245, 345, 2345\}$ .

*In the list  $\{23, 24, 25, 234, 235, 245, 2345\}$ , could you tell me the first coalition that may make change your opinion if they would say 'no' ?*

Answer: 234. Then  $C_1^{\text{yes}} = \{1, 234\}$ ,  $\mathcal{N}_1^{\text{yes}} = \{5, 35, 45, 235, 245, 345\}$ .

*In the list  $\{235, 245\}$ , could you tell me the first coalition that may make change your opinion if they would say 'no' ?*

Answer: 235. Then  $C_1^{\text{yes}} = \{1, 234, 235\}$ ,  $\mathcal{N}_1^{\text{yes}} = \{45, 245, 345\}$ .

*In the list  $\{245\}$ , could you tell me the first coalition that may make change your opinion if they would say 'no' ?*

Answer: 245. Then  $C_1^{\text{yes}} = \{1, 234, 235, 245\}$ ,  $\mathcal{N}_1^{\text{yes}} = \{45, 345\}$ .

(ii) Agent 3:

*Suppose that your opinion on some question is 'no'. Would you be inclined to change your opinion if Agent 3 would say 'yes' ?*

Answer: No.

*In the list  $\{345\}$ , could you tell me the first coalition that may make change your opinion if they would say 'no' ?*

Answer: 345. Then  $C_1^{\text{yes}} = \{1, 234, 235, 245, 345\}$ ,  $\mathcal{N}_1^{\text{yes}} = \emptyset$ . STOP.

We give now another example to illustrate how the reduced transition matrix  $\tilde{\mathbf{B}}$  can be obtained through Theorem 1 from the influential coalitions. To this end, we suppose that the foregoing algorithm has been applied to each agent in order to obtain all influential coalitions. The condition that every agent is self-influential permits to simplify the application of the theorem for the determination of every term  $\tilde{b}_{ST}$ . Indeed, the following facts are easy to show.

**Fact 2** *Suppose that  $\{i\} \in C_i^{\text{yes}}$  and  $\{i\} \in C_i^{\text{no}}$  for every  $i \in N$ . Then:*

- (i)  $\tilde{b}_{SS} = 1$  for every  $S \in 2^N$ ;
- (ii) If  $T \subseteq S$ , condition (i) in Theorem 1 is always satisfied for checking  $\tilde{b}_{ST} = 1$ ; moreover, if  $\tilde{b}_{ST} = 0$ , then  $\tilde{b}_{ST'} = 0$  for every  $T' \subset T$ ;
- (iii) If  $T \supseteq S$ , condition (ii) in Theorem 1 is always satisfied for checking  $\tilde{b}_{ST} = 1$ ; moreover, if  $\tilde{b}_{ST} = 0$ , then  $\tilde{b}_{ST'} = 0$  for every  $T' \supset T$ ;
- (iv) If in addition,  $\{i\}$  is the only singleton in  $C_i^{\text{yes}}$ ,  $\tilde{b}_{\{i\}S} = 0$  for every  $S \neq \{i\}$ .

*Example 2.* Consider a society  $N = \{1, 2, 3, 4\}$  of 4 agents. Suppose that the following collections have been obtained (braces are omitted for coalitions):

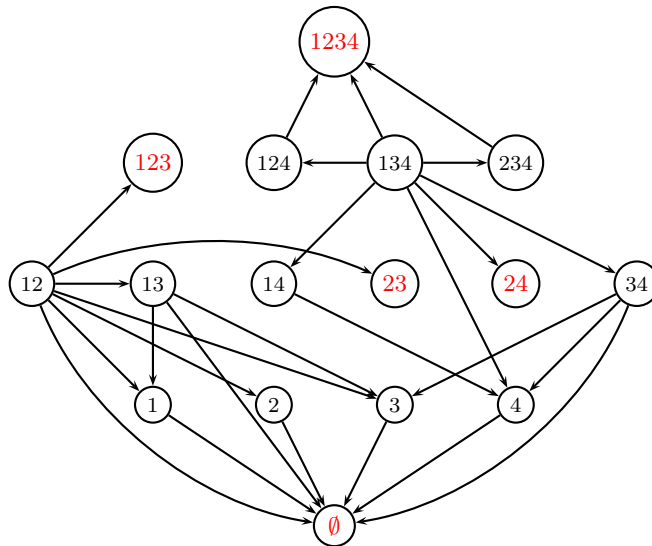
$$\begin{aligned} C_1^{\text{no}} &= \{1, 2, 34\}, & C_1^{\text{yes}} &= \{1, 234\} \\ C_2^{\text{no}} &= \{2, 34\}, & C_2^{\text{yes}} &= \{2, 134\} \\ C_3^{\text{no}} &= \{2, 3\}, & C_3^{\text{yes}} &= \{3, 12\} \\ C_4^{\text{no}} &= \{12, 4\}, & C_4^{\text{yes}} &= \{4\} \end{aligned}$$

Observe that agent 4 is stubborn for 'yes' (no influence is possible when agent 4 thinks 'yes').

Let us apply Theorem 1. Using Fact 2, one easily finds that  $\tilde{b}_{ST} = 1$  only for the following  $S, T$  (braces omitted):

- $S = 1 : T = \emptyset, 1$
- $S = 2 : T = \emptyset, 2$
- $S = 3 : T = \emptyset, 3$
- $S = 4 : T = \emptyset, 4$
- $S = 12 : T = \emptyset, 1, 2, 3, 12, 13, 23, 123$
- $S = 13 : T = \emptyset, 1, 3, 13$
- $S = 14 : T = 4, 14$
- $S = 23 : T = 23$
- $S = 24 : T = 24$
- $S = 34 : T = \emptyset, 3, 4, 34$
- $S = 123 : T = 123$
- $S = 124 : T = 124, 1234$
- $S = 134 : T = 4, 14, 24, 34, 124, 134, 234, 1234$
- $S = 234 : T = 234, 1234.$

One can check that Fact 1 is satisfied. The corresponding transition graph  $\Gamma$  is given on Figure 1 below. It is seen that, apart from the trivial terminal classes, 23, 24 and 123 are



**Fig. 1.** Transition graph (loops are omitted). Terminal states are in red and larger font

terminal states. There is no regular nor cyclic class.

We show now that it is possible to get conclusions on the convergence without computing  $\tilde{\mathbf{B}}$ , by the sole examination of the hypergraphs, thanks to results shown in Grabisch and Rusinowska (2013). To this end, we need the notion of ingoing hyperarc. We say that a coalition  $S$  has an *ingoing hyperarc*  $(T', T'')$  in hypergraph  $H$  if  $T' \subseteq N \setminus S$  and  $T'' \subseteq S$  (and vice versa for *outgoing*).

Now, Theorem 3 in the aforementioned paper establishes that a nonempty  $S \neq N$  is a terminal state if and only if  $S$  has no ingoing arc in the hypergraph  $(\hat{H}_{\mathbf{A}}^{\text{yes}})^* \cup \hat{H}_{\mathbf{A}}^{\text{no}}$ , where  $()^*$  indicates that the hyperarcs have been inverted, and  $\hat{H}$  indicates that only normal hyperarcs are considered<sup>3</sup>.

This result can be translated in terms of influential collections as follows:

**Fact 3** *A nonempty  $S \neq N$  is a terminal state if and only if*

- (i) *For every  $i \notin S$ , there is no  $T \in \mathcal{C}_i^{\text{yes}}$  such that  $T \subseteq S$ ;*
- (ii) *For every  $i \in S$ , there is no  $T \in \mathcal{C}_i^{\text{no}}$  such that  $T \cap S = \emptyset$ .*

Applying this fact to Example 2, we find indeed that the only terminal states are 23, 24 and 123.

## 4 Concluding remarks

We have shown how in a practical situation one can determine an influence model based on aggregation functions. An exact determination, yielding the type and parameters of the aggregation function of every agent, appears to be out of reach without using heavy procedures. What we show is that, on the contrary, it is easy to obtain the “qualitative part” of the model, which permits a full qualitative analysis of the convergence of opinions, that is, to determine all terminal classes. This is sufficient to predict if a consensus will occur, or if on the contrary, a dichotomy of the society will appear, or a cycle will appear, etc. Simple criteria are available to detect terminal states or the presence of regular terminal classes, without even determining the reduced transition matrix. We believe that this study will make the use of aggregation functions-based models of influence more familiar and easier to use.

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<sup>3</sup> A hyperarc  $(T', T'')$  is normal if  $T' \cap T'' = \emptyset$ . Note that due to our assumption that every player is self-influential, all hyperarcs are normal.

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