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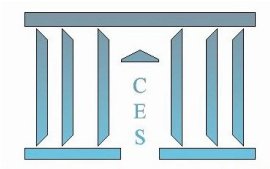
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The Principle of Minimum Differentiation Revisited: Return of the Median Voter*

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Abstract

We study a linear location model (Hotelling, 1929) in which n (with $n \geq 2$) boundedly rational players follow (noisy) myopic best-reply behavior. We show through numerical and mathematical analysis that such players spend almost all the time clustered together near the center, re-establishing the *"Principle of Minimum Differentiation"* that had been discredited by equilibrium analyses. Thus, our analysis of the best-response dynamics shows that when considering market dynamics as well as their policy and welfare implications, it may be important to look beyond equilibrium analyses.

JEL Classification: C72, D72, L13, R30

Keywords: Hotelling location model, Principle of Minimum Differentiation, Nash equilibrium, Best-response dynamics, Stochastic stability, Invariant measures

1 Introduction

Hotelling (1929) showed in a linear location model, in which two players independently choose a location, with their payoffs depending on the distance from individuals distributed on the line, that both players will locate together in the center. This result led to the notion of the *"Principle of Minimum Differentiation"*.

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Hotelling's model, where individuals have preferences over locations, provided a theoretical explanation for some casual observations that he reported about a widespread tendency for decision makers to choose only slight deviations from each other's location in the most diverse fields of competitive activity, even quite apart, as he put it, from what is typically called economic life. Besides geographical locations, he also discussed locations in product characteristics space, mentioning a *"tremendous standardization of our furniture, our houses, our clothing, our automobiles and our education"* (p. 54), as well as the choice of platforms by political parties in policy space.

Building on Hotelling's results, Black (1948) and Downs (1957) established the *"Median Voter Theorem"*, i.e., that a plurality voting rule will select the outcome preferred by the median voter. Also related is *"Duverger's Law"* (Duverger, 1954), which suggests that with plurality voting, one should expect a two-party system. This has given rise to an extensive empirical and theoretical literature (see e.g. Riker (1982), Rowley (1984), Osborne (1995), Grofman (2004), and many others). In this paper, we will go back to Hotelling (1929), and focus on one single aspect of this literature: The Principle of Minimum Differentiation.

Hotelling's result that in equilibrium the players will locate together in the center turns out to be correct for the special case of $n = 2$ players. As we will briefly sketch in Section 2, equilibrium analyses have made clear that the Principle of Minimum Differentiation is not robust to changing the number of players n in the basic location model to any number $n > 2$. In fact, for any $n > 2$, the Nash equilibria look almost the opposite of what is described by the Principle of Minimum Differentiation, with all players spread out relatively evenly, leaving only the extreme fringes unoccupied, as the most extreme players will be tugging in with their neighbors, and strict bounds on the spatial differences between the players.

Experimental evidence for Hotelling's location game in the laboratory with $n = 3$

(Collins and Sherstyuk, 2000) and $n = 4$ (Huck, Müller, and Vriend, 2002) suggests that one may expect many non-equilibrium outcomes in this game, with in particular more choices near the center than predicted by the theory.[1]

Given the apparently widespread centripetal tendency of decision makers in a range of different settings, as argued e.g. by Hotelling (1929), typically also in cases where there are more than two decision makers, and given also the experimental evidence for the basic Hotelling game, the game-theoretic analysis of the basic location game raises the important question as to how one may explain such a centripetal tendency. In the industrial organization literature, Palma, Ginsburgh, Papageorgiou, and Thisse (1985) showed that one way to restore the Principle is to assume that there is sufficient heterogeneity in consumers' tastes combined with uncertainty by the firms about these preferences. Another approach is to consider a different equilibrium concept. Shino and Kawasaki (2012) characterize the farsighted stable set of the Hotelling game, and show that this set contains location profiles that reflect the minimum differentiation.

In this paper, we will follow an alternative approach. Instead of an equilibrium analysis, we will analyze the behavior of myopic best-response dynamics. We will show, through numerical as well as formal, mathematical analysis, that the players will be located almost all the time close to the center if their location choices are governed by noisy, myopic best-responses to the other players. We show for $n \geq 3$ that players spend significantly much more time in locations that are closer to the center (and close to each other) than predicted by the Nash equilibria. What is more, we show that by refining the discrete space (thus approximating a continuous space), we can get the players located arbitrarily close to the center almost all the time, catering essentially just to the median voter.

Thus, while the Principle of Minimum Differentiation is consistent with Nash

[1] See also Brown-Kruse, Cronshaw, and Schenk (1993) and Brown-Kruse and Schenk (2000) for related duopoly experiments.

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equilibrium behavior only for $n = 2$, to the extent that there may be a tendency for decision makers to choose (noisy) myopic best-responses, our analysis suggests a possible explanation as to why this popular notion is so ubiquitous, acquiring almost "folk wisdom" status, as well as for the empirical evidence this is based on.

Providing a possible explanation for some empirical phenomenon is not the only reason why our analysis may be interesting. That best-response dynamics may lead to outcomes that are so different from the Nash equilibrium analyses seems an intriguing feature of the Hotelling game from a theoretical point of view, and relates to the literature on best-response dynamics and Nash equilibria. See, e.g., Hofbauer (1995), Hopkins (1999), and Balkenborg, Hofbauer, and Kuzmics (2013).

The rest of the paper is organized as follows. Section 2 presents the basic Hotelling location model as well as a brief overview of the equilibrium analysis. Section 3 presents our numerical analysis, to gain some insight into the behavior of best-response dynamics in our model. We, then, formally characterize the long-run behavior for the case of $n = 3$ in Section 4, followed by the case of $n = 4$ in Section 5. Section 6 presents some concluding remarks.

2 Equilibrium and minimum differentiation

In this section, we will give a brief overview of the various equilibria found in Hotelling's location game. We focus our attention on a basic, linear location model, which we will present for convenience as a model of spatial voting where the players are parties who choose a platform x in a one-dimensional, continuous policy space (normalized to $[0, 1]$). Following Eaton and Lipsey (1975), assume that no two parties can choose exactly the same location, with the minimum distance being δ (with δ close to 0). An infinite number of voters with unit mass, whose preferences are distributed uniformly over this space, vote for the player who are closest to their

preferred location.[2] The spatial distribution of voters is known to the players.

The payoff for a player from choosing location x is simply the share of votes received at that location. Therefore, if the two nearest players to the left and to the right are located in $L(x)$ and $R(x)$ respectively, with $L(x) < x < R(x)$, the payoff of choosing location x is given by:

$$\pi\Big(x|L(x),R(x)\Big) = \begin{cases} \frac{R(x)-L(x)}{2} \text{ if } 0 \leq L(x) < R(x) \leq 1 \\ \left(x + \frac{R(x)-x}{2}\right) \text{ if no player to the left of } x \text{ and } R(x) \leq 1 \\ \left(1 - x + \frac{x-L(x)}{2}\right) \text{ if } 0 \leq L(x) \text{ and no player to the right of } x \end{cases}$$

Eaton and Lipsey (1975) characterized pure strategy Nash equilibria for games with any number of players but three. If $n = 2$, the unique Nash equilibrium is when both players locate at $x = \frac{1}{2}$, giving a payoff $\pi = \frac{1}{2}$, as any other location would imply a lower payoff. For $n = 3$ there is no Nash equilibrium in pure strategies. Any of the peripheral players will want to move in towards the interior player as much as possible, but that interior player will always want to jump out of this squeeze. We will turn our attention to mixed strategy equilibria in a moment, but first we will consider equilibria in pure strategies for $n > 3$. For $n = 4$, the Nash equilibrium configuration has two players located at $\frac{1}{4}$ and the other two at $\frac{3}{4}$. Each player will get a payoff of $\frac{1}{4}$. When $n = 5$, there will be three locations that will be occupied in equilibrium: one in the center (occupied by one player), and two peripheral ones equally distanced from the center, located at $\frac{1}{6}$ and $\frac{5}{6}$ respectively, and each occupied by two players. Note that this means that the interior player will get a payoff of $\frac{1}{3}$, whereas the four peripheral players will each get a payoff of $\frac{1}{6}$.

Eaton and Lipsey show that there is multiplicity of equilibria when there are more than 5 players (for any $n > 5$). For $n = 6$, one equilibrium configuration

[2] When a voter finds that the best option to the left and the best option to the right are equally distant, they choose by a fair coin toss.

involves three locations, with $\frac{1}{6}$, $\frac{3}{6}$ and $\frac{5}{6}$ each occupied by two players, and each of them getting a payoff of $\frac{1}{6}$. An alternative equilibrium configuration involves the four locations $\frac{1}{8}$, $\frac{3}{8}$, $\frac{5}{8}$ and $\frac{7}{8}$, where the two peripheral locations are occupied by two players and the two interior ones by one player each. In this case the two interior players will get a payoff of $\frac{1}{4}$ and the four peripheral ones a payoff of $\frac{1}{8}$. Between these two equilibria there is an infinite number of additional equilibria. Think of the second equilibrium as a stretched version of the first one. As the interval between the two peripheral locations gets gradually stretched, for any distance one can compute the required distance for the interior players such that no player has an incentive to deviate. The same logic, and hence multiplicity of equilibria, applies for any $n > 6$.

Eaton and Lipsey present an informal proposition (for any $n \geq 6$) computing the bounds for the locations of the peripheral pairs of players, as well as for the vote shares of each of the players as a function of n. The upper bound for the distance from the boundary of the space to the leftmost (or rightmost) player is $\frac{1}{n}$. If the space left at the extreme by the peripheral player were greater than that, this player would get a payoff greater than $\frac{1}{n}$, and hence the average payoff of the other players must be strictly less than $\frac{1}{n}$, which means that at least one of them could profitably deviate to this space left by the peripheral player and get a payoff of at least $\frac{1}{n}$. Thus, in any equilibrium the distance between the leftmost and rightmost player will be at least $\frac{n-1}{n}$, which will approach 1 as n increases. Within this range the players must be spread relatively evenly. Towards the two boundaries there must be a pair of peripheral players, as the outermost players always have an incentive to move towards their neighbours as far as possible. Interior players may be located either individually or in pairs. If they appear as a pair their distance will be δ, but otherwise the upper bound for the intervals between players will be $\frac{2}{n}$. What is more, in equilibrium no player can get more votes than twice the number of votes for any of the other players.

We now consider mixed strategy Nash equilibria. Although no pure strategy equilibrium exists for $n = 3$, there is a doubly symmetric mixed strategy Nash equilibrium characterized by Shaked (1982): All players choose locations x such that $\frac{1}{4} \leq x \leq \frac{3}{4}$ with equal probabilities. Ewerhart (2014) characterized the set of mixed Nash equilibria for $n \geq 4$, showing that as n increases this leads to a more dispersed distribution of individual locations. The distributions are sharply M-shaped, with most weight at locations at the periphery of the support interval. The support increases as n increases, and for the reported values of n exceeds that of the maximum distance between leftmost and rightmost players in the corresponding pure strategy Nash equilibria.

We can summarize these findings by noting that for any $n > 2$ the equilibrium analyses seem to discredit the Principle of Minimum Differentiation as the locations chosen in the Nash equilibria are spread out considerably.

Another noteworthy characteristic coming out of the equilibrium analysis is the generic multiplicity of Nash equilibria (for any $n > 5$ in pure strategies, and any $n > 3$ if we consider mixed strategies too). Besides this multiplicity of equilibria, another characteristic of these Nash equilibria, to which we will return in our analyis of the best-response dynamics, is that all these Nash equilibria are weak. While this is trivially true for any mixed Nash equilibrium, it is also the case in the pure strategy equilibria for any $n > 3$, where there are always some players who could deviate from an equilibrium location to any two locations that are between two other players without loss of payoffs.

Besides varying the number of players n, a number of alternative variations of the basic model and their effect on the equilibrium predictions have been considered in the literature. For example, Eaton and Lipsey (1975) consider one-dimensional spaces without bounds, or two-dimensional spaces, showing that the Principle of Minimum Differentiation will normally not hold. Apart from considering other

spatial dimensions, equilibrium locations may contradict the Principle also when extending the basic Hotelling game in other dimensions. For example, in industrial organization, (d'Aspremont, Gabszewicz, and Thisse, 1979) show that in a two-stage game where two firms choose their locations in the first stage, and then compete in terms of prices in the second stage, when the cost of transportation is quadratic for consumers, firms will locate in the opposite ends of the line to soften the competition in terms of prices.[3] Irmen and Thisse (1998) consider product differentiation in a multi-characteristics space, and show that firms will choose to maximize differentiation in some dominant (salient) characteristic while minimizing differentiation in the others. Similarly, in political economy, the Principle may not apply if political parties care not only about winning elections but are also ideologically motivated, i.e., they care about the policy actually implemented. Even with only two parties, in equilibrium we may see diverging platforms if there is sufficient uncertainty about the location of the median voter (Drouvelis, Saporiti, and Vriend, 2014).

3 Best-response dynamics: numerical analysis

Before turning to the formal, mathematical analysis of the behavior of best-response dynamics in the Hotelling model in sections 4 and 5, where we characterize the invariant distribution of the players in the long run, in this section we will present a numerical analysis of the dynamics of the system.

For the remainder of the paper we will focus on a slight variation of the basic, linear location model sketched in section 2. Instead of a continuous strategy space, we will consider a discrete space. We discretize the interval $[0, 1]$ into $2M+1$ equally

[3] Matsumura, Matsushima, and Yamanori (2010) analyze the consequences of the evolutionary dynamics of two firms competing in the setting of d'Aspremont, Gabszewicz, and Thisse (1979), and show that such dynamics restore the minimum differentiation in that, under the unique stochastically stable equilibrium, the two firms will locate in the center and set their prices equal to their marginal costs.

spaced locations $x \in \{0, 1, .., M-1, M, M+1, ..., 2M\}$, where $x = 0$ and $x = 2M$ correspond to the two boundaries, 0 and 1 respectively, and $x = M$ is the median. Higher values of M correspond to a finer discretization of the space.[4] As before, the spatial distribution of voters (uniform with full support) is known to all players, and it will stay constant.

Each player can only occupy one of these discretized locations at any point in time. But any location x can be selected by any of the players simultaneously.[5] As the payoff for a player from choosing location x is simply the number of votes received at that location, when the number of players in location x is n_x, and two nearest players to the left and to the right are located in $L(x)$ and $R(x)$ respectively, with $L(x) < x < R(x)$, the payoff of choosing location x is given by:

$$\pi\Big(x|n_x, L(x), R(x)\Big) = \begin{cases} \frac{1}{n_x}\frac{R(x)-L(x)}{2} \text{ if } 0 \leq L(x) < R(x) \leq 1 \\ \frac{1}{n_x}\left(x + \frac{R(x)-x}{2}\right) \text{ if no player to the left of } x \text{ and } R(x) \leq 1 \\ \frac{1}{n_x}\left(1 - x + \frac{x-L(x)}{2}\right) \text{ if } 0 \leq L(x) \text{ and no player to the right of } x \end{cases}$$

The discreteness of the space and the opportunity for more than one player to select the same location induce some slight changes to the equilibria sketched in section 2, but the main qualitative features, with all Nash equilibria for any $n > 2$ looking strongly at odds with the Principle of Minimum Differentiation, are not affected. We will report on the dynamics for values of $M \in \{1500, 15,000, 150,000\}$, and values of $n \in \{3, 4, 5, 6, 7, 8\}$.[6]

[4]Note that this implies that the number of locations will be odd, ensuring that the space has a median, location M, that can actually be chosen by the players. Results for the numerical analysis of the best-response dynamics for even numbers of locations are available from the authors upon request.

[5]As before, when a voter finds that the best option(s) to the left and the best option(s) to the right are equally distant, they choose between going left or right by a fair coin toss. If, then, there is more than one player at location x, votes will be equally divided among the players located there.

[6]We chose the values of M so that the number of intervals will be a multiple of 60, accommodating many of the Nash equilibria in the discretized system for the numbers of players n that we consider. Of course, as noted already, in a continuous strategy space there is an infinite

In each period, all players simultaneously decide where to locate themselves, except in the very first period when they are all located randomly. We consider noisy myopic best-replies. Each player chooses a myopic best-reply with probability $1-\epsilon$, and chooses a location uniformly randomly with probability ϵ. We report the results for $\epsilon = 0.001$. When myopically best-replying, each player takes the positions of the other players in the previous period as given and selects a position that maximizes his payoff. When there are multiple such locations, one of them will be chosen (uniform) randomly.[7]

Best-response dynamics are related to a broad class of learning dynamics and evolutionary dynamics (see, e.g., Hopkins, 1999). The underlying idea of considering such a plausible class of dynamics is to shed some light on the question as to whether one should expect (boundedly rational) players to play a Nash equilibrium. If best-response dynamics converge, it can only be at a strict Nash equilibrium. But if they exhibit endless cycling, one question to consider will be where the system spends most of the time.

As we saw above, for any $n > 3$ the location game is characterized by a multiplicity of equilibria, and all these Nash equilibria are weak, as there are always some players who are indifferent between their equilibrium locations and some alternative locations. As players choose randomly among their best-replies, for any equilibrium there may be some individual players who move out of their equilibrium location, rendering these equilibria unstable as other players may be

number of Nash equilibria for any $n > 5$, and thus any discretization will be able to relate to only a subset of these. The minor effect of the discreteness of the strategy space is illustrated e.g. in Huck, Müller, and Vriend (2002), where we see that with $n = 4$ in equilibrium the paired players may stay either in the same location or in two neighboring ones. Results for the analysis of the dynamics for different values of M and n are available from the authors upon request.

[7]We also considered a version with inertia, in which players for whom the current location is part of their best-response correspondence will stay put, and a version with a preference for near-by locations, in which, in case of indifference, the location closest to the current location will be chosen. Note that these versions rely on additional specific assumptions about moving costs. Instead of simultaneous moves we also considered a version with sequential moves, where in each period one randomly chosen player decides where to locate himself. Results for these variants are available from the authors upon request.

affected. In addition to this effect of the equilibria being weak, the best-response dynamics that we consider are characterized by a small amount of noise. This prevents the system getting stuck in simple periodic trajectories.[8] Moreover, the presence of noise ensures that the location dynamics will be an ergodic Markov chain. It is then well known that their long run behavior will be described by an invariant distribution on the states that is reached regardless of the initial conditions of the system.

The question, then, is what outcomes one should expect in these simple linear location models when players follow a noisy myopic best-reply. Although one should not expect perfect convergence to a weak Nash equilibrium, one possible outcome would be that they spend most of the time near or approximating Nash equilibrium locations.

Before we will turn to an examination of some statistics of the system later in this section, we start our numerical analysis with a relatively close-up look at a number of representative runs of the model, examining how individual players move around from period to period. This will also provide some insights that may be helpful in our formal, mathematical analysis in sections 4 and 5.

Figure 1 shows some representative snapshots of locations chosen by all n players over a 1000 period interval for $M = 1500$, and $n = 3, 4, 5, 6, 7, 8$, with $\epsilon = 0.001$. Each location chosen by each player in each period is represented by a dot. The 1000 periods were taken after $10,000,000$ periods had passed from the beginning of each run to reduce the possible effects of the initial, random allocations. Time is shown on the horizontal axis, while the vertical axis shows the locations. The left-hand side column shows the 1000 locations near the center for the graphs for $n = 3, 4, 6$ and 8, while the right-hand side column shows the same runs, but now with the vertical axis zoomed into the most relevant parts of the strategy space for

[8] It is also effective in the variant with inertia (where players stay put in case of indifference).

each graph to optimize the display of the locations.

Figure 1 reveals some interesting features. For $n = 3$, the players are basically staying within a small set of locations around the center. For $n \geq 4$, we can see more clearly how the dynamics are dominated by waves of outward expansion of two clusters of players that are equally distanced from the center and located in the opposite sides of the center, alternated with waves of single clusters slowly moving inward to the center. While riding the outward waves, when it comes to choosing a best-response all players are indifferent between these two clusters, as long as there is at least one other player in their current cluster. As a result, the numbers involved within each of the clusters may vary from period to period, and at some point it will happen that all the players locate themselves in the same cluster, and hence in the same half of the strategy space. The single cluster, then, starts moving step-by-step towards the center.[9] Once the cluster reaches the center, there are two possibilities. Either all players stay together, moving to either $M-1$ or to $M+1$ before returning to M. Or the cluster splits into two, with some players moving to $M-1$ and some others to $M+1$, followed by a step-by-step outwards movement of both clusters, with varying memberships, until all players happen to choose the same side of the strategy space again. To start these waves all that is needed is that all players are located in the same half of the strategy space.[10]

Note that in none of the snapshots do we see convergence to the corresponding Nash equilibrium. Instead, we tend to see the wrong clustering at the wrong locations and moving into the wrong direction. At first sight, the most likely candidate to be reached by best-response dynamics is the Nash equilibrium for the case of $n = 4$, as it consists of exactly two clusters of two players at locations $\frac{M}{2}$ and $\frac{3M}{2}$. Taking a closer look at this case provides an interesting illustration for the

[9]Note that during their move to the center at every step all players select the same location. That is, there is minimum differentiation even when they are still on their way to the center.

[10]This takes, depending on n, typically only a few periods from a random initial allocation.

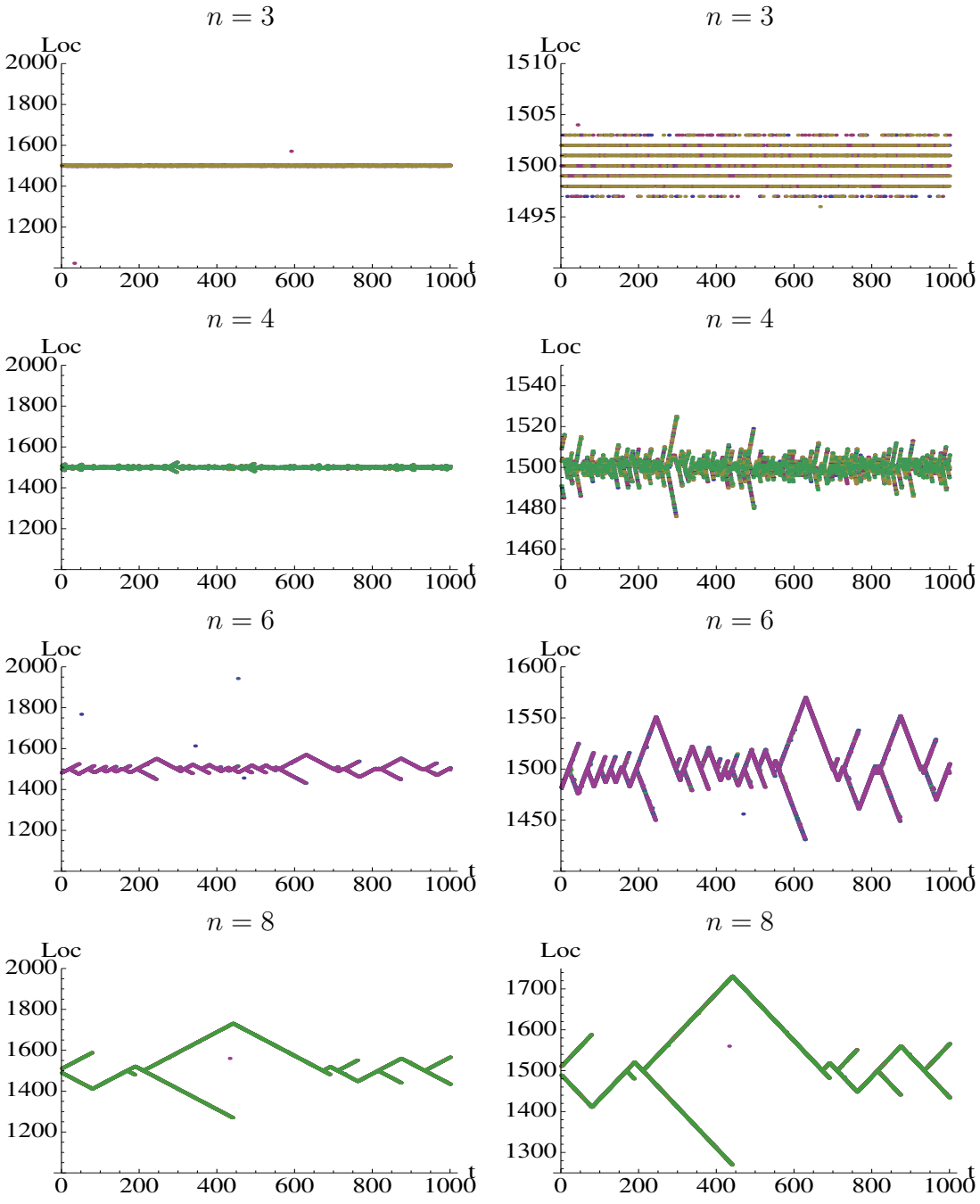


Figure 1: Locations over 1000 period interval for $M = 1500$, $n = 3, 4, 6, 8$, with $\epsilon = 0.001$

general lack of convergence to equilibrium. Note that, for any n, the farthest the outward-moving waves can reach is a distance $\frac{M}{2}$ from the center, as the clusters are shattered at that point because players prefer to move to anywhere between the two clusters rather than continue their outward movement. Thus, if this Nash equilibrium were reached, it may be shattered immediately. But as Figure 1 shows, the outward-moving clusters may disintegrate already long before the equilibrium locations are reached. For other values of n the dynamics look similar, but on top of that the Nash equilibrium configurations are not characterized by two clusters. What is more, for any $n > 4$ the Nash equilibria require some players to move beyond the points where clusters would stop moving outwards and start disintegrating with some players moving back inwards.

Next, what about the long-run properties in terms of the average distances of players from the center for various n? How do they vary when we increase the number of locations, i.e, when we increase M? To answer such questions, we now turn to some statistics of these best-response dynamics, moving beyond these representative runs.

We analyze the myopic best-replies for $11 \times T$ periods where $T = 10,000,000$. We drop, again, the data from the first T periods to reduce the possible effects of the initial, random allocations, and keep the outcomes from the remaining $10 \times T$ periods. For a given run, for each period we compute the average distance of the n players from the center, and we then check for that run the distances from the center below which this average distance is found 90%, 95% and 99% of the 100 million periods. Table 1 reports these distances for $n = 3, 4, 5, 6, 7$ or 8 and for $M = 1500$, $15,000$ and $150,000$, and with $\epsilon = 0.001$ throughout. The reported mean distances (with the standard deviations in parentheses) are taken over 30 runs.

Table 1 also reports the distances for the Nash equilibrium predictions that minimize the average distance from the center, i.e., the equilibria with as much

clustering of players in the middle as possible, to compare the outcomes of the noisy best-replies dynamics with the best Nash equilibrium predictions.[11]

Table 1 shows that when $n = 3$, regardless of the number of locations, the players are within distance 2 from the center almost all the time. Moving beyond $n = 3$, the average distance from the center increases with the number of players n. This is true for all values of M. Focussing on M, we note that the average distance below which the system spends time increases only very slightly as we move from $M = 1500$ to $M = 15,000$, and to $M = 150,000$, with these increases being detectable only for larger values of n. Thus, for example, for $M = 1500$ and $n = 8$, we see that the distance from the center below which the system (i.e., the average distance of the n players in the period concerned) spends 95% of the time is 265.5, and this increases to only 271.9 when we increase the number of locations to 150,000 (a hundredfold increase).

We can compare this with how the average distances in the Nash equilibria increase with the number of locations. As we see in Table 1, the predictions of the Nash equilibria are not affected by the refinements of the strategy space in the sense that they are scaled up proportionally with the number of locations and hence stay away from the median at a constant relative distance. For example, the average distances from the center for $n = 4$ are 750, 7500, and 75,000 for M =1500, 15,000 and 150,000, respectively.[12]

Table 1 also reflects the fact that as n increases, the average distance of the Nash equilibrium that is closest to the center is essentially constant.[13] As n increases the peripheral players get closer to the boundaries, but this is exactly offset by the most central ones getting closer to the center, and thus the average distance stays

[11]For $n = 3$ this is based on the mixed strategy Nash equilibrium, and therefore we report the average distance below which the system will spend 90% of the time.

[12]If, for $n = 3$, we consider the 99% criterion instead of the 90% reported in the Table, these will be 652, 6521, and $65,213$, for $M = 1500$, $15,000$, and $150,000$, respectively.

[13]Apart from some minor variances as different values of n lead to slightly different types of Nash equilibrium configurations.

$M = 1500$						
	n=3	n=4	n=5	n=6	n=7	n=8
90%	2.0	8.0	30.0	66.0	123.7	203.7
	(0.0)	(0.0)	(0.0)	(0.0)	(0.5)	(0.6)
95%	2.0	10.0	40.0	86.9	161.3	265.5
	(0.0)	(0.0)	(0.0)	(0.3)	(0.5)	(0.9)
99%	2.0	15.0	65.9	139.9	247.4	409.3
	(0.0)	(0.0)	(0.4)	(0.3)	(0.8)	(1.4)
NE	539	750	800	667	804	750

$M = 15,000$						
	n=3	n=4	n=5	n=6	n=7	n=8
90%	2.0	8.0	30.0	66.1	124.1	204.6
	(0.0)	(0.0)	(0.0)	(0.3)	(0.4)	(0.6)
95%	2.0	10.0	40.0	87.3	164.2	271.1
	(0.0)	(0.0)	(0.0)	(0.5)	(0.4)	(0.9)
99%	2.0	15.0	67.2	152.0	291.5	481.9
	(0.0)	(0.0)	(0.4)	(0.5)	(1.4)	(2.4)
NE	5392	7500	8000	6667	8036	7500

$M = 150,000$						
	n=3	n=4	n=5	n=6	n=7	n=8
90%	2.0	8.0	30.0	66.1	124.2	204.7
	(0.0)	(0.0)	(0.0)	(0.3)	(0.4)	(0.6)
95%	2.0	10.0	40.0	87.5	164.6	271.9
	(0.0)	(0.0)	(0.0)	(0.5)	(0.5)	(0.8)
99%	2.0	15.0	67.9	154.0	302.0	525.6
	(0.0)	(0.0)	(0.3)	(0.5)	(1.4)	(3.8)
NE	53,914	75,000	80,000	66,667	80,357	75,000

Table 1: Mean distances from the center (with standard deviations in parentheses), and NE distances, for $M = 1500, 15,000, 150,000$, $n = 3, 4, 5, 6, 7, 8$, with $\epsilon = 0.001$

constant. A good example of this is a comparison the average distance for the Nash equilibria reported for $n = 4$ and 8.

We can conclude that Table 1 confirms what we saw in Figure 1, i.e. that the best-response dynamics do not converge to the Nash equilibria.[14] Our numerical analysis also shows that for any $n > 2$, as M, and hence the number of locations, increases, the players tend to spend their time farther and farther away from the Nash equilibrium predictions and within smaller and smaller regions around the center. In other words, as we refine the discrete strategy space, approximating a continuous strategy space, we can get the players locating arbitrarily close to the median when they follow best-response dynamics.

Thus, the best-response dynamics restore something close to the Principle of Minimum Differentiation and as a result, the preferences of the median voter will tend to rule.

4 Long run behavior: analyzing the invariant measure with three players

With the numerical analysis presented in Section 3 in mind, in this section and in Section 5, we characterize the long-run properties of the system. For any number of locations M and any level of noise ϵ, the long run behavior of the location dynamics is described by the invariant distribution $\mu_{M,\epsilon}$ of an ergodic Markov chain. The dynamics that we observe in the numerical analysis after a sufficiently large number of time steps, are in fact sample paths from the distribution $\mu_{M,\epsilon}$. Our analytical results concerning the behavior of the location dynamics are based on an asymptotic analysis (w.r.t. M and ϵ) of the invariant distribution $\mu_{M,\epsilon}$. In other words, we study this distribution when the number of locations tends to infinity and/or the level of

[14] That the best-response dynamics lead to outcomes that are in some sense the 'opposite' of the Nash equilibrium predictions was also observed in Pancs and Vriend (2007).

noise is driven to zero. The asymptotic analysis is relevant for understanding the behavior of the dynamics when the number of locations is finite but large and for positive but small levels of noise, which is precisely the setting we are interested in here.

For the case of $n = 3$ players analyzed in this section, we will let the number of locations M be fixed and drive noise down to zero, as is a standard approach in the equilibrium selection literature. The states in which the invariant measure is concentrated as noise goes to zero are referred to as the stochastically stable states, and correspond to outcomes that remain stable when players make errors with a small probability.

In the case with four players, presented in Section 5, we will use the fact that M is large and we will study the behavior of $\mu_{M,\epsilon}$ driving M to infinity and noise down to zero simultaneously, while imposing some conditions on the relative speed of convergence of these quantities. Although by doing so we lose the connection to the equilibrium selection approach, such an asymptotic analysis, as we argued above, does indeed capture the behavior of our system when the number of locations is large and the noise level small.

The reason we present two distinct approaches for $n = 3$ and $n = 4$ is related to the difference in the dynamics between these two cases as we have seen in Section 3. For $n = 3$, as observed in the numerical analysis and as we will prove in this section, the agents spend almost all their time at locations whose average distance from the center is strictly less than three. Thus, the typical distance to the center is bounded independently of the system size. For $n \geq 4$, instead, there are repeated waves of two-sided expansion and one-sided contraction centered around the middle. The distance of these waves from the center is typically very small compared to the number of locations, but it is not bounded independently of the latter. When the number of locations increases, the maximal distance of the waves from the center

also increases but at a speed that is some orders of magnitude smaller. Thus, as the number of locations increases, the sizes of the waves will become negligible compared to the size of the system. We will make such statements more precise in the asymptotic framework presented in Section 5. In the remainder of this section we will characterize the stochastically stable steady states for the case of three firms.

4.1 Absorbing classes

It is well known that when noise is driven down to zero, the invariant distribution will concentrate on some union of absorbing classes. A set of states is an absorbing class if we remain within the class under best reply without noise ($\epsilon = 0$). Thus, we begin by characterizing the absorbing classes of our system.

Recall that we consider a system where the center is unique and denoted by M. The leftmost location is then 0 and the rightmost location $2M$. Let (a, b, c) denote the locations of the players, adopting the convention that $a \leq b \leq M < c$. By symmetry, we consider, without loss of generality, the case where two players are to the left of the center and the third player on the other side.[15] The number of positions to the left of a is then a and the number of positions to the right of c is $2M - c$. If $a > 2M - c$ it is preferable to locate to the left of a than to the right of c and conversely. If $a = 2M - c$ these two choices are indifferent. We will also consider the distance between the peripheral, i.e. the most extremely located, players, which under the above assumptions is the length of the interval $[a, c]$, i.e., $c - a$. Also, a and $2M - c$ are the distances of the peripheral players to the boundaries. For given M, when a and $2M - c$ increase, the distance to the center decreases. We will denote the set of best replies to (a, b, c) by $BR(a, b, c)$. Since all players are identical we do not care who is located where so we will take $BR(a, b, c)$ to be an unordered set. We will denote by BR^l, BR^m, BR^r the best responses of the leftmost,

[15] The case where all players are on the same side will be treated separately.

middle and rightmost player respectively. We denote by BR_t^l, BR_t^m, BR_t^r the t-th best response of the leftmost etc. player.[16] We denote by $BR_t(a,b,c)$, or sometimes for convenience simply BR_t, the locations of the players after t periods, determined by best replies, starting from (a,b,c). Best replies may be non-unique.

The following proposition characterizes the absorbing classes of the system.

Proposition 1 *Let* $C = \{(M-3, M-1, M+1), (M-1, M+1, M+3), (M, M+2, M+2), (M, M-2, M-2), (M-1, M+1, M+1), (M-1, M-1, M+1), (M-2, M, M+2),\}$ *and* $C_k = \{(M-k, M-k, M+k-1), (M-k+1, M+k, M+k)\}$. *Then,* C *is an absorbing class, and* C_k *is an absorbing class for* $1 \leq k \leq K$, *where* $K =: max\{k|2M-(M+k-1) > \frac{2k-1}{2}\}$.

Proof: Immediate verification. The condition on k ensures that the players on the left will prefer location $M+k$ to an interior location. ■

Although it is possible to show that C and $(C_k)_{k=1}^{k=K}$ are the only absorbing classes, we will not need this result. In fact, in our analysis we will focus on the following union of absorbing classes: Define $S = C \bigcup C_1 \bigcup C_2$.

4.2 Basins of attraction

If an element that is not in the absorbing class S will eventually lead to S by best replies with probability one, it is said to belong to the basin of attraction of S. The basins of attraction of the absorbing classes play an important role in determining their stochastic stability. We recall that the basin of attraction $B(S)$ of an absorbing class S is: $B(S) = \{z|Prob\ (\exists\ T\ st\ BR_t(z) \in S\ \forall\ t > T) = 1\})$.

As the following two lemmas show, a large subset of configurations is in the basin of attraction $B(S)$ of the union of absorbing classes S defined above.

[16] Note that BR_t^l refers to the t-th best reply of the leftmost player at time $t-1$, which is not necessarily the leftmost player in the initial configuration.

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Lemma 1 *Any configuration* (a, b, c) *where* $a \leq b \leq c \leq M$ *(or* $M \leq a \leq b \leq c$*), i.e. with all players on the same side of the center, is in the basin of attraction of* S*.*

Proof: We make explicit the possible BR sequences. If $a \leq b < c \leq M$, $BR(a, b, c) = (b + 1, c + 1, +c + 1)$. If $b < 2M - c - 2$, $BR_2 = (c + 2, c + 2, c + 2)$, which is considered below. If $b \geq 2M - c - 2$, which is only possible if $c = M$ and $b \in \{M - 1, M - 2\}$, these cases correspond to $BR \in \{(M, M + 1, M + 1), (M - 1, M + 1, M + 1)\}$. The first is in C_1 and the second in C. If $a \leq b = c < M$, we get $(c+1, c+1, c+1), (c+2, c+2, c+2)$, and then all the way to (M, M, M). The configuration (M, M, M) leads either to $(M-1, M-1, M+1)$ or $(M-1, M+1, M+1)$, which are both in C or to $(M-1, M-1, M-1)$ and then (M, M, M) or similarly to $(M+1, M+1, M+1)$ and then (M, M, M). In this case we cycle, but with positive probability we will eventually reach C from (M, M, M). ■

Lemma 2 *Any configuration that lies within* $[M - 3, M + 3] - C_3$ *is in the basin of attraction of* S*.*

Proof: Given in Appendix A. It simply requires checking a number of configurations. ■

Combined, Lemmas 1 and 2 characterize the basin of attraction $B(S)$ of the union of absorbing classes S that is the focus of our analysis.

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4.3 Stochastically stable steady states: Radius, coradius results

To show that the invariant distribution will be concentrated on S, we will use the radius-coradius theorem of Ellison (2000). The main idea behind this theorem is to associate a cost to any path of transitions. Loosely speaking, this cost is proportional to the number of random events (i.e. choices that are not best replies) required on the path. The stochastic stability of a set is then determined by comparing the cost of leaving its basin of attraction and the cost of entering its basin of attraction from the outside.

The Hotelling model with noise lies within the general setting considered by Ellison (2000) since a player locates with probability $1-p$ on a best reply, and with probability p uniformly at random. In the Hotelling model, we can define a cost function as follows: the cost of the transition from configuration z_1 to z_2 is $c(z_1, z_2) = 3 - m$, where m is the number of players in z_2 who play a best reply to z_1. In other words, the cost of a transition is the number of random events it requires. The cost of a path $(z_1, ..., z_t)$ can then be defined as $\sum_{i=1}^{i=t-1} c(z_i, z_{i+1})$. It can be verified that this cost function satisfies the properties required to use Ellison's radius-coradius theorem.

The crucial quantities defined by Ellison (2000) are the radius and coradius of S, $R(S)$ and $CR(S)$ respectively. The coradius of S basically measures the number of random events required to reach $B(S)$ from outside S. The radius of a union of stochastically stable sets measures the random events required to leave $B(S)$ starting from S.

The main result in Ellison (2000) is that the invariant measure is concentrated on a union of absorbing classes such that the number of random events required to leave this set (the radius) is greater than the number of random events required to reach it from the outside (the coradius):

Theorem 1 *(Ellison, 2000) In a model of evolution with noise, let Ω be a union of limit states. If $R(\Omega) > CR(\Omega)$ then the long run stochastically stable set is contained in Ω.*

We will use this theorem in order to show that the stochastically stable steady state is contained in $S = C \bigcup C_1 \bigcup C_2$. We first compute the radius and coradius of S.

Proposition 2 *Let $S = C \bigcup C_1 \bigcup C_2$, then the coradius of S, $CR(S) = 1$.*

To see that a single random event is sufficient to enter the basin of attraction of S, consider some C_k with $k > 2$. To move from such a C_k into S, it is sufficient that one player, the one who is alone on one side, relocates at random to the other side.[17]

Proposition 3 *The radius of S, $R(S) > 1$.*

We show that a single random event is not enough to leave the basin of attraction of S with Lemma 3.

Lemma 3 *Take a configuration $s \in S$. Relocate one player. Call this configuration ω. Then there exists $0 < t < 3$ such that $BR_t(\omega) \in ([M-3, M+3] - C_3) \bigcup B(S)$.*

Proof: Given in Appendix A. It simply requires checking a number of configurations. ■

Lemma 3 shows that after a single random event, we will return to $([M-3, M+3] - C_3) \bigcup B(S)$, which by Lemma 2 is also in $B(S)$. Consequently, any path from S to a configuration that is not in the basin of attraction of S requires at least two random events, and hence the cost is strictly greater than 1.

[17] Note that this applies to any configuration outside $B(S)$, as they are all of the two on one side versus one on the other side type.

Combining Proposition 2 and 3 shows that the radius $R(S)$ is strictly greater than the coradius $CR(S)$, and together with the theorem in Ellison (2000) this gives us the main result for our analysis of the case of three players:

Corollary 1 *The long run stochastically stable set is a subset of* $S = C \bigcup C_1 \bigcup C_2$. *All configurations in* S *are such that the average distance to the center is strictly smaller than* 3*, and for none of the players is the distance to the center greater than* 3.

5 Long run behavior: analyzing the invariant measure with four players

We now move to the analysis of cases with four or more players. As mentioned already, the case with more than three players differs qualitatively from the case with only three players and needs to be studied using a different approach.

With only three players, in the absence of random noise, the central location belongs to an absorbing class. This is not the case with more than three players. In this case, players will move away from the center in waves and then return to it. However, we will show that the distance to the center will typically remain small compared to the system size as a whole when the latter is large. To capture this intuition formally, we let the number of locations grow and impose some restrictions on the level of random noise compared to the number of locations. We then show that the invariant measure is concentrated on the center and on some absorbing classes close to the center. We first provide some notation needed to state our main result.

5.1 Description and notation

We denote by N the $2M+1$ locations indexed by $\{0,1,2,...,M,M+1,...,2M\}$. Let $\omega \in \{0,1,2,...,M,M+1,...,2M\}^4$ be a configuration, i.e. giving the locations of the four players. For convenience, we will write $\omega = (a*k,...)$, when the configuration ω has k ($k \geq 3$) players located at a.

We denote by $\bar{a}$ the configuration where all players are at location a. The distance between two locations a and b is $d(a,b) = |a-b|$. For a configuration $\omega = (a,b,c,d)$, we define the distance from the center M as $d(\omega, M) =: max_{i=a,b,c,d} d(i,M)$. Given a configuration ω at date t, the configuration at $t+1$ is given by the realization of the following random variables: (1) First, we determine the number of players who do not relocate at one of their best replies. This number is given by a binomial random variable $X_t = Bin(\epsilon, 4)$. (2) Given the realization of X_t, we draw a variable Y_t following a uniform law on all the subsets of $\{1,2,3,4\}$ of size X_t. If $X_t = 1$, each player has probability $\frac{1}{4}$ of being selected for relocation. (3) Finally, for any element $e \in \{1,2,3,4\}$ such that $e \in Y_t$, we draw its location uniformly at random. Any player who is not drawn for random relocation, picks each of his best replies according to a uniform probability.

This description, where we draw first the number of uniformly relocated players, then their identity, then their location, is convenient for our proof and obviously equivalent to relocating each player uniformly at random with i.i.d. probability ϵ. Let us also introduce some notation for some particular subsets of configurations that will be of interest. Let A be the set of configurations where all players are on the same side (i.e. $\geq M$ or $\leq M$). Analogous to the case of three players, we note that the classes consisting of the two states $\{((M-a)*3, M+a-1), (M-a+1, (M+a)*3)\}$, with the same restriction on a as we saw for k in Proposition 1, are absorbing classes for the dynamics. Let us denote by Ω_{AC} the set of such absorbing classes. Moreover, let Ω_{AC}^l be the subset of such classes for which $d(a,M) < l$, in other words absorbing

classes containing configurations that are closer than l to the center.

5.2 Setting and main result

In the case of $n = 3$ players, the number of locations was fixed and the level of noise ϵ converged to zero asymptotically. In other words, we studied the behavior of the invariant distribution $\mu_{N,\epsilon}$, holding N fixed and taking the noise ϵ down to zero. For the case with $n = 4$ players, we will analyze the behavior of the invariant distribution $\mu_{N,\epsilon}$ when N goes to infinity and ϵ goes down to zero. We will put some restrictions on the level of noise compared to the number of locations. To this effect, assume that the level of noise in a Hotelling model with N locations is ϵ_N[18] The invariant measure μ_{N,ϵ_N} will be denoted simply by μ^N. Our main result characterizes the asymptotic behavior of μ^N under some assumptions on the speed at which ϵ_N goes down to zero as N increases.

Theorem 2 *In the Hotelling model with N locations, let I_N be an interval centered at M containing l_N locations. Let S_N be a subset of locations such that $\omega \in S_N \iff \omega \in [M-3, M+3] \bigcup \Omega_{AC}^{l/2} \subset I_N$. Let B_N be any set of states $B_N \subset I^c$ such that $card(B_N) \leq card(S_N)$. Suppose that the Condition 1 below is verified, then $lim_{N\to\infty} \frac{\mu^N(B_N)}{\mu^N(S_N)} = 0$.*

Condition 1 *There exist constants $\alpha \in]0, \frac{1}{7}[$, $\beta_M < \infty$ and N_0 such that for every $N \geq N_0$, $l_N = N^\alpha$ and the level of random noise ϵ_N verifies $\epsilon_N = N^{-\beta}$, where $\beta \in [1, \beta_M]$.*

Theorem 2 states that when we consider a certain set of states S_N included in a 'small' (some orders of magnitude smaller than N) interval around the center, I_N, any set containing the same number of locations as S_N (or a smaller number) and that is located outside of the interval I_N, i.e. farther from the center, has an

[18] Later on, when N is fixed, for convenience we will omit the index N, writing simply ϵ.

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invariant measure that is vanishingly small compared to the measure of S_N when the number of locations is large. In particular, it follows from this result that states close to the center have more weight than the Nash equilibrium, and also more weight than absorbing classes farther from the center.

How demanding are assumptions under which this asymptotic result holds? The most important restriction imposed by Condition 1 concerns the speed at which ϵ_N goes down to zero as the number of locations increases, since we must have $\epsilon_N \leq \frac{1}{N}$. The upper bound of the size of the interval l_N is not a demanding condition in our context. Since we want to show that the invariant measure is concentrated close to the center, we are only interested in intervals l_N that are small compared to the total number of locations N. The lower bound of the size of l_N and the lower bound on the noise are not very restrictive. Together they guarantee that $\epsilon_N^2 \geq p^{l_N}$ for any $0 < p < 1$ when N is large. We should note that Condition 1 is a sufficient but not necessary condition.

5.3 Analysis of the dynamics of the system and intuition for our results

Before turning to the proof of Theorem 2 itself, we will provide some intuition for the result based on the behavior of the system's dynamics. A central part of our proof is to show that starting from the center, we will return to it with high probability. As we have seen in the numerical analysis shown in Figure 1, starting from the center, in absence of random noise, the dynamics would be as follows. If two players locate at $M-1$ and two at $M+1$, we move farther away from the center. However, there is also a positive probability that all players locate on the same side. As we have established previously, the dynamics will then bring them back to the center. Similarly, in the next step, all players may locate on $M-2$ and $M+2$ but again, with positive probability, they may all end up on the same side. In each period, we

can move farther away from the center but there is also a positive probability that all players end up on the same side. The probability that the latter does not occur in a long sequence of time steps is very small.[19] This description of the dynamics ignores the possibility of random relocations. However, with high probability only few random relocations occur (as shown in Lemma 4 in subsection 5.3.2). This makes the dynamics somewhat more complex but not significantly different from what is described above.

An occasional random relocation may lead to various configurations within the interval I. However, we show (Lemma 5 in subsection 5.3.3) that the configurations we reach are always either contracting, in the sense that best replies bring us closer to the center, or, if the state is such that best replies can be farther away from the center with positive probability, there is also a positive probability that all players end up on the same side. This is because such configurations involve indifference on the part of some players. The typical, but not unique, example of such a state is the one described previously where two players are at $M-k$ and two players at $M+k$. Thus, we can only move far away from the center if we pass through configurations of the second type a large number of times. However, each time we do, there is a positive probability that all players end up on the same side and the probability of avoiding this event for a long time is very small.

So far, we have ignored the possibility of entering one of the absorbing classes in Ω^l_{AC}. This case is analyzed separately in subsection 5.3.4. If such a class is reached, we remain there for a long time. The most probable way to exit is with a single random relocation. When a single random relocation occurs, with positive probability we move to another absorbing class farther from the center, or, also with positive probability, we enter a state where all players are on the same side. Again, it is highly unlikely to have a long sequence of realizations where we move farther

[19] The conditions we impose in Theorem 2 guarantee that the probability of not returning in this way is in fact smaller than the probability of a random relocation.

away from the center without an occurrence of an event where all players end up on the same side.

Finally, the proof also involves showing that with a single random relocation we can move back from any state to the center. This is the object of subsection 5.3.5.

5.3.1 Proof of Theorem 2

In what follows, we often do not need to establish exact values of quantities but only their order of magnitude with respect to N when N is large. We will thus use the notation $g(N) = O(f(N))$ if $0 < \lim_{N\to\infty} \frac{f(N)}{g(N} < \infty$. Moreover, from now on we will write l for l_N and ϵ for ϵ_N.

Our proof will make use of the following property of the invariant distribution of a Markov chain: $\mu(S) = \mu(b)E[V(S, b, b)]$, where $V(S, b, b)$ is a random variable that counts the number of times that the process reaches an element in S before it reaches b, starting from state b.[20]

To bound the value of this expectation, we will bound on the one hand the expected number of times we return to S starting from S, and also bound the probability of reaching S from a state in I^c. These two bounds are the object of the following two propositions:

Proposition 4 *The probability of not returning to the set S when starting from a configuration in S is bounded above by $1 - q = O(\frac{\epsilon}{l^2})$.*

Proposition 5 *Let $b \in I^C$, the probability of reaching S from b without passing again through b is bounded below by $\tilde{q} = O(\epsilon)$.*

The event $V(S, b, b) = k$, requires first that we move from $b \in I^C$ to S without passing again through b, which occurs with a probability greater than $\tilde{q}$, and then that starting from S, we return again to S exactly $k - 1$ times before reaching

[20] See, e.g., Kemeny and Snell (1960).

b again. Due to propositions 4 and 5, we can bound the expectation as follows: $E[V(S,b,b)] = \sum_{k=1}^{\infty} \tilde{q}kq^k(1-q) = \tilde{q}E[Z]$, where Z is a geometric law of parameter q. Thus $E[Z] = \frac{1}{1-q}$ and $E[V(S,b,b)] = \frac{\tilde{q}}{1-q} = O(\frac{\epsilon l^2}{\epsilon})$. The number of elements in the set S is at most $2l$, because we can index the states in Ω^l_{AC} by the location of the three players who are at the same location. Therefore if $card(B) \leq card(S) \leq 2l$, $lim_{N\to\infty}\frac{\mu^N(S)}{\mu^N(B)} = lim_{N\to\infty}\frac{\mu^N(S)}{\sum_{i=1}^{i=card(B)}\mu^N(b)} = lim_{N\to\infty}2lE[V(S,b,b)] = \infty$. ■

The rest of the section will provide proofs of Propositions 4 and 5. A number of lemmas needed for these proofs will be presented in the next subsections.

5.3.2 Typical sequences: bounding the probability of rare events

Scenarios where two or more random relocations occur close to each other are very unlikely. It will be useful to bound the probability of such an event to show that it is very small. This is the objective of Lemma 4. When we study the dynamics, we will then condition on the fact that there is at most one random relocation in a given number of periods.

Definition 1 *We will say that a sequence is typical if it contains at most one random relocation and this relocation is not located in I.*

Lemma 4 *The probability that a sequence of length l is not typical is at most $O(\frac{\epsilon l^2}{N})$.*

Proof: The probability of at least two random relocations in a sequence of length l is $\sum_{k=2}^{k=4l}\epsilon^k(1-\epsilon)^{4l-k}C_k^{4l} \leq \sum_{k=2}^{k=4l}\epsilon^k(4l)^k = \frac{(4l\epsilon)^2-(4l\epsilon)^{4l+1}}{1-4l\epsilon} = O(\epsilon^2 l^2)$. Note that we have $4l$ because at each date, there are four players who each relocate with probability ϵ. The probability that there is exactly one random relocation and that it is in I is smaller than $4l\epsilon\frac{l}{N} = O(\frac{l^2\epsilon}{N})$. ■

5.3.3 Best reply dynamics starting from a state close to the center

In what follows, a state will be $\omega =: (a, b, c, d)$ with $a \leq b \leq c \leq d$ and $l(\omega) = d - a$.

Lemma 5, proven in Appendix B.1 together with some other lemmas, tells us that starting from a configuration close to the center and assuming the sequence is typical, best replies will either take us closer to the center than before, or, if there is a positive probability of moving farther away from the center, there is also a positive probability that all the agents end up on the same side, or, as a final possibility, we may enter an absorbing class close to the center.

Lemma 5 *Assume that the evolution is 'typical' as in Definition 1 above. Any state* $\omega \in I - S$ *belongs to* $\bigcup_{i=1}^{i=4} \Omega_i$ *where:*

- *If* $\omega \in \Omega_1$, *Then* $P((BR^{t+4}(\omega) \bigcup BR^{t+5}(\omega)) = \bar{x} \in I) \geq p$ $(p \geq \frac{1}{8^3})$*and if this does not occur,* $d(BR^2(\omega), M) \leq d(\omega, M) + 2$.

- *If* $\omega \in \Omega_2$, *then* $d(BR(\omega), M) \leq d(\omega, M)$ *and* $l(BR(\omega)) \leq l(\omega)$, *with at least one strict inequality.*

- *If* $\omega \in \Omega_3$, *then* $BR(\omega)$ *is at the same distance to the center as* ω *and* $BR(\omega) \in \Omega_1 \bigcup \Omega_2$.

- *If* $\omega \in \Omega_4$, *then* $BR^2(\omega)$ *belongs to an absorbing class in* Ω^l_{AC} *with* $l < d(\omega, M) + 2$.

We will also need a lemma that tells us that when the sequence is typical, a state of the type $\bar{x} \in I$ leads to $[M - 3, M + 3]$.

Lemma 6 *If we are in* $\bar{x} \in I$ *at date* t *and the sequence is typical until* $t + l$, *we reach a state in* $[M - 3, M + 3]$ *within* l *periods.*

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5.3.4 Best reply dynamics starting from an absorbing class $\Omega_{AC}^{l/2}$

We also need to determine what happens when we exit an absorbing class in $\Omega_{AC}^{l/2}$.

Lemma 7 *Consider a date t at which the deterministic best reply is $\omega = ((M-a)*3, (M+a-1)) \in \Omega_{AC}^{l/2}$ (without loss of generality). Conditioning on the fact that a single random relocation outside of I occurs at t, and no random relocation occurs at $t+1$, the state at $t+1$ can be*

- *$\omega_{t+1} = ((M-a)*3, (M+a-1)) \in \Omega_{AC}^{l/2}$, i.e. we remain in the same absorbing class at the same distance from the center.*
- *$\omega_{t+1} \in A$. This happens with probability at least $p = \frac{1}{8}$, and consequently, the probability that $\omega_{t+3} = \bar{x} \in I$ is at least $p = 1/8$ times the probability that no random relocation occurs at $t+2$ or $t+3$.*
- *$\omega_{t+1} \in \{(M-a-1)*3, (M+a)\}$, that is in an absorbing class one step further from the center than before, which occurs with at most probability $1-p$.*

Lemma 8 *Starting from any state $\omega \in \Omega_{AC}^{l/2}$, the probability of returning to S before reaching a configuration in I^C is greater than $1 - O(\frac{\epsilon}{l^2})$.*

Lemma 7 are 8 are proven in Appendix B.1. Now, we can prove Proposition 4.

5.3.5 Proof of Proposition 4

We want to bound the probability of returning to S from a state in $S = [M-3, M+3] \bigcup \Omega_{AC}^{l/2}$. First, consider the case where we start from an $\omega \in [M-3, M+3]$. Let us bound the probability of returning in at most $l/4$ periods, assuming that the sequence is typical over these periods. We can assume that we do not enter an absorbing class because if we do we have returned to S, so $|\Omega_4|$=0. Note that by Lemma 5, we have $|\Omega_3| = |\Omega_1| + |\Omega_2|$ since any successor of an element in Ω_3 is

in $\Omega_1 \bigcup \Omega_2$. Also note that if $|\Omega_1| < 2|\Omega_2|$, then we have returned to the center M because the configurations in Ω_2 contract one step and those in Ω_1 grow at most two steps. Therefore we must have $|\Omega_3| + |\Omega_1| + |\Omega_2| = \frac{l}{4}$, where $|\Omega_1| \geq 2|\Omega_2|$. Suppose that $|\Omega_1| < \frac{l}{24}$. Then $|\Omega_2| \leq \frac{2l}{24}$ and $|\Omega_3| < \frac{3l}{24}$. But this contradicts $|\Omega_3| + |\Omega_1| + |\Omega_2| = \frac{l}{4}$. Consequently, $|\Omega_1| \geq \frac{l}{24}$. Each time we are in Ω_1, we enter a state $\bar{x} \in I$ with probability p. From such a state we then return to $[M-3, M+3]$ if the sequence is typical by Lemma 6. The probability of not reaching a $\bar{x} \in I$ in $l/24$ trials is $(1-p)^{l/24} \leq O(\epsilon^2)$, which is a consequence of Condition 1 since $0 < (1-p)^{\frac{1}{24}} < 1$. Consequently, the probability of returning to S is bounded by $1-(1-p)^{l/24}$ times the probability that the sequence is typical for $l/4+l$ periods. Using Lemma 4 this probability is greater than $(1-O(\epsilon^2))[1-\epsilon^2 l^2] = 1-O(\epsilon^2 l^2)$. In the case which was not considered before, where we start from a state $\omega \in \Omega_{AC}^{l/2}$, it is sufficient to apply Lemma 8. ■

5.3.6 Proof of Proposition 5

Proposition 5 is a consequence of Lemma 9 below which is proved in Appendix B.2:

Lemma 9 *For any configuration $\omega \in A^C$, there exists a random relocation of a player, which occurs with probability $O(\epsilon)$ such that A is reached in at most three steps.*

The proposition follows almost immediately from this lemma. Indeed, once in A, the configuration will move towards the center at the pace of one location per time step. To reach the center, it is thus sufficient that no random relocation occurs in N steps. The probability that this is the case is $(1-p)^N$, where $p = 1-(1-\epsilon)^4$. Since $\epsilon < 1/N$, we have $(1-p)^N = O(1)$. Consequently, the probability of reaching M from $b \in A^C$ is given by the probability of entering A, which is $O(\epsilon)$, since a single random relocation takes us to A, times the probability that no random relocation

occurs for N periods after entering A, which has probability $O(1)$. The result in Proposition 5 follows. ■

5.4 A remark regarding extensions to $n > 4$

In this section, we have provided a proof for the case $n = 4$ players. For cases $n > 4$, the dynamics starting from the center are similar: we move away from the center as long as all players do not locate on the same side. With probability $(\frac{1}{2})^{n-1}$ all players end up on the same side and we return to the center. It would seem possible to generalize the proof regarding the probability of returning to the center to the case $n > 4$. However, proving that a single random relocation takes us back from any location to the center becomes less manageable, because there is a large number of cases to consider. We conjecture that the behavior of the system for $n > 4$ is similar to the case $n = 4$, as is suggested by the numerical analysis.

Analyzing the dynamics also allows us to understand why the average distance to the center is larger when we increase the number of players. Indeed, the probability that all players end up on the same side and then return to the center is $p_n =: (\frac{1}{2})^{n-1}$ at each time step, and thus the expected value of the first time that this occurs is $1/P_n$.

6 Concluding remarks

We considered a linear location model (Hotelling, 1929) in which players follow noisy myopic best-replies. We asked what are the likely configurations in terms of numbers of players in each location in such a case.

We analyzed numerically how the average distance from the center depends on the number of players n and the number of locations $2M + 1$, showing that by refining the discrete strategy space we can get the players locate arbitrarily close to

the center almost all the time.

In our formal, mathematical analysis we prove, in the case of $n = 3$ players, that all the players are located in close proximity of the center in the stochastically stable steady states. For the case of $n = 4$ we prove that the players will tend to be located near the center if the noise is small (in a sense made precise) relative to the number of locations. The logic of the proof for $n = 4$ seems applicable to any $n \geq 5$ as well.

Although our analyses show that we do not necessarily always have all players located precisely in the center, and thus we do not always have the minimum differentiation as the principle of minimum differentiation would suggest, our analyses suggest that if players are myopic and adaptive, we may tend to observe outcomes that conform rather closely to the principle of minimum differentiation, with the difference becoming negligible as the space is refined and approximates a continuum. Thus, we re-established Hotelling's principle for this class of boundedly rational players, and provided a possible explanation for the relatively common perception that decision makers in a wide range of situations tend to cater to the median voter: The return of the median voter.

As mentioned already in Section 1, and emphasized by Hotelling (1929) to start with, interpreting the location model as one of electoral competition is only one of the many possibilities, and the Principle of Minimum Differentiation seems applicable in a wide range of situations of players competing in some discrete or continuous strategy space. Thus, our analysis of the best-response dynamics seems relevant in particular also for market dynamics, with the firms competing e.g. in geographical space or product characteristics space.[21]

Focussing on geographical space, the total distance between the locations chosen by the firms and the preferred locations of the consumers corresponds to the total

[21] Firms may compete in other dimensions too. For example, Ewerhart (2014) discuss competition between professional forecasters with reputational concerns.

travel distance for the consumers, and we can use this to do some welfare analysis. As Eaton and Lipsey (1975) already indicated, one important aspect of the multiplicity of the Nash equilibria in the Hotelling model is that these travel costs may differ from equilibrium to equilibrium. Our analysis adds a new dimension to this, as the best-response dynamics lead to outcomes that stand in stark contrast to these Nash equilibria. It is not just that we do not get perfect convergence to the Nash equilibria. What we see is that these equilibria are not even approximated, as the system moves into other directions, with minimum differentiation quickly emerging. This implies an important welfare loss as less differentiation means substantially increased travel costs compared with any of the Nash equilibria.

Thus, to the extent that firms may be inclined to adopt behavior resembling myopic best-replies, our analysis suggests that from a policy (welfare) point of view, it may be important to look beyond an equilibrium analysis of such models.

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A Appendix: Proofs for the three player case

A.1 Lemma 2

Proof: Because we have shown that when all the three players are on the same side (AOS, for 'all one side'), we are in $B(S)$, we consider the remaining cases.

There are 19 cases that we need to consider (up to symmetry). And, for each case, we will show the best response path either reaches the state where all the players are on the same side AOS or S.

(i) $[M-3, M-3, M+3] \rightarrow$ either

- $[M-4, M-4, M-2] \in AOS$
- $[M+4, M+4, M-2] \rightarrow [M-1, M-1, M+3]$, which will be considered below (xi)
- $[M-4, M+4, M-2] \rightarrow$ either
  - $[M-5, M-1, M-5] \in AOS$
  - $[M-3, M-1, M+5] \rightarrow [M-2, M-4, M] \in AOS$

(ii) $[M-3, M-2, M+3] \rightarrow$ either

- $[M-3, M-4, M-1] \in AOS$
- $[M-3, M+4, M-1] \rightarrow [M-2, M, M-4] \in AOS$

(iii) $[M-3, M-1, M+3] \rightarrow$ either

- $[M-2, M-4, M] \in AOS$
- $[M-2, M+4, M] \rightarrow [M-1, M-3, M+1] \in S$

(iv) $[M-3, M, M+3] \rightarrow$ either

- $[M-1, M+4, M+1] \rightarrow$ either $[M, M+2, M-2] \in S$ or $[M+2, M+2, M] \in S$
- $[M-1, M-4, M+1] \rightarrow$ either $[M+2, M+2, M] \in S$ or $[M+2, M-2, M] \in S$

(v) $[M-3, M-3, M+1] \rightarrow [M+2, M+2, M-2]$ to be considered below (xiv)

(vi) $[M-3, M-2, M+1] \rightarrow [M+2, M+2, M-1] \in S$

(vii) $[M-3, M-2, M+2] \rightarrow [M+3, M+3, M-1]$ considered above (v)

(viii) $[M-3, M-1, M+2] \rightarrow [M+3, M+3, M] \in AOS$

(ix) $[M-3, M, M+1] \rightarrow [M+2, M+2, M+1] \in AOS$

(x) $[M-3, M, M+2] \rightarrow [M+3, M+3, M+1] \in AOS$

(xi) $[M-3, M+1, M+1] \rightarrow [M, M-2, M-2] \in S$

(xii) $[M-3, M+1, M+2] \rightarrow [M, M+3, M+2] \in AOS$

(xiii) $[M-3, M+2, M+2] \rightarrow [M+1, M-2, M-2] \in S$

(xiv) $[M-2, M-2, M+2] \rightarrow$ either

- $[M-3, M-3, M-1] \in AOS$
- $[M-3, M+3, M-1]$ considered above (iii)

(xv) $[M-2, M-1, M+1] \rightarrow$ either

- $[M-2, M+2, M] \in S$
- $[M+2, M+2, M] \in S$

(xvi) $[M-2, M-1, M+2] \rightarrow$ either

- $[M-2, M+3, M]$ considered above (x)
- $[M-2, M-3, M] \in AOS$

(xvii) $[M-2, M, M+1] \rightarrow$ either

- $[M-1, M+2, M+1]$ considered above (xv)
- $[M, M+2, M-2] \in S$

(xviii) $[M-2, M+1, M+1] \rightarrow [M-2, M-2, M] \in S$

(xix) $[M-1, M, M+1] \rightarrow$ either

- $[M-1, M-2, M+1]$ considered above (xv)
- $[M-1, M+2, M+1]$ considered above (xv)

■

A.2 Lemma 3

Proof: Let $s = (a, b, c) \in S$, and consider all possible cases.

- First case, we relocate the middle player b: If he is relocated in $[0, M-4]$, then $BR^r = a+1, BR^m = c+1$ and the best reply of the left player is $a-1$ or $c+1$. We are in $[M-3, M+3]-C_3$ unless $c = M+3$ or $a = M-3$. In the first case $s = (M-1, M+1, M+3)$ and $BR = (M-2, M, M+4)$ but then $BR_2 = (M-3, M-1, M+1) \in S$. In the second case $s = (M-3, M-1, M+1)$ and $BR_2 = (M-2, M, M+2) \in S$. The case where he is relocated on the right is identical by symmetry.

- Second, we relocate the rightmost player c: If we relocate him on the left we are in $[0, M]$. If he is relocated on $[M+4, 2M]$, then $BR = (a-1, b-1, b+1)$. This is in $[M-3, M+3]-C_3$ unless $a = M-3$. But then $s = (M-3, M-1, M+1)$ and $BR = (M-4, M-2, M) \in [0, M]$.

- Third, we relocate the leftmost player a: If he is relocated in $[0, M-4]$, then $BR^r = b+1, BR^m = c+1$ and BR^l is either $b-1$ or $c+1$. We are in $[M-3, M+3]-C_3$ unless $c = M+3$. If this is the case $s = (M-1, M+1, M+3)$ and $BR = (M, M+2, M+4) \in [M, 2M]$. If we relocate the leftmost player at a in $[M+4, 2M]$, then $BR^l = c-1, BR^m = b-1$ and BR^r is either $c+1$ or $b-1$. We are in $[M-3, M+3]-C_3$ unless $c = M+3$ but in this last case, $BR = (M+2, M, M) \in [M, 2M]$.

- Finally, we need to check that we cannot get to C_3 by relocating a single player in a configuration $s \in S$. Indeed, we need to have two players at $M-3$ or at $M+3$. Consider $s = (M-1, M+1, M+3)$ and $(M-3, M-1, M+1)$. We cannot get from these to C_3 by moving a single player.

■

B Appendix: Proofs for the four player case

We begin by proving Lemma 5, 6, 7, and 8 used in Proposition 4. We then provide a proof of Proposition 5.

B.1 Lemma 5 to 8

Proof: (of Lemma 5) First, we prove that Lemma 5 holds at a date t where no random perturbation occurs. Let us prove that any state $(a, b, c, d) \in I$ belongs to one of the aforementioned categories.

- First, consider $a < b \leq c < d$. Note that there are two possible cases, either there are two players on each side of the center, or three players on one side and one player alone on the other side. Sometimes, but not always, it is necessary to treat these cases separately.

  - First, consider the case $d(a, M) = d(d, M)$. If $c < M$, the best reply $(a-1, a-1, b-1, c+1) \in A$ has positive probability, so we are in Ω_1. Suppose from now on that $c \geq M$. If $d(b-1, M) \geq d(c+3, M)$, with positive probability $BR^t = (a-1, a-1, b-1, c+1)$, $BR^2 = (b, (c+2)*3)$, and with positive probability $BR^3 = (c+1, (c+3)*3) \in A$. If $d(b-1, M) < d(c+3, M)$, with positive probability $BR^t = (d+1, d+1, b-1, c+1)$, $BR^2 = (c, (b-2)*3)$, and $BR^3 = (b-1, (b-3)*3) \in A$.

  - Suppose now without loss of generality that $d(d, M) < d(a, M)$, that is players prefer location $d+1$ to $a-1$.

    * If $d(d+1, M) < d(a, M)$, then the maximal endpoint has contracted, we are in Ω_2.

    * If $d(d+1, M) = d(a, M)$, then if $d(b, M) > d(a+2, M)$, $BR^t = (b-1, d+1, d+1, c+1)$ and $l(BR^t(\omega)) = (d+1)-(b-1) < d-a = l(\omega)$.

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Thus, $\omega \in \Omega_2$.

- If $d(b, M) = d(a + 2, M)$, $BR^t = (a + 1, c + 1, d + 1, d + 1)$ and $BR^{t+1} = (a, a, a, c)$, which has contracted.

- If $d(b, M) = d(a+1, M)$, we can have $BR^t = (a, c+1, d+1, d+1)$, as a is indifferent between staying put and $d+1$. In the next step, a will relocate at c. All others are indifferent between $a - 1$ and $d + 2$, so with positive probability they end up on the same side as c so that A is reached. Thus we are in Ω_1.

- Now consider the case (a, b, c, c) with $a < b < c$, where there are two players at one of the endpoints.

  - If $d(a, M) = d(c, M)$, we are in Ω_1 because due to indifference, there is a positive probability that all players end up on the same side as $b - 1$.

  - If $d(a, M) \geq d(c + 1, M)$, $BR^t = ((c + 1) * 3, b - 1)$. Thus, we assume $b \leq M$. Since $c + 1 - (b - 1) = c - b + 2$, the interval has contracted unless $b \in \{a + 1, a + 2\}$. If $d(b, M) = d(a + 1, M)$, $\omega = (a, a + 1, c, c)$ and $(c + 1) * 4 \in A \in BR^t$. If $b = a + 2$, and if $d(a, M) > d(c + 1, M)$, with positive probability $BR^t = (c+1)*4 \in A$. If $d(a, M) = d(c+1, M)$, $BR^t = (a+1, (c+1)*3)$ which is an absorbing state. We are then in Ω_4.

  - If $d(a, M) \leq d(c - 1, M)$, $BR^t = ((a - 1) * 3, b - 1)$, and since $b - 1 - (a - 1) = b - a < c - a$, we are in Ω_2.

- If the configuration is (a, c, c, c), if $d(c, M) \leq d(a, M)$, with positive probability all players locate on the same side and we reach A. If $d(a, M) < d(c, M)$, $BR^t = ((a - 1) * 3, c - 1)$. If $d(a - 1, M) = d(c, M)$, we are in an absorbing state. Otherwise the maximal endpoint has decreased and we are in Ω_2.

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- Finally if the configuration is (a, a, c, c), without loss of generality $d(a, M) \leq d(c, M)$ and with positive probability, $BR^t = (a-1) * 4 \in A$.

Now, consider a date t at which exactly one player is relocated randomly. Let L_p and R_p be the leftmost and rightmost endpoint at time t. At $t+1$, the players who best reply are not further from the center than $max\{-L_p - 1, R_p + 1\}$. Suppose that $BR^{t+1} = (a, b, c, r)$. The player who locates at random at r locates outside the interval I. At $t+2$, the player located at random will locate at $a-1$ or $c+1$ and the remaining players at $a-1$ if r is to the right and at $c+1$ if r is to the left. Thus at $t+2$, no player locates further away than $max\{-a-2, d+2\}$.

When exactly one player relocates at random, each player is chosen to be the one who does with probability $\frac{1}{4}$. His probability of being on the right/left side respectively is $\frac{1}{2}$.

Let us show that the probability of entering A is at least $\frac{1}{8}$, in other words there is at least one player whose random relocation leads to BR in A. Let a, b, c, d be (one of) the best replies without random relocation at time $t+1$. Note that now (a, b, c, d) is the configuration we reach by best replies (in the absence of random relocations) and not as before the configuration we start from. It is obvious that if at least three players are on the same side it is sufficient to relocate the last one. Therefore suppose $a \leq b < 0 \leq c \leq d$. Suppose first that $d(a, M) \leq d(d, M)$ (without loss of generality). Relocate the player whose deterministic best reply was c to the right of d (such a relocation occurs with probability $\frac{1}{8}$), so that the best reply with random relocation is $BR^{t+1} = (a, b, d, r)$. Since $r >> d$, $BR^{t+2} = ((a-1) * 3, b-1) \in A$. It is now sufficient to note that whenever we are in $A \bigcap I$, if no random relocation occurs, we reach a configuration of the form $\bar{x} \in I$ in two steps. Therefore, if we reach a state in $A \bigcap I$ in two or three steps starting from ω, we reach a configuration of the form $\bar{x} \in I$ in four or five steps. ■

Proof: (of Lemma 6) Suppose without loss of generality that $x < M$. If a random relocation outside I occurs, the configuration can be $(r, x*3)$ and then $BR = \overline{x+1}$. Given that this is the only random relocation in l periods, best replies take us into $[M-3, M+3]$ within l periods. If the configuration with random location is $(x*3, r)$, $r > M$, then $BR = ((x-1)*3, x+1)$, and since all players are on the same side, again best replies take us into $[M-3, M+3]$ within l periods. ■

Proof: (of Lemma 7) Consider without loss of generality $\omega = ((M-a)*3, (M+a-1))$ (the other case is similar by symetry). If the player who is alone is randomly relocated outside of I, if it is on the right we are done because all players are on the same side. This choice has probability $\frac{1}{8}$, as each player is chosen with probability $\frac{1}{4}$, and a location on the right is chosen with probability $\frac{1}{2}$. If it is on the left, the configuration with random relocation is $(r, (M-a)*3)$ and $BR^t = (M-a-1, (M-a+1)*3) \in A \bigcap I$. If one of the three players on the same side relocates at random on the left, the configuration with random relocation is $(r, (M-a)*2, M+a-1)$ and $BR^t = (M-a+1, (M+a)*3) \in \Omega_{AC}^{l/2}$. If one of the three players on the same side relocates at random on the right, the configuration with random relocation is $((M-a)*2, (M+a-1), r)$, hence $BR^t = ((M-a-1)*3, (M+a)) \in \Omega_{AC}$. Note that we are not necessarily in $\Omega_{AC}^{l/2}$ because we have moved to an absorbing state one step farther from the center. ■

Proof: (of Lemma 8) Consider the probability of returning to S from a state in $\Omega_{AC}^{l/2}$. This probability is minimized when $\omega = \{(M-a)*3, M+a-1)\}$ with $a = l/2$, the state in S that is the farthest from the center. Let q be the probability that exactly one random relocation occurs and that it leads to $\{(M-a-1)*3, (M+a)\}$. With probability $1 - q - O(\epsilon^2)$ we return to S. With probability $q = O(\epsilon)$ we are in $\{(M-a-1)*3, (M+a)\}$, an absorbing class that is neither in S nor in I^C.

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From this class, we can reach I^C in two ways. Either by gradually moving from one absorbing class to another one step farther from the center, i. e. repeating $l/2$ times case 3 in Lemma 7, or we can exit rapidly because of 'atypical' random relocations. Consider the sequence of random variables drawn at times $t, ...t+L$, where t is the date of entry in $((M-a-1)*3,(M+a))$ and $L =: \frac{l^4}{\epsilon}$. Let us define three events E_1, E_2, E_3 such that if these three events are realized we return to S.

- E_1: Let A_k be the event that there is no random relocation in I at date $t+k$ or $t+k+1$, and at most one random relocation in I^C (either at $t+k$ or at $t+k+1$ but not both), and let E_1 be the event $\bigcap_{k=1}^{k=L-1} A_k$).
- E_2 is the event that there are at least $O(l)$ random relocations in I^C.
- E_3 is the event that at least one random relocation in I^C leads to a state $\bar{x} \in I$.

If the three events above are realized, we return to S. Consequently the probability that we do not return to S is bounded by $P(E_1^C \bigcup E_2^C \bigcup E_3^C) \leq P(E_1^C) + P((E_2^C \bigcup E_3^C))$.

We begin by bounding $P(E_1)$. At each date, the probability that there is no random relocation is $(1-\epsilon)^4 =: 1-\pi$, with $\pi = 4\epsilon + O(\epsilon_N^2)$. The probability that exactly one random relocation occurs at a given date is $q =: 4\epsilon(1-\epsilon)^3 = 4\epsilon + O(\epsilon^2)$. The probability that the required property is satisified by A_1 is thus $(1-\pi)^2 + 2(1-\pi)q(N-l)/N = 1 - O(\frac{l\epsilon}{N})$. Either we have no random relocations in A_1 or there is a random relocation outside of I at one of the dates and no random relocation at the other date. The probability that A_k satisfies the property is not independent of the probability that it is satisfied by A_{k-1} since there is one date in common. However the probability $P(\bigcap_{k=1}^{k=L-1} A_k) = \prod_{k=2}^{k=L-1} P(A_k|A_{k-1})P(A_1)$. The conditional probability $P(A_k|A_{k-1})$ is not smaller than $P(A_k)$. Indeed the dependence occurs through the common location k and the probability of a random relocation at this date is lower conditioning on A_{k-1}. We

have $P(\bigcap_{k=1}^{k=L-1} A_k) \geq (1 - O(\frac{l\epsilon}{N}))^L = (1 - O(\frac{l\epsilon}{N}))^{(\frac{N}{l\epsilon})(\frac{l^5}{N})} = \exp -O(\frac{l^5}{N}) = 1 - O(\frac{l^5}{N})$. Thus $P(E_1^C) = O(\frac{l^5}{N}) < O(\frac{1}{l^2})$ since $lim_{N\to\infty}\frac{l^7}{N} = 0$.

Next, we prove that with high probability the number of random relocations in I^C is approximately $L\epsilon$, their expected value. The number of periods at which we obtain random relocations is given by a binomial random variable $Bin(q, L)$. This variable has mean $qL = O(l^4)$ and standard deviation $\sigma = \sqrt{Lq(1-q)} = O(l^2)$. By the Chebycheff inequality, the probability $P(|Bin(q, L) - qL| \geq k\sigma) \leq \frac{1}{k^2}$. Taking for example $k = l$, with probability at least $1 - \frac{1}{l^2}$, the number of random relocations is $O(l^4)$. Thus, $P(E_2) \geq 1 - \frac{1}{l^2}$.

Finally, each time a random relocation occurs, it is such that we enter A with probability $p = \frac{1}{8}$, and if no random relocation occurs in the next two steps, which has a probability close to 1, we then reach $\bar{x} \in I$. Consequently, at each random relocation, the probability of not reaching $\bar{x} \in I$ is smaller than, say, $\frac{8}{9}$. If we condition on the fact that there are $O(l^4)$ random relocations in I^C, the probability that none leads to $\bar{x} \in I$ is $P(E_3^C|E_2) = (\frac{8}{9})^{O(l^4)} < O(\epsilon^2)$ (as a result of Condition 1). Therefore $P((E_2^C \bigcup E_3^C)) = 1 - P(E_2 \bigcap E_3) = 1 - P(E_3|E_2)P(E_2) = 1 - [1 - P(E_3^C|E_2)]P(E_2) \leq 1 - [1 - O(\epsilon^2)](1 - \frac{1}{l^2}) = O(\frac{1}{l^2})$

Thus, we conclude that $P(E_1^C \bigcup E_2^C \bigcup E_3^C) = O(\frac{1}{l^2})$. Since the probability of entering $((M - a - 1) * 3, (M + a))$ instead of returning to S immediately is q, the probability of not returning to S is $O(\frac{q}{l^2}) = O(\frac{\epsilon}{l^2})$. ■

B.2 Lemma 9

Proof: The proof is based on a number of lemmas dealing with all possible cases of configurations in A^C.

The first lemma deals with configurations where three players are on the same side:

Lemma 10 *Suppose that ω is a configuration such that $a \leq b \leq c < M < d$. Then either $BR(\omega)$ is a configuration such that $a \leq b < M < c \leq d$, or a single random relocation will place $BR(\omega)$ in A.*

Proof: Indeed, suppose that the deterministic best replies place three players on the same side, it is then sufficient to relocate at random the last player. If he is relocated on the same side as the others, which occurs with probability $\frac{1}{2}$, we are in A. ■

Therefore, from now on, we restrict attention to configurations with two players on each side of the center: $a \leq b < M < c \leq d$.

In Lemma 11 below, we consider the case where both endpoints of the configuration are close to the center so that no player has an interior best reply.

Lemma 11 *Suppose that max $\{d(a, M), d(d, M)\} < \frac{N}{4}$, then the set A is reached within at most two steps with a probability at least equal to ϵc_1, where c_1 is a constant independent of N.*

Proof: If max $\{d(a, M), d(d, M)\} < \frac{N}{4}$, the interior players prefer to relocate at the endpoints. We have $a \leq b < M < c \leq d$, and $d(d, M) \leq d(a, M)$ (without loss of generality), then with probability at least $\frac{1}{4}$ both interior players locate at $d + 1$, and the player at d locates at $c + 1$. It is sufficient to randomly relocate the player at a on the right side to be in A. ■

When the endpoints of a configuration are far from the center, best replies can be interior. We summarize some useful properties of interior best replies. We denote by $U_{[a,b]}$ a discrete uniform law on the set of locations strictly between a and b.

Property 1

- *If the players at a or d prefer an interior location it is given by $U_{[b,c]}$*
- *If the player at b has an interior best reply, it is given by the uniform random variable $U_{[a,c]}$ or $U_{[c,d]}$*
- *If the player at c has an interior best reply, it is given by the uniform random variable $U_{[b,d]}$ or $U_{[a,b]}$*
- *If the best reply of the player at b is $U_{[c,d]}$, the best reply of the player at c is $U_{[b,d]}$*
- *If the best reply of the player at c is $U_{[a,b]}$, the best reply of the player at a is $U_{[a,c]}$*

Lemma 12 *Suppose that $U \in \{U_{[a,c]}, U_{[b,c]}, U_{[b,d]}, U_{[c,d]}, U_{[a,b]}\}$ is an interior best reply of some player, then $P(d(U,M) < \frac{N}{4}) \geq c_2$, where $c_2 > 0$ is a constant independent of N.*

Proof: For uniform variables on the intervals $[a,c]$, $[b,c]$ and $[b,d]$, the result is obvious since these intervals contain M. For example $[a,c] = [a,M] \bigcup (M,c]$ and at least one of the two intervals must have a strictly positie probability to make $[a,c]$ a best reply, and similarly for $[b,c]$ and $[b,d]$. Let us show that the best reply $U_{[c,d]}$ (and similarly $U_{[a,b]}$ by symetry) can only occur if $d(c,M) \leq [\frac{1}{4} - K]N$, implying $P(d(U_{[c,d]}, M) < \frac{N}{4}) \geq K$. If $d(c,M) > [\frac{1}{4} - K]N$, the utility of location $U_{[c,d]}$ is inferior to $N/8 + KN$, but then if $d(a,M) < \frac{N}{4}$, the player at b prefers location $a-1$, which gives a utility of at least $\frac{N}{4}$. And if $d(a,M) > \frac{N}{4}$, he prefers $U_{[a,c]}$ which provides a utility superior to $\frac{N}{4} - KN$ (with K assumed small but positive). Thus, $U_{[c,d]}$ cannot be a best reply. ■

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Lemma 13 *Let* $c_3 = \frac{1}{10}$. *If* $d(b, M) > kN$ *and* $d(c, M) > c_3 N$, *then any configuration* ω *where at least one player has an interior best reply reaches* A *with a single random relocation.*

Proof: Suppose at least one player has an interior best reply $U \in \{U_{[a,c]}, U_{[b,c]}, U_{[b,d]}\}$. By the assumptions, $P(U > M) > \frac{1}{10}$ and $P(U < M) > \frac{1}{10}$. Such a player can end up on any side of the center with positive probability. If one of the remaining three players is alone on his side, it suffices to relocate him at random. We note that by Property 1, if the interior reply (of player b) is $U_{[c,d]}$, then the best reply of the player at c is $U_{[b,d]}$, and similarly if the interior reply (of player c) is $U_{[a,b]}$ then that of player b is $U_{[a,c]}$. ■

The cases that were not covered by Lemma 13 are covered by Lemma 14.

Lemma 14 *If* $d(b, M) \leq c_3 N$ *or* $d(c, M) \leq c_3 N$, *where* $c_3 = \frac{1}{10}$, *then any configuration* ω *reaches* A *in at most three steps involving a single random relocation.*

Proof: Suppose without loss of generality $d(b, M) \leq d(c, M)$.

(i) Suppose $d(a, M) > \frac{N}{4}$ and $d(d, M) < \frac{N}{4}$. Since the point $d+1 \leq \frac{N}{4}$ is available, no player locates at $a-1$. If some player has an interior best reply, by lemma 12, the realized location is closer than $\frac{N}{4}$ to the center with probability at least $\frac{1}{4}$, and we reach a configuration where both endpoints are inferior to $\frac{N}{4}$, a case already treated in Lemma 11.

(ii) The case $d(a, M) < \frac{N}{4}$ and $d(d, M) > \frac{N}{4}$ is the same as the one above, by symmetry, since the positions of b and d did not intervene in the argument above.

(iii) Suppose $d(a, M) > \frac{N}{4}$ and $d(d, M) > \frac{N}{4}$.

(a) If $d(b,M) \leq d(c,M) < c_3 N$, then the best replies are $b-1, c+1$ and the interior players either locate at the smallest endpoint or in the interior. In both cases, with positive probability, the smallest endpoint in $BR^t(\omega)$ is closer than $\frac{N}{4}$ to the center, and we are in one of the cases considered above.

(b) Suppose $d(c,M) \geq c_3 N$, and $d(a,M) > d(d,M)$. If the best reply of the player at b is $U_{[a,c]}$, he has positive probability of ending up on either side. We can then apply the argument in the proof of Lemma 13: at least two of the other players locate on the same side. Relocate the remaining one on this side. Thus suppose the player at b locates at $d+1$ or at $U_{[c,d]}$. The players at c and d locate on the right with positive probability.[22] The player at a relocates at $b-1$. It is sufficient to relocate the player whose best reply is $b-1$ on the right side.

(c) Suppose $d(c,M) \geq c_3 N$, and $d(a,M) \leq d(d,M)$. If the best reply of the player at b is $U_{[a,c]}$, he has positive probability of ending up on either side, and we are done by the argument in the proof of Lemma 13. Thus, suppose the player at b locates at $a-1$ or $U_{[c,d]}$ and the player at a relocates at $b-1$. Note that $d(c,M) < N/5$, because otherwise the player at b would prefer the interval $[a,c]$. The possible locations to the right of c are $U_{[b,d]}$ and $U_{[c,d]}$ but $P(d(U_{[c,d]},M) < \frac{N}{4}) \geq \frac{N/4-N/5}{d-c}$. With positive probability, we reach a configuration whose rightmost endpoint is smaller than $\frac{N}{4}$, a case that has been analyzed before.

■

Combining Lemma 10 to 14 proves Lemma 9. ■

[22] Indeed $U_{[a,b]}$ is not a best reply for the player at c if $U_{[a,c]}$ is not a best reply for the player at b.