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Measuring risks in the extreme tail: The extreme VaR and its confidence interval

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Abstract

Contrary to the current regulatory trend concerning extreme risks, the purpose of this paper is to emphasize the necessity of considering the Value-at-Risk (VaR) with extreme confidence levels like 99.9%, as an alternative way to measure risks in the "extreme tail". Although the mathematical definition of the extreme VaR is trivial, its computation is challenging in practice, because the uncertainty of the extreme VaR may not be negligible for a finite amount of data. We begin to build confidence intervals around the unknown VaR. We build them using two different approaches, the first using Smirnov 's result (Smirnov, 1949 [24]) and the second Zhu and Zhou 's result (Zhu and Zhou, 2009 [25]), showing that this last one is robust when we use finite samples. We compare our approach with other methodologies which are based on bootstrapping techniques, Christoffersen et al. (2005) [7], focusing on the estimation of the extreme quantiles of a distribution. Finally, we apply these confidence intervals to perform a stress testing exercise with historical stock returns during financial crisis, for identifying potential violations of the VaR during turmoil periods on financial markets.

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1. Introduction

The Value-at-Risk (VaR) summarizes the worst potential loss over a target horizon within a given confidence level. The Expected Shortfall (ES) is defined as the expected loss beyond the VaR. In the new standards of the Basel Committee on Banking Supervision (BCBS) for minimum capital requirements for market risk (BCBS, 2016 [4]), the BCBS proposes to shift from VaR to an ES measure of risk under stress. In this standards, the objective of using the ES is to ensure a more prudent capture of "tail risk" (or equivalently extreme risk) and capital adequacy during periods of significant financial market stress. Unfortunately, from its mathematical definition, the interpretation of the ES as a measure of risks in the "extreme tail" has to be treated carefully, since the ES only computes the conditional expectation of losses beyond the VaR. Intuitively, it is just the expected losses in the tail, equivalently and mathematically the average of the VaR values beyond a threshold. So when we compute the ES, some potential extreme losses in the "extreme tail" may be averaged by those potential losses beyond the VaR but close to it. Thus using the ES can provide an inappropriate evaluation of the extreme risks, Koch-Medina et al. (2016) [15].

The question is to find an adequate way to measure the risks in the "extreme tail". An intuitive way is to compute the extreme VaR, which is a VaR with a confidence level close to 1 like 99.9%, and to use it as a measure of risks. Although this definition is quite simple, the computation of an extreme VaR is quite challenging, because of the uncertainty in the fitting of the extreme tail of the distribution used to computed this extreme VaR. This uncertainty comes from the finite amount of data we use in practice. Especially, the lack of observed data in the extreme tail leads to large uncertainty for the extreme

VaR. From another point of view, the reported VaR values are point estimates, which are also sources of errors and uncertainty. Thus to get a robust understanding of the risks, we need to associate to each point estimate of the VaR its confidence interval. For example, a bank may have two portfolios with the same point VaR estimate equals to 100 million euros. However, the confidence interval for the first portfolio ranges between 80 and 120 million euros, and for the second portfolio it ranges between 50 and 150 million euros. It implies that the second portfolio is more risky than the first one, because the potential losses of the second portfolio are more likely to deviate above 100 million euros than the first one's losses. Therefore, the bank should hold higher amount of capital for the second portfolio than the amount for the first one. Rather than finding a point VaR estimate without uncertainty (for a discussion in the independent case, we refer to Francioni et al. (2012) [12]), in this paper, we attempt to build confidence intervals around the extreme VaR with accurate coverage probability, in order to quantify ex ante its uncertainty.

In the literatures, when researchers build confidence intervals they generally use asymptotic Gaussian results or bootstrapping which is time consuming. We can distinguish two kinds of papers: some consider independent and identically distributed (i.i.d.) random variables, other dependent variables. Under the i.i.d assumption on the financial time series, Jorion (1996) [13] points out the necessity of considering the uncertainty of the historical simulation VaR when the underlying distribution is Gaussian or Student-t distribution. He builds confidence bands for quantiles based on its asymptotic Gaussian result. Instead of using asymptotic result to build the confidence interval, Pritsker (1997) [21] assumes a distribution on the returns and draws i.i.d returns by Monte-Carlo simulation, computes the empirical quantiles and builds the confidence intervals. To avoid assuming a specific distribution on the returns, Christoffersen et al. (2005) [7] use a bootstrapping approach, that is generating i.i.d returns by resampling with replacement from historical return data, to compute the empirical quantiles and builds the confidence intervals. Both Christoffersen et al. (2005) [7] and Gao et al.,(2008) [9] find that the bootstrap intervals for a 99% VaR have nearly correct coverage probability. When the financial return time series are not i.i.d. McNeil et al. (2000) [16] and Chan et al. (2007) [6] propose GARCH models (Bollerslev, 1986 [5]) to capture the heteroscedasticity property of the financial return series. Based on these modellings, they derive formulas for the conditional distribution of the returns and the conditional VaR. The confidence intervals are obtained either by bootstrapping (Christoffersen et al., 2005 [7]) or using the asymptotic behaviour of the GARCH residuals (Chan et al., 2007 [6]; Gao et al., 2008 [9]). Although these dynamical models may be closer to the nature of data, they suffer from some problems such as estimation errors, choices of initial values, model misspecification and overfitting. Furthermore, the use of dynamical models and the conditional VaR for market risk measurement may lead to the regulatory capitals fluctuate widely over time and are therefore difficult to carry out.

It exists also a huge literature on the choice of the risk models for the financial data, but it mainly considers the point estimates of the VaR. McAleer et al. (2013) [17] discuss how to chose optimal risk models among different conditional volatility models like Exponentially Weighted Moving Average (EWMA) by Riskmetrics^{TM} (1996) [22], GARCH and EGARCH (Nelson, 1991 [18]), in terms of minimizing daily capital charges before and during the global financial crisis. Kellner et al. (2016) [14] compare eight different models for the innovations' distribution assuming that the financial returns follow ARMA-GARCH process. They find the ES's level of model risk is higher than the VaR's which is inherent to the definitions of the risk measures and the models they chose.[1]

The approach in this paper is quite different from the previous cited papers. In order to associate to each point estimate of the VaR a confidence

[1] In this paper we do not enter in the discussion along the question of coherency and suggest the readers to look at Artzner et al. (1999) [1] and Guégan and Hassani (2016) [10].

interval, we first consider the asymptotic distribution of this estimate provided by two results, the theorems of Smirnov (1949) [24] and of Zhu and Zhou (2009) [25]. These two theorems provide us the asymptotic distributions of the VaR estimates under different assumptions, which permit to build two kinds of confidence intervals around the unknown VaR. We extend the results by providing the exact bounds of these confidence intervals, as soon as the unknown distribution which characterizes the financial data is estimated. We show that one of these approaches is robust with finite samples. Consequently it is interesting to use it in practice. In order to determine a robust framework for the extreme VaR to answer to the actual demands of the regulator, we show that our approach is definitively interesting when we want to estimate the VaR for $p = 99.9\%$. We compare our approach with classical bootstrapping approach. And we exhibit that the confidence intervals of the extreme VaR built with the plug-in corollary of Zhu and Zhou's theorem outperform the confidence intervals obtained from bootstrapping. This means that we reduce the bias coming from the bootstrapping approach when we use our approach to build the confidence interval of the extreme VaR.

The remainder of the paper is structured as follows. Section 2 presents the plug-in version of the asymptotic Gaussian and saddlepoint results, for approximation the distribution of the point estimate of the VaR. We compare these two results quantitatively and graphically by simulation. Then we build confidence intervals from these results and compare them with bootstrapping intervals. Section 3 presents a stress testing implementation applying these confidence intervals with historical stock returns during financial crisis, for identifying potential violations of the VaR during periods of financial market stress. Section 4 concludes.

2. Confidence intervals of the VaR

All along this section we consider the VaR with one day horizon. Let X be a random variable (r.v.), which is the daily loss of a given financial asset or portfolio. And let F_θ be the cumulative distribution function (c.d.f.) of X with a parameter θ. Let $F_\theta^{-1}(x)$ be its left continuous inverse, i.e. $F_\theta^{-1}(x) = min\{u : F_\theta(u) \geq x\}$. For a given confidence level $0 \leq p \leq 1$, we define (see Pflug, 2000 [20] for instance) the Value-at-Risk VaR_p as the p-quantile :

$$VaR_p = F_\theta^{-1}(p), \tag{1}$$

and for the same confidence level p, the Expected Shortfall ES_p is equal to:

$$ES_p = \frac{1}{1-p}\int_p^1 F_\theta^{-1}(u)du = \frac{1}{1-p}\int_p^1 VaR_u du. \tag{2}$$

Equation (2) implies that ES_p is just an average quantity of the potential losses beyond the VaR_p. Therefore, to measure the extreme risks, we consider the extreme VaR as an alternative measure and discuss its uncertainty in the following. In practice, we cannot compute the true values of VaR_p, since they depend on the unknown distribution F_θ. We need to estimate F_θ to derive the confidence interval of the VaR. This is the goal of section 2.

2.1. Asymptotic distribution of an estimate of the VaR

Let $X_1, ..., X_n$ be a sequence of losses corresponding to the previous random variable X. We consider the Historical Simulation (HS) estimate of the VaR_p, which is the most commonly used method for VaR computation (Pérignon, 2010 [19]). It provides the VaR estimate using the order statistics of past losses. We rank $X_1, ..., X_n$ and obtain $X_{(1)} \leq \cdots \leq X_{(n)}$ and define the HS estimate as:

$$VaR_p^{HS} = X_{(m)}, \tag{3}$$

where $m = np$ if np is an integer and $m = [np]+1$ otherwise. $[x]$ denotes the largest integer less than or equal to x.[2] Furthermore, VaR_p^{HS} is a consistent estimator of VaR_p (Serfling, 2009 [23]). It means that the VaR_p^{HS} converges to VaR_p in probability when n tends to infinity.

In order to build confidence intervals of the VaR, we introduce two plug-in corollaries for the theorems of Smirnov (1949) [24] and Zhu et al. (2009) [25]. They provide two asymptotic approximations for the distribution of VaR_p^{HS}. We call the first approximation, the Asymptotic Normality (AN) approximation and the second one, the saddlepoint (SP) approximation. We derive confidence intervals of the VaR from them. Theoretically, the convergence speed of the SP approximation is $O(\frac{1}{n})$, which is faster than the speed of convergence of the AN approximation ($O(\frac{1}{\sqrt{n}})$). Consequently, we may construct a robust confidence interval of the VaR from the SP approximation, even if we only use a small sample. In practice it is a trade-off to use long historical data which may containing more information, or to use short historical data which may be more related to the current market (Halbleib et al., 2012 [11]). Note that, in order to build these confidence intervals we need to estimate F_θ [3].

Corollary 1 (Plug-in AN approximation). *Given a r.v. X whose c.d.f F_θ and density f_θ are continuous functions with respect to (w.r.t) θ, and $\hat{\theta}$ is a consistent estimator of θ, then*

$$\sqrt{n}\frac{VaR_p^{HS} - VaR_p}{\sqrt{\widehat{V}}} \rightarrow_{(d)} N(0,1), \quad as \quad n \longrightarrow \infty, \tag{4}$$

where $\widehat{V} = \frac{p(1-p)}{[f_{\hat{\theta}}(F_{\hat{\theta}}^{-1}(p))]^2}$.

[2] $X_{(m)}$ is also called the mth order statistic, which is a fundamental tool in nonparametric statistics.

[3] The estimates of the parameters θ are obtained by maximum likelihood approach. We do not restrict the choice of F_θ here: it is chosen through a panel of distributions and the best fit is decided using for instance Akaike criterion.

The proof is postponed in Appendix A and derived from Smirnov (1949) theorem.

Corollary 2 (Plug-in and linear transformed SP approximation). *Assume F_θ and f_θ are continuous functions with respect to (w.r.t) θ, and $\hat{\theta}$ is an consistent estimator of θ. For $\forall \epsilon > 0$ and $\forall x$ in the domain of F_θ, we assume $\epsilon < p < 1-\epsilon$. Let $r_0 = \frac{m}{n}$, $F_{\hat{\theta}}(x + F_{\hat{\theta}}^{-1}(p)) = t$, then for $t \neq p$*

$$VaR_p^{HS} - VaR_p \rightarrow_{(d)} \Psi(\sqrt{n}\widehat{\omega}^\sharp) \quad as \quad n \longrightarrow \infty, \tag{5}$$

where

$$\begin{aligned} &\Psi(\sqrt{n}\widehat{\omega}^\sharp) = 1 - \Phi(\sqrt{n}\widehat{\omega}^\sharp) && \widehat{\omega}^\sharp = \widehat{\omega} + \frac{1}{n\widehat{\omega}} ln \frac{1}{\psi(-\widehat{\omega})} \\ &\psi(-\widehat{\omega}) = \frac{\widehat{\omega}(t-1)}{t - r_0}(\frac{r_0}{1-r_0})^{\frac{1}{2}} && \widehat{\omega} = -\sqrt{2h(t)} sign(t - r_0) \\ &h(t) = r_0 ln \frac{r_0}{t} + (1-r_0) ln \frac{1-r_0}{t}. \end{aligned} \tag{6}$$

For $t = p$

$$P(VaR_p^{HS} \leq x) \approx \frac{1}{2} + \frac{1}{\sqrt{2\pi n}} \frac{1+r_0}{3r_0} (\frac{r_0}{1-r_0})^{\frac{1}{2}}. \tag{7}$$

The proof is postponed in Appendix B and derived from Zhou and Zhu (2000) theorem.

The two results given in these two corollaries are asymptotic results. In order to verify how they perform on finite samples, we propose the following experiment to check whether they provide reliable approximations for the distribution of the VaR_p^{HS} when the data are finite. We consider samples of sizes $n = 11, 121, 241, 501, 1001, 10001, 30001$ and different $p = 0.05, 0.01, 0.005, 0.95, 0.99, 0.995$. We use a panel of distributions for F_θ: (i) a $N(0,1)$ distribution; (ii) a NIG_0 (Normal-inverse Gaussian, Barndorff-Nielsen (1978) [2]) distribution with tail parameter 0.3250, skewness parameter equal to $5.9248e-04$, location parameter equal to $-1.6125e-04$ and the scale parameter equal to 0.0972); (iii) a GEV (Generalized extreme value, Embretch et al. (1997) [8]) distribution with shape parameter 0.8876698, scale parameter equal to 2049.7625278 and the location parameter equal to 245.7930751.[4]

We simulate $n{*}1000$ independent daily return or loss data $\left\{X_i^j : i = 1, ..., n\right\}_{j=1,...,1000}$ using the three previous distributions ($N(0,1)$, NIG_0 and GEV). For a given p, we obtain one realization of the VaR_p^{HS} from $\left\{X_i^j : i = 1, ..., n\right\}_j$ and in fine we have 1000 realizations. Then we build the empirical cdf (ecdf) of VaR_p^{HS} realizations. It is a proxy for the true distribution of the VaR_p^{HS}. For comparison, we compute the Kolmogorov–Smirnov (K-S) statistic and the Anderson–Darling (A-D) statistic between the ecdf of the VaR_p^{HS} and the AN (or SP) approximations. The results of K-S statistic are in Table (1) [5].

From Table (1), we observe that the values of the K-S statistic vary between 0.0139 and 0.0449 for the SP approximations, and for the AN approximations, the values vary between 0.0204 and 0.3959. It appears that the SP approximations are closer to the true distributions of VaR_p^{HS} than the AN approximations, and that this result is mainly observed when $n \leq 1000$. Another interesting result is that the difference of these two approximations is important for the very high values pf p, which correspond to the extreme VaR. For example, the value of the K-S statistic is 0.0266 for the SP approximation but 0.0744 for the AN approximation, where $n = 241$, $p = 0.005$ and F_θ is NIG_0 distribution. Indeed, when the sample size is small and we consider the extreme VaR, even for a thin-tailed distribution, the SP approximation is more reliable than the AN approximation. For instance, the value of the K-S statistic is 0.0213 for the SP approximation but 0.1711 for the AN approximation, where $n = 121$, $p = 0.005$ and F_θ is $N(0,1)$ distribution. Therefore, the confidence intervals from the SP approximations may be trusted and the uncertainty of the extreme VaR need to be quantified carefully.

[4] These distributions represent financial data with different features.

[5] The results of A-D statistic are similar to those obtained with the K-S statistic, so we do not provide them in the paper.

	$N(0,1)$		NIG_0			GEV	
	AN	SP	AN	SP		AN	SP
$p = 0.05$					$p = 0.95$		
$n = 11$	0.2183	0.0361	0.1933	0.0260	$n = 241$	0.0847	0.0265
$n = 121$	0.0767	0.0192	0.0876	0.0358	$n = 501$	0.0503	0.0265
$n = 241$	0.0509	0.0274	0.0621	0.0197	$n = 1001$	0.0756	0.0449
$n = 1001$	0.0435	0.0248	0.0276	0.0228	$n = 10001$	0.0371	0.0363
$n = 10001$	0.0322	0.0259	0.0208	0.0207	$n = 30001$	0.0368	0.0327
$p = 0.01$					$p = 0.99$		
$n = 11$	0.0757	0.0211	0.1963	0.0191	$n = 241$	0.1155	0.0205
$n = 121$	0.0847	0.0318	0.1072	0.0222	$n = 501$	0.1409	0.0370
$n = 241$	0.0948	0.0330	0.0564	0.0265	$n = 1001$	0.0988	0.0266
$n = 1001$	0.0456	0.0191	0.0546	0.0213	$n = 10001$	0.0459	0.0238
$n = 10001$	0.0204	0.0195	0.0227	0.0187	$n = 30001$	0.0327	0.0238
$p = 0.005$					$p = 0.995$		
$n = 11$	0.3930	0.0245	0.3959	0.0138	$n = 241$	0.1584	0.0278
$n = 121$	0.1711	0.0213	0.1418	0.0278	$n = 501$	0.1213	0.0253
$n = 241$	0.0653	0.0225	0.0744	0.0266	$n = 1001$	0.1461	0.0311
$n = 1001$	0.1071	0.0296	0.0976	0.0308	$n = 10001$	0.0515	0.0149
$n = 10001$	0.0367	0.0139	0.0359	0.0163	$n = 30001$	0.0392	0.0227

Table 1: We simulate $n * 1000$ independent daily return or loss data from $N(0,1)$, NIG_0 and GEV distributions. For a given p, we obtain n realizations of VaR_p^{HS}. Then we compute the Kolmogorov–Smirnov statistic between the AN (or SP) approximation and the empirical cdf (ecdf) of VaR_p^{HS} realizations.

We illustrate the results provided in Table (1) graphically, exhibiting some figures for $n = 241$, which represents the number of one year daily stock return data. For Figure (1), we do the simulations for $p = 0.995$ and F_θ a $N(0,1)$ distribution. For Figure (2) we perform the simulations with $p = 0.005$ and F_θ a NIG_0 for the left graph, and for the right graph, using $p = 0.995$ and F_θ a GEV distribution. In both figures, the solid curve is the ecdf of the VaR_p^{HS} realizations, the dash curve is the SP approximation and the dash-dot curve is the AN approximation. We find that the SP approximations are closer to the ecdf of the VaR_p^{HS} realizations than the AN approximations. In particular, the VaR_p^{HS} realisations simulated from a GEV distribution is asymmetric ($skewness$ = 14.6562) and leptokurtic ($kurtosis$ = 247.1869). Apparently, the AN approximation cannot capture these behaviours, since it is symmetric ($skewness = 0$) and thin-tailed ($kurtosis = 3$). Consequently, it is reasonable to use SP approximation instead of AN approximation as soon as the data sets exhibit these kinds of behaviour.

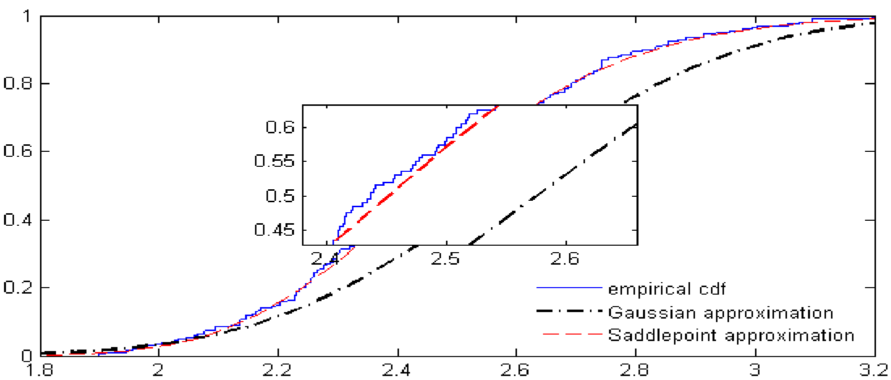

Figure 1: We do the same simulations for Table (1), where $p = 0.995$ and F_θ is a $N(0,1)$ distribution. The solid curve is the ecdf of the VaR_p^{HS} realizations, the dash curve is the SP approximation and the dash-dot curve is the AN approximation.

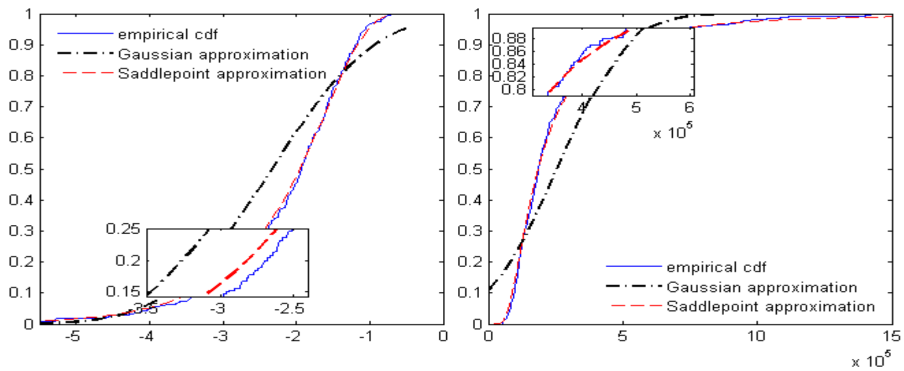

Figure 2: We do the same simulations for Table (1), where $p = 0.005$ and F_θ is a NIG_0 for the left graph; for the right graph, $p = 0.995$ and F_θ is a GEV distribution. For both graphs, the solid curve is the ecdf of the VaR_p^{HS} realizations, the dash curve is the SP approximation and the dash-dot curve is the AN approximation.

2.2. Confidence intervals around the VaR

Using the results of Corollary 1 and Corollary 2 we can derive two confidence intervals from them around the VaR. For that we consider another confidence level $0 < q < 1$ and the confidence interval around the VaR obtained from Corollary 1 is:

$$\left[VaR_p^{HS} + z_{\frac{1-q}{2}}\hat{\sigma}, \quad VaR_p^{HS} + z_{\frac{1+q}{2}}\hat{\sigma}\right], \tag{8}$$

where $z_{\frac{q}{2}}$ is a quantile of N(0,1) and $\hat{\sigma}$ is a standard deviation equal to $\frac{\widehat{V}}{n}$. We call this confidence interval CI-AN. It is symmetric, because N(0,1) is symmetric.

We provide now the confidence interval around the VaR obtained from Corollary 2:

$$\left[VaR_p^{HS} + Z_{\frac{1-q}{2}}, \quad VaR_p^{HS} + Z_{\frac{1+q}{2}}\right]. \tag{9}$$

where $Z_q = \Psi_q^{-1}(\sqrt{n}\widehat{\omega}^\sharp)$ and Ψ and $\widehat{\omega}^\sharp$ are provided in Corollary 2. The bound Z_q is generally obtained numerically. This CI may be symmetric or not, we call it CI-SP .

In order to see how our methodology outperforms the bootstrapping confidence intervals of the VaR in the literatures, we propose an experiment which permits to compare the two previous confidence intervals and a confidence interval built by bootstrapping following the methodology developed by Christoffersen et al. (2005) [7]. We call this bootstrapping confidence interval CI-BT. The experiment is in the following:

1. Let $q = 0.9$ and $p = 0.05, 0.01, 0.005, 0.001$, and $n = 250, 500, 1000$ samples of independent daily return data $\{R_1, ..., R_n\}$ respectively from: (i) a $N(0,1)$ distribution, (ii) a Student-t distribution with mean zero and variance $20^2/252$ with degrees of freedom 8 or 500 denoted as $t(8)$ and $t(500)$ [6], (iii) a GEV distribution denoted as $GEV1$, with the shape parameter equal to -0.4144, the scale parameter equal to 0.0361 and the location parameter equal to -0.0083. The true values of the VaR is derived from the distributions, for instance, when $p = 0.05$ and we consider a $N(0,1)$, the true VaR value is equal to -1.6449.
2. For each data $\{R_1, ..., R_n\}$, we fit the distribution F_θ on $\{R_1, ..., R_n\}$ and use the fit to build the CI-AN and CI-SP.
3. To build the CI-BT for each $\{R_1, ..., R_n\}$ we proceed in the following way. We generate pseudo returns $\left\{R^*_{1,j}, ..., R^*_{n,j}\right\}_{j=1,...,999}$ using resampling with replacement. We compute the VaR_p^{HS} for each j and get $\left\{VaR_p^{HS*,j}\right\}_{j=1}^{999}$. Then the $100q\%$ CI-BT of the VaR is equal to:

$$\left[Q_{\frac{1-q}{2}}\left(\left\{VaR_p^{HS*,j}\right\}_{j=1}^{999}\right), \quad Q_{\frac{1+q}{2}}\left(\left\{VaR_p^{HS*,j}\right\}_{j=1}^{999}\right)\right], \tag{10}$$

   where $Q_q(\cdot)$ is the $q-quantile$ of the ecdf of $\left\{VaR_p^{HS*,j}\right\}_{j=1}^{999}$.
4. Step 1-3 are repeated one thousand times and count how many times the true values of the VaR are inside the confidence intervals, for each

[6] The parameters we use here for Student-t distribution are the same as those in Christoffersen et al. (2005) [7]. The mean zero and variance $20^2/252$ imply a volatility of 20% per year.

distribution, n and p. We divide each number of times by 1000 and call it the nominal coverage rate (NCR). If the confidence interval is reliable, the NCR should be close to the chosen $q = 0.9$.

The results of NCR are provided in Table (2). Consistent with the results found in Christoffersen et al. (2005) [7], the CI-BT have NCR close to the promised $q = 0.9$ when p equals 0.05 and 0.01. The CI-AN and CI-SP also have NCR close to $q = 0.9$ in these cases. However, when we consider the extreme VaR like $p = 0.005$ or $p = 0.999$, the CI-BT have considerable low NCR. For example, when $n = 250$ and $p = 0.001$, the NCR of the CI-BT ranges from 0.207 to 0.241 which are far less than 0.9. But the NCR of the CI-AN and CI-SP ranges from 0.804 to 0.984.

Now we focus on the confidence intervals obtained from the AN approximation and SP approximation for the extreme VaR. Comparing the NCR of the CI-AN and CI-SP, we observe from Table (2) that we obtain better results when we use the CI-SP, as soon as $p = 0.01$ and 0.005 and $p = 0.001$: indeed the NCR is close to $q = 0.9$ in a greater number of cases than those observed when we use the CI-AN. When $p = 0.05$, the results are similar whatever the interval we use: the number of times where the true VaR value is outside the confidence intervals is nearly the same for each interval. Thus, the CI-SP is more informative than the CI-AN when we quantify the violations of the extreme VaR.

	$n = 250$				$n = 500$				$n = 1000$			
	$N(0,1)$	$t(8)$	$t(500)$	$GEV1$	$N(0,1)$	$t(8)$	$t(500)$	$GEV1$	$N(0,1)$	$t(8)$	$t(500)$	$GEV1$
$p = 0.05$												
$CI - BT$	0.887	0.894	0.917	0.889	0.893	0.888	0.880	0.894	0.902	0.893	0.899	0.913
$CI - AN$	0.907	0.897	0.921	0.895	0.901	0.896	0.910	0.908	0.893	0.896	0.898	0.902
$CI - SP$	0.906	0.898	0.921	0.891	0.892	0.893	0.905	0.903	0.891	0.893	0.893	0.900
$p = 0.01$												
$CI - BT$	0.909	0.903	0.892	0.906	0.877	0.879	0.881	0.868	0.896	0.899	0.878	0.893
$CI - AN$	0.906	0.898	0.915	0.896	0.907	0.901	0.927	0.898	0.909	0.914	0.923	0.908
$CI - SP$	0.900	0.890	0.912	0.887	0.884	0.883	0.905	0.889	0.898	0.906	0.909	0.899
$p = 0.005$												
$CI - BT$	0.713	0.712	0.692	0.685	0.902	0.898	0.897	0.893	0.888	0.888	0.872	0.874
$CI - AN$	0.932	0.899	0.959	0.926	0.919	0.891	0.926	0.893	0.914	0.910	0.918	0.914
$CI - SP$	0.901	0.868	0.938	0.891	0.910	0.887	0.911	0.886	0.891	0.899	0.899	0.891
$p = 0.001$												
$CI - BT$	0.207	0.209	0.233	0.241	0.393	0.391	0.386	0.394	0.631	0.610	0.649	0.635
$CI - AN$	0.959	0.913	0.984	0.959	0.943	0.904	0.963	0.935	0.941	0.905	0.959	0.946
$CI - SP$	0.881	0.804	0.918	0.874	0.897	0.849	0.928	0.893	0.854	0.813	0.869	0.848

Table 2: Let $q = 0.9$. We simulate $n = 250, 500, 1000$ independent daily return data $\{R_1, ..., R_n\}$ respectively from a $N(0,1)$ distribution, or a Student-t distribution with mean zero and variance $20^2/252$ for degree of freedom equal to 8 or 500 denoted as $t(8)$ and $t(500)$, or a GEV distribution denoted as $GEV1$, with the shape parameter equal to -0.4144, scale parameter equal to 0.0361 and location parameter equal to -0.0083. We derive the true values of the VaR. For each data $\{R_1, ..., R_n\}$, we build the CI-AN, CI-SP and CI-BT by definitions and check if the true values of the VaR are inside the confidence intervals. We repeat this procedure one thousand times and count how many times the true values of the VaR are inside the confidence intervals and obtain the nominal coverage rates. The results are provided in the table.

3. A stress testing application

For identifying ex ante the potential violations of the VaR during periods of financial market stress, in this section, we apply the CI-AN and CI-SP to perform a stress testing with historical stock returns during financial crisis. The data we consider are: the daily returns of S&P 500 from 03/01/2008 to 31/12/2008 (252 data, denoted as SP1); the daily returns of S&P 500 from 03/01/1987 to 31/12/1987 (252 data, denoted as SP2); the daily returns of HSI from 03/01/1997 to 31/12/1997 (244 data, denoted as HSI1); the daily returns of HSI from 03/01/1987 to 31/12/1987 (245 data, denoted as HSI2); All the data are obtained from Bloomberg. These data correspond to three financial crisis: Black Monday (1987), the Asian financial crisis (1997) and the global financial crisis (2007-08). In Table (3), we provide the first four empirical moments and the number of observations of the data.

		Empirical moments			
	points	mean	variance	skewness	kurtosis
SP1 (03/01/2008-31/12/2008)	252	-0.0015	0.0007	0.1841	6.8849
SP2 (03/01/1987-31/12/1987)	252	0.0003	0.0004	-4.0440	45.5834
HSI1 (03/01/1997-31/12/1997)	244	-0.0005	0.0006	0.7616	18.7190
HSI2 (03/01/1987-31/12/1987)	245	0.0000	0.0008	-6.7209	78.8165

Table 3: In this table, we provide the first four empirical moments of the data and the number of observations.

	p=0.05	p=0.01	p=0.005	p=0.001
NIG fit (SP1)				
VaR	-0.0415	-0.0706	-0.084	-0.1171
ES	-0.0597	-0.0904	-0.1047	-0.1378
NIG fit (HSI1)				
VaR	-0.0349	-0.0724	-0.0921	-0.1440
ES	-0.0589	-0.1027	-0.1246	-0.1778

Table 4: For $p = .05, 0.01, 0.005, 0.001$, we compute the VaR and the ES for SP1 and HSI1 using the NIG fit.

To perform the stress testing exercise, first we fit a NIG distribution on the data sets SP1 and HSI1 using moments method (it is the best fit since they are asymmetric and leptokurtic). The estimates are provided in Appendix C. Second we follow the demands of the regulator (BCBS, 2016 [4]) suggesting to use the ES for $p = 0.01$ to measure the risks in the extreme tail. In Table (4), for $p = .05, 0.01, 0.005, 0.001$, we compute the VaR and the ES for SP1 and HSI1 using the NIG fit. For SP1, we obtain $ES_p = -0.0904$ for $p = 0.01$, but if we now consider the extreme VaR (with the approach privileged in this paper) to quantify the risk in the extreme tail, this means using the VaR for $p = 0.001$ we get $VaR_p = -0.1171$. For HSI1, we get $ES_p = -0.127$ for $p = 0.01$ whereas we get $VaR_p = -0.1440$ for $p = 0.001$. The results illustrate the fact that the extreme VaR provides very interesting information for measuring the risk in the extreme tail. Thus it is more the choice of the value of the confidence level which is important to control, measure and understand the risks in the tails than the shift from one measure to another one, as proposed inside Basel guidelines. Moreover, this discussion lies on point estimates whatever the risk measure we chose (VaR or ES). Following our previous discussion in section 2 on the variability of the point estimates, we come back to the use of confidence intervals of the VaR for stress testing purposes.

Following the methodology presented in section 2, we build the CI-AN and CI-SP for these two sets of data with $q = 0.01$, and n equals to 252 and 244 respectively, $p = 0.001, 0.0055, 0.0099, 0.0144, 0.0188, 0.0233, 0.0277, 0.0322, 0.0366, 0.0411, 0.0455, 0.05$. We consider p close to 0 because for the returns the losses appear in the left tail. Then, we compute the empirical quantiles of SP2 and HSI2 using the same p. Although in Basel II (BCBS, 2005 [3]), the value of p is suggested as 99%, we show in the following examples that it is interesting to consider different values of p. We focus on $p = 99.9\%$ to get information on the extreme VaR, in order to analyse the extreme losses.

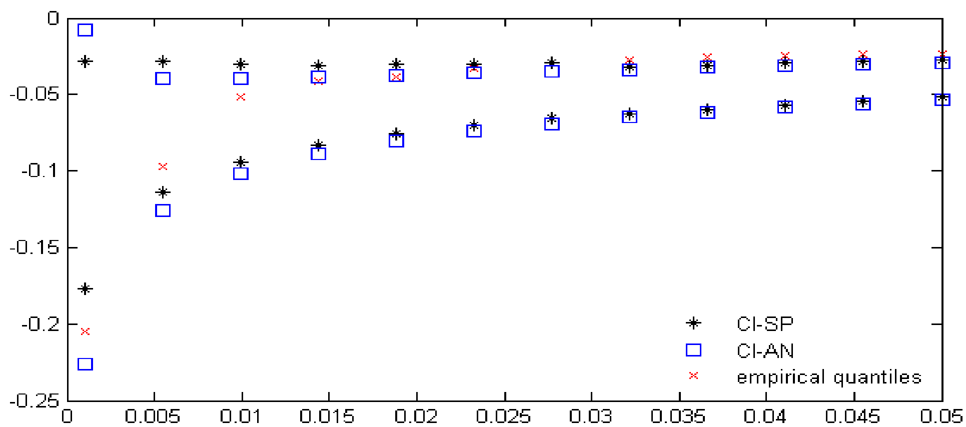

Figure 3: We fit a NIG distribution using moments method on the data SP1 and compute the CI-AN and CI-SP use the fits, where $q = 0.01$, $n = 252$ and $p = 0.001, 0.0055, 0.0099, 0.0144, 0.0188, 0.0233, 0.0277, 0.0322, 0.0366, 0.0411, 0.0455, 0.05$. At last, we compute the empirical quantiles of SP2 for the same p. In the figure, the squares are the bounds of the CI-AN; the stars are the bounds of the CI-SP; the crosses are the empirical quantiles of the data.

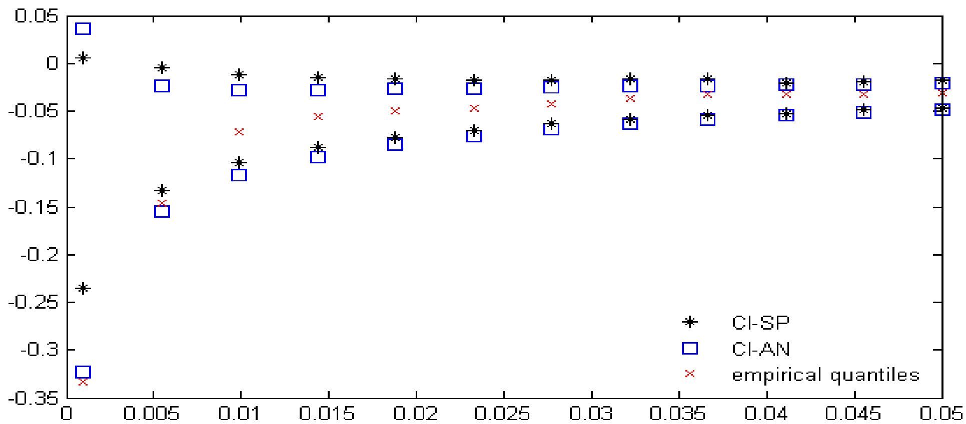

Figure 4: We fit a NIG distribution using moments method on the data HSI1 and compute the CI-AN and CI-SP use the fits, where $q = 0.01$, $n = 252$ and $p = 0.001, 0.0055, 0.0099, 0.0144, 0.0188, 0.0233, 0.0277, 0.0322, 0.0366, 0.0411, 0.0455, 0.05$. At last, we compute the empirical quantiles of HSI2 for the same p. In the figure, the squares are the bounds of the CI-AN; the stars are the bounds of the CI-SP; the crosses are the empirical quantiles of the data.

In Figure (3) and Figure (4) we represent the bounds of the confidence intervals: the squares are the bounds of the CI-AN; the stars are the bounds of the CI-SP; the crosses are the empirical quantiles of the data. In Figure (3), for $p = 0.001$, the empirical quantile equals -0.2047, which is outside the lower bound of the CI-SP -0.1774. But it is inside the lower bound of the CI-AN -0.2261. In Figure (4), for $p = 0.0055$, the empirical quantile equals -0.1464, which is outside the lower bound of the CI-SP -0.1337, but it is inside the lower bound of the CI-AN -0.1550. For $p = 0.001$, the empirical quantile equals -0.3333, which is outside of the both lower bounds. Nevertheless the deviation is larger comparing with the lower bound of CI-SP -0.2351, than the lower bound of CI-AN -0.3237: that means the confidence interval from the saddlepoint approximation may be more sensitive to the uncertainty of the extreme VaR. Notice that the wider confidence interval is not the better. For example, the interval $[-\infty, \infty]$ containing all the possibility of the VaR, but it does not provide any information. In both figures, when $p \geq 0.0099$, the empirical quantiles are inside the lower bounds of CI-AN and CI-SP. Thus, we may not find the potential violations of the VaR when we just consider $p = 0.01$. Finally the extreme VaR have higher possibility to violate the confidence intervals than the regular VaR.

Notice that in Figure (3), we observe when $0.0366 \leq p \leq 0.05$, the empirical quantiles are outside the upper bounds of CI-AN and CI-SP. In this case, the parameters estimates of the distribution are likely to be inappropriate. Therefore, instead of using moments method, we fit a NIG distribution on the data set SP1 using maximum likelihood approach. The estimates are provided in Appendix C. Then we perform the stress testing exercise again using these estimates and the results are provided in Figure (5). We observe in this figure that all the empirical quantiles are inside the bounds of CI-AN and CI-SP. Moreover, for $p = 0.001$, the empirical quantile equals -0.2047. The lower bound of the CI-AN equals -0.3444 and the lower bound of the CI-SP equals -0.2565. That means CI-AN is over conservative and may lead to unnecessary high regulatory capital.

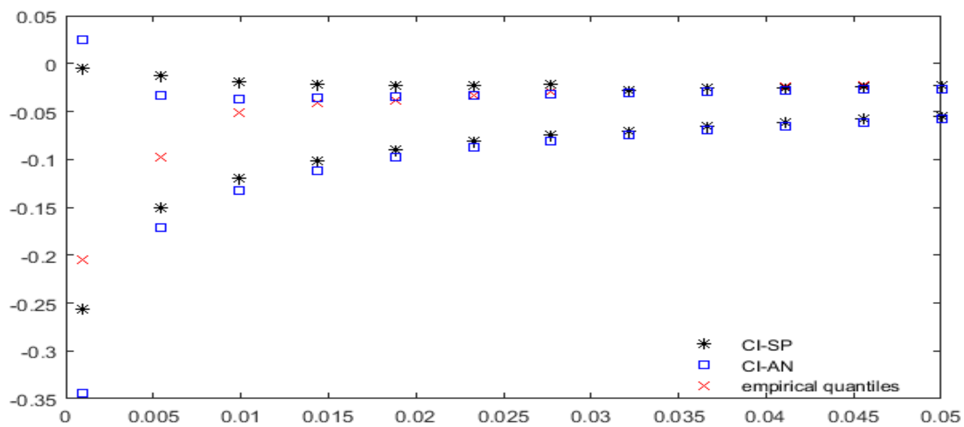

Figure 5: We fit a NIG distribution using maximum likelihood approach on the data SP1 and compute the CI-AN and CI-SP use the fits, where $q = 0.01$, $n = 252$ and $p = 0.001, 0.0055, 0.0099, 0.0144, 0.0188, 0.0233, 0.0277, 0.0322, 0.0366, 0.0411, 0.0455, 0.05$. At last, we compute the empirical quantiles of SP2 for the same p. In the figure, the squares are the bounds of the CI-AN; the stars are the bounds of the CI-SP; the crosses are the empirical quantiles of the data.

For robustness, we also fit a NIG distribution on the data set SP1 using maximum likelihood approach. The estimates are provided in Appendix C. We perform the stress testing exercise again using these estimates and the results are provided in Figure (6). Not like the result of Figure (4), there is only one outlier: when $p = 0.001$, the lower bound of CI-SP is and the empirical quantile is. It confirms our opinion that the CI-SP is more sensitive to the variation of the extreme VaR than the CI-AN.

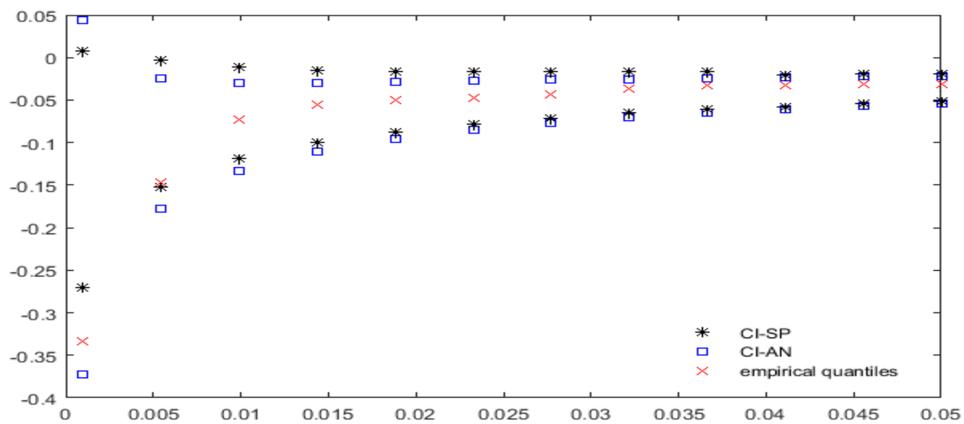

Figure 6: We fit a NIG distribution using maximum likelihood approach on the data HSI1 and compute the CI-AN and CI-SP use the fits, where $q = 0.01$, $n = 252$ and $p = 0.001, 0.0055, 0.0099, 0.0144, 0.0188, 0.0233, 0.0277, 0.0322, 0.0366, 0.0411, 0.0455, 0.05$. At last, we compute the empirical quantiles of SP2 for the same p. In the figure, the squares are the bounds of the CI-AN; the stars are the bounds of the CI-SP; the crosses are the empirical quantiles of the data.

4. Conclusion

In this paper, we first propose the extreme VaR as an alternative way to the Expected Shortfall to measure the extreme risks. Second we associate to this extreme VaR confidence intervals, to identify the variability of the point estimates of the VaR. We use two asymptotic results to construct the confidence intervals for the extreme VaR. By performing simulation experiments, we compare our approach to others, exhibiting the fact that our confidence intervals outperform the others in terms of nominal coverage ratio. In particular, in these experiments, we show that the uncertainty of the extreme VaR is not negligible, but the confidence interval built using bootstrapping approach cannot quantify this uncertainty correctly. It may be because when we work with a finite amount of data, we do not have enough observations of extreme losses for resampling techniques to build reliable confidence intervals for the extreme VaR. Instead of this non-parametric tool, our confidence intervals based on the parametric, asymptotic results are robust for the extreme VaR and less time consuming than the bootstrapping approach. Moreover, the CI-SP provides more accurate nominal coverage ratio than the CI-AN for the extreme VaR. Finally, we apply these confidence intervals to perform a stress testing exercise using historical stock returns during financial crisis, for identifying potential violations of the VaR during turmoil periods on financial markets. In this application the viability of our approach is demonstrated, and the quality of the information provided is illustrated. We find that the extreme VaR contains more uncertainty and is more volatile than the VaR with confidence levels like 95%. Thus it is necessary to perform the stress testing exercise with robust confidence intervals for the point estimates of the extreme VaR. Between the two confidence intervals, the CI-SP is more sensitive and informative than the CI-AN. Furthermore, We find that the distributions and the methods we use to get the estimates of the distribution parameters have influence on the results of the exercise. We suggest to use the distribution and the fit which can capture the properties of data objectively.

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Appendix A. Proof of Corollary 1

At first, we introduce the Slutsky's theorem and the theorem of Smirnov (1949) [24]:

Theorem 1 (Slutsky's theorem). *Let* $\{X_n\}$*,* $\{Y_n\}$ *be sequences of r.v., If* $\{X_n\}$ *converges in distribution* ($\rightarrow_{(d)}$) *to a r.v.* X *and* $\{Y_n\}$ *converges in probability to a constant* c ($\rightarrow_{(p)}$)*, then*

$$X_n Y_n \rightarrow_{(d)} cX. \tag{A.1}$$

Theorem 2 (Asymptotic normality approximation (Smirnov, 1949)). *Given a r.v.* X *with a continuous and differentiable cdf* F_θ *and a density* f_θ *strictly positive at* $F_\theta^{-1}(p)$*, then*

$$\sqrt{n}\frac{X_{(m)} - F_\theta^{-1}(p)}{\sqrt{V}} \rightarrow_{(d)} N(0,1), \quad as \quad n \longrightarrow \infty \tag{A.2}$$

where $\rightarrow_{(d)}$ *means convergence in distribution,* $V = \frac{p(1-p)}{f_\theta(F_\theta^{-1}(p))^2}$. $N(0,1)$ *represents the standard Gaussian distribution.*

To prove Corollary 1, we begin with

$$
\begin{aligned}
&\frac{X_{(m)} - F_\theta^{-1}(p)}{\sqrt{\widehat{V}}} \\
&= \sqrt{\frac{V}{\widehat{V}}} \frac{X_{(m)} - F_\theta^{-1}(p)}{\sqrt{V}} \\
&= \frac{f_{\hat{\theta}}(F_{\hat{\theta}}^{-1}(p))}{f_\theta(F_\theta^{-1}(p)} \frac{X_{(m)} - F_\theta^{-1}(p))}{\sqrt{V}}.
\end{aligned}
\tag{A.3}
$$

Since convergence in probability is preserved under continuous transformations, from $\hat{\theta} \rightarrow_{(P)} \theta$ we have

$$
f_{\hat{\theta}}(F_{\hat{\theta}}^{-1})(p) - f_\theta(F_\theta^{-1}(p)) \rightarrow_{(P)} 0. \tag{A.4}
$$

From Smirnov's theorem, we know that $\frac{X_{(m)} - F_\theta^{-1}(p))}{\sqrt{V}} \rightarrow_{(d)} N(0,1)$, then from Slutsky's theorem we have finally

$$
\frac{X_{(m)} - F_\theta^{-1}(p)}{\sqrt{\widehat{V}}} \rightarrow_{(d)} N(0,1). \tag{A.5}
$$

Appendix B. Proof of Corollary 2

At first, we introduce Zhu and Zhou's theorem [25]:

Theorem 3 (Saddlepoint approximation (Zhu and Zhou, 2009)). *Given a r.v. X with cdf F_θ, for $\forall \epsilon > 0$ and $\forall x$ in the domain of F_θ, we assume $\epsilon < p < 1 - \epsilon$. Then, if $r_0 = \frac{m}{n}$, $F_\theta(x) = t$ and $t \neq p$, we have:*

$$
X_{(m)} \rightarrow_{(d)} \Psi(\sqrt{n}\omega^\sharp) \quad as \quad n \longrightarrow \infty \tag{B.1}
$$

where the convergence speed is $O(\frac{1}{n})$ uniformly w.r.t x. Φ denotes the cdf of standard Gaussian distribution ($N(0,1)$)

$$\begin{aligned}
&\Psi(\sqrt{n}\omega^{\sharp}) = 1 - \Phi(\sqrt{n}\omega^{\sharp}) \qquad && \omega^{\sharp} = \omega + \frac{1}{n\omega} ln \frac{1}{\psi(-\omega)} \\
&\psi(-\omega) = \frac{\omega(t-1)}{t-r_0}\left(\frac{r_0}{1-r_0}\right)^{\frac{1}{2}} && \omega = -\sqrt{2h(t)}sign(t-r_0) \\
&h(t) = r_0 ln \frac{r_0}{t} + (1-r_0) ln \frac{1-r_0}{t}
\end{aligned} \tag{B.2}$$

where $sign(x) = 1$ if $x \geq 0$ and $sign(x) = -1$ otherwise. For $t = p$

$$P(X_{(m)} \leq x) \approx \frac{1}{2} + \frac{1}{\sqrt{2\pi n}} \frac{1+r_0}{3r_0}\left(\frac{r_0}{1-r_0}\right)^{\frac{1}{2}} \quad as \quad n \longrightarrow \infty. \tag{B.3}$$

In order to this theorem to build a CI of $F_{\theta}^{-1}(p)$, we do a linear transformation in the following way: for $\forall x$ fixed,

$$X_{(m)} - F_{\theta}^{-1}(p) \rightarrow_{(d)} \Psi(\sqrt{n}\omega^{\sharp}) \quad as \quad n \longrightarrow \infty \tag{B.4}$$

where we substitute t in Theorem 2 by $F_{\theta}(x + F_{\theta}^{-1}(p))$. Then we plug in $F_{\hat{\theta}}$ and $f_{\hat{\theta}}$ and we have the Corollary 2. The following proof is similar as the proof of Corollary 1 using Slutsky's theorem.

Appendix C. Estimates of the NIG distribution for SP1 and HSI1

We fit the data SP1 and HSI1 with NIG distribution using moments method and maximum likelihood approach. The estimates are provided in Table (C.5).

	Parameters estimates (*NIG*)			
	tail	skewness	location	scale
Moments method				
SP1	23.7081	-7.3294	0.0015	0.0029
HSI1	17.7947	-9.0184	0.0029	0.0034
Maximum likelihood approach				
SP1	21.3110	-1.6069	-0.0005	0.0150
HSI1	19.6534	-2.4435	0.0010	0.0119

Table C.5: We provide the fitted parameters (NIG) for data sets SP1 and HSI1