Secular Stagnation, Liquidity Trap and Rational Asset Price Bubbles

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First draft: 30/03/2016
This version: 31/05/2016

Abstract

I build a model of liquidity trap and secular stagnation of arbitrary duration caused by shortages of liquid assets and propose a new mechanism through which rational bubbles sustain investment and output in the liquidity trap. As opposed to general shortages of assets or safe assets shortages, local shortages of assets emphasize the macroeconomic importance of microeconomic portfolio constraints and liquidity differentials across markets. Financial frictions that hamper the well-functioning of financial markets preclude some agents to invest in capital and introduce a liquidity premium between the interest rate and the stock market returns. A negative shock on the supply of assets available to the constrained agents raises the liquidity premium, decreases the interest rate and pushes the economy toward the liquidity trap. Once at the ZLB, such a shock cannot be accommodated by a drop in the interest rate. Instead, output must fall below its potential to restore equilibrium between the supply and demand for liquidity. Financial bubbles that increase the supply of assets close the output gap and restore efficiency of the equilibrium, at the possible cost of a drop in potential output.

Keywords: Bubbles, Monetary policy, Zero lower bound, Liquidity trap, Secular stagnation, Asset shortage, Liquidity, Financial frictions, Dynamic inefficiency

JEL Classification Numbers: E21, E24, E520.

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*Address: Maison des Sciences Economiques 106-122, Boulevard de l'Hôpital, 75013 Paris, France, e-mail: *mathieu.boullot@psemail.eu* I would to thank Bertrand Wigniolle, as well as all the participants at the macrodynamics seminar at PSE - Paris 1 for useful comments and suggestions.

1 Introduction

Since 2007, it has often been advanced in that US GDP growth during the years that predate the sub-prime crisis has been sustained by rising levels of private debt and housing prices due to an overwhelmingly lax monetary policy. As the central bank eventually recognized the unsustainable nature of this "speculative" growth, the theory goes, it "leaned against the wind" and increased its policy rate, which burst the bubble and provoked the Great Recession that followed [1]. A similar chain of events may be thought of as the origins of the Japanese malaise since the 90's. In both cases, the burst of the bubble that triggered the crisis – on the US housing market, Japanese housing and stock markets, has let the economy with falling inflation rates and rising output gap despite zero nominal interest rate policies, forward guidance, fiscal stimulus, quantitative easing etc. However, the US – or Japan – did not show any signs of a booming economy during the bubbly episode. In particular, GDP growth was modest and inflation under control.

In this paper, I propose a different approach. I formalize the idea advanced by Summers (2013) and argue that a large financial bubble, instead of putting the economy on an unsustainable growth path, was necessary to maintain GDP at its potential. I build a simple macroeconomic model of secular stagnation (Hansen, 1939), that is a depression induced by a chronic shortfall in aggregate demand of arbitrary duration, and I identify conditions under which (*i*) bubbles exist; (*ii*) bubbles are expansionary. Roughly speaking, the existence of rational bubbles is conditional on the stance of fiscal policy while their macroeconomic effects are conditional on the stance of monetary policy. My contributions are threefold: first, I uncover another force behind the liquidity trap and the demand side secular stagnation that comes with, which stems from a liquid assets shortage [2], as opposed to a general shortage of assets (Eggertsson and Mehrotra, 2014) or a shortage of safe assets (Caballero and Farhi, 2016). Second, I provide a new mechanism to explain how rational bubbles sustain investment, current and potential output in the liquidity trap. Third, I clarify how the existence and macroeconomic effects of rational bubbles are conditional on respectively the fiscal and monetary regimes.

As in the standard New Keynesian model, the output gap – the gap between current and

[1] Gali (2014) questions from a theoretical point of view, and Gali and Gambetti (2015) provides empirical evidences against, the "conventional" view that a "leaning against the wind" monetary policy could cause the bubble to burst.

[2] Asriyan et al. (2015) and Bacchetta et al. (2015) study supply side secular stagnation driven by liquid assets shortages.

potential output – is an increasing function of the interest rate gap – the gap between the current interest rate set by the central bank and the natural interest rate [3]. In turn, the natural interest rate is determined by the ratio of the supply to the demand of liquidity. Away from the zero lower bound (ZLB), monetary policy accommodates a drop in the supply or an increase in the demand of liquidity by adjusting the policy rate downward so as to maintain the interest rate gap nil. However, if fundamental liquidity – the sum of public and private debt – becomes increasingly scarce, the natural interest rate turns negative: the central bank hits the ZLB and the economy enters the liquidity trap. Since the interest rate is fixed at its lower bound, the output gap grows as the natural interest rate falls.

If, when the interest rate is equal to the growth rate of the economy, the demand of liquidity exceeds its supply, this excess demand can be channeled to create rational bubbles. Outside of the liquidity trap, bubbles absorb some savings that would have otherwise financed the accumulation of capital: output and potential output are negatively affected. But in the liquidity trap, bubbles also contribute to sustain aggregate demand as they raise the natural interest rate. Which of the two effects, supply or demand side, dominates depends non-linearly on the degree to which monetary policy is constrained at the fundamental equilibrium, i.e on the interest rate gap without bubbles. In particular, financial bubbles have sizable expansionary effects if the fundamental output gap is large.

A negative natural interest rate is symptomatic of a general shortage of assets (Caballero, 2006), which is akin to dynamic inefficiency: in equilibrium, agents want to hold more assets than the economy can produce. Despite being theoretically possible, a general shortage of assets is usually thought of as unrealistic and empirically irrelevant (Abel et al., 1989) [4]. This automatically rules out the possibility of long-lasting liquidity traps and rational bubbles altogether [5], since both exist only if the equilibrium is dynamically inefficient.

However, if the agents have heterogeneous investment opportunities and face borrowing constraints, one may re-conciliate dynamic efficiency and permanently negative interest rates (Martin and Ventura, 2012): only a subset of agents face an assets shortage, i.e the shortage is local but not general. If for example participation in the stock market is restricted as I assume in the paper, the rental rate of capital carries a liquidity premium over the interest

[3] Except stated otherwise, the interest rate refers to the *real* interest rate, not to the *nominal* interest rate.
[4] This last point has been recently challenged by Geerolf (2013).
[5] Dynamic inefficiency arises, and bubbles are possible, when $r^* < g$, where r^* is the natural interest rate and g the growth rate of the economy. A liquidity trap arises when $r^* < \underline{r}$, where $\underline{r} \leq 1$ is the real lower bound that depends on the ZLB and inflation expectations.

rate [6]. Recent empirical evidences confirm that illiquid assets pay a participation or liquidity premium over near-money assets (Krishnamurthy and Vissing-Jorgensen, 2012). Furthermore, the equity premium as well as the level of the safe interest rate have proven exceedingly difficult, if not impossible, to rationalize in terms of pure risk aversion (Mehra and Prescott (1985), Weil (1989)) and households' liquidity needs are barely explained without relying on financial constraints (Ragot, 2014).

The model has three key ingredients: an OLG demographic structure that allows the natural interest rate to deviate permanently from agents' discount factor; financial frictions that introduce a wedge between the rental rate of capital and the interest rate; nominal rigidities and the ZLB that create a gap between the natural interest rate and the current interest rate, or output and potential output, and open the door to liquidity traps. Households live for two periods. Young investors, who have access to the stock market, sell inside liquidity (debt) to the young savers, who don't have access to the stock market, in order to finance purchases of capital that they rent to the firms. Inside liquidity must be backed by imperfectly pledgeable future returns – dividends and profits. Because the capital market is illiquid relative to the debt market, the rental rate of capital carries a liquidity premium over the interest rate. As inside liquidity becomes scarcer, the natural interest rate falls and the liquidity premium rises. Facing a lower cost of debt, investors choose a higher level of leverage, that is they accumulate more capital and create more assets.

As long as the interest rate can absorb changes in the scarcity of liquidity, the financial frictions and sticky prices are irrelevant on aggregate: their effects are purely distributional. But once the economy enters the liquidity trap with a binding ZLB, the central bank cannot adjust the interest rate downward anymore after a shock on the natural interest rate. Since the borrowers have no incentives to leverage more and issue more debt, there is an excess demand for liquidity. To restore equilibrium in the financial markets, an output gap endogenously emerges which redistributes wealth – from the savers to the investors as well as from the young agents to the old one – and acts as a substitute for a fall in the policy rate. This alternative equilibrating mechanism has an efficiency cost in the short run: output drops below its potential. In the long run, potential output also shrinks because the aggregate savings rate decrease: an aggregate demand shortage has supply side effects in the long run.

[6] "According to the last [2013] Survey of Consumer Finances, fewer than 15 percent of U.S. households own stocks directly, and only about 50 percent of households own stocks either directly or indirectly through mutual funds or retirement accounts." (Bricker et al., 2014).

If agents succeed in coordinating their expectations, rational bubbles are another way to reach equilibrium in the liquidity trap. As rational bubbles raise the supply of assets available to the young savers, they increase the natural interest rate and decrease the liquidity premium. This helps the economy escape the liquidity trap, and close the interest rate and output gaps. Bubbles have benefits: they minimize the efficiency loss associated to a fundamental shortage of assets, but they also have costs: they reduce the aggregate savings rate – with respect to the fundamental economy outside of the liquidity, and as a consequence potential output. Even if the same trade-off applies to public debt, either in the form of bonds or money, macro-prudential policies that directly transfer wealth from the young savers to the young investors can enhance investment and output at virtually no cost.

Related literature The two closest papers are Kocherlakota (2013) and Caballero and Farhi (2016). The present paper builds extensively on, and complement, theirs. Like Caballero and Farhi (2016), I study an OLG economy with financial frictions and nominal rigidities. But while they introduce aggregate risk, heterogeneity in risk-aversion, and they restrict their analysis to equilibria with pure risk premia – safe assets shortages, I make the opposite choices. Indeed, I abstract from any risk and focus solely on segmented markets, liquidity premia and liquid assets shortages. Through a very simple static OLG model with fixed wages, Kocherlakota (2013) provides intuitions as of why a fall in housing prices – either fundamental or bubbly – generates an endogenous drop in employment if monetary policy is unresponsive. I extend his analysis to a fully-fledged model with optimal monetary policy, sticky prices and financial frictions. Furthermore, unlike both papers, I provide microfoundations for the supply of assets and the accumulation of capital. Thereby, my model can characterize not only the dynamic of the current output, but also potential output, in a demand side secular stagnation, and may provide an explanation to the drop in measured potential output after the Great Recession (Bosworth, 2015) [7].

Eggertsson and Mehrotra (2014) and Bacchetta et al. (2015) develop models of secular stagnation and permanent liquidity traps. On the contrary to the general assets shortages which are the focus of Eggertsson and Mehrotra (2014), shortages of liquid assets emphasize the distribution of wealth within a generation as well as the role of financial frictions rather

[7] At least with respect to labor market participation and capital accumulation. The present model is silent about TFP growth.

than life-cycle savings and a rapidly decreasing marginal product of capital. Bacchetta et al. (2015) demonstrate how a liquid assets shortage can provoke a supply side secular stagnation, where money crowds out capital in the liquidity trap [8]. In the presence of nominal rigidities, I show that the reverse conclusion may hold: a higher supply of public liquidity crowds in capital accumulation.

Despite the flourishing literature on rational bubbles [9] [10] that builds on the seminal papers by Samuelson (1958) and Tirole (1985), the macroeconomic consequences of rational bubbles on aggregate demand have barely been investigated – two aforementioned papers, Caballero and Farhi (2016) and Kocherlakota (2013), being, to the best of my knowledge, the only exceptions [11]. As numerous recent papers (e.g Santos and Woodford (1997), Caballero and Krishnamurthy (2006), Kocherlakota (2010), Arce and Lopez-Salido (2011), Kunieda et al. (2014), Martin and Ventura (2014), Clain-Chamosset-Yvrard and Seegmuller (2015), Ikeda and Phan (2015)), I consider an economy with financial frictions and heterogeneous agents. A few papers already provide conditions for bubbles to be expansionary in such an environment (e.g Kocherlakota (2009), Farhi and Tirole (2012), Martin and Ventura (2012), Miao and Wang (2015a), Miao and Wang (2015b)): bubbles reallocate wealth toward productive investment, rising the stock of capital and potential output. Unlike those papers, I focus on the demand side effects of rational bubbles in the liquidity trap.

My paper also relates to the literature on liquidity traps (e.g Krugman (1998), Eggertsson and Woodford (2003), Christiano et al. (2011), Correia et al. (2013)). While this literature usually relies on an exogenous shock on a structural parameter [12] to generate a temporary shortage of assets and negative interest rates, e.g a discount rate shock, tightened borrowing constraints (Lorenzoni and Guerrieri (2011), Eggertsson and Krugman (2012), Korinek and Simsek (2016)) or overbuilding with irreversible investment (Rognlie et al., 2015) [13], my model underlines how a non-fundamental shock, a shock on the stock of bubbles, can induce a liquidity trap of arbitrary duration.

[8] Asriyan et al. (2015) show that bubbles creation shocks are expansionary in such an environment.

[9] See Miao (2014) for a good introduction to that literature.

[10] Alternatives theories based on asymmetric information, agency problems and behavioral biases have been developed. See Brunnermeier and Oehmke (2013) for a literature review.

[11] Caballero and Farhi (2016) show that bubbles creation shocks can be expansionary in a liquidity trap driven by a general assets shortage. This is somewhat artificial as each new generation benefits from a wealth effect.

[12] One exception is Schmitt-Groh and Uribe (2012), where expectations of low inflation drive the economy into the liquidity trap.

[13] If we interpret capital in the present model as housing, bubbles provide an explanation for the initial overbuilding in Rognlie et al. (2015).

The macro-finance literature (e.g Woodford (1990), Bernanke et al. (1996), Kiyotaki and Moore (1997), Holmstrom and Tirole (1998), Kiyotaki and Moore (2012), Gertler and Kiyotaki (2013)) as well as the New Monetarist literature (e.g Lagos and Wright (2005), Williamson (2012), Andolfatto and Williamson (2015)) stress the the role of fundamental stores of value and liquidity to support respectively investment and consumption in environments with financial frictions. Unlike them, I focus on bubbly liquidity in an environment with both financial frictions and nominal rigidities.

Finally, my model is reminiscent of the old literature on disequilibrium [14] (e.g Barro and Grossman (1971), Benassy (1976), D'Autume and Michel (1985)), which explains how an excess supply in one market translates into an excess demand in another – because of rigid prices or more generally demand constraints. In the liquidity trap, the interest rate is fixed at its lower bound – the ZLB minus expected inflation: an excess demand of liquidity implies an excess supply of goods: the supply of goods as well as the demand of liquidity are rationed.

The paper is organized as follows. Section 2 exposes the basic model and characterizes the equilibrium. Section 3 studies the bubbly and fundamental steady states. Section 4 introduces fiscal policy. Section 5 introduces inflation. Section 6 briefly discusses and contrasts the various types of assets shortages. Appendix A introduces risky bubbles. Appendix B studies the dynamic of the models with rigid prices or inflation.

2 A model of liquidity shortages and bubbles

2.1 The basic environment and equilibrium

The basic setup is an overlapping generations model with a single consumption good, two factors of production, capital and labor, and three assets: private bonds, capital and bubbles. Time is discrete, $t \in \{0, 1, ...\}$, and the horizon infinite.

Demography and preferences At each date t a mass 1 of households indexed by $j \in [0, 1]$ is born. In each generation, a fraction $\omega \in (0, 1)$ of agents are investors, whereas the remaining $1 - \omega$ are savers. Investors own the firms and have access to all the financial markets, whereas savers can trade only in bubbles and real bonds. Each agent lives for two

[14] I thank my PhD advisor, Bertrand Wigniolle, who directed me to this literature.

periods and has preferences over consumption and labor,

$$U(c^y_{j,t}, c^o_{j,t+1}, n_{j,t}) = u(c^y_{j,t} - v(n_{j,t})) + \beta u(c^o_{j,t+1}) \tag{1}$$

Here, $\beta \in (0,1)$ is the discount factor, $c^y_{j,t}$ denotes young-age consumption, $c^o_{j,t+1}$ old-age consumption and $n_{j,t}$ is the labor supply. Such a non-separable utility function shuts down the wealth effect on the labor supply (Greenwood et al., 1988) and allows aggregation. Indeed, the labor supply solves a static problem,

$$e_{j,t} \equiv \max_{n_{j,t}} W_t n_{j,t} - v(n_{j,t}) \tag{2}$$

Here, W_t denotes the real wage and $e_{j,t}$ denotes net labor income, i.e labor income net of disutility of work. For the sake of tractability, I consider the following functional forms, $u(c) = \log(c)$ and $v(n) = \frac{n^{1+\eta}}{1+\eta}$, where $\eta > 0$ is the (inverse) Frisch elasticity [15]

Assets, budget and borrowing constraints Young households must choose a portfolio that consists of three assets: bubbles, real bonds and capital. Bubbles are intrinsically worthless assets akin to Ponzi-schemes: entrepreneur j buys $q_{j,t+1}$ units of bubbles today at a price B_t because he expects to re-sell tomorrow at a price B_{t+1}. The returns on riding bubbles, $\frac{B_{t+1}}{B_t}$, are pure capital gains. In the basic model, I follow Tirole (1985) and consider safe bubbles. However, appendix A introduces risky bubbles in the spirit of Weil (1987). Agents cannot short the bubbles,

$$q_{j,t+1} \geq 0 \tag{3}$$

And the supply is exogenous and normalized to 1. Real bonds are promises to pay one unit of the consumption good the next period. The price of a real bond in period t is $\frac{1}{r_t}$, where r_t is the interest rate. Household j's net supply of bonds at date t is denoted by $d_{j,t+1}$, that is he is a net borrower in period t if $d_{j,t+1} > 0$. Debt issuance is subject to a standard borrowing constraint,

$$d_{j,t+1} \leq \phi \left(\Pi_{j,t+1} + R_{t+1} k_{j,t}\right), \quad \phi \in (0,1) \tag{4}$$

[15] The results in the present paper are rather intuitive, and not particularly conditional on the functional forms which I assume. More general specifications would however unnecessarily obscure the analysis without providing much insights.

Here, $\Pi_{j,t+1}$ denotes the profit received by household j from the firms described below, R_{t+1} is the rental rate of capital – I assume full capital depreciation – and $k_{j,t+1}$ is household j's stock of capital. Because agents cannot commit to repay their debt, inside liquidity must be backed by claims on pledgeable future returns – dividends or profits. Each agent j has access to the bubbles and debt markets at each date t, however participation in the stock market is restricted. Specifically, I assume that households have at their disposal the following technology to transform the consumption good into capital,

$$k_{j,t+1} = i_{j,t}\theta_{j,t} \tag{5}$$

Where $i_{j,t}$ is investment made out at date t and $\theta_{j,t}$ indexes whether household j has an investment opportunity – $Pr(\theta_{j,t} = 1) = \omega$ and $Pr(\theta_{j,t} = 0) = 1 - \omega$ [16]. Households know their type before taking any decision in t. Since those shocks are $i.i.d$ across agents, ω households are active in the stock market at each date. I will call the households with an investment opportunity "investors", and the remaining households "savers".

The young- and old-age budget constraints are respectively

$$c^y_{j,t} + i_{j,t} + q_{j,t+1}B_t - \frac{d_{j,t+1}}{r_t} = W_t n_{j,t} \tag{6}$$

$$c^o_{j,t+1} + d_{j,t+1} = \Pi_{j,t+1} + R_{t+1}k_{j,t+1} + q_{j,t+1}B_{t+1} \tag{7}$$

Firms and sticky prices As in the standard New Keynesian model, a continuum of monopolistically competitive firms indexed by $v \in [0, 1]$ produces each a differentiated intermediate good. The final good is produced by a competitive firm that aggregates the intermediate varieties,

$$Y_t = \left(\int y_{v,t}^{\frac{\epsilon-1}{\epsilon}} dv \right)^{\frac{\epsilon}{\epsilon-1}} \tag{8}$$

Where $\epsilon > 1$ is the elasticity of substitution between varieties. Each monopolist $v \in [0, 1]$ has a standard constant-returns-to-scale production function,

$$y_{v,t} = F(k_{v,t}, n_{v,t}) \tag{9}$$

[16] We can think of this as a reduced-form for transaction costs or legal constraints.

I will consider the Cobb-douglas production function, $F(k,n) = k^\alpha n^{1-\alpha}, \alpha \in (0,1)$. In the basic model, I follow Caballero and Farhi (2016) and assume that prices are fully rigid: $Pv,t = P$ for each v and t. As this assumption is quite extreme, I relax it in section 5 and introduce inflation. Each monopolist $v \in [0,1]$ maximizes the net profits under a demand constraint and a price rigidity constraint,

$$\Pi_{v,t} = \max_{P_{v,t},k_{v,t},n_{v,t}} \frac{P_{v,t}}{P_t} F(k_{v,t}, n_{v,t}) - (1-\tau(N_t))\left(R_t k_{v,t} + W_t n_{v,t}\right) - \Gamma_{v,t} \tag{10}$$
$$\text{s.t} \quad F(k_{v,t}, n_{v,t}) \leq \left(\frac{P_{v,t}}{P_t}\right)^{-\epsilon} Y_t, \quad P = P_{v,t}$$

Monopolistic competition introduces a distortion with respect to the efficient allocation: the price includes a markup over the marginal cost. To preserve the efficiency of the equilibrium with an unconstrained monetary policy, I assume that the government subsides both labor and capital, $\tau(N_t) = \frac{1}{\epsilon}$ if $N_t \leq N_t^*$ and $\tau(N_t) = 0$ otherwise [17], where N_t^* is the efficient level of employment described below. The subsidies are financed by lump-sum taxes levied on the gross profits, $\Gamma_{v,t} = \tau(N_t)(W_t n_{v,t} + R_t k_{v,t})$. The aggregate profits net of taxes, Π_t, are rebated to the investors, $\Pi_{j,t} = \theta_{j,t}\frac{\Pi_t}{\omega}$ [18].

Since monopolists face the same marginal cost and price, they are symmetric and choose the same capital-labor ratio and output. Furthermore, the aggregate price level is constant, $P_t = \left(\int P_{v,t}^{1-\epsilon} dv\right)^{\frac{1}{1-\epsilon}} = P = 1$ and normalized to one. In equilibrium, the representative monopolist faces an aggregate demand constraint, $F(K_t^d, N_t^d) \leq Y_t = C_t + I_t$, where $N_t^d \equiv \int N_{v,t} dv$ and $K_t^d \equiv \int K_{v,t} dv$ are the aggregate demand of labor and capital, $I_t \equiv \int i_{j,t} dj$ is aggregate investment and $C_t \equiv \int (c_{j,t}^y + c_{j,t}^o) dj$ is aggregate consumption.

Equilibrium Let $Q_t \equiv \int q_{j,t} dj$, $D_t \equiv \int d_{j,t} dj$, $N_t \equiv \int N_{j,t} dj$ and $K_t \equiv \int K_{j,t} dj$ be respectively the aggregate demands of bubbly and inside liquidity, aggregate supplies of labor and capital. The market clearing conditions for the bubbles, liquidity, labor, capital and good are then,

[17] As Korinek and Simsek (2016) notes, with GHH preferences, another equilibrium would exist if the firms continue to receive the subsidy while producing more than the efficient level.

[18] This is minor and innocuous simplification. In equilibrium, savers and investors do not discount profits at the same rate, which complicates the analysis.

$$Q_{t+1} = 1 \tag{11}$$

$$D_{t+1} = 0 \tag{12}$$

$$N_t^d = N_t \tag{13}$$

$$K_t^d = K_t \tag{14}$$

$$C_t + I_t = Y_t \tag{15}$$

Given initial assets holdings $\{k_{j,0}, d_{j,0}, q_{j,0}\}$ and a path for the interest rate, $\{r_t\}_{t=0}^{\infty}$, a competitive equilibrium is defined as a sequence of prices $\{B_t, W_t, R_t\}_{t=0}^{\infty}$ and aggregates $\{Q_t, D_t, C_t, I_t, N_t, K_t, Y_t\}_{t=0}^{\infty}$ such that: (i) the market clearing conditions (11) - (15) are satisfied; (ii) each household maximizes (1) under (3) - (7); each monopolist solves (10).

Monetary policy Since the prices are fully rigid, I follow Rognlie et al. (2015) and assume that monetary policy replicates the efficient allocation. Given the stock of capital, we can define the efficient level of output, Y_t^* as,

$$Y_t^* \equiv F(K_t, N_t^*), \quad \text{where} \quad N_t^* = \arg\max_N F(K_t, N) - v(N) \tag{16}$$

Here, N_t^* is the efficient level of employment which equalizes the marginal disutility of work to the marginal product of labor. Y_t^* can be thought of as the supply-determined level of output, i.e the level of output that would prevail in an economy with flexible prices. Therefore, the natural interest rate, r_t^*, is defined as the unique interest rate such that output is efficient at the competitive equilibrium in period t, $Y_t = Y_t^*$. Given the natural interest rate, monetary policy sets the nominal interest rate according to the following rule,

$$r_t^n = \max\{r_t^*, \underline{r}\}, \quad \underline{r} = \underline{r}^n \leq 1 \tag{17}$$

Because the inflation rate is nil at each period, $\pi_{t+1} \equiv \frac{P_{t+1}}{P_t} = 1$, the ZLB constraint on the *nominal* interest rate, $r_t^n \geq \underline{r}^n$, translates into a constraint on the *real* interest rate, $r_t = r_t^n \geq \underline{r} = \underline{r}^n$ [19]. The recent experience in some OECD countries with negative nominal interest rates has shown that the money demand does not explode at $r^n = 1$, strictly speaking

[19] The nominal and real interest rates must satisfy the Fisher equation.

the zero lower bound (ZLB) is somewhat lower than 0 [20]. However, major central banks have been quite reluctant to implement negative nominal interest rates; furthermore, there is great deal of uncertainty surrounding the "true" ZLB. Thus one may think of $\underline{r}^n$ as resulting from either an exploding money demand at $r^n_{t+1} < \underline{r}^n$, or a "timidity trap" (Krugman (1998), Bernanke (1999)). The economy is in the liquidity trap when the ZLB constraint in (11) binds, i.e when $r_t = \underline{r} > r^*_t$. Even through in most of the paper I consider the cashless limit, I introduce money and public bonds in subsection 4.2: conditional on a fiscal policy that targets a constant debt-to-GDP ratio, money is irrelevant.

It is well-known since Diamond (1965) that the equilibrium of the standard OLG model with production is Pareto-efficient if and only if the steady state rental rate of capital is weakly higher than the growth rate of the economy, $R \geq g$. As long as $r^*_t \geq \underline{r}$, the aggregate dynamic of the present model is reproduces the dynamic of the standard frictionless OLG model. In particular, the sole effect of financial frictions and sticky prices is to redistribute consumption within a generation. Intuitively, the equilibrium is Pareto-efficient if and only if $Y_t = Y^*_t$ for each t and $R \geq g = 1$. As I assume the latter, the monetary rule in (11) is constrained efficient [21].

2.2 Characterizing the equilibrium

In this subsection I derive the main equations that govern the dynamic of the model. To begin with, we will have a look at the supply side of the economy. In particular, we will discuss how the liquidity trap provokes an endogenous drop in output. Then we will focus on the households' portfolio choices and the determination of the natural interest rate. Finally, we will study the roots of the liquidity trap and analyze how rational bubbles can alleviate shortfalls in aggregate demand induced by liquid assets shortages.

Output gap and factors' shares in income To understand the macroeconomic consequences of price rigidities and the ZLB, i.e of the liquidity trap, consider the representative monopolist's optimal factor demands for problem (10),

[20] See Rognlie (2015) for theoretical results on optimal monetary policy with negative nominal interest rates.

[21] If the equilibrium is dynamically inefficient, monetary policy faces a trade-off between aggregate demand management and consumption smoothing. As this is behind the scope of this paper, I rule this case out by assumption.

$$(1-\xi_t)F_k(K_t^d, N_t^d) = R_t \quad \text{and} \quad (1-\xi_t)F_n(K_t^d, N_t^d) = W_t \tag{18}$$

Here, $\xi_t \equiv \frac{\delta_t}{Y_t}\frac{1}{1-\epsilon}$ is the labor wedge, where $\delta_t \geq 0$ is the Lagrange multiplier on the price rigidity constraint in (10). The labor wedge is endogenously determined in equilibrium as the difference between the price, $P = 1$, and the marginal cost of production, Λ_t,

$$\xi_t = (1-\Lambda_t), \quad \Lambda_t \equiv \left(\frac{R_t}{\alpha}\right)^{\alpha}\left(\frac{W_t}{1-\alpha}\right)^{1-\alpha} \tag{19}$$

When monetary policy is unconstrained, $r_{t+1}^n = r_{t+1}^*$, it drives aggregate demand such that the price rigidity constraint in (10) never binds: firms would not set another price even if they could. Output is efficient and supply-determined, $Y_t = Y_t^*$ and $\xi_t = 0$.

However, if monetary policy is constrained by the ZLB, $r_t^n = \underline{r} > r_t^*$, aggregate demand is inefficiently low, $C_t + I_t < Y_t^*$. In this case, the price rigidity constraint binds: firms must keep an inefficiently high level of markup and they produce just enough to meet the demand, $Y_t < Y_t^*$. As the monopolists cut production with respect to the efficient level, they ask for less labor and capital. This induces the demand curves to shift, which in equilibrium reduces the productive factors share in income but raise the share of rent,

$$\frac{W_t N_t}{Y_t} = (1-\alpha)(1-\xi_t) \quad \text{and} \quad \frac{R_t K_t}{Y_t} = \alpha(1-\xi_t) \tag{20}$$

The labor wedge acts as a markup between the price and the marginal cost. If the labor supply is endogenous, $\eta < \infty$, a shift in the labor demand curve induces households to work less in equilibrium,

$$N_t = W_t^{\frac{1}{\eta}} = ((1-\alpha)(1-\xi_t)K_t^{\alpha})^{\frac{1}{\alpha+\eta}} \quad \text{and} \quad E_t \equiv \int e_{j,t} dj = \frac{\eta}{1+\eta} N_t W_t \tag{21}$$

Which creates a gap between actual and potential output,

$$\frac{Y_t}{Y_t^*} = (1-\xi_t)^{\frac{1-\alpha}{\alpha+\eta}} \tag{22}$$

I will call ξ_t the labor wedge or output gap indifferently [22].

When monetary policy lets the aggregate demand fall below potential output, firms facing rigid prices must cut their production and their demand of factors relative to the efficient level.

[22] With more general specifications for the preferences, the two concepts do not need to overlap.

Consequently, the share of rent increases, labor supply drops and output is inefficiently low. We know from (17) that this happens if and only if monetary policy is constrained by the ZLB. The next natural question is: what determines the natural interest rate in this economy? As we will see, the natural interest rate depends on households' portfolio choices, and in particular the supply and demand of liquid assets.

Individual consumption, savings and Euler equation A standard non-arbitrage equation implies that in equilibrium, $\frac{B_{t+1}}{B_t} = r_{t+1} \leq R_{t+1}$: as the capital market is much less liquid than the two others, the rental rate of capital may carry a liquidity premium over the interest rate. Taking this into account as well as the static labor choice (2), we can rewrite the problem of household j as:

$$\begin{aligned}
&\max_{i_{j,t},q_{j,t+1},c^y_{j,t},c^o_{j,t+1}} u(c^y_{j,t} - v(n_{j,t})) + \beta u(c^o_{j,t+1}) \\
&\text{s.t} \quad c^y_{j,t} - v(n_{j,t}) + \frac{c^o_{j,t+1}}{r_t} = e_{j,t} + \frac{\Pi_{j,t+1}}{r_t} + \left(\frac{R_{t+1}}{r_t}\theta_{j,t} - 1\right) i_{j,t} \\
&i_{j,t} \leq \left(1 + \theta_{j,t}\left(\varphi_t - 1\right)\right)\left(e_{j,t} - c^y_{j,t} - v(n_{j,t}) + \frac{\phi \Pi_{j,t+1}}{r_t}\right)
\end{aligned} \tag{23}$$

Where $\varphi_t \equiv \frac{1}{1 - \frac{\phi R_{t+1}}{r_t}}$ is the leverage, and $n_{j,t}$ solves (2). A sort of distorted Euler equation holds for each household,

$$u'(c^y_{j,t} - v(n_{j,t})) = \beta R^W_{j,t+1} u'(c^o_{j,t+1}) \tag{24}$$

Here, $R^W_{j,t+1}$ is the rate of returns household j obtains on his savings,

$$R^W_{j,t+1} \equiv \theta_{j,t} R_{t+1} \varphi_t \left(1 - \phi\right) + \left(1 - \theta_{j,t}\right) r_t \tag{25}$$

Given the discounted logarithmic preferences, young households consume a constant fraction of their discounted wealth,

$$c^y_{j,t} - v(n_{j,t}) = \frac{1}{1+\beta}\left(e_{j,t} + \frac{\Pi_{j,t+1}}{R_{t+1}}\right) \tag{26}$$

And old-age consumption is given by the Euler equation (24).

Demand and supply of assets Households are always indifferent about the type of liquidity they buy, bubbles or private bonds. However, some of them, the savers, cannot hold any stocks – capital – in equilibrium. Assuming that $R_{t+1} > r_t$ the borrowing constraint (4) binds for productive households,

$$\frac{d_{j,t+1}}{r_t} = \begin{cases} \frac{\phi R_{t+1}}{r_t}(k_{j,t+1} + \frac{\Pi_{j,t+1}}{R_{t+1}}) > 0 & \text{if} \quad \theta_{j,t} = 1 \\ -\frac{\beta}{1+\beta} e_{j,t} + q_{j,t} B_t < 0 & \text{otherwise} \end{cases} \tag{27}$$

And

$$k_{j,t+1} = i_{j,t} = \begin{cases} \varphi_t \frac{\beta}{1+\beta} \left(e_{j,t} + \frac{\Pi_{j,t+1}}{R_{t+1}} \right) - \frac{\Pi_{j,t+1}}{R_{t+1}} & \text{if} \quad \theta_{j,t} = 1 \\ 0 & \text{otherwise} \end{cases} \tag{28}$$

Investors face an arbitrage opportunity: the rental rate of capital, R_{t+1}, is greater than the cost of debt, r_t. Thus, they issue bonds in order to finance capital purchases until their borrowing constraint binds. Savers have no choice but to buy bubbles and inside liquidity issued by the leveraged borrowers. While the demand of liquidity by the savers is inelastic, the supply of liquidity is not: a higher interest rate reduces the leverage of the investors and their investment, which has a negative effect on the supply of liquidity. Bubbles, which are a form of outside liquidity, crowd out the demand of inside liquidity by the savers.

Bubbles, borrowing limits and the interest rate Given binding borrowing constraints (4), we can compute the real interest rate consistent with equilibrium in the assets markets, (11) and (12),

$$r_t = \min \left\{ \frac{\phi \left(R_{t+1} I_t + \Pi_{t+1} \right)}{(1-\omega) \frac{\beta}{1+\beta} E_t - B_t}; R_{t+1} \right\} \tag{29}$$

The interest rate is equal to the ratio of the the supply of inside liquidity relative to the demand. The binding borrowing constraints put an upper limit on the creation of inside liquidity by the leveraged investors, which must be backed by pledgeable returns. As a consequence, the agents who are not active on the capital market, the savers, may face a shortage of assets in equilibrium. The scarcity of inside liquidity induces the rental rate of capital to carry a liquidity premium over the interest rate. This breaks the link between dynamic efficiency of the equilibrium and the interest rate. Throughout the paper, I only consider equilibria where inside liquidity is scarce, i.e $r_t < R_{t+1}$.

Using the factors' share in income, (18),

$$r_t = \frac{\phi\left(\alpha(1-\xi_{t+1})+\xi_{t+1}\right)}{(1-\omega)\frac{\beta}{1+\beta}\frac{\eta}{1+\eta}(1-\alpha)(1-\xi_t)-b_t}\frac{Y_{t+1}}{Y_t} \tag{30}$$

Here, $b_t \equiv \frac{B_t}{Y_t}$ is the ratio of the stock of bubbles to output. The natural interest rate obtains in (30) when the current output gap is nil, $\xi_t = 0$ and $Y_t = Y_t^*$. Rational bubbles and output gaps tend to lower the liquidity premium and prevent a drop in the interest rate. The former because they provide the savers with a higher supply of liquidity; the latter because the current output gap reduces the labor income of the savers while the expected output gap raises the supply of pledgeable returns through the share of rent.

As the returns on riding the bubbles must be equal to interest rate, $b_{t+1} = \frac{r_t}{g_{t+1}} b_t$, where $g_{t+1} \equiv \frac{g_{t+1}}{g_t}$ is the growth rate of output.

$$b_{t+1} = \frac{\phi(\alpha(1-\xi_{t+1})+\xi_{t+1})}{(1-\omega)\frac{\beta}{1+\beta}\frac{\eta}{1+\eta}(1-\alpha)(1-\xi_t)-b_t} b_t \tag{31}$$

ZLB, interest rate and output gaps We have already computed the interest rate that clears the asset markets, (29). The nominal interest rate in the economy is set by central bank according to (17). In equilibrium, the Fisher equation must hold: $r_t = \frac{r_t^n}{\pi_{t+1}} = \max\{r_t^*, \underline{r}\}$ [23]. If the shortage of asset is severe, the central bank hits the ZLB and the economy enters the liquidity trap. In this case, the output gap ξ_t is proportional to the interest rate gap, and endogenously determined in equilibrium by setting $r_t = \underline{r}$ in (29),

$$1 - \xi_t = \min\left\{\left(\frac{r_t^*}{\underline{r}}\right)^{\frac{\alpha+\eta}{1+\eta}}, 1\right\} \tag{32}$$

The economy has a powerful tool to restore equilibrium in the financial markets: adjustments in the interest rate that increase the leverage, stimulate aggregate investment, and hence liquidity creation. However, once the central bank hits the ZLB, the price mechanism cannot operate anymore. The fall in the labor share in income and the rise in the share of rent that

[23] It is a standard non-arbitrage condition between liquidity and some nominal asset in zero net supply.

come with the labor wedge acts as a substitute for a fall in the interest rate: the demand of liquidity decreases – because the wealth of the savers is down – while the supply of liquidity increases – because the pledgeable returns are up.

Another way to think of the preceding analysis is through the lens of the good market. By Walras' law, an excess demand of liquidity is equivalent to a shortfall in the aggregate demand for goods (investment).

$$Y_t = \min\{Y_t^*, C_t + I_t\} \tag{33}$$

Given $r_t = \underline{r}$, we can compute the aggregate investment, I_t, and aggregate consumption, C_t. If aggregate demand is inefficiently low, the output gap grows so as to reduce production and restore equilibrium in the good market, (29).

Aggregate dynamic and potential output So far we have considered the static properties of the equilibrium, taking the capital and potential output as given. But the capital stock at date $t+1$ is built on investment at date t,

$$K_{t+1} = I_t = \frac{\beta}{1+\beta}\left(E_t + \frac{\Pi_{t+1}}{R_{t+1}}\right) - \frac{\Pi_{t+1}}{R_{t+1}} - B_t \tag{34}$$

Bubbles and profits not only transfer wealth *infra*-generation, but also *inter*-generations from the young to the old agents, which reduces the incentives to save. Since profits depend on the stock of capital in $t+1$,

$$K_{t+1} = \frac{\frac{\beta}{1+\beta}\frac{\eta}{1+\eta}(1-\alpha)(1-\xi_t) - b_t}{1 + \frac{1}{1+\beta}\frac{\xi_{t+1}}{\alpha(1-\xi_{t+1})}} Y_t \tag{35}$$

In a production economy, a (static) labor wedge translates into a (dynamic) investment wedge: an aggregate demand shortfall has supply side effects in the long run through the accumulation of capital. In the liquidity trap, bubbles alleviate the efficiency loss due to a positive output gap, but they tend to reduce the aggregate savings rate. Without nominal rigidities and the ZLB, the former effect would disappear and output would unambiguously shrink with rational bubbles.

Given the initial stock of capital, K_0, the dynamic of the economy is governed by a three dimensional system in $\{K_{t+1}, b_t, \xi_t\}$, with a predetermined variable, K_t, and two non-predetermined variables, b_t and ξ_t.

$$1-\xi_t = \min\left\{\left(\frac{r^*_{t+1}}{\underline{r}}\right)^{\frac{\alpha+\eta}{1+\eta}}, 1\right\} \tag{36}$$

$$b_{t+1} = \frac{\max\{r^*_t, \underline{r}^n\}}{\left(\frac{1-\xi_{t+1}}{1-\xi_t}\right)^{\frac{1-\alpha}{\alpha+\eta}}\left(\frac{K_{t+1}}{K_t}\right)^{\alpha\frac{1+\eta}{\alpha+\eta}}} \tag{37}$$

$$K_{t+1} = \frac{\frac{\beta}{1+\beta}\frac{\eta}{1+\eta}(1-\alpha)(1-\xi_t)-b_t}{1+\frac{1}{1+\beta}\frac{\xi_{t+1}}{\alpha(1-\xi_{t+1})}}\left((1-\alpha)(1-\xi_t)\right)^{\frac{1-\alpha}{\alpha+\eta}} K_t^{\alpha\frac{1+\eta}{\alpha+\eta}} \tag{38}$$

Where the natural interest rate is a function of the future output gap as well as the growth rate of the stock of capital,

$$r^*_t = \frac{\phi\left(\alpha(1-\xi_{t+1})+\xi_{t+1}\right)}{\frac{\beta}{1+\beta}\frac{\eta}{1+\eta}(1-\alpha)-b_t}(1-\xi_{t+1})^{\frac{1-\alpha}{\alpha+\eta}}\left(\frac{K_{t+1}}{K_t}\right)^{\alpha\frac{1+\eta}{\alpha+\eta}} \tag{39}$$

I analyze the dynamic in appendix B. The bubbly steady state, if it exists, is always locally determinate, whereas the fundamental steady state may be indeterminate if in the liquidity trap. Global indeterminacy arises if the bubbly steady state exists, that is if the natural interest rate is negative [24]. In particular, there is a unique saddle-path that converges to the bubbly steady state. Along that path, the supply of bubbly liquidity as a share of output is constant and just enough to equalize the natural interest rate to the growth rate of output. This implies that as long as the economy grows, i.e $g \geq 1$ and $K_0 < K$, the output gap is nil at each period, $\xi_t = 0$ for each t. This is case even if all other equilibria feature a positive output gap: as we will explain in the next section, bubbles sustain aggregate demand in an economy which suffers from a shortage of liquid assets.

3 Expansionary bubbles in the liquidity trap

In this section, I consider the steady states of the basic model. I focus only on dynamically efficient constrained equilibria, that is equilibria where $R > \max\{r, 1\}$. First, I analyze the fundamental roots of the liquidity trap and the secular stagnation. Second, I study the conditions under which bubbles exist and their macroeconomic effects.

[24] There are a continuum of bubbly equilibria that converge to the fundamental steady state. This form of global indeterminacy exists in virtually all models with negative interest rates in steady state.

3.1 Scarcity of fundamental liquidity, liquidity trap and secular stagnation

Without bubbles, $b_t = b = 0$, the only source of liquidity is the debt supplied by the borrowers, i.e fundamental inside liquidity. Since the output gap and the interest rate are constant over time, the economy is either in the liquidity trap or at full employment forever. Thus the "long run" natural interest rate is given by (30) with $\xi_{t+1} = \xi_t = \xi = 0$,

$$r^* = \frac{\phi}{1-\omega} \frac{1}{\frac{\beta}{1+\beta} \frac{\eta}{1+\eta} \frac{1-\alpha}{\alpha}} \tag{40}$$

In contrast to standard models featuring infinitely-lived agents and perfect financial markets, the natural interest rate in steady state is not equal to the time discount rate but it is a function of the aggregate supply and demand of assets.

Assumption 1 *[Fundamental liquid assets shortage]* $\frac{\phi}{1-\omega} < 1 < \frac{1}{\frac{\beta}{1+\beta} \frac{\eta}{1+\eta} \frac{1-\alpha}{\alpha}}$

The binding borrowing constraints place an upper limit on the supply of inside liquidity by the leveraged investors, whereas the portfolio constraints put a lower limit on the demand by the savers. The two constraints are crucial for the existence of shortages of liquid assets. If both are severe, i.e $\phi + \omega < 1$, inside liquidity is scarce in equilibrium – it commands a liquidity premium under the stock market returns. The second inequality is equivalent to $R^* > 1$ and ensures that the economy does not suffer from a general shortage of assets.

The interest rate in steady state is given by the monetary rule (17) together with the Fisher equation, $r^n = r$. In turn, the output gap solves (32) for $\xi_{t+1} = \xi_t = \hat{\xi}$,

$$\hat{\xi} = \max \left\{ 0, \frac{\frac{r}{r^*} - 1}{\frac{r}{r^*} + \frac{1-\alpha}{\alpha}} \right\} \tag{41}$$

A severe liquid assets shortage depresses the long run natural interest rate, which induces the central bank to hit the ZLB. Consequently, monetary policy becomes impotent: it cannot accommodate further negative shocks on the natural interest rate nor manage aggregate demand efficiently anymore. As the interest rate gap expands, a growing output gap is necessary to restore equilibrium in the financial and good markets. The liquidity trap and the output gap that comes with are permanent in this economy, i.e the steady state has the secular stagnation property.

From the law of motion of capital, (35), and the production function, (8), the capital-labor ratio is negatively affected,

$$\kappa^{\xi} \equiv \frac{K^{\xi}}{N^{\xi}} = \left(\frac{\frac{\beta}{1+\beta}\frac{\eta}{1+\eta}(1-\alpha)(1-\hat{\xi})}{1+\frac{1}{1+\beta}\frac{\hat{\xi}}{\alpha(1-\hat{\xi})}} \right)^{\frac{1}{1-\alpha}} \tag{42}$$

This is true even if the labor supply is fully inelastic, $\eta \to \infty$: a higher share of rent transfers wealth from the old to the young households, which reduces the incentives to save. And since households generally supply less labor, (21), the steady state stock of capital decreases even more than the capital-labor ratio,

$$K^{\xi} = \Big((1-\alpha)(1-\hat{\xi})\Big)^{\frac{1}{\eta}} \kappa^{\xi\,\frac{\eta+\alpha}{\eta}} \tag{43}$$

Since potential output, $Y^{*\xi}$, is an increasing function of the stock of capital, an aggregate demand shortage has negative effects on potential output in the long run.

Proposition 1 *[Fundamental steady state] There exists a unique fundamental steady state.*

(i) The natural interest rate, r^*, is increasing in ϕ and ω.

(ii) The efficient level of output, Y^*, does not depend on ϕ or ω.

(iii) In the liquidity trap, the output gap, $\hat{\xi}$, expands with the interest rate gap, $\frac{r}{r^*}$.

(iv) Output, the capital-labor ratio, as well as the stocks of capital and labor are decreasing in the output gap, $\hat{\xi}$.

The economy can experience two regimes with dramatically different properties:

- Outside of the liquidity trap, if aggregate demand is inefficiently low, the central bank cuts the policy rate to stimulate aggregate investment. The economy behaves on aggregate as if prices were flexible and the financial markets frictionless. This equilibrium is neoclassical.

- In the liquidity trap, the central bank has no room to offset a negative shock on the supply (positive shock on the demand) of liquidity. Since the prices are fixed, quantities must adjust to restore equilibrium. An output gap emerges that is proportional to the degree to which the central is constrained, i.e the interest rate gap. More severe financial frictions go hand-in-hand with bigger output gaps and lower stocks of labor and capital. This secular stagnation equilibrium is Keynesian.

In the Keynesian - secular stagnation case, the economy reaches equilibrium at the cost of an efficiency loss: factors of production are underemployed. Note that in this model as in Caballero and Farhi (2016) or Eggertsson and Mehrotra (2014), the secular stagnation steady state is the unique fundamental steady state. Nevertheless, in the next subsection we will see that there exists another non-fundamental steady state, a unique bubbly steady state.

3.2 Bubbly liquidity, current and potential output

Now, the supply of liquidity has two components: bubbles and debt. As Tirole (1985) was the first to prove, bubbles are stable in the long run only if the long run interest rate is equal to the growth rate, $r^b = g$. This is rather intuitive: because the bubbles are not productive, they cannot grow faster than the economy – this cannot constitute an equilibrium. Furthermore, if they grow slower than the economy, they disappear in the long run as a share of output. Thus,

$$r = r^{*,b} = 1 > \underline{r} \tag{44}$$

The bubbly steady state is unique and outside of the liquidity trap, $\xi^b = 0$ and $Y^b = Y^{*,b}$. The size of the bubble in steady-state is just enough to equalize the supply and the demand of liquidity when $r^{*,b} = g$,

$$b = \hat{b} \equiv (1-\omega)\frac{\beta}{1+\beta}\frac{\eta}{1+\eta}(1-\alpha) - \phi\alpha \tag{45}$$

Here, $\hat{b}$ is the maximal sustainable stock of bubbles as a share of output. If $b_t > \hat{b}$ at some date t, (31) implies that the bubbles grow infinitely big. In fact, $\hat{b}$ equalizes the interest rate to the growth rate not only in the long run, but at each period. If the financial frictions are severe, a bigger stock of bubbles can be sustained because the fundamental supply of liquidity (the demand) is very low (high). Note that $\phi\alpha$ is the supply of inside liquidity as a share of output: a higher supply of fundamental liquidity crowds out one-for-one the supply of bubbly liquidity.

Lemma 1 *[Existence of bubbles] Bubbles are possible, $\hat{b} > 0$, if and only if the long run natural interest rate is negative in the fundamental steady-state, $r^* < 1$.*

On the contrary to the standard model of Tirole (1985), a general shortage of assets is not a necessary condition for bubbles to exist. If the liquidity premium is sufficiently high, that is, if the liquid assets shortage is pronounced, the interest rate is negative in an otherwise dynamically efficient economy. In the bubbly economy, bubbles absorb the excess of savings over investment, and not the output gap as in the fundamental economy. The capital-labor ratio depends on the severity of the financial frictions through the size of the bubble, $\hat{b}$,

$$\kappa^b = \left(\frac{\beta}{1+\beta} \frac{\eta}{1+\eta}(1-\alpha) - \hat{b} \right)^{\frac{1}{1-\alpha}} \tag{46}$$

As the savers have more assets available, they buy less inside liquidity and put some of their wealth in bubbles. This represents a negative shock on the supply of productive savings, which tends to decrease the capital-labor ratio as well as the stock of capital,

$$K^b = (1-\alpha)^{\frac{1}{\eta}} {\kappa^b}^{\frac{\eta+\alpha}{\eta}} \tag{47}$$

As a consequence, potential output in the bubbly economy is lower than in the fundamental economy away from the liquidity trap, $Y^{*,b} < Y^*$. However, it is not clear whether potential output in the liquidity trap is higher or lower than in the bubbly economy.

Proposition 2 *[Bubbly steady state] There exists a unique bubbly steady state with $b = \hat{b}$.*

(i) The natural interest rate, $r^{*,b}$, is constant and equal to the growth rate of output.

(ii) The economy is outside of the liquidity trap and output efficient.

(iii) The size of the bubble, $\hat{b}$, is decreasing in ϕ and ω.

(iv) Output, the capital-labor ratio, as well as the stocks of capital and labor are decreasing in the size of the bubble, $\hat{b}$.

When agents coordinate their expectations to create rational bubbles, the supply of liquidity goes up, the natural interest rises and the economy escapes the liquidity trap. The interest rate and output gaps are filled, but at the possible cost of a drop in potential output. Indeed, bubbles divert some productive savings away from capital accumulation. If the shortage of asset is severe, the steady state stock of bubbles is large. If monetary policy is unconstrained at the fundamental steady state, i.e the fundamental steady state is not in the liquidity trap, bubbles have unambiguously negative effects on the stock of capital and output. Things are not so clear in an economy which suffers from an aggregate demand shortage because we do not know which of the supply or demand side effect dominates.

Proposition 3 *[Expansionary bubbles] There exist $r^*_Y(\underline{r})$ and $r^*_K(\underline{r})$ such that,*

(i) Bubbles expand aggregate demand and output if and only if $r^* < r^*_Y(\underline{r})$.

(ii) Bubbles expand the stock of capital and potential output if and only if $r^* < r^*_K(\underline{r})$.

(iii) $r^*_Y(\underline{r})$ and $r^*_K(\underline{r})$ satisfy $r^*_K(\underline{r}) < r^*_Y(\underline{r}) < \underline{r}$ and $r^{*\prime}_X(\underline{r}) > 0, X = K, Y$.

The macroeconomic effects of rational bubbles depend non-linearly on the severity of the fundamental assets shortage:

- If it is benign – with or without binding ZLB, rational bubbles represent a negative shock on the aggregate savings rate: output and potential output drop.
- If it is severe, case (i), bubbles expand current output through a positive effect on the labor supply which overcomes the negative effect on the aggregate savings rate.
- If it is very severe, case (ii), bubbly episodes are positively correlated with both investment and the labor supply.

On the contrary to popular beliefs, the existence of rational bubbles is largely independent on the stance of monetary policy. Their macroeconomic effects, however, crucially depend on whether monetary policy is constrained or not. In the next section, we will see that fiscal policy is the right tool to prevent the formation of rational bubbles. By rising the supply of liquidity available to the savers, public debt can prevent negative interest rates in the first place.

4 Public liquidity, bubbly liquidity and assets shortages

Now, let us introduce a government that issues debt, either in the form of bonds or money. On the contrary to monetary policy, fiscal policy has a direct control of the supply of liquidity available to the savers. Therefore, the government can cure the shortage of liquid assets. Indeed, public liquidity is a perfect substitute for scarce private liquidity. As inside liquidity, public liquidity perfectly crowds out bubbly liquidity: by providing the savers with enough assets, the government can eliminate the bubbly steady-state and push the economy outside of the liquidity trap. This extension also allows to understand the monetary - fiscal nexus in a world with permanently negative natural interest rates. On the contrary to the conventional view (e.g Sargent and Wallace (1981)) an *insufficient* supply of public debt may threaten the ability of the central bank to manage aggregate demand properly.

4.1 Public liquidity and liquidity trap

Assume that the government issues real bonds at each date t, L_{t+1}, which are perfect substitutes for the private bonds issued by the borrowers. The government budget constraint is given by

$$\frac{L_{t+1}}{r_t} = T_t + L_t \tag{48}$$

Here, $T_t^y \equiv \int T_{j,t}^y dj$ is an aggregate lump-sum transfer from the government to the young households. I assume that fiscal policy targets a constant debt-to-GDP ratio, $\bar{l} \geq 0$, at date, i.e $L_{t+1} = \bar{l}Y_{t+1}$. The lump-sum transfers adjust to ensure that the government always meets his per-period budget constraint.

The market clearing condition for bonds, (12), is replaced by

$$L_{t+1} + D_{t+1} = 0 \tag{49}$$

Assuming binding borrowing constraint, (4), we can compute the interest rate consistent with equilibrium in the financial markets,

$$r_t = \min\left\{\frac{\phi\left(R_{t+1}I_t + \Pi_{t+1}\right) + L_{t+1}}{(1-\omega)\frac{\beta}{1+\beta}E_t + \frac{\beta}{1+\beta}T_{L,t}^y - B_t}; R_{t+1}\right\} \tag{50}$$

Here, $T_{L,t}^y \equiv \int(1-\theta_{j,t})T_{j,t}^y dj$ is the aggregate transfer to the savers. Public debt raises the supply of liquidity by providing the savers with another asset, but it also increases their wealth because of the lump-sum transfers that if finances. If we substitute the government budget constraint, (48), and the factors' share in income, (18), in (50),

$$r_t = \frac{\phi\left(\alpha(1-\xi_{t+1}) + \xi_{t+1}\right) + \frac{1+\beta\tau_\omega}{1+\beta}\bar{l}}{(1-\omega)\frac{\beta}{1+\beta}\frac{\eta}{1+\eta}(1-\alpha)(1-\xi_t) - \frac{\beta}{1+\beta}(1-\tau_\omega)\bar{l} - b_t}\frac{Y_{t+1}}{Y_t} \tag{51}$$

Here, $\tau_\omega \equiv 1 - \frac{T_{L,t}}{T_t}$ is the fraction of the aggregate lump-sum transfer that accrues to the investors. The interest rate is an increasing function of the supply of public liquidity, $\bar{l}$. The effects of τ_ω on r_t is non-linear. Indeed, public debt is a way for the government to redistribute wealth both between and within generations. If $r_t < g_t$, public liquidity is a bubble, i.e the government can roll over its debt ad infinitum. However, if $r_t > g_t$, the government must tax the households to make interest payments on its debt. In the first case, $T_t > 0$: a lower τ_ω

tends to increase the wealth of the savers, which raises their demand of liquidity. In the second, $T_t < 0$: a greater τ_ω increases the burden of the tax levied on the savers, which lowers their demand of liquidity.

Given the level of output, public debt crowds out capital as it divers some savings away from capital accumulation,

$$K_{t+1} = I_t = \frac{\beta}{1+\beta}\left(E_t + T_t^y + \frac{\Pi_{t+1}}{R_{t+1}}\right) - \frac{\Pi_{t+1}}{R_{t+1}} - \frac{L_{t+1}}{r_t} - B_t \tag{52}$$

Or equivalently,

$$K_{t+1} = \frac{\frac{\beta}{1+\beta}\frac{\eta}{1+\eta}(1-\alpha)(1-\xi_t) - \frac{\frac{g_{t+1}}{r_t}+\beta}{1+\beta}\bar{l} - b_t}{1 + \frac{1}{1+\beta}\frac{\xi_{t+1}}{\alpha(1-\xi_{t+1})}} Y_t \tag{53}$$

Public debt raises the same trade-off as rational bubbles: from the one hand, a higher supply of public debt stimulate investment and output in the liquidity trap; on the other hand, it decreases capital accumulation and potential output outside of the trap.

Proposition 4 *[Public liquidity and liquidity trap]*

(i) The natural interest rate, r^*, is increasing in $\bar{l}$.

(ii) If $r^* < 1$, the natural interest is increasing in τ_ω. It is decreasing otherwise.

(iii) Public liquidity crowds out bubbly liquidity one-for-one.

(iv) The results of proposition 3 apply to public debt.

A higher stock of public debt implies a lower stock of bubbles in steady state: fiscal policy can eliminate the bubbly steady state by supplying the savers with enough liquidity.

$$b = \hat{b}^{\bar{l}} \equiv (1-\omega)\frac{\beta}{1+\beta}\frac{\eta}{1+\eta}(1-\alpha) - \phi\alpha - \bar{l} \tag{54}$$

Note that one of the big differences between liquid assets and general assets shortage is that the latter are conditional on the distribution of wealth between generations, whereas the former are conditional on the distribution of wealth within generation. A fiscal policy that redistributes wealth from the young savers to the young investors can prevent the economy to fall in a liquidity trap induced by a shortage of liquid assets, without any cost in terms of potential output.

4.2 Money and the cashless limit

As a robustness check, I introduce money in the previous model. On the contrary to most most monetary models, I find that money and bubbles are not perfect substitutes for each other but co-exist. Furthermore, if the central bank follows a Taylor rule which takes the ZLB into account and the government targets some real variables, e.g real debt-to-GDP ratio, money is irrelevant. Both results cannot be understood without carefully analyzing fiscal policy, which is usually not included in monetary models. Since money is a form of government debt, it is implicitly backed by future taxes (Cochrane, 2005), whereas bubbles are not [25]. If the government targets some real values, the composition of its debt between money, nominal or real bonds, does not matter, even at the ZLB – especially at the ZLB.

Assume that the old savers are subject to the following cash-in-advance (CIA) constraint,

$$(1-\theta_{j,t})c^o_{j,t+1} \leq \mathrm{V} M_{j,t+1} \tag{55}$$

Where $\mathrm{V} \in (0,1)$ is the share of old-age consumption that must be purchased with money. If the economy is away from the liquidity trap, the CIA binds for the savers: because money is a dominated asset, they hold as few of it as possible. However, when the economy enters the liquidity trap, money becomes a store of value as any other forms of liquidity: the portfolio of each individual saver is indeterminate.

The central bank still sets the nominal interest rate according to (17), with $\underline{r}^n = 0$ [26]. Given the policy rate, the central bank adjusts the money supply to meet the demand, $M_{t+1} = M^d_t \equiv \int M_{j,t+1} dj$. The government debt is the sum of the money and bonds issued,

$$M_{t+1} + \frac{L_{t+1}}{r_t} = T_t + M_t + L_t \tag{56}$$

Again, the government follows a constant real debt-to-GDP ratio at each period, $M_t + L_t = \bar{l}Y_t$, and the lump-sum transfers adjust to ensure that (56) holds. The supply of real bonds by the government at each period is then,

$$L_t = \bar{l}Y_t - M^d_t \tag{57}$$

[25] The government *can* engineer a rational bubble by rolling over its debt ad infinitum, but does not necessarily *want* to do so.

[26] With a CIA, households would never choose to hold debt if $r^n_t < \underline{r}^n = 0$. Empirically, this is not observed.

In most monetary models, the central bank sets a path for the money supply, $\{M_t\}_{t=0}^{\infty}$, the government a path for the bonds and taxes, $\{L_t, T_t\}_{t=0}^{\infty}$, the only restriction on policies being that (56) holds. This is not, however, how monetary policy is implemented in modern economies. To paraphrase Sargent (1989), since the central bank does not have the power to levy taxes, its only role is to manage the portfolio of debts of the government, i.e it changes the *composition* of government debt – money versus bonds, but *not its level*.

The equilibrium condition on the assets markets is still given by (49), where the the supply of public bonds is now given by (57). The interest rate is increasing in the demand of money,

$$r_t = \frac{\phi\left(\alpha(1-\xi_{t+1})+\xi_{t+1}\right)+\frac{1+\beta\tau_\omega}{1+\beta}\left(\bar{l}+r_t m_{t+1}\right)+b_{t+1}}{(1-\omega)\frac{\beta}{1+\beta}\frac{\eta}{1+\eta}(1-\alpha)(1-\xi_t)-\frac{\beta}{1+\beta}(1-\tau_\omega)\bar{l}}\frac{Y_{t+1}}{Y_t} \tag{58}$$

Here, $m_{t+1} \equiv \frac{M_{t+1}}{Y_{t+1}}$ is the stock of money as a share of output. The savers hold some of their wealth in the form of money, which reduces their demand of liquidity. Note however that the maximum bubble is still given by (50), because in steady state $r^b = 1$. As public bonds, money crowds out capital accumulation,

$$K_{t+1} = \frac{\frac{\beta}{1+\beta}\frac{\eta}{1+\eta}(1-\alpha)(1-\xi_t)-\frac{\frac{g_{t+1}}{1+r_t}+\beta}{1+\beta}\left(\bar{l}+r_t m_{t+1}\right)-b_t}{1+\frac{1}{1+\beta}\frac{\xi_{t+1}}{\alpha(1-\xi_{t+1})}}Y_t \tag{59}$$

Two things are to note: first, money and bubbles can co-exist. Intuitively, money is not a "pure" bubble as it is implicitly backed by taxes through the government budget constraint, (54). Second, as V $\rightarrow \infty$ and conditional on the interest rate and fiscal rules, money is irrelevant [27] – especially at the ZLB [28].

5 Inflation, liquidity trap and expansionary bubbles

In the introduction, I mentioned the puzzling behavior of inflation during the housing "boom". Obviously, the basic framework with fixed prices cannot provide any explanation. Now, I relax the assumption of fully rigid prices and consider an economy where firms can re-set their price at each period, however subject to a downward price rigidity constraint that puts a lower limit on the rate of inflation – or deflation. The results are qualitatively similar to

[27] Of course, the supply of money relative to public bonds determines the nominal interest rate. However, it is innocuous to consider the cashless limit.

[28] As Bacchetta et al. (2015) underline, this need not to be true if prices are somewhat flexible and the government has a target in nominal terms.

those obtained with fixed prices, with the novelty that rational bubbles also prevent inflation from falling short of the central bank's target in the liquidity trap.

Monopolists can change their prices at each period, however subject to a downward rigidity constraint: the current price, $P_{v,t}$, can fall at most by a factor $\underline{\pi}(\xi_t)$ with respect to last period's price level, P_{t-1}.

Assumption 2 *[Downward price rigidity]* $\underline{\pi}(\xi) \in [0,1]$, $0 < -\frac{\underline{\pi}'(\xi)}{\underline{\pi}(\xi)} \leq 1$ *for all* $\xi \in [0,1]$.

Prices become more flexible as the output gap becomes larger. Full price flexibility obtains when $\underline{\pi}(\xi) = 0$, and we recover the results from the last sections when prices cannot fall, $\underline{\pi}(\xi) = 1$. The program of a monopolist $v \in [0,1]$ is similar to (9) with a minor modification,

$$\Pi_{v,t} = \max_{P_{v,t},k_{v,t},n_{v,t}} \frac{P_{v,t}}{P_t} F(k_{v,t}, n_{v,t}) - (1-\tau(N_t))\left(R_t k_{v,t} + W_t n_{v,t}\right) - \Gamma_{v,t} \tag{60}$$
$$\text{s.t} \quad F(k_{v,t}, n_{v,t}) \leq \left(\frac{P_{v,t}}{P_t}\right)^{-\epsilon} Y_t, \quad \underline{\pi}(\xi_t) P_{t-1} \leq P_{v,t}$$

The first-order conditions for labor and capital with partial price flexibility are similar to those with fully rigid prices, (18) and (19). Facing a demand shortage, firms try to cut their price until it is equal to their nominal marginal cost,

$$P_{v,t} = \max\{\Lambda_t P_t, \underline{\pi}(\xi_{v,t}) P_{t-1}\} \tag{61}$$

If they are prevented to do so by the binding downward price rigidity constraint in (60), the production is inefficiently low whereas the markup is inefficiently high. As in Caballero and Farhi (2016), the supply side of the economy aggregates into a Philips curve that relates the output gap to the inflation rate,

$$(\pi_t - \underline{\pi}(\xi_t))\xi_t = 0 \tag{62}$$

To understand this Philips curve, note that if we impose symmetry across firms in (61), $P_{v,t} = P_t$ for each $v \in [0,1]$, either $\Lambda_{t+1} = 1$ or $\pi_t = \underline{\pi}(\xi_t)$. In the former case, firms are unconstrained: they set the price equal to the marginal cost and produce the efficient level. In the latter, monopolists are constrained and they adjust their production downward to meet the demand.

Since the inflation rate is endogenous, I replace the monetary rule (17) by a standard Taylor rule which includes the ZLB,

$$r_t^n = \max\{r_t^* \pi^* \left(\frac{\pi_t}{\pi^*}\right)^{\gamma}, \underline{r}^n\} \tag{63}$$

Here, π^* is the inflation target of the central bank, and π_t is the inflation rate. The central bank reacts more than-to-one with inflation, $\gamma > 1$, to ensure the local determinacy of the intended steady state according to the Taylor principle.

Assumption 3 *[Inflation target, nominal rigidities and the ZLB]* $\pi^* > \underline{\pi}(0) > \underline{r}^n$.

Assumption 3 ensures that (i) in any liquidity trap steady state, the ZLB binds; (ii) the steady state is in the liquidity only if the natural interest rate is negative.

Given the stock of capital, K_0, the dynamic of the economy is governed by a four dimensional system in K_t, b_t, ξ_t, π_t, with a predetermined variable, K_t, and three non-predetermined variables, b_t, ξ_t and π_t.

$$1 - \xi_t = \left(\frac{r_t^* \pi_{t+1}}{\max\left\{r_t^* \pi^* \left(\frac{\pi_t}{\pi^*}\right)^{\gamma}, \underline{r}^n\right\}}\right)^{\frac{\alpha+\eta}{1+\eta}} \tag{64}$$

$$0 = \left(\pi_t - \underline{\pi}(\xi_t)\right)\xi_t \tag{65}$$

$$b_{t+1} = \frac{\max\left\{r_t^* \pi^* \left(\frac{\pi_t}{\pi^*}\right)^{\gamma}, \underline{r}^n\right\}}{\pi_{t+1}\left(\frac{1-\xi_{t+1}}{1-\xi_t}\right)^{\frac{1-\alpha}{\alpha+\eta}}\left(\frac{K_{t+1}}{K_t}\right)^{\alpha\frac{1+\eta}{\alpha+\eta}}} b_t \tag{66}$$

$$K_{t+1} = \frac{\frac{\beta}{1+\beta}\frac{\eta}{1+\eta}(1-\alpha)(1-\xi_t) - b_t}{1 + \frac{1}{1+\beta}\frac{\xi_{t+1}}{\alpha(1-\xi_{t+1})}}\left((1-\alpha)(1-\xi_t)\right)^{\frac{1-\alpha}{\alpha+\eta}} K_t^{\alpha\frac{1+\eta}{\alpha+\eta}} \tag{67}$$

Where the natural interest rate is still given by,

$$r_t^* = \frac{\phi\left(\alpha(1-\xi_{t+1}) + \xi_{t+1}\right)}{(1-\omega)\frac{\beta}{1+\beta}\frac{\eta}{1+\eta}(1-\alpha) - b_t}(1-\xi_{t+1})^{\frac{1-\alpha}{\alpha+\eta}}\left(\frac{K_{t+1}}{K_t}\right)^{\alpha\frac{1+\eta}{\alpha+\eta}} \tag{68}$$

A formal analysis of the dynamic can be found in appendix B, but let us make a few remarks. First, given a path for the output gap and the inflation rate, the dynamic of the stocks of capital and bubbles are similar to those of the model with fully rigid prices. This implies that the model is globally indeterminate if the bubbly steady state exists. Furthermore, there is a unique saddle-path that converges to the bubbly steady state, along which the output gap is nil at each period and inflation on target, $\xi_t = 0$ and $\pi_t = \pi^*$ for each t.

But the ZLB and together with nominal rigidities introduce another form of global indeterminacy if the secular stagnation steady state exists [29]. Indeed, if the agents expect aggregate demand to be low in the future, they will expect future prices to be low as well. Given current prices, inflation expectations fall and the the interest rate rises. In turn, a higher interest rate induces investment and aggregate demand to shrink. Facing a demand-shortage, the firms reduce their prices, and the inflation expectations drop further. The central bank can stop this deflationary spiral if it acts fast and strong, i.e if it follows the Taylor principle. However, if the ZLB binds, the interest rate is (asymmetrically) pegged at its lower bound, and expectations about secular stagnation are self-fulfilling .

Proposition 5 *[Inflation target and policy-induced liquidity trap] There is a unique liquidity trap steady state which exists if and only if* $r^* \underline{\pi}(0) < \underline{r}^n$.

(i) It is the unique fundamental steady state if $r^* \pi^* < \underline{r}^n$.

(ii) If $r^* \pi^* > \underline{r}^n$, *there are two other fundamental steady states with* $\xi = 0$, *and* $\pi \in \{\frac{\underline{r}^n}{r^*}, \pi^*\}$.

(iii) In the liquidity trap, the inflation rate is at its lower bound, $\pi^\xi = \underline{\pi}(\hat{\xi}^\pi)$, *and the output gap,* $\hat{\xi}^\pi$, *is bigger than in the economy with rigid prices,* $\underline{\pi}(\hat{\xi}^\pi) < 1 \Rightarrow \hat{\xi}^\pi > \hat{\xi}$.

The extension with an endogenous inflation rate does not question the conclusions relative to the existence of a permanent liquidity trap, which again is possible in case of a severe shortage of liquid assets, but it underlines the possibility of policy-induced convergence to this particular steady state. Indeed, by picking a too low inflation target, the central bank suppresses the two efficient steady states.

A liquidity trap occurs if, at the ZLB, the central bank cannot engineer enough inflation to maintain the interest rate gap nil. In anticipation of an aggregate demand shortage, firms cut their prices. The falling inflation rate that results raises the interest gap, which further reduces aggregate demand. Etc. At one point, the firms cannot reduce their prices any further, and an output gap grows that is bigger than in the economy with fixed prices. Nominal rigidities mitigate this deflationary spiral as they reduce the volatility of the inflation rate in the short run and anchor inflation expectations, which helps to stabilize the economy and may even prevent the liquidity trap in the first place.

[29] Schmitt-Groh and Uribe (2012) study this global indeterminacy in a slightly different model without financial frictions, where the wages, not the prices, are downwardly rigid.

Remark 1 (Price flexibility and the liquidity trap) *As Kocherlakota (2016) notices, there is a discontinuity between a vertical Philips curve, $\underline{\pi}(\xi) = 0$, and a Philips curve that tends to be vertical, $\underline{\pi}(\xi) \to 0$. If prices immediately adjust, the output gap is always nil, $\xi = 0$, even at the ZLB: the liquidity trap has no real effects. However, if there is a vanishingly small amount of price rigidity, $\underline{\pi}(\xi) \to 0$, (i) the liquidity trap steady state always exists and (ii) in the liquidity trap, the output gap becomes arbitrarily large, $\xi \to 1$.*

Remark 2 (Global indeterminacy with inflation) *The possible multiplicity of equilibria is standard in New Keynesian models with a Taylor rule that includes a ZLB (Benhabib et al., 2001). In contrast to the model with fully rigid prices, sunspots can occur that induce the economy to jump from one equilibrium to another: the deflationary secular stagnation steady state can be of arbitrary duration with endogenous inflation. This multiplicity may be desirable because it could potentially explain the divergent long run trajectories of Japan and the US after the bubbles burst (Aruoba et al., 2013).*

Proposition 6 *[Bubbly steady state and inflation] There is a unique bubbly steady state that exists if and only if $r^* < 1$.*

(i) The economy is outside of the liquidity trap, $r^{*,b} = 1 > \frac{\underline{r}^n}{\pi^*}$.

(ii) Inflation is on target, $\pi^b = \pi^*$, and the output gap nil, $\xi^b = 0$.

(iii) The results of proposition 3 apply for $r_X^{\pi}(\underline{r}^n) > r_X(\underline{r}^n), X = Y, K$.

The intuitions for proposition 6 with downward price rigidities are the same as for propositions 2 and 3. Bubbles increase the supply of assets available to the savers, raise the natural interest rate, close the interest rate gap and push the economy out of the liquidity trap. The demand side expansionary effects of rational bubbles are magnified by price flexibility, because of the deflationary spiral that occurs in the fundamental economy. Despite the presence of bubbles, or actually thanks to them, the inflation rate is exactly on target.

6 General, safe or liquid assets shortage?

The macroeconomic effects of assets shortages are not conditional on their type. Indeed, the mechanism can be briefly summarized as follows: because assets are scarce, the natural interest rate turns negative. If the central bank faces the ZLB constraint and the economy features nominal rigidities – e.g sticky prices or wages, the vector of price cannot adjust to ensure equilibrium on all markets simultaneously. This leads to a disequilibrium, or equilibrium with rationing on some particular markets – here, the good market.

However, the policy implications of the various types of assets shortages are vastly heterogeneous. Whereas a general shortage of assets (Kocherlakota (2013), Eggertsson and Mehrotra (2014)) calls for supply side policies to create more investment opportunities and raise the marginal product of capital, other types of assets shortages rather emphasize policies that enhance the supply of very specific classes of assets – safe assets in Caballero and Farhi (2016), liquid assets in the present paper. Indeed, in a general shortage of assets, agents want to save so much that the marginal product of capital turns negative. In a liquid or safe assets shortage, agents want to hold a lot of a particular class of assets which are scarce because of financial frictions that limit their creation. As those assets pay a premium – safety or liquidity – under the stock market returns, a general shortage of assets implies a liquid shortage of assets, but the converse is not true.

The contrast between liquid and safe assets shortages is more subtle. Safe assets shortages as developed in Caballero and Farhi (2016) are an acute form of liquid assets shortage where, because they are infinitely risk-averse, the constrained – "Knightian" – agents do not want to hold any liquid but risky assets. For this reason, stochastic bubbles (see appendix B) do not stimulate aggregate demand and output in the safety trap, while they do in the liquidity trap.

7 Conclusion

I have presented a model of liquidity trap and demand side secular stagnation driven by liquid assets shortages; I also provided a new mechanism to explain expansionary bubbles. As in the standard New Keynesian model, the output gap is proportional to the interest rate gap. If some agents face a shortage of assets, a liquid assets shortage, the natural interest rate turns negative in an otherwise dynamically efficient economy. Because of the binding ZLB and nominal rigidities, the economy enters the liquidity trap and an output gap emerges that restores equilibrium in the financial markets. Outside of the liquidity trap, bubbles decrease output and potential output as they divert some savings away from capital accumulation. However, in the liquidity trap, bubbles are expansionary as they increase the supply of assets available to the constrained agents, raise the natural interest rate and close the interest rate gap.

The predictions of the model are consistent with the broad trends of GDP, investment, interest and inflation rates during two of the recent crises triggered by the bust of financial bubbles, the US in 2007 and Japan in the 90's, as well as during the decade that predate the crash. In particular, the model can explain why: (i) a large financial bubble did not create an economic boom; and (ii) the burst pushed the economy in a liquidity trap of arbitrary duration, with a growing output gap and a falling inflation rate despite a zero nominal interest rate policy.

One obvious limitation of the present model, which it shares with the rest of the literature on rational asset price bubbles, is that it cannot explain how bubbles emerge and crash. Another limitation is that it takes the financial frictions as exogenous. It could be interesting to study the coordination failures at the origin of liquid assets shortages, and how agents can overcome those by creating rational bubbles. I leave an elaboration of these questions for future research.

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Appendix A Stochastic bubbles

Following Weil (1987), I extend the analysis to stochastic bubbles. Specifically, conditional on a positive bubble in period t, the bubble bursts – its price becomes zero – with a probability $1-\lambda \in [0,1)$ in period $t+1$. A standard non-arbitrage argument implies that once they burst, bubbles never recover.

The policy functions of the investors are unchanged because they do not hold any bubbles in equilibrium. Indeed, since I consider constrained equilibria, the marginal buyer of the bubbles must be a saver – it would otherwise lead to a contradiction. Now, since bubbles are risky and savers are risk-averse, bubbles must pay a risk premium over the interest rate. Let $c^{o,f}_{j,t+1}$ be the consumption of an old saver if the bubble burst in $t+1$, and $c^{o,b}_{j,t+1}$ if it does not. The following non-arbitrage condition between the safe bonds and risky bubbles must hold if j is a saver,

$$\left(\lambda u'(c^{o,b}_{j,t+1}) + (1-\lambda)u'(c^{o,f}_{j,t+1})\right) r_t = \lambda u'(c^{o,b}_{j,t+1}) \frac{B_{t+1}}{B_t} \tag{69}$$

The old agents consume the proceed of their savings, thus $c^{o,b}_{j,t+1} = q_{j,t+1}B_{t+1} - d_{j,t+1}$ and $c^{f}_{j,t+1} = -d_{j,t+1}$. Under the assumption of logarithmic preferences, households save a fraction $\frac{\beta}{1+\beta}$ of their net wealth while young, such that $q_{t+1}B_t - \frac{d_{j,t+1}}{r_{t+1}} = \frac{\beta}{1+\beta}e_{j,t}$. We can solve (69) for the demand of bubbly liquidity,

$$B_t q_{j,t+1} = \begin{cases} 0 & \text{if} \quad \theta_{j,t+1} = 0 \\ \frac{\lambda \frac{B_{t+1}}{B_t} - (r_t)}{\frac{B_{t+1}}{B_t} - r_t} \frac{\beta}{1+\beta} e_{j,t} & \text{if} \quad \text{otherwise} \end{cases} \tag{70}$$

The policy functions for debt and capital are still given by (27) and (28). Note that the only difference is the demand of bubbles by the savers. Thus, the equilibrium equations from the basic model are not affected, with the exception of the law of motion of the stock of bubbles. If we aggregate (70) and impose equilibrium on the financial markets (11) and (12),

$$\frac{B_{t+1}}{B_t} = \begin{cases} \frac{(1-\omega)\frac{\beta}{1+\beta}E_t - B_t}{\lambda(1-\omega)\frac{\beta}{1+\beta}E_t - B_t} r_t & \text{with a probability } \lambda \text{ if } B_t > 0 \\ 0 & \text{otherwise} \end{cases} \tag{71}$$

Of course, (71) is conditional on a positive stock of bubbles in period t. Risk-averse savers ask for a liquidity premium to hold rational bubbles if they may burst. As a consequence, the bubbles grow faster than the real interest rate. Again, it is more convenient to study the dynamic of the supply of bubbly liquidity as a share of output.

$$b_{t+1} = \begin{cases} \frac{\phi(\alpha(1-\xi_{t+1})+\xi_{t+1})}{\lambda(1-\omega)\frac{\beta}{1+\beta}\frac{\eta}{1+\eta}(1-\alpha)(1-\xi_t)-b_t} b_t & \text{with a probability } \lambda \text{ if } b_t > 0 \\ 0 & \text{otherwise} \end{cases} \tag{72}$$

In the long run, the maximal sustainable bubble is decreasing in its riskiness. Indeed, since risky bubbles grow faster than the interest rate, the interest rate must be lower than the growth rate of the economy.

$$\hat{b}^s \equiv \lambda(1-\omega)\frac{\beta}{1+\beta}\frac{\eta}{1+\eta}(1-\alpha)(1-\xi) - \phi(\alpha(1-\xi)+\xi) \tag{73}$$

Note that the output gap is not necessarily zero when bubbles are stochastic: if bubbles are very risky, they command a very high risk premium which reduces the long run natural interest rate,

$$r^{*s} = \frac{r^*}{r^* + 1 - \lambda} \tag{74}$$

If $\lambda < 1$, then $r^{*s} < 1 = r^b$. Again, bubbles exist if and only if they increase the assets, which is equivalent to $\lambda > r^*$. If there exist a steady state with risky bubbles, the steady state with safe bubbles also exist, but the converse is not true. The steady state interest rate is given by the monetary rule, (17). If $r^{*s} < \underline{r}$, the bubbly steady state has the secular stagnation property, where the output gap equalizes $r^{*,b}$ to $\underline{r}$,

$$\xi^s = \frac{\frac{\underline{r}}{r^{*,b}} - 1}{\frac{\underline{r}}{r^{*,b}} + \frac{1-\alpha}{\alpha} - \frac{\underline{r}}{\alpha}} \tag{75}$$

Risky and safe bubbles increase the supply of liquidity, raise the interest rate and lower the liquidity premium. However, the former are less likely to exist and also less likely to increase output and the stock of capital. Indeed, the relation between output and the aggregate supply of liquidity is non-linear: when the aggregate supply of liquidity is low, the expansionary effects are limited; when it is high, the expansionary effects are much stronger.

Appendix B Dynamic analysis

B.1 Fundamental dynamic

First, consider the case where the supply of bubbles is equal to zero at each period, $b_t = 0$ for all $t \geq 0$. The dynamic of the stock of capital is given (38). We can substitute (39) in (36) and rearrange to get the dynamic of the output gap.

$$1 - \xi_t = \min\left\{1, \left(\frac{1 + \frac{\xi_{t+1}}{\alpha(1-\xi_{t+1})}}{1 + \frac{\hat{\xi}}{\alpha(1-\hat{\xi})}}\right)^{\frac{\alpha+\eta}{1+\eta}} \left(\frac{K_{t+1}}{K_t}\right)^{\alpha} (1 - \xi_{t+1})\right\} \tag{76}$$

$$K_{t+1} = \frac{1 + \frac{1}{1+\beta}\frac{\hat{\xi}}{\alpha(1-\hat{\xi})}}{1 + \frac{1}{1+\beta}\frac{\xi_{t+1}}{\alpha(1-\xi_{t+1})}} \left(\frac{1-\xi_t}{1-\hat{\xi}}\right)^{\frac{1+\eta}{\alpha+\eta}} \left(\frac{K^{\xi}}{K_t}\right)^{1-\alpha\frac{1+\eta}{\alpha+\eta}} K_t \tag{77}$$

The initial stock of capital, K_0, is given. Let us introduce the following functions defined on $[0, 1]$,

$$D(\xi) = 1 + \frac{\xi}{\alpha(1-\xi)}$$

$$E(\xi) = (1-\xi)\frac{1 + \frac{\xi}{\alpha(1-\xi)}}{1 + \frac{1}{1+\beta}\frac{\xi}{\alpha(1-\xi)}}$$

$$F(\xi) = (1-\xi)D(\xi)^{\frac{\eta}{1+\eta}} E(\xi)^{\frac{\alpha}{1-\alpha}}$$

Assumption 4 *[Capital share in income]* $\alpha < \frac{1}{1+\beta}$

Because $\beta < 1$, a sufficient condition for assumption 4 to hold is $\alpha < 0.5$, which is equivalent to the dynamic efficiency of the model with a fully inelastic labor supply.

Lemma 2 *$D(0) = E(0) = F(0) = 1$, $F(1) = 0$, $D'(\xi) > 0$, $E'(\xi) < 0$ and $\frac{F(\xi)}{D^{\frac{\eta}{1+\eta}}} < 0$ $\forall \xi \in [0, 1]$. $F'(\xi) > 0$ iff $\xi \in [0, \bar{\xi}]$, where $\bar{\xi} \in (0, 1)$ is implicitly defined by*

$$\bar{\xi} = \frac{\eta}{1+\eta} - \frac{\alpha}{1-\alpha}\frac{1 - \frac{\bar{\xi}}{\alpha(1-\bar{\xi})}}{1 + \frac{1}{1+\beta}\frac{\bar{\xi}}{\alpha(1-\bar{\xi})}}$$

Proof.

$$\gamma(\xi) = \frac{1}{1-\xi}\frac{1}{\alpha(1-\xi)+\xi}$$
$$\mu(\xi) = \frac{1}{1-\xi}\frac{\beta}{1+\beta}\frac{1}{\alpha(1-\xi)+\xi}\frac{1}{1+\frac{1}{1+\beta}\frac{\xi}{\alpha(1-\xi)}} - \frac{1}{1-\xi}$$
$$\chi(\xi) = \frac{\eta}{1+\eta}\gamma(\xi) + \frac{\alpha}{1-\alpha}\mu(\xi) - \frac{1}{1-\xi}$$

$D'(\xi) = \gamma(\xi)D(\xi) > 0$, where $\gamma(\xi)$ is strictly increasing and $\min_{x\in[0,1]}\gamma(\xi) = \frac{1}{\alpha}$. $E'(\xi) = \mu(\xi)E(\xi)$, where, under assumption 1, $\mu(\xi)$ is deceasing and and $\max_{x\in[0,1)}(1-\xi)\mu(\xi) = \frac{\beta}{1+\beta}\frac{1}{\alpha} - 1 < 0$. $F'(\xi) = \chi(\xi)F(\xi)$. $\chi(\xi) > 0 \Leftrightarrow \frac{1}{1-\xi}\frac{1}{\alpha(1-\xi)+\xi}\left(\frac{\eta}{1+\eta} - \frac{\alpha(1-\xi)+\xi}{1-\alpha} + \frac{\alpha}{1-\alpha}\frac{\beta}{1+\beta}\frac{1}{1+\frac{1}{1+\beta}\frac{\xi}{\alpha(1-\xi)}}\right) > 0$, where the term between parenthesis is decreasing in ξ. ■

If we substitute (76) into (77), a little algebra reveals that in the liquidity trap, the future stock of capital is a function of the future output gap only,

$$K_{t+1} = \min\left\{\frac{1}{1+\frac{1}{1+\beta}\frac{\xi_{t+1}}{\alpha(1-\xi_{t+1})}}\left(\frac{K^*}{K_t}\right)^{\frac{\eta(1-\alpha)}{\alpha+\eta}}K_t, \left(\frac{E(\xi_{t+1})}{E(\hat{\xi})}\right)^{\frac{\alpha+\eta}{\eta(1-\alpha)}}\left(\frac{1-\xi_{t+1}}{1-\hat{\xi}}\right)^{\frac{1}{\eta}}K^{\xi}\right\} \tag{78}$$

A higher expected output gap lowers capital accumulation through two channels: it decreases the income of the households as well as their savings rate. In turn, if we substitute (78) back in (76), we obtain the dynamic of the output gap,

$$1-\xi_t = \min\left\{1, \left(\frac{F(\xi_{t+1})}{F(\hat{\xi})}\right)^{\frac{\alpha+\eta}{\eta}}\left(\frac{K^{\xi}}{K_t}\right)^{\alpha}(1-\hat{\xi})\right\} \tag{79}$$

The output gap depends on the expected growth rate of output, which is proportional to the growth rate of the stock of capital. Given K_0, every sequence $\{K_{t+1},\xi_t\}_{t=0}^{\infty}$ that satisfies (78) and (79) is an equilibrium with perfect foresight. In the liquidity trap, it is possible to simplify by suppressing K_{t+1}. Indeed, if $\xi_t > 0$ and $t > 0$, K_t is a function of ξ_t only,

$$\xi_t = \max\{0, X_t\}, \quad \text{where} \quad \frac{F(X_t)}{D(X_t)^{\frac{\eta}{1+\eta}}} = \frac{F(\xi_{t+1})}{D(\hat{\xi})^{\frac{\eta}{1+\eta}}} \tag{80}$$

X_t is a "fictive" output gap which is realized if and only if it is positive, since ξ_t cannot take negative values. Let $\tilde{\xi}$ be the the expected output gap for which the current output gap is nil. Given $\hat{\xi}$, it is implicitly defined by

$$F(\tilde{\xi}) = D(\hat{\xi})^{\frac{\eta}{1+\eta}} \tag{81}$$

Either $F(\bar{\xi}) < D(\hat{\xi})^{\frac{\eta}{1+\eta}}$ such that $\tilde{\xi}$, if it exists, does not belong to $[0,1]$: the expected output gap is never big enough to fill the current output gap; or $F(\bar{\xi}) > D(\hat{\xi})^{\frac{\eta}{1+\eta}}$, and equation (81) admits two solutions, $\tilde{\xi}^h \in (\bar{\xi}, 1)$ and $\tilde{\xi}^l \in (\hat{\xi}, \bar{\xi})$: the current output gap is nil if and only if $\xi_{t+1} \in [\tilde{\xi}^l, \tilde{\xi}^h]$. If the equilibrium output gap is small, the current output gap is a U-shaped function of the expected output gap. Indeed, the expected output gap raises the share of rent from the one hand, but on the other hand it lowers the growth rate of output.

Proposition 7 (Convergence to the fundamental steady state) *The fundamental steady state is locally determinate if either:*

(i) $\hat{\xi} \in \{0\} \bigcup [\bar{\xi}, 1)$

Or

(ii) $\frac{1}{\eta} > \frac{1-\alpha}{\alpha}\frac{1+\beta}{2} - 1$

Under (i) or (ii) and given K_0*, there is a unique fundamental equilibrium that converges. Furthermore, there exists* $T \geq 0$ *such that* $\xi_t = 0$ *if* $t < T$*,* $\xi_T > 0$ *and* $\xi_t = \hat{\xi}$ *if* $t > T$*.* $T = 0$ *if and only if* $K_0 \geq K^*$ *and* $T < \infty$ *if and only if* $\hat{\xi} > 0$*.*

Proof. First, we will study the dynamic in the liquidity trap, i.e $t = T + 1 \geq 1$. Because $\xi_T > 0$, the whole dynamic of the economy is governed by equation (...) which has one unique fixed point on $[0,1)$,$\xi = \hat{\xi}$. This fixed point is locally unstable if $|\frac{d\xi_t}{d\xi_{t+1}}|_{\xi_{t+1}=\hat{\xi}} < 1$, where

$$|\frac{d\xi_t}{d\xi_{t+1}}| = \begin{cases} \frac{-\chi(\xi_{t+1})}{\frac{\alpha}{1-\alpha}\mu(\xi_t) - \frac{1}{1-\xi_t}} & \text{if} \quad \xi_{t+1} < \bar{\xi} \\ \frac{\chi(\xi_{t+1})}{\frac{\alpha}{1-\alpha}\mu(\xi_t) - \frac{1}{1-\xi_t}} & \text{otherwise} \end{cases}$$

When $\hat{\xi} > \bar{\xi}$, $|\frac{d\xi_t}{d\xi_{t+1}}|_{\xi_{t+1}=\hat{\xi}} < 1$ is equivalent to $\frac{\eta}{1+\eta}\gamma(\hat{\xi}) > 0$, or $\hat{\xi} < 1$. When $\hat{\xi} < \bar{\xi}$, is equivalent to

$$2\left(\frac{\alpha(1-\hat{\xi}) + \hat{\xi}}{1-\alpha} - \frac{\alpha}{1-\alpha}\frac{\beta}{1+\beta}\frac{1}{1 + \frac{1}{1+\beta}\frac{\hat{\xi}}{\alpha(1-\hat{\xi})}}\right) - \frac{\eta}{1+\eta} > 0 \tag{82}$$

Note that the term in parenthesis is strictly increasing in $\hat{\xi}$. If $\frac{1}{\eta} > \frac{1-\alpha}{\alpha}\frac{1+\beta}{2} - 1$, (82) holds when $\hat{\xi} = 0$. Thus, if $\xi_T > 0$, the equilibrium converges only if $\xi_{T+1} = \hat{\xi}$, which implies $\xi_{T+s} = \hat{\xi}$ and $K_{T+s} = K^{\xi}$, $s \geq 1$. Now, let us study the dynamic in period T and before. The stock of capital in $K_{T-s}, s \geq 0$ is given by (77), which we can iterate backward:

$$\frac{K_{T-s}}{K^{\xi}} = \prod_{t=1}^{T-s} \left(\frac{1 + \frac{1}{1+\beta}\frac{\hat{\xi}}{\alpha(1-\hat{\xi})}}{1 + \frac{1}{1+\beta}\frac{\xi_t}{\alpha(1-\xi_t)}} \left(\frac{1-\xi_{t-1}}{1-\hat{\xi}} \right)^{\frac{1+\eta}{\alpha+\eta}} \right)^{\left(\alpha\frac{1+\eta}{\alpha+\eta}\right)^{T-s+1-t}} \left(\frac{K_0}{K^{\xi}} \right)^{\left(\alpha\frac{1+\eta}{\alpha+\eta}\right)^{T-s}}$$

We know that if the economy enters the liquidity trap in period T, i.e $\xi_{T-s} = 0, s \geq 1$ and $\xi_T > 0$, the equilibrium is convergent iff $\xi_{T+s} = \hat{\xi}, s \geq 1$. Thus, T must satisfy:

$$\left(\prod_{t=1}^{T} \left(\left(1 + \frac{1}{1+\beta}\frac{\hat{\xi}}{\alpha(1-\hat{\xi})} \right) \left(\frac{1}{1-\hat{\xi}} \right)^{\frac{1+\eta}{\alpha+\eta}} \right)^{\left(\alpha\frac{1+\eta}{\alpha+\eta}\right)^{T+1-t}} \left(\frac{K_0}{K^{\xi}} \right)^{\left(\alpha\frac{1+\eta}{\alpha+\eta}\right)^{T}} \right)^{\alpha} > 1 - \hat{\xi}$$

And

$$\left(\prod_{t=1}^{T-s} \left(\left(1 + \frac{1}{1+\beta}\frac{\hat{\xi}}{\alpha(1-\hat{\xi})} \right) \left(\frac{1}{1-\hat{\xi}} \right)^{\frac{1+\eta}{\alpha+\eta}} \right)^{\left(\alpha\frac{1+\eta}{\alpha+\eta}\right)^{T-s+1-t}} \left(\frac{K_0}{K^{\xi}} \right)^{\left(\alpha\frac{1+\eta}{\alpha+\eta}\right)^{T-s}} \right)^{\alpha} < 1 - \hat{\xi}, s \geq 1$$

The first condition ensures that $\xi_T > 0$, whereas the second ensures that $\xi_{T-s} < 0, s \geq 1$. Since $\alpha\frac{1+\eta}{\alpha+\eta} < 1$ and $\hat{\xi} \in [0, 1)$, $T \to \infty$ iff $\hat{\xi} = 0$. Furthermore, $T > 0$ iff $K_0 < K^{\xi}$. ■

If the steady state is outside of the liquidity trap, secular stagnation is nevertheless possible for an arbitrary but finite duration. Indeed, if the economy must disinvest because it has accumulated too much capital in the past, households expect GDP to fall. Thus, they have strong incentives to save today, which can induce a temporary shortage of assets. This is the focus of Rognlie et al. (2015) in a model with two sectors and two types of capital.

B.2 Bubbly dynamic

Now, I consider the global dynamic when the economy is always outside of the liquidity trap, i.e $r^*_{t+1} \geq \underline{r}$ at each date t, which implies that $\xi_t = \xi = 0$.

$$K_{t+1} = \left(\frac{1}{1+\beta}\frac{\eta}{1+\eta} - b_t\right)(1-\alpha)^{\frac{1+\eta}{\alpha+\eta}} K_t^{\alpha\frac{1+\eta}{\alpha+\eta}} \tag{83}$$

$$b_{t+1} = \frac{\phi\alpha}{(1-\omega)\frac{1}{1+\beta}\frac{\eta}{1+\eta}(1-\alpha) - b_t} b_t \tag{84}$$

Note that the dynamic of the stock of bubbles as a share of output does not depend on the stock of capital. We can rewrite (84) as:

$$b_{t+1} = b_t + \frac{\hat{b} - b_t}{\phi\alpha + \hat{b} - b_t} b_t \tag{85}$$

Every $b_0 \in (0, \hat{b}]$ is a bubbly equilibrium: the model displays global indeterminacy. The economy converges to the bubbly steady state if and only if $b_0 = \hat{b}_0$: there is unique path which converges to the bubbly-state. However, there are a continuum of equilibria indexed by $b_0 \in (0, \hat{b})$ that converge to the fundamental steady state. Despite the local determinacy of both steady states, the coexistence of those steady states makes the dynamic globally indeterminate.

B.3 The economy with partially flexible prices

B.4 Fundamental dynamic

First, consider the case where the supply of bubbles is equal to zero at each period, $b_t = 0$ for all $t \geq 0$.

$$1 - \xi_t = \left(\frac{r_t^* \pi_{t+1}}{\max\left\{r_t^* \pi^* \left(\frac{\pi_t}{\pi^*}\right)^\gamma, \underline{r}^n\right\}}\right)^{\frac{\alpha+\eta}{1+\eta}} \tag{86}$$

$$0 = (\pi_t - \underline{\pi}(\xi_t)\,\xi_t \tag{87}$$

$$K_{t+1} = \frac{\frac{\beta}{1+\beta}\frac{\eta}{1+\eta}(1-\alpha)(1-\xi_t) - b_t}{1 + \frac{1}{1+\beta}\frac{\xi_{t+1}}{\alpha(1-\xi_{t+1})}} \left((1-\alpha)(1-\xi_t)\right)^{\frac{1-\alpha}{\alpha+\eta}} K_t^{\alpha\frac{1+\eta}{\alpha+\eta}} \tag{88}$$

Where the natural interest rate is still given by,

$$r_t^* = \frac{\phi\left(\alpha(1-\xi_{t+1}) + \xi_{t+1}\right)}{(1-\omega)\frac{\beta}{1+\beta}\frac{\eta}{1+\eta}(1-\alpha)}(1-\xi_{t+1})^{\frac{1-\alpha}{\alpha+\eta}} \left(\frac{K_{t+1}}{K_t}\right)^{\alpha\frac{1+\eta}{\alpha+\eta}} \tag{89}$$

Assumption 5 *[Intended and secular stagnation equilibria]* $\pi^* > \frac{\underline{r}^n}{r_t^*} > \underline{\pi}(0)$ *for all* $t \geq 0$.

Assumption 5 implies that (i) the inflation target can be implemented as an equilibrium; (ii) the liquidity trap steady state exists. Together with assumption 3, this means that we are considering an environment with negative interest rates.

Proposition 8 (Convergence to a fundamental steady state) *The intended steady state is locally determinate, the unintended steady state is locally indeterminate, and the liquidity trap steady state is locally determinate if either:*

(i) $\hat{\xi}^\pi \in \{0\} \bigcup [\bar{\xi}^\pi, 1)$*, where* $\bar{\xi}^\pi < \bar{\xi}$

Or

(ii) $\frac{1}{\eta} > \frac{1-\alpha}{\alpha}\frac{1+\beta}{2} - 1$

Given K_0*, if the economy converges to the liquidity trap steady state,* $\xi_t > 0$ *and* $\pi_t = \underline{\pi}(\xi_t)$ *for all* $t \geq 0$*. If it converges to the intended steady state,* $\xi_t = 0$ *and* $\pi_t = \pi^*$ *for all* $t \geq 0$*.*

Proof. Determinacy of the intended steady state: Take any arbitrary period t. We will show that if $\pi_t < (>)\pi^*$, $\pi_{t+s} < (>)\pi^*$ for all $s \geq 1$. If $\pi_t > \pi^*$, assumption 5 implies that $\xi_t = 0$ and the ZLB does not bind. Thus, $\pi_{t+1} = \pi^* \left(\frac{\pi_t}{\pi^*}\right)^\gamma > \pi_t > \pi^*$. In the long run, $\lim_{T\to\infty} \pi_T = +\infty$. If $\pi_t < \pi^*$, either $\pi_t = \underline{\pi}(\xi_t)$ and $\xi_t > 0$ or $\pi_t \geq \underline{\pi}(0)$ and $\xi_t = 0$. In the first case, $\pi_{t+1} = (1-\xi_t)^{\frac{1+\eta}{\alpha+\eta}} \frac{r^n}{r_t^*}$. In the second case, $\pi_{t+1} = \max\left\{\pi^*\left(\frac{\pi_t}{\pi^*}\right)^\gamma, \frac{r^n}{r_t^*}\right\} \geq \underline{\pi}(0)$, where the last inequality comes from assumption 3. Thus, $\pi_{t+1} \leq \max\left\{\pi^*\left(\frac{\pi_t}{\pi^*}\right)^\gamma, \frac{r^n}{r_t^*}\right\} < \pi^*$ because $\pi_t < \pi^*$ and assumption 5. Thus, $\pi_t < \pi^* \Rightarrow \pi_{t+1} < \pi^*$. We can iterate for any arbitrary number of periods, $\pi_t < \pi^* \Rightarrow \pi_T < \pi^*$ for any $T \geq 0$.

Determinacy of the liquidity trap steady state: If $\pi_t \geq \underline{\pi}(0)$, $\xi_t = 0$. Thus, $\pi_{t+1} = \max\left\{\pi^*\left(\frac{\pi_t}{\pi^*}\right)^\gamma, \frac{r^n}{r_t^*}\right\} \geq \underline{\pi}(0)$, which implies $\xi_{t+1} = 0$. As $\pi_{t+1} \geq \underline{\pi}(0)$ and $\xi_{t+1} = 0$, $\pi_{t+2} \geq \underline{\pi}(0)$ and $\xi_{t+2} = 0$. Etc. Note that we can rewrite the dynamic of the output gap in a similar fashion as in the economy with rigid prices. Assume that $\xi_t > 0$ for all t. Then, the dynamic of the output gap is given by:

$$1 - \xi_t = \left(\frac{1 + \frac{\xi_{t+1}}{\alpha(1-\xi_{t+1})}}{1 + \frac{\hat{\xi}^\pi}{\alpha(1-\hat{\xi}^\pi)}} \frac{\underline{\pi}(\xi_{t+1})}{\underline{\pi}(\hat{\xi}^\pi)} \right)^{\frac{\alpha+\eta}{1+\eta}} \left(\frac{K_{t+1}}{K_t}\right)^\alpha (1 - \xi_{t+1}) \tag{90}$$

$$K_{t+1} = \frac{1 + \frac{1}{1+\beta}\frac{\hat{\xi}^\pi}{\alpha(1-\hat{\xi}^\pi)}}{1 + \frac{1}{1+\beta}\frac{\xi_{t+1}}{\alpha(1-\xi_{t+1})}} \left(\frac{1-\xi_t}{1-\hat{\xi}^\pi}\right)^{\frac{1+\eta}{\alpha+\eta}} \left(\frac{K^\xi}{K_t}\right)^{1-\alpha\frac{1+\eta}{\alpha+\eta}} K_t \tag{91}$$

Let $D^\pi(\xi) = \underline{\pi}(\xi)D(\xi)$, $E^\pi(\xi) = \underline{\pi}(\xi)E(\xi)$ and $F^\pi(\xi) = (1-\xi)D^\pi(\xi)^{\frac{\eta}{1+\eta}}E^\pi(\xi)^{\frac{\alpha}{1-\alpha}}$. We can rearrange equations (90) and (91) to obtain the dynamic of the output gap:

$$\xi_t = \max\{0, X_t\}, \quad \text{where} \quad \frac{F^\pi(X_t)}{D^\pi(X_t)^{\frac{\eta}{1+\eta}}} = \frac{F^\pi(\xi_{t+1})}{D^\pi(\hat{\xi}^\pi)^{\frac{\eta}{1+\eta}}}$$

Let $\bar{\xi}^\pi$ be implicitly defined by $\chi^\pi(\bar{\xi}^\pi) = 0$.

$$|\frac{d\xi_t}{d\xi_{t+1}}| = \begin{cases} \frac{-\chi^\pi(\xi_{t+1})}{\frac{\alpha}{1-\alpha}\mu^\pi(\xi_t) - \frac{1}{1-\xi_t}} & \text{if} \quad \xi_{t+1} < \bar{\xi}^\pi \\ \frac{\chi^\pi(\xi_{t+1})}{\frac{\alpha}{1-\alpha}\mu^\pi(\xi_t) - \frac{1}{1-\xi_t}} & \text{otherwise} \end{cases}$$

Where $\mu^\pi(\xi) = \mu(\xi) + \frac{\underline{\pi}'(\xi)}{\underline{\pi}(\xi)}$, $\gamma^\pi(\xi) = \gamma(\xi) + \frac{\underline{\pi}'(\xi)}{\underline{\pi}(\xi)}$ and $\chi^\pi(\xi) = \frac{\eta}{1+\eta}\gamma^\pi(\xi) + \frac{\alpha}{1-\alpha}\mu^\pi(\xi) - \frac{1}{1-\xi}$. When $\hat{\xi}^\pi > \bar{\xi}^\xi$, $|\frac{d\xi_t}{d\xi_{t+1}}|_{\xi_{t+1}=\hat{\xi}^\pi} < 1$ is equivalent to $\frac{\eta}{1+\eta}\gamma^\pi(\hat{\xi}) > 0$, which, under assumption 2, holds for all $\hat{\xi}^\pi < 1$. When $\hat{\xi}^\pi < \bar{\xi}^\pi$, it is equivalent to:

$$2\left(\frac{\alpha(1-\hat{\xi}^\pi)+\hat{\xi}^\pi}{1-\alpha} - \frac{\alpha}{1-\alpha}\frac{\beta}{1+\beta}\frac{1}{1+\frac{1}{1+\beta}\frac{\hat{\xi}^\pi}{\alpha(1-\hat{\xi}^\pi)}} - \right) - \frac{\eta}{1+\eta}$$
$$- \left(2\frac{\alpha}{1-\alpha} + \frac{\eta}{1+\eta}\right)(1-\hat{\xi}^\pi)(\alpha(1-\hat{\xi}^\pi)+\hat{\xi}^\pi)\frac{\underline{\pi}'(\hat{\xi}^\pi)}{\underline{\pi}(\hat{\xi}^\pi)} > 0 \quad (92)$$

Because the last term is strictly positive, if (82) holds, then (92) holds as well. ■

Appendix C Omitted proofs

Proof of proposition 3. In equilibrium, we know that $Y = K^{\alpha}N^{1-\alpha}$, where aggregate employment is given by (21) and the aggregate capital stock by (39). Thus, the ratios of output and capital in the fundamental steady state to those in the bubbly steady state are both functions of the ratio of the capital-labor ratios,

$$\frac{Y^{\xi}}{Y^{b}} = (1-\hat{\xi})^{\frac{1}{\eta}} \left(\frac{\kappa^{\xi}}{\kappa^{b}}\right)^{\frac{\alpha(1+\eta)}{\eta}}$$
$$\frac{K^{\xi}}{K^{b}} = (1-\hat{\xi})^{\frac{1}{\eta}} \left(\frac{\kappa^{\xi}}{\kappa^{b}}\right)^{\frac{\alpha+\eta}{\eta}}$$

The capital-labor ratios are given by (38) and (42). If we plug in the output gap, (37), and the bubbles stock as a share of output, (41),

$$\frac{\kappa^{\xi}}{\kappa^{b}} = \left(\frac{\frac{\underline{r}}{r^{*}}+\beta}{1+\beta}\left(\omega+(1-\omega)r^{*}\right)\left(\alpha\frac{\underline{r}}{r*}+(1-\alpha)\right)\right)^{-\frac{1}{1-\alpha}}$$

Thus,

$$\frac{X^{\xi}}{X^{b}} > 1 \Leftrightarrow \frac{\frac{\underline{r}}{r^{*}_{X}}+\beta}{1+\beta}\left(\omega+(1-\omega)r^{*}_{X}\right)\left(\alpha\frac{\underline{r}}{r^{*}_{X}}+(1-\alpha)\right)^{1+\frac{1-\alpha}{c_X}} > 1, X = Y, K$$

Where $c_Y \equiv \alpha(1+\eta) < c_K \equiv \alpha+\eta$. By the implicit function theorem, there exists $r^{*}_{X}(\underline{r})$ such that $\frac{X^{\xi}}{X^{b}} = 1$ holds true, and ${r^{*}_{X}}' > 0$. It is immediate to verify that $r^{*}_{K}(\underline{r}) < r^{*}_{Y}(\underline{r}) < \underline{r}$. ∎

Proof of proposition 4. Points (i) and (ii) are already discussed in the main text. Now, let us prove point (iii). First, we can use (51) to compute the output gap,

$$\hat{\xi}^{l} = \max\left\{0, \frac{\frac{\underline{r}}{r^{*}} - 1 - \left(\frac{\beta}{1+\beta}(1-\tau_{\omega})\underline{r} + \frac{1+\beta\tau_{\omega}}{1+\beta}\right)\frac{\bar{l}}{\phi\alpha}}{\frac{\underline{r}}{r^{*}}+\frac{1-\alpha}{\alpha}}\right\}$$

Which is strictly decreasing in the supply of public liquidity, $\bar{l}$. Next, the capital-labor ratio,

$$\kappa^{l} = \left(\frac{\frac{\beta}{1+\beta}\frac{\eta}{1+\eta}(1-\alpha)(1-\hat{\xi}^{l}) - \frac{\frac{1}{r}+\beta}{1+\beta}\bar{l}}{1+\frac{1}{1+\beta}\frac{\hat{\xi}^{l}}{\alpha(1-\hat{\xi}^{l})}}\right)^{\frac{1}{1-\alpha}}$$

Where the interest is given by the monetary rule, $r = \max\{\underline{r}, r^{*l}\}$, and the natural interest is increasing the supply of public liquidity,

$$r^{*l} = r^* \frac{1 + \frac{1+\beta\tau_\omega}{1+\beta}\frac{\bar{l}}{\phi\alpha}}{1 - \frac{\beta}{1+\beta}(1-\tau_\omega) r^* \frac{\bar{l}}{\phi\alpha}}$$

If we let $\tau_\omega = 1$ and $\bar{l} = \hat{b}$, we recover the results of proposition 3.

■

Proof of proposition 5. If the economy is in the liquidity trap with an output gap of $\hat{\pi}^\pi$, the inflation rate is at its lower bound, $\pi = \underline{\pi}(\hat{\xi}^\pi)$. Under assumption 2, this implies that the ZLB binds, $r^n = \underline{r}^n$. Thus, the output gap solves:

$$\hat{\xi}^\pi = \max\left\{0, \frac{\frac{\underline{r}^n}{r^*\underline{\pi}(\hat{\xi}^\pi)} - 1}{\frac{\underline{r}^n}{r^*\underline{\pi}(\hat{\xi}^\pi)} + \frac{1-\alpha}{\alpha}}\right\}$$

Under assumption 2, this equation has a unique fixed point in $[0, 1)$. Clearly, $\hat{\xi}^\pi$ is greater the more flexible the prices. Furthermore, $\hat{\xi}^\pi = 0$ if $\underline{r}^n < r^*\underline{\pi}(0)$. Now, if the economy is not in the liquidity trap, $\xi = 0$ and the inflation rate solves $r^*\pi = \max\left\{r^*\pi^*\left(\frac{\pi}{\pi^*}\right)^\gamma, \underline{r}^n\right\}$. If the ZLB does not bind, $\pi = \pi^*$; if the ZLB binds, $\pi = \frac{\underline{r}^n}{r^*}$. In turn, the ZLB binds if and only if $\pi < \tilde{\pi} \equiv \left(\frac{\underline{r}^n}{r^*\pi^*}\right)^{\frac{1}{\gamma}}\pi^*$. If $r^*\pi^* < \underline{r}^n$, $\tilde{\pi} \in (\pi^*, \frac{\underline{r}^n}{r^*})$: the liquidity trap steady state, if it exists, is unique. If $r^*\pi^* > \underline{r}^n$, $\tilde{\pi} \in (\frac{\underline{r}^n}{r^*}, \pi^*)$: the economy features two steady states outside of the liquidity trap. ■

Proof of proposition 6. In any bubbly steady state, $b = \hat{b}$ and $r^{*,b} = 1$. Assumption 3 implies that the economy must be outside of the liquidity trap, $\xi^b = 0$. The inflation rate solves $\pi^b = \max\left\{\pi^*\left(\frac{\pi^b}{\pi^*}\right)^\gamma, \underline{r}^n\right\}$: either $\pi^b = \pi^*$ or $\pi^b = \underline{r}^n$. But the second solution leads to a contradiction. Indeed, under assumption 3, $\underline{r}^n < \underline{\pi}(0)$, which contradicts the fact that $\xi^b = 0$. The macroeconomic effects of rational bubbles are enhanced by price flexibility because $\hat{\xi}^\pi > \hat{\xi}$: the proof of point (iii) is similar to the proof of proposition 3. ■