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# A comparison of the GAI model and the Choquet integral with respect to a $k$-ary capacity

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**Abstract**

Two utility models are classically used to represent interaction among criteria: the Choquet integral and the Generalized Additive Independence (GAI) model. We propose a comparison of these models. Looking at their mathematical expression, it seems that the second one is much more general than the first one. The GAI model has been mostly studied in the case where attributes are discrete. We propose an extension of the GAI model to continuous attributes, using the multi-linear interpolation. The values that are interpolated can in fact be interpreted as a $k$-ary capacity, or its extension – called $p$-ary capacity – where $p$ is a vector and $p_i$ is the number of levels attached to criterion $i$. In order to push the comparison further, the Choquet integral with respect to a $p$-ary capacity is generalized to preferences that are not necessarily monotonically increasing or decreasing on the attributes. Then the Choquet integral with respect to a $p$-ary capacity differs from a GAI model only by the type of interpolation model. The Choquet integral is the Lovász extension of a $p$-ary capacity whereas the GAI model is the multi-linear extension of a $p$-ary capacity.

*Keywords:* Multiple criteria analysis; Generalized Additive Independence;

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Choquet integral; interpolation.

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## 1. Introduction

Multiattribute utility theory (MAUT) [1] is a commonly used framework for dealing with decision with multiple criteria, of which the additive utility model is one of its best-known representatives. In the additive utility model, however, the contributions of criteria to the overall utility are added independently, so that it is not possible to represent any interaction effect between the criteria. So far in the MAUT literature, two main models have been proposed that are able to deal with interactive criteria, namely the Choquet integral model [2, 3], and the generalized additive independence (GAI) model [4]. The Choquet integral model is a particular instance of the decomposable model, where marginal utility functions defined on each attribute are aggregated by some aggregation function. It is based on the Choquet integral w.r.t. a capacity (a.k.a. fuzzy measure, nonadditive measure, etc.). Decomposable models are characterized by weak separability, which allows to induce from the preference relation of the decision maker on alternatives a preference relation on the values taken by each attribute. In particular, weak separability entails that a preference between two values of an attribute is unconditional of the value taken by the other attributes. By contrast, the GAI model does not necessarily satisfy this condition, so that the well-known menu example, where white wine is preferred to red wine if the main dish is fish, and the converse preference holds in case of meat, can be easily dealt with.

Inspecting the mathematical expression of the GAI model reveals that it is much more general than the Choquet integral, and the above property is an evidence of it. However, we will see that by considering $k$-ary capacities (i.e., capacities considering $k$ different levels, like multichoice games) [5] instead of capacities, the two models are in fact much closer than expected. When attributes are discrete, they even coincide. It is the main aim of the paper to study the exact relation between the two models.

So far, most of the work done on GAI models has been done under the assumption that the attributes are discrete [4, 6, 7, 8, 9]. As this is a limitative assumption, the second aim of the paper is to consider GAI models with continuous attributes. Our approach will be to consider a multilinear interpolation over a discrete set of values for the attributes (called the grid), similarly to what is done in the UTA method [10]. We show that this type of interpolation satisfies reasonable properties (continuity, stability regarding additivity, and an extension property saying that the interpolated model gives back the discrete model on the grid).

In order to make the Choquet integral model be as close as possible to the above continuous GAI model, we introduce $p$-ary capacities ($p \in \mathbb{N}^n$ defines reference levels for each attribute) together with an adequate Choquet integral, based on the Choquet integral w.r.t. $k$-ary capacities and the Choquet integral w.r.t bi-capacities. Our general definition can cope with the case where levels are not necessarily arranged in increasing or decreasing order of preference. As a result, we get a model corresponding to a parsimonious interpolation (like the Lóvasz extension). This general Choquet integral is no more a decomposable model, in the sense that there is no marginal utility functions on attributes. Instead, the attributes are taken modulo the grid of the reference levels, and these values are directly used in the interpolation à la Lóvasz. We show that the interpolation performed satisfies the required properties (continuity, stability regarding additivity, and extension property).

The paper is organized as follows. Section 2 introduces the necessary background on GAI models, the Choquet integral, $k$-ary capacities and bi-capacities. The continuous GAI model is addressed in Section 3, together with interpolation properties. Section 4 defines the general Choquet integral w.r.t $p$-ary capacities and studies its properties. Lastly, it is shown how in the discrete case both models coincide and hence differ only by the interpolation method in the continuous case.

## 2. Background on the GAI model and the Choquet integral

We are given a set of $n$ attributes indexed by $N = \{1, \ldots, n\}$. Each attribute $i \in N$ is represented by a set $X_i$ which is supposed to be an interval $[\underline{b}_i, \overline{b}_i]$ of $\mathbb{R}$. The alternatives are characterized by a value on each attribute, and are thus represented by an element in $X = X_1 \times \cdots \times X_n$. We assume that we are given a preference relation $\succsim$ over $X$. It is supposed to be represented by an overall utility function

$$U : X \to \mathbb{R}, \tag{1}$$

i.e., such that $x \succsim y$ iff $U(x) \geq U(y)$.

For $x, y \in X$ and $A \subseteq N$, we denote by $X_A$ the set $\prod_{i \in A} X_i$, by $x_A$ the restriction of $x$ on attributes $A$, and by $(x_A, y_{N \setminus A}) \in X$ the compound alternative taking value $x_i$ for attribute $i$ in $A$, and value $y_i$ otherwise. We also denote by $(x_A, y_B, z_{N \setminus (A \cup B)})$ the alternative taking value $x_i$ for attribute $i$ in $A$, value $y_i$ for $i$ in $B$, and value $z_i$ otherwise.

Preference relation $\succsim$ is said to satisfy *weak separability* [11] if: for all $i \in N$, all $x_i, y_i \in X_i$, and all $a_{N \setminus \{i\}}, b_{N \setminus \{i\}} \in X_{N \setminus \{i\}}$

$$(x_i, a_{N \setminus \{i\}}) \succsim (y_i, a_{N \setminus \{i\}}) \iff (x_i, b_{N \setminus \{i\}}) \succsim (y_i, b_{N \setminus \{i\}}). \tag{2}$$

Under this assumption, one can derive, for every $i \in N$, a preference relation $\succsim_i$ over attribute $i$ from $\succsim$: for all $x_i, y_i \in X_i$

$$x_i \succsim_i y_i \quad \Leftrightarrow \quad (x_i, a_{N \setminus \{i\}}) \succsim (y_i, a_{N \setminus \{i\}}) \tag{3}$$

for some $a_{N \setminus \{i\}} \in X_{N \setminus \{i\}}$. We denote by $\succ_i$ and $\sim_i$ the asymmetric and symmetric parts of $\succsim_i$ respectively.

Under weak separability, utility $U$ shall fulfil the following monotonicity conditions, which states that it should be consistent with each relation $\succsim_i$:

$$\forall x, y \in X \text{ with } y_i \succsim_i x_i \text{ for every } i \in N, \qquad U(y) \geq U(x) \tag{4}$$

There exist many different utility models of the form (1). In the rest of this section, we focus on two models: ($k$-ary) capacities and the Choquet integral (section 2.1), and the GAI model (section 2.2).

### 2.1. (k-ary) Capacities and the Choquet integral

When $\succsim$ satisfies weak separability and other properties, then function $U$ takes the decomposition form [1]

$$U(x) = F(u_1(x_1), \ldots, u_n(x_n)), \tag{5}$$

where $u_i : X_i \to \mathbb{R}$ is the *utility function* (also called *value function*) on $X_i$ and $F : \mathbb{R}^n \to \mathbb{R}$ is an *aggregation function*. Utility function $u_i$ shall be consistent with $\succsim_i$ (i.e. $u_i(x_i) \geq u_i(y_i)$ whenever $x_i \succsim_i y_i$).

The Choquet integral is one of the most versatile aggregation function as it is able to capture various decision strategies representing interaction among criteria [2, 12, 3]. It is based on a capacity (Section 2.1.1) defined from two reference levels on each criteria. It has been extended by using more reference levels on the criteria, leading to the concept of $k$-ary capacity (Section 2.1.2). With three reference levels, a bi-capacity yields a bipolar approach, where the aggregation is not the same for positive or negative values (Section 2.1.3).

#### 2.1.1. Capacities, the multi-linear extension and the Choquet integral.

**Definition 1.** *A* fuzzy measure *[13] or* capacity *[2] on $N$ is a set function $\mu : 2^N \to \mathbb{R}$ satisfying (1) the monotonicity conditions: $\mu(A) \leq \mu(B)$ for every $A \subseteq B$, and (2) the normalization conditions: $\mu(\emptyset) = 0$, $\mu(N) = 1$.*

Capacities are related to the concept of pseudo-Boolean function. A *pseudo-Boolean function* is any function $f : \{0,1\}^N \to \mathbb{R}$ [14]. Writing $2^N \equiv \{0,1\}^N$, there is a one-to-one correspondence between set functions and pseudo-Boolean functions: $f(1_A) = \mu(A)$ for all $A \subseteq N$. From this correspondence, the problem of defining an aggregation function from a capacity is similar to the one of extending a pseudo-Boolean function on $[0,1]^N$. We note that the two reference levels 0 and 1 correspond to two reference elements $a_i^0$ and $a_i^1$ on each attribute $i \in N$, through the utility function $u_i$:

$$u_i(a_i^0) = 0 \quad \text{and} \quad u_i(a_i^1) = 1. \tag{6}$$

Any pseudo-Boolean function can be written in the *multi-linear form* [15]

$$f(t) = \sum_{A \subseteq N} m^\mu(A) \cdot \prod_{i \in A} t_i \qquad \forall t \in \{0,1\}^N \tag{7}$$

where $m^\mu$ is the Möbius transform [16] of $\mu$ corresponding to $f$, defined by

$$m^\mu(A) = \sum_{B \subseteq A} (-1)^{|A \setminus B|} \mu(B). \tag{8}$$

Conversely, $\mu$ can be derived from $m^\mu$ by the formula $\mu(A) = \sum_{B \subseteq A} m^\mu(B)$. We note that expression (7) could have been replaced by

$$f(t) = \sum_{A \subseteq N} m^\mu(A) \cdot \bigwedge_{i \in A} t_i \qquad \forall t \in \{0,1\}^N. \tag{9}$$

In order to extend $f$ (or equivalently $\mu$) to $[0,1]^N$, one can simply use relations (7) and (9). They are denoted by

$$f^\Pi(t) = \sum_{A \subseteq N} m^\mu(A) \cdot \prod_{i \in A} t_i \tag{10}$$

$$f^\wedge(t) = \sum_{A \subseteq N} m^\mu(A) \cdot \bigwedge_{i \in A} t_i. \tag{11}$$

The first expression is the multi-linear extension of $f$ or $\mu$, and the second one is the *Lovász* extension of $f$ or $\mu$. Yet $f^\wedge$ corresponds to the Choquet integral w.r.t. $\mu$ [2]. For a permutation $\tau$ on $N$, we define $\Omega_\tau = \{t \in \mathbb{R}^N \ : \ t_{\tau(1)} \leq t_{\tau(2)} \leq \cdots \leq t_{\tau(n)}\}$. The Choquet integral of $t \in [0,1]^N$ can be written in terms of the capacity $\mu$ [2]

$$C_\mu(t) = \sum_{i=1}^{n} \left(t_{\tau(i)} - t_{\tau(i-1)}\right) \mu\left(\{\tau(i), \cdots, \tau(n)\}\right), \tag{12}$$

where $t_{\tau(0)} := 0$ and $\tau$ is a permutation on $N$ such that $t \in \Omega_\tau$. The Choquet integral is clearly a simple weighted sum in each domain $\Omega_\tau$. In Figure 1, the shaded area represents $\Omega_\tau$ with $n = 2$, $\tau(1) = 2$ and $\tau(2) = 1$.

*2.1.2. k-ary capacities and Choquet integral.*

The concept of a capacity is based on reference levels $\{0,1\}$ on each criterion. It has been generalized to an arbitrary number of reference levels over each

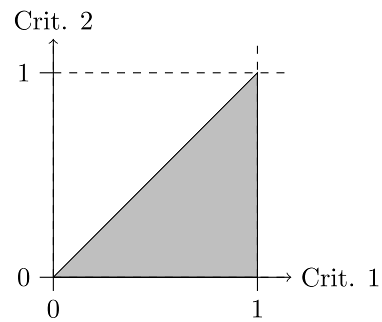


Figure 1: Set of vectors $t = (t_1, t_2)$ such that $t_1 \geq t_2$.

criterion. For $k \in \mathbb{N}_*$, we define

$$\mathcal{Q}_k(N) = \{0, 1, 2, \dots, k\}^N \tag{13}$$

when there are $k+1$ reference levels, and $\leq$ on $\mathcal{Q}_k(N)$ by

$$q \leq q' \quad \text{iff} \quad q_i \leq q'_i \;\; \forall i \in N. \tag{14}$$

We may call $\mathcal{Q}_k(N)$ the *grid* of reference levels. In view of form (6) , the $k+1$ reference levels $\{0, 1, 2, \dots, k\}$ on the criteria correspond to $k+1$ reference elements denoted by $a_i^0, a_i^1, \dots, a_i^k$ on each attribute $X_i$, and the utility functions map the grid of reference levels on the attributes onto the grid of reference levels on the criteria:

$$\forall i \in N \;\; \forall l \in \{0, 1, 2, \dots, k\} \qquad u_i(a_i^l) = l. \tag{15}$$

We can now define $k$-ary capacities, where a usual capacity is a 1-ary capacity.

**Definition 2 ([5, 17]).** *A* $k$-ary capacity *on* $N$ *is a function* $v : \mathcal{Q}_k(N) \to \mathbb{R}$ *satisfying the monotonicity conditions:*

$$\forall q, q' \in \mathcal{Q}_k(N) \text{ s.t. } q \leq q' \quad , \quad v(q) \leq v(q'), \tag{16}$$

*and the normalization conditions:* $v(0, \dots, 0) = 0$, $v(k, \dots, k) = 1$.

A $k$-ary capacity is said to be *non-normalized* if we relax the normalization condition $v(k, \dots, k) = 1$. In the context of cooperative game theory, this concept is similar to *multichoice games* [18].

Let $t \in \Omega := [0,k)^N$. We define $q \in \mathcal{Q}_k(N)$ by $q_i = \lfloor t_i \rfloor$ (the floor integer part of $t_i$) if $t_i < k$, and $q_i = k-1$ if $t_i = k$. We also define a non-normalized capacity given $q$ by

$$\mu_q(S) = v((q+1)_S, q_{-S}) - v(q). \tag{17}$$

Then the Choquet integral w.r.t. $v$ at point $t$ is defined by

$$\mathcal{C}_v(t) = v(q) + C_{\mu_q}(\phi) \tag{18}$$

where

$$\forall i \in N \qquad \phi_i = t_i - q_i \in [0,1]. \tag{19}$$

The Choquet integral w.r.t. a $k$-ary capacity will be called later *$k$-ary Choquet integral* by abuse of language.

Let $\Omega_{q,\tau} = \{t \in [q,q+1]^N,\ t_{\tau(1)} - q_{\tau(1)} \leq \cdots \leq t_{\tau(n)} - q_{\tau(n)}\}$. The Choquet integral w.r.t. $v$ is clearly a simple weighted sum in each domain $\Omega_{q,\tau}$. In Figure 2, the shaded areas are the sets $\Omega_{q,\tau}$ for all values of $q$, with $\tau(2) = 1$ and $\tau(1) = 2$.

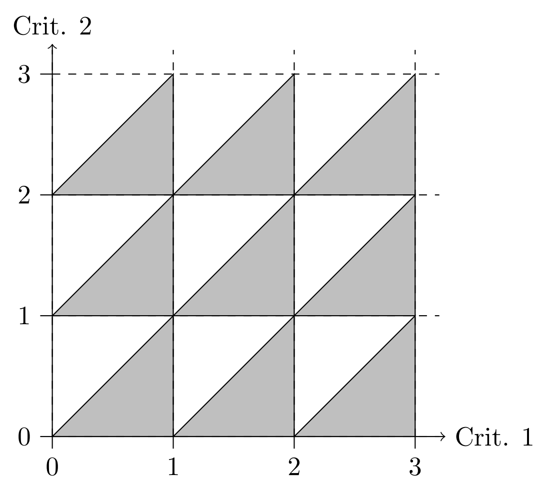


Figure 2: Example with $k = 3$. The shaded triangles represent the areas where $\phi_1(x) \geq \phi_2(x)$.

*2.1.3. Bi-capacity and Choquet integral.*

A bi-capacity is based on 3 reference levels. But contrarily to 2-ary capacities, the middle reference level has a special meaning as it corresponds to the neutral level. The latter demarcating attractive and repulsive elements, it characterizes a bipolar scale [19]. As the neutral level is usually assigned to 0, the three reference levels on the criteria are thus $-1$, 0 and 1. The associated reference elements on the attributes are denoted by $\mathbb{B}_i$ (unsatisfactory element in $X_i$), $\mathbb{O}_i$ (neutral element in $X_i$) and $\mathbb{G}_i$ (satisficing element in $X_i$ [20]) respectively. Thus we have

$$u_i(\mathbb{B}_i) = -1 \quad , \quad u_i(\mathbb{O}_i) = 0 \quad , \quad u_i(\mathbb{G}_i) = 1. \tag{20}$$

Let $\mathcal{Q}(N) = \{(S,T) \in \mathcal{P}(N) \times \mathcal{P}(N) \mid S \cap T = \emptyset\}$. A *bi-capacity* is a function $\nu : \mathcal{Q}(N) \to \mathbb{R}$ satisfying [21]: (i) $S \subseteq S' \Rightarrow \nu(S,T) \leq \nu(S',T)$; (ii) $T \supseteq T'$ $\Rightarrow \nu(S,T) \leq \nu(S,T')$; (iii) $\nu(\emptyset,\emptyset) = 0$, $\nu(N,\emptyset) = 1$, $\nu(\emptyset,N) = -1$.

The first two properties depict increasingness. $\nu(S,T)$ is interpreted as the overall assessment of the ternary act $(1_S, -1_T, 0_{-S\cup T})$ taking value 1 on attributes in $S$, value $-1$ on attributes in $T$ and value 0 on the remaining attributes.

The Choquet integral w.r.t. a bi-capacity has been proposed in [21, 22]. But contrarily to the case of a $k$-ary capacity, it aggregates the values of the alternatives in a "polar" way around the neutral level 0. Let $t \in \mathbb{R}^N$, $N^+ = \{i \in N, t_i \geq 0\}$ and $N^- = N \setminus N^+$. Define the capacity $\mu$, for all $S \subseteq N$, by $\mu(S) := \nu\left(S \cap N^+, S \cap N^-\right)$. Then the Choquet integral w.r.t. $\nu$ is defined by:

$$BC_\nu(t) := C_\mu\left(t_{N^+}, -t_{N^-}\right) \ . \tag{21}$$

Note that this concept is similar to the bipolar Choquet integral [23]. Let $\Omega_{S,\tau} = \{t \in \mathbb{R}_+^S \times \mathbb{R}_-^{N\setminus S} \ : \ |t_{\tau(1)}| \leq |t_{\tau(2)}| \leq \cdots \leq |t_{\tau(n)}|\}$. The Choquet integral w.r.t. a bi-capacity is clearly a simple weighted sum in each domain $\Omega_{S,\tau}$. In Figure 3, the shaded areas represent the four sets $\Omega_{\emptyset,\tau}$, $\Omega_{\{1\},\tau}$, $\Omega_{\{2\},\tau}$, and $\Omega_{\{1,2\},\tau}$, with $n = 2$, $\tau(1) = 2$ and $\tau(2) = 1$.

### *2.2. GAI model*

The use of the decomposable model (5) with a Choquet integral implies that the partial utility functions $u_i$ return evaluations on the same scale. This

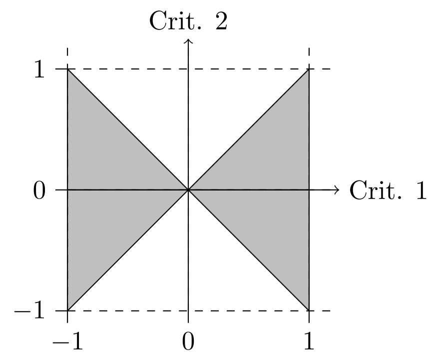


Figure 3: Illustration of $\Omega_{S,\tau}$.

means that if $u_i(x_i) = u_j(x_j)$, then value $x_i$ on attribute $X_i$ has the same satisfaction/attractiveness as value $x_j$ on attribute $X_j$. This strong assumption is called *commensurability*.

There are alternative utility models which do not need the commensurability assumption. The best-known model of this class is the *additive utility* model [1]

$$U(x) = \sum_{i \in N} u_i(x_i) \tag{22}$$

where $u_i : X_i \to \mathbb{R}$. This model has been generalized to allow some interaction among criteria – under the name of the Generalized Additive Independence (GAI) model [4, 24, 6]. The GAI model takes the form of the sum of utilities over subsets of attributes:

$$U(x) = \sum_{S \in \mathcal{S}} u_S(x_S) \tag{23}$$

where $\mathcal{S}$ is a collection of subsets of $N$, $x_S \in X_S$ is the restriction of $x$ over attributes in coalition $S$, and

$$u_S : X_S \to \mathbb{R}.$$

The set $\mathcal{S}$ contains all subsets of attributes that interact one another. Hence the additive model (22) is a particular case of the GAI model where $\mathcal{S}$ is composed of only singletons. One may consider for instance that $\mathcal{S}$ is the collection of all singletons and pairs of criteria, as in the utility model underlying $UTA^{GMS} - INT$ [25], which have strong connections with the GAI model.

It is important to note that the GAI does not necessarily satisfy the weak separability condition [26]. A well-known example of such a violation is the following [27]: the problem is to choose menus in a restaurant on the basis of two attributes $X_1$ (main course: 'meat' or 'fish') and $X_2$ (wine: 'white' or 'red'). Then 'red wine' is preferred to 'white wine' if the main course is 'meat', but 'white wine' is preferred to 'red wine' if the main course is 'fish'.

### *2.3. Comparison for discrete attributes*

Our previous review allows us to make a straightforward comparison of these models in the case where the attributes are discrete, with $X_i = \{a_i^0, \ldots, a_i^{p_i}\}$, $i = 1, \ldots, n$. Let us define a $k$-ary capacity on $N$, with $k = \max_{i \in N} p_i$[1]. Then by (18), for any point $q \in \mathcal{Q}_k(N)$, we have $\mathcal{C}_v(q) = v(q)$. Hence the Choquet integral w.r.t. a $k$-ary capacity $v$ reduces to $v$, if $v$ is defined on the same number of levels as the attributes contain elements.

A GAI model $U : X \to \mathbb{R}$ and a $k$-ary capacity $v : \mathcal{Q}(N) \to \mathbb{R}$ can be made equivalent, thanks to the correspondence $U(x) =: v(\varphi(x))$, where $\varphi$ is given by $(a_1^{j_1}, \ldots, a_n^{j_n}) \mapsto \varphi(a_1^{j_1}, \ldots, a_n^{j_n}) = (j_1, \ldots, j_n)$. The only difference that may appear is the decomposition property (23) of a GAI model $U$ that $v$ does not necessarily fulfill. Due to this clear correspondence, we will focus only on continuous attributes in the rest of this paper.

## 3. GAI model on continuous attributes

It is interesting to note that, so far, the GAI model has never been defined as a general model on continuous attributes. In most of references, the attributes are assumed to take a finite number of values, both in Decision Theory [4, 24] and in AI [6, 7, 8, 9]. In [25], the attributes are not assumed to be discrete,

[1] We will extend in Section 4 the notion of $k$-ary capacity to situations where criteria do not have the same number of levels.

but the authors are interested only in the values of the attributes taken by the alternatives appearing in the training set and the recommendation set (as for the UTA (UTility Additive) approach [10]). Hence they need not make any interpolation. The aim of this section is to propose an interpolation approach of a GAI model, similar to what is done in the UTA approach.

### *3.1. Unknowns of the model*

As in the UTA approach, each attribute is discretized and the unknowns of the model are the values of the utility functions at the discrete elements of $X_i$. For attribute $i \in N$, we consider a finite subset $D_i = \{a_i^0, a_i^1, \ldots, a_i^{p_i}\}$ of $X_i$ with $\underline{b}_i \leq a_i^0 < a_i^1 < \cdots < a_i^{p_i} \leq \overline{b}_i$ (recall that $X_i = [\underline{b}_i, \overline{b}_i]$). The unknowns of the GAI model are $\{u_S(z_S) \ : \ S \in \mathcal{S}\ ,\ z_S \in D_S\}$, where $D_S = \prod_{i\in S} D_i$. The unknowns are denoted by $u_S^D(z_S)$ in order to distinguish them from the general utility model $u_S$ which interpolates $u_S^D$. We also set:

$$\mathcal{U}^D := \{u_S^D(z_S) \ : \ S \in \mathcal{S}\ ,\ z_S \in D_S\}. \tag{24}$$

The number of unknowns is $\sum_{S\in\mathcal{S}} \prod_{i\in S} p_i$.

### *3.2. Condition on the interpolation*

We wish to define an interpolation operator $\mathcal{I}^S$, for any non-empty $S \subseteq N$, which maps a function $f_S^D : D_S \to \mathbb{R}^N$ to its interpolation $f_S : X_S \to \mathbb{R}^N$. Before defining an expression for $\mathcal{I}^S$, we first give some wished properties.

The following property states that $\mathcal{I}^S$ shall be an extension.

**Interpolation**. For all $x \in D_S$, $\mathcal{I}^S\left(f_S^D\right)(x) = f_S^D(x)$.

We note that this is a generalization of the *Properly Weighted* property of the Choquet integral [28].

Continuity is an essential property of the interpolation.

**Continuity**. Function $x \mapsto \mathcal{I}^S\left(f_S^D\right)(x)$ is continuous in $X_S$.

We are interested in interpolating a utility model satisfying the GAI form (23). We must thus perform two operations: interpolation and addition over the subsets in $\mathcal{S}$. The next axiom says that whatever the order in which these two operations are done, the result shall be the same.

**Stability of interpolation regarding additivity**. For all $x \in X$,

$$\sum_{S\in\mathcal{S}} \mathcal{I}^S\left(f_S^D\right)(x_S) = \mathcal{I}^N\left(\sum_{S\in\mathcal{S}} f_S^D\right)(x).$$

*3.3. Interpolation from $\mathcal{U}^D$*

The aim of this section is to determine for every $S \in \mathcal{S}$ the utility function $u_S$ on $X_S$, given $\mathcal{U}^D$.

Let us start with the simple case where $S$ is a single attribute $i$. In order to deduce the value of $u_i$ for all elements of $X_i$ from $u_i^D(a_i^0), \ldots, u_i^D(a_i^{p_i})$, we assume that $u_i$ is piecewise affine, as it is done for the UTA method. Hence we set

$$u_i(x_i) = \begin{cases} u_i^D(a_i^0) & \text{if } x_i \le a_i^0 \\ \frac{x_i - a_i^k}{a_i^{k+1} - a_i^k}\, u_i^D(a_i^{k+1}) + \frac{a_i^{k+1} - x_i}{a_i^{k+1} - a_i^k}\, u_i^D(a_i^k) & \text{if } a_i^k \le x_i \le a_i^{k+1} \\ u_i^D(a_i^{p_i}) & \text{if } x_i \ge a_i^{p_i} \end{cases} \tag{25}$$

When $S$ contains more than one attribute, the idea is to perform a multi-linear interpolation (see (10)). The following set

$$I = \{i \in N \ : \ x_i \in [a_i^0, a_i^{p_i}] \text{ and } x_i \notin D_i\}$$

contains all attributes for which an interpolation is required. If $x_i < a_i^0$, we set $\underline{x}_i = \overline{x}_i = a_i^0$. If $x_i > a_i^{p_i}$, we set $\underline{x}_i = \overline{x}_i = a_i^{p_i}$. Otherwise, we set

$$\underline{x}_i = \max\{z_i \in D_i \ : \ z_i \le x_i\}$$
$$\overline{x}_i = \min\{z_i \in D_i \ : \ z_i \ge x_i\}$$

Note that $\overline{x}_i = \underline{x}_i$ iff $i \in N \setminus I$. We wish to generalize (25) and the multi-linear extension (10). For this reason, $u_S$ will be denoted by $u_S^\Pi$. Function $u_S^\Pi$ shall interpolate $u_S^D$ (see Property **Interpolation**):

$$u_S^\Pi(z_S) = u_S^D(z_S) \quad \forall z_S \in D_S. \tag{26}$$

We write for every $x_S \in X_S$

$$u_S^\Pi(x_S) = \sum_{A\subseteq I\cap S} \left[ \prod_{i\in A} \frac{\overline{x}_i - x_i}{\overline{x}_i - \underline{x}_i} \times \prod_{i\in (I\cap S)\setminus A} \frac{x_i - \underline{x}_i}{\overline{x}_i - \underline{x}_i} \times u_S^D(\underline{x}_A, \overline{x}_{(I\cap S)\setminus A}, x_{S\setminus I}) \right] \tag{27}$$

where $(\underline{x}_A, \overline{x}_{(I\cap S)\setminus A}, x_{S\setminus I})$ is an alternative that is equal to $\underline{x}_k$ if $k \in A$, to $\overline{x}_k$ if $k \in (I \cap S) \setminus A$, and to $x_k$ if $k \in S \setminus I$. The GAI model becomes

$$U^\Pi(x) = \sum_{S\in\mathcal{S}} u_S^\Pi(x_S). \tag{28}$$

Note that the choice of the multi-linear interpolation is motivated by the fact that it is a barycentric interpolation among all extreme points $\prod_{i\in I\cap S}\{\underline{x}_i, \overline{x}_i\}$. Moreover, this is the usual interpolation. Finally it satisfies the *weak difference independence* [29], which is a well-known property in MAUT [1]. This property implies that one can construct interval scales over each attribute independently of the remaining attributes. The scales over each attribute need not be commensurate as expression (27) uses only sums of products (there is no comparison of the values of attributes).

The next lemma shows that the two properties are also fulfilled by expression (26).

**Lemma 1.** *The multi-linear extension satisfies* **Interpolation**, **Continuity** *and* **Stability of interpolation regarding additivity**.

**Proof :** Property **Interpolation** is clearly fulfilled by (26).

On the other hand, we have

$$\begin{aligned}
&\mathcal{I}^N\left(\sum_{S\in\mathcal{S}} u_S^D\right)(x) \\
&= \sum_{A\subseteq I} \left[ \prod_{i\in A} \frac{\overline{x}_i - x_i}{\overline{x}_i - \underline{x}_i} \times \prod_{i\in I\setminus A} \frac{x_i - \underline{x}_i}{\overline{x}_i - \underline{x}_i} \times \left( \sum_{S\in\mathcal{S}} u_S^D(\underline{x}_{A\cap S}, \overline{x}_{(S\cap I)\setminus A}, x_{S\setminus I}) \right) \right] \\
&= \sum_{S\in\mathcal{S}} \sum_{A\subseteq I\cap S} \left[ \prod_{i\in A} \frac{\overline{x}_i - x_i}{\overline{x}_i - \underline{x}_i} \times \prod_{i\in (S\cap I)\setminus A} \frac{x_i - \underline{x}_i}{\overline{x}_i - \underline{x}_i} \; T_S \; u_S^D(\underline{x}_{A\cap S}, \overline{x}_{(S\cap I)\setminus A}, x_{S\setminus I}) \right]
\end{aligned}$$

where

$$T_S = \sum_{B \subseteq I \setminus S} \prod_{i \in B} \frac{\overline{x}_i - x_i}{\overline{x}_i - \underline{x}_i} \times \prod_{i \in I \setminus (S \cup B)} \frac{x_i - \underline{x}_i}{\overline{x}_i - \underline{x}_i} = \prod_{i \in I \setminus S} \left[ \frac{\overline{x}_i - x_i}{\overline{x}_i - \underline{x}_i} + \frac{x_i - \underline{x}_i}{\overline{x}_i - \underline{x}_i} \right] = 1$$

Hence

$$\mathcal{I}^N \left( \sum_{S \in \mathcal{S}} u_S^D \right) (x) = \sum_{S \in \mathcal{S}} \mathcal{I}^S \left( u_S^D \right) (x_S)$$

Hence **Interpolation** is fulfilled.

To show the continuity, we only have to prove continuity w.r.t. an attribute, say $k \in N$. Let then $x, x', x'' \in X$ such that

$$x_i = x'_i = x''_i \quad \forall i \in N \setminus \{k\}$$

$$x'_k < \overline{x}'_k = x_k = \underline{x}''_k < x''_k \quad \text{and} \quad x_k \in D_k$$

Let $S \ni k$ and $I = \{i \in N \ : \ x_i \in [a_i^0, a_i^{p_i}] \text{ and } x_i \notin D_i\}$. We have

$$u_S(x'_S) = \sum_{A \subseteq I \cap S} \left[ \prod_{i \in A} \frac{\overline{x}_i - x_i}{\overline{x}_i - \underline{x}_i} \times \prod_{i \in (I \cap S) \setminus A} \frac{x_i - \underline{x}_i}{\overline{x}_i - \underline{x}_i} \times \frac{x'_k - \underline{x}'_k}{\overline{x}'_k - \underline{x}'_k} \times u_S^D(\underline{x}_A, \overline{x}_{(I \cap S) \setminus A}, \overline{x}'_k, x_{S \setminus (I \cup k)}) \right]$$

$$+ \sum_{A \subseteq I \cap S} \left[ \prod_{i \in A} \frac{\overline{x}_i - x_i}{\overline{x}_i - \underline{x}_i} \times \prod_{i \in (I \cap S) \setminus A} \frac{x_i - \underline{x}_i}{\overline{x}_i - \underline{x}_i} \times \frac{\overline{x}'_k - x'_k}{\overline{x}'_k - \underline{x}'_k} \times u_S^D(\underline{x}_A, \underline{x}'_k, \overline{x}_{(I \cap S) \setminus A}, x_{S \setminus (I \cup k)}) \right]$$

$$\xrightarrow[x'_k \to x_k]{} \sum_{A \subseteq I \cap S} \left[ \prod_{i \in A} \frac{\overline{x}_i - x_i}{\overline{x}_i - \underline{x}_i} \times \prod_{i \in (I \cap S) \setminus A} \frac{x_i - \underline{x}_i}{\overline{x}_i - \underline{x}_i} \times u_S^D(\underline{x}_A, \overline{x}_{(I \cap S) \setminus A}, x_{S \setminus I}) \right] = u_S(x_S)$$

and

$$u_S(x''_S) = \sum_{A \subseteq I \cap S} \left[ \prod_{i \in A} \frac{\overline{x}_i - x_i}{\overline{x}_i - \underline{x}_i} \times \prod_{i \in (I \cap S) \setminus A} \frac{x_i - \underline{x}_i}{\overline{x}_i - \underline{x}_i} \times \frac{x''_k - \underline{x}''_k}{\overline{x}''_k - \underline{x}''_k} \times u_S^D(\underline{x}_A, \overline{x}_{(I \cap S) \setminus A}, \overline{x}''_k, x_{S \setminus (I \cup k)}) \right]$$

$$+ \sum_{A \subseteq I \cap S} \left[ \prod_{i \in A} \frac{\overline{x}_i - x_i}{\overline{x}_i - \underline{x}_i} \times \prod_{i \in (I \cap S) \setminus A} \frac{x_i - \underline{x}_i}{\overline{x}_i - \underline{x}_i} \times \frac{\overline{x}''_k - x''_k}{\overline{x}''_k - \underline{x}''_k} \times u_S^D(\underline{x}_A, \underline{x}''_k, \overline{x}_{(I \cap S) \setminus A}, x_{S \setminus (I \cup k)}) \right]$$

$$\xrightarrow[x''_k \to x_k]{} \sum_{A \subseteq I \cap S} \left[ \prod_{i \in A} \frac{\overline{x}_i - x_i}{\overline{x}_i - \underline{x}_i} \times \prod_{i \in (I \cap S) \setminus A} \frac{x_i - \underline{x}_i}{\overline{x}_i - \underline{x}_i} \times u_S^D(\underline{x}_A, \overline{x}_{(I \cap S) \setminus A}, x_{S \setminus I}) \right] = u_S(x_S)$$

■

**Remark 1.** *Expression (27) can be put into the linear form*

$$u_S(x_S) = \sum_{z_S \in D_S} coef_{S,x_S}(z_S)\, u_S^D(z_S) \tag{29}$$

*where* $coef_{S,x_S}(z_S) := \prod_{i\in A} \frac{\overline{x}_i - x_i}{\overline{x}_i - \underline{x}_i} \times \prod_{i\in(I\cap S)\setminus A} \frac{x_i - \underline{x}_i}{\overline{x}_i - \underline{x}_i}$ *if there exists* $A \subseteq I \cap S$ *s.t.* $z_S = (\underline{x}_A, \overline{x}_{(I\cap S)\setminus A}, x_{S\setminus I})$, *and* $coef_{S,x_S}(z_S) := 0$ *otherwise, are non-negative coefficients. Hence if we are given a training data set composed of pairwise comparisons of alternatives, we can learn unknowns* $\mathcal{U}^D$ *using Linear Programming, as for the UTA method or for value function handling interaction [25].*

## 4. Extension of *k*-ary Choquet integral

In order to make a connection between $k$-ary Choquet integrals and the GAI model (27), we need to generalize the concept of $k$-ary Choquet integral in order to integrate the utility functions.

In the GAI model, the counterpart of the reference levels $\{0, 1, 2, \dots, k\}$ over each criterion are the elements $a_i^0, \dots, a_i^{p_i}$ in $X_i$ that are used to discretize the attribute space. We set $p = (p_1, \dots, p_n)$. We need to generalize the $k$-ary Choquet integral to use the elements on the attribute space that are chosen by the DM, rather than reference points $\{0, 1, 2, \dots, k\}$. When doing so, there are two main issues. First of all, the elements in $D_i$ are not commensurate across the criteria. In other words, there is no reason to say that $a_i^l$ should have the same satisfaction degree than $a_j^l$, for a level $l$. Secondly, the elements in $D_i$ are not necessarily ordered from the worst one to the best one according to the preferences of the DM. In other words, the ordering $\succsim_i$ is not necessarily non-decreasing in $X_i$. We will handle these two difficulties.

As the attributes do not necessarily have the same number of reference elements, equation (13) is generalized as follows:

$$\mathcal{Q}_p(N) = \{0, \dots, p_1\} \times \cdots \times \{0, \dots, p_n\}. \tag{30}$$

Relation $\leq$ on $\mathcal{Q}_p(N)$ is defined by (see (14))

$$q \leq q' \quad \text{iff} \quad a_i^{q_i} \precsim_i a_i^{q'_i} \quad \forall i \in N. \tag{31}$$

We note that $\mathcal{Q}_p(N)$ equipped with $\leq$ is a lattice. Generalizing (16), we introduce the concept of a *p-ary capacity*.

**Definition 3.** *Let $p \in \mathbb{N}^N$. A $p$-*ary capacity* is a function $v : \mathcal{Q}_p(N) \to \mathbb{R}$ such that*

$$\forall q, q' \in \mathcal{Q}_p(N) \ s.t. \ q \leq q' \quad , \quad v(q) \leq v(q'). \tag{32}$$

Note that we do not impose any normalization condition.

We use the same wording for $k$-ary (with $k \in \mathbb{N}$) and $p$-ary (with $p \in \mathbb{N}^N$) capacities. We will avoid any confusion by using for $k$ only scalar values and for $p$ only vector values. The $k$-ary capacities are particular $p$-ary capacities, with $p = (k, \ldots, k)$.

As for decomposable decision models, we assume that the overall preference relation $\succsim$ satisfies weak separability, so that there exists preference relations $\succsim_i$ on the attributes.

### 4.1. *Case where $a_i^0 \precsim_i \cdots \precsim_i a_i^{p_i}$*

We first consider the classical (and simpler) case where the larger the value of the attributes, the better, so that $a_i^0 \precsim_i \cdots \precsim_i a_i^{p_i}$.

In order to use a Choquet integral, we need to "*normalize*" the attributes. Here we do not enforce strong condition such as commensurability. We just need to define a function (utility function) $u_i : X_i \to \mathbb{R}$ such that

$$u_i(a_i^l) = l \quad \forall l \in \{0, \ldots, p_i\}. \tag{33}$$

We consider here the simplest utility function fulfilling this condition. It performs linear interpolation between two successive points $a_i^l$ and $a_i^{l+1}$ in $D_i$, as for (28). Hence we "*normalize*" the attribute in the following way (see Figure 4):

$$u_i(x_i) = \begin{cases} 0 \quad \text{if } x_i \leq a_i^0 \\ \frac{x_i - a_i^l}{a_i^{l+1} - a_i^l}\,(l+1) + \frac{a_i^{l+1} - x_i}{a_i^{l+1} - a_i^l}\,l = l + \frac{x_i - a_i^l}{a_i^{l+1} - a_i^l} \quad \text{if } a_i^l \leq x_i < a_i^{l+1} \\ p_i \quad \text{if } x_i \geq a_i^{p_i} \end{cases} \tag{34}$$

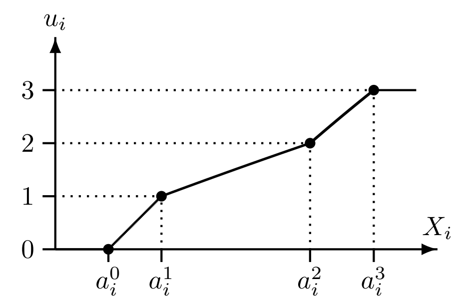


Figure 4: Piecewise affine utility function $u_i$ with $p_i = 3$.

Let $x \in X$. We define $q(x) \in \mathcal{Q}_p(N)$ by

$$q_i(x) = \begin{cases} 0 \quad \text{if } x_i < a_i^1 \\ l \quad \text{if } a_i^l \le x_i < a_i^{l+1} \ (l \in \{1, \dots, p_i - 2\}) \\ p_i - 1 \quad \text{if } x_i \ge a_i^{p_i - 1} \end{cases} \tag{35}$$

for all $i \in N$.

According to this expression, we have $u_i(x_i) \in [q_i(x), q_i(x) + 1]$. An illustration of $q(x)$ can be found in Figure 5 and 6.

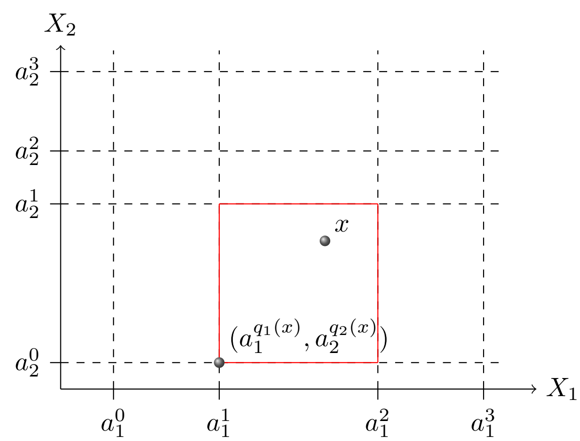


Figure 5: Point $x$ belongs to the square $[a_1^1, a_1^2] \times [a_2^0, a_2^1]$. Hence $q_1(x) = 1$ and $q_2(x) = 0$.

Generalizing (19), we define

$$\forall i \in N \qquad \phi_i(x) = u_i(x_i) - q_i(x) \in [0, 1]. \tag{36}$$

Figure 6 illustrates the concepts of utilities and $\phi_i$.

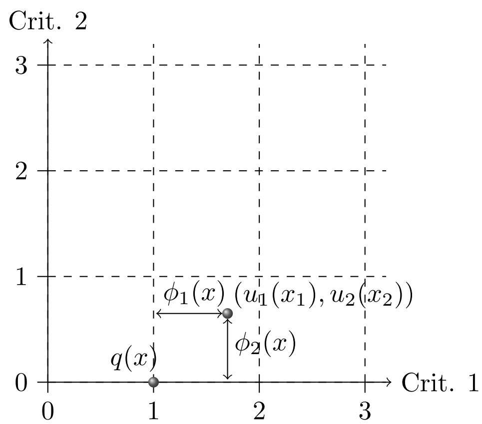


Figure 6: Point $x$ in the space of utilities.

**Remark 2.** *There is a clear connection between the previous definitions and some notions used in signal compression [30, 31]. In this theory,* nearest neighbour quantizer *of a point is the element of the lattice that is the closest to this point. This is close to $q(x)$, which can be defined as the largest element of lattice $\mathcal{Q}_p(N)$ that is smaller than vector $(u_1(x_1), \ldots, u_n(x_n))$. The Voronoi cell of the lattice associated with $q \in \mathcal{Q}_p(N)$ is the set of points for which $q(x) = q$. In our case, the set of point $x$ for which $q(x) = q$ is the hypercube $\times_{i\in N}[a_i^{q_i}, a_i^{q_i+1}]$. Lastly, the* modulo-$\mathcal{Q}_p(N)$ operation w.r.t. the lattice *is defined as the difference between the point and the associated nearest neighbour quantizer. We note that $\phi_i(x)$ is the modulo-$\mathcal{Q}_p(N)$ operation w.r.t. the lattice of $u_i(x_i)$ (and could be denoted by $u_i(x_i) \bmod \mathcal{Q}_p(N)$). The following table compares the concepts used in this paper and in signal compression.*

| *Our definitions* | *Counterpart in signal compression* |
|---|---|
| $\mathcal{Q}_p(N)$ | *lattice* |
| $q(x)$ | *nearest neighbour quantizer* |
| $\times_{i\in N}[a_i^0, a_i^1]$ | *basic Voronoi cell of the lattice* |
| $\times_{i\in N}[a_i^{q_i}, a_i^{q_i+1}]$ | *Voronoi cell associated to q* |
| $\phi_i(x)$ | *modulo-$\mathcal{Q}_p(N)$ operation w.r.t. the lattice* |

Then the Choquet integral w.r.t. $v$ for alternative $x$ is given by (similar to

(18) and (17))

$$U_v^\wedge(x) = C_{\mu_{q(x)}}(\phi(x)) + v(q(x)) \tag{37}$$

where the capacity $\mu_{q(x)}$ is given by, for every $S \subseteq N$

$$\mu_{q(x)}(S) = v((q(x)+1)_S, q_{N\setminus S}(x)) - v(q(x)), \tag{38}$$

where $((q(x)+1)_S, q_{N\setminus S}(x))$ takes value $q_i(x)+1$ if $\in S$ and value $q_i(x)$ otherwise. Note that notation $U_v^\wedge$ comes from the fact that it corresponds to the Lovász extension of $v$.

Clearly, **Interpolation** is fulfilled (see Figure 7):

$$U_v^\wedge(x) = v(q(x)) \qquad \forall x \in D.$$

Moreover if a $p$-ary capacity $v$ on $N$ takes the form $v(A) = \sum_{S\in\mathcal{S}} v_S(A \cap S)$, where $v_S$ is a $p_S$-ary capacity on $S$, then for every $x \in X$

$$U_v^\wedge(x) = \sum_{S\in\mathcal{S}} U_{v_S}^\wedge(x_S).$$

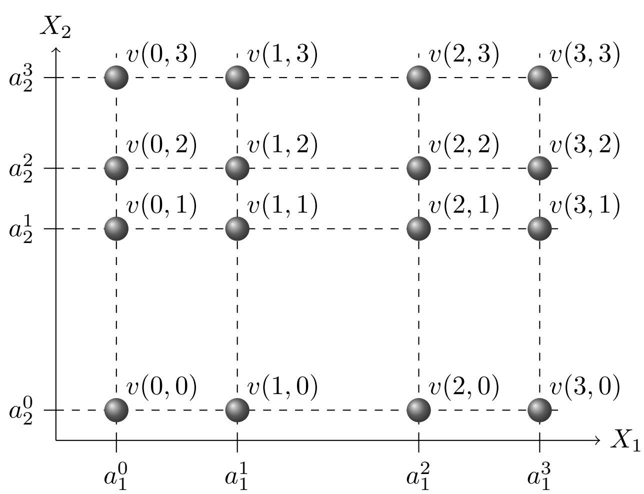


Figure 7: Illustration of the fact that the evaluation through $U_v^\wedge$ at a point $x \in \widehat{X}$ is equal to $v(q(x))$. In other words, $U_v^\wedge(t_1^l, t_2^h) = v(l,h)$.

### 4.2. Case of more general monotonicity conditions

We now consider more general monotonicity condition, e.g. when the preferred element in $X_i$ is not the largest value.

The problem with the previous expression (37) is that it does not discriminate the cases where $\succsim_i$ is locally increasing or decreasing. For instance, in the definition of the Choquet integral w.r.t. a bi-capacity, we take the absolute value of the integrand $t$, see Section 2.1.3. This amounts to reversing the sign of the utilities. We propose to use a similar approach for attributes that are locally decreasing around $x$. Consider $q = q(x)$, see (35). More precisely, for $q \in \mathcal{Q}_p(N)$, we first identify the attributes that are increasing, decreasing or constant in $[a_i^{q_i}, a_i^{q_i+1}]$:

$$A^+(q) = \{i \in N \ , \ a_i^{q_i+1} \succ_i a_i^{q_i}\}$$
$$A^-(q) = \{i \in N \ , \ a_i^{q_i+1} \prec_i a_i^{q_i}\}$$
$$A^=(q) = \{i \in N \ , \ a_i^{q_i+1} \sim_i a_i^{q_i}\}$$

For simplicity, we assume that $A^=(q) = \emptyset$.

We proceed as for the definition of the Choquet integral w.r.t. bi-capacities. For the elements in $\times_{i\in N}[q_i, q_i+1]$, we need to define two reference points $\mathbb{O}(q)$ and $\mathbb{G}(q)$ in $\times_{i\in N}\{q_i, q_i+1\}$ such that $\mathbb{G}_i(q) \succ_i \mathbb{O}_i(q)$ for all $i \in N$. We obtain $\mathbb{O}_i(q) = q_i$ and $\mathbb{G}_i(q) = q_i + 1$ if $i \in A^+(q)$ and $\mathbb{O}_i(q) = q_i + 1$ and $\mathbb{G}_i(q) = q_i$ if $i \in A^-(q)$.

Figure 8 illustrates these concepts, in the square $[a_1^1, a_1^2] \times [a_2^0, a_2^1]$, with $q = (1, 0)$. If the preference is increasing between $a_1^1$ and $a_1^2$ (on attribute $X_1$), and is decreasing between $a_2^0$ and $a_2^1$ (on attribute $X_2$), then $\mathbb{O}(q) = (a_1^1, a_2^1)$ and $\mathbb{G}(q) = (a_1^2, a_2^0)$.

Then the Choquet integral w.r.t. $v$ for alternative $x$ is defined by

$$U_v^\wedge(x) = C_{\mu^B_{q(x)}}(\phi^B(x)) + v(\mathbb{O}(q(x))) \tag{39}$$

where $B$ stands for bipolar and the capacity $\mu^B_{q(x)}$ is given by

$$\mu_q^B(S) = v(\mathbb{G}_S(q), \mathbb{O}_{N\setminus S}(q)) - v(\mathbb{O}(q)) \tag{40}$$

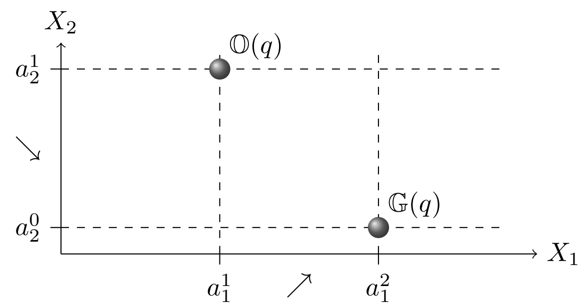


Figure 8: Example of the values of $\mathbb{O}(q)$ and $\mathbb{G}(q)$ with $q = (1, 0)$. The arrows indicate the monotonicity on each attribute: $\nearrow$ (resp. $\searrow$) when increasing (resp. decreasing) in the interval according to $\succsim_i$.

and

$$\forall i \in N \qquad \phi_i^B(x) = \begin{cases} \frac{x_i - a_i^{q_i(x)}}{a_i^{q_i(x)+1} - a_i^{q_i(x)}} & \text{if } i \in A^+(x) \\ \frac{a_i^{q_i(x)+1} - x_i}{a_i^{q_i(x)+1} - a_i^{q_i(x)}} & \text{if } i \in A^-(x) \end{cases} \tag{41}$$

One can readily see that expression (39) collapses to (37) when $\succsim_i$ is strictly increasing. This is why we keep the same notation $U_v^\wedge$ in Sections 4.1 and 4.2. However, there is a major difference between these two relations.

In (37), we explicitly use a utility function $u_i$ defined on $X_i$ from which the local utility $\phi_i(x)$ is defined. Utility function $u_i$ is consistent with $\succsim_i$ in the sense that $u_i(x_i) \geq u_i(y_i)$ iff $x_i \succsim_i y_i$. On the other hand, there is no reference to a utility function $u_i$ in the definition of $\phi^B$. Function $\phi^B$ is defined only locally in each interval $[a_i^{q_i(x)}, a_i^{q_i(x)+1}]$, whereas we have seen that $u_i$ is defined on the whole set $X_i$.

### *4.3. Some properties of the Choquet integral w.r.t. p-ary capacities*

We show some properties for the Choquet integral in this section.

**Lemma 2.** $U_v^\wedge$ *satisfies* **Interpolation** *and* **Continuity**.

The *Interpolation* property was illustrated in Figure 7.

**Proof :** Let us first show that $U_v^\wedge$ satisfies **Continuity**. It is already known that the Choquet integral w.r.t. a usual capacity is continuous. This implies $U_v^\wedge$

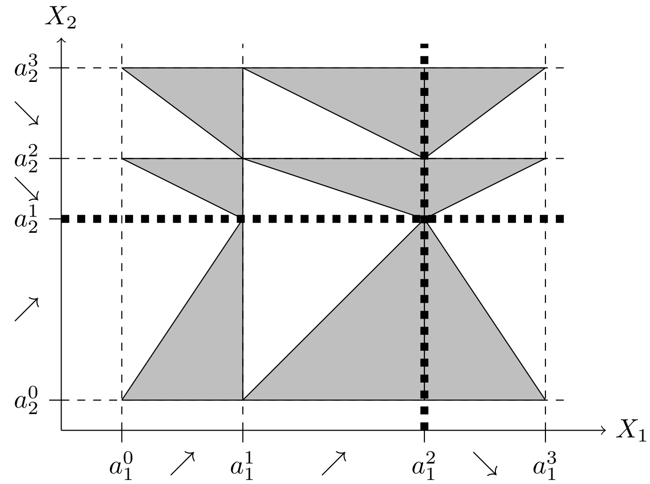


Figure 9: Example with $p_1 = p_2 = 3$. The arrows indicate the monotonicity on each attribute: $\nearrow$ (resp. $\searrow$) when increasing (resp. decreasing) in the interval. Hence $a_1^0 \prec_1 a_1^1 \prec_1 a_1^2 \succ_1 a_1^3$ and $a_2^0 \prec_2 a_2^1 \succ_2 a_2^2 \succ_2 a_2^3$. Accordingly $a_1^2$ and $a_2^1$ are the most preferred values on the two attributes respectively. The shaded parts represent the areas where $\phi_1^B(x) \geq \phi_2^B(x)$.

is continuous in the hypercube $\prod_{i\in N}[a_i^{q_i}, a_i^{q_i+1}]$ for every $q \in \mathcal{Q}_p(N)$. We just need to show that $U_v^{\wedge}$ is continuous across the boundaries of these hypercubes.

Let $x \in X$ such that $J := \{i \in N \ : x_i \in D_i \setminus \{a_i^{p_i}\}\} \neq \emptyset$. We wish to show that $U_v^{\wedge}$ is continuous around $x$. Set $q = q(x)$. Hence $x_i = a_i^{q_i}$ for every $i \in J$. Consider $S \subseteq J$. We define $\widehat{q} \in \mathcal{Q}_p(N)$ by

$$\text{For } i \in S \qquad \widehat{q}_i + 1 = q_i$$
$$\text{For } i \in N \setminus S \qquad \widehat{q}_i = q_i$$

Then $u_i(x_i) = \widehat{q}_i$ if $i \in J \setminus S$ and $u_i(x_i) = \widehat{q}_i + 1$ if $i \in S$. Moreover $\phi_i^B$ depends on $\widehat{q}$. We have

$$\phi_i^B(x) = \begin{cases} 1 & \text{if } i \in (S \cap A^+) \cup ((J \setminus S) \cap A^-) \\ 0 & \text{if } i \in (S \cap A^-) \cup (J \setminus S) \cap A^+) \end{cases}$$

We can compute $U_v^\wedge(x)$ from index vector $\widehat{q}$:

$$
\begin{aligned}
U_v^\wedge(x) &= C_{\mu^B_{\widehat{q}}}(\phi^B_{N\setminus J}(x), 1_{(S\cap A^+)\cup((J\setminus S)\cap A^-)}, 0_{(S\cap A^-)\cup(J\setminus S)\cap A^+)}) + v(\mathbb{O}) \\
&= \sum_{j=1}^{l} \left(\phi^B_{\tau(j)}(x) - \phi^B_{\tau(j-1)}(x)\right) \mu^B_{\widehat{q}} \left(\{\tau(j),\ldots,\tau(l)\} \cup (S\cap A^+) \cup ((J\setminus S)\cap A^-)\right) \\
&+ \left(1 - \phi^B_{\tau(l)}(x)\right) \mu^B_{\widehat{q}} \left((S\cap A^+) \cup ((J\setminus S)\cap A^-)\right) + v(\mathbb{O})
\end{aligned}
$$

where $l = |N \setminus J|$ and $N \setminus J = \{\tau(1), \ldots, \tau(l)\}$, $\phi^B_{\tau(0)}(x) := 0$ and $\phi^B_{\tau(1)}(x) \leq \cdots \leq \phi^B_{\tau(l)}(x)$. The terms appearing in the previous relation are of the form $\mu^B_q(K \cup (S \cap A^+) \cup ((J \setminus S) \cap A^-))$ with $K \subseteq N \setminus J$. We have with $L = K \cup (S \cap A^+) \cup ((J \setminus S) \cap A^-)$

$$
\begin{aligned}
&\mu^B_{\widehat{q}}(L) = v(\mathbb{G}_L(\widehat{q}), \mathbb{O}_{N\setminus L}(\widehat{q})) - v(\mathbb{O}(\widehat{q})) \\
&= v\left(\widehat{q}_{(L\cap A^+)\cup((N\setminus L)\cap A^-)} + 1, \widehat{q}_{(L\cap A^-)\cup((N\setminus L)\cap A^-)}\right) - v(\mathbb{O}(\widehat{q})) \\
&= v\left(\widehat{q}_{S\cup(K\cap A^+)\cup((N\setminus(K\cup J))\cap A^-)} + 1, \widehat{q}_{(J\setminus S)\cup(K\cap A^-)\cup((N\setminus(K\cup J))\cap A^+)}\right) - v(\mathbb{O}(q)) \\
&= v\left(q_J, q_{(K\cap A^+)\cup((N\setminus(K\cup J))\cap A^-)} + 1, q_{(K\cap A^-)\cup((N\setminus(K\cup J))\cap A^+)}\right) - v(\mathbb{O}(q))
\end{aligned}
$$

Hence $\mu^B_{\widehat{q}}(L)$ does not depend on $S$. Therefore $U_v^\wedge(x)$ is also independent of $S$. Hence $U_v^\wedge$ is continuous around $x$.

Taking the previous expressions with $x \in D$, we obtain that $U_v^\wedge(a_1^{q_1}, \ldots, a_n^{q_n}) = v(q)$ for every $q \in \mathcal{Q}_p(N)$. Hence $U_v^\wedge$ satisfies **Interpolation** ■

In this respect, this expression looks like the Lovász extension of the $p$-ary capacity, whereas the GAI model appears as the multilinear extension of the $p$-ary capacity.

The next result shows monotonicity of $U_v^\wedge$.

**Lemma 3.** *For any $i \in N$, $U_v^\wedge$ is monotone relatively to $\succsim_i$.*

**Proof :** It is sufficient to show that $U_v^\wedge$ is monotone w.r.t. each attribute $X_i$ in each interval $[x_i^{q_i}, x_i^{q_i+1}]$.

Assume first that $i \in A^+(q)$. Then $\mathbb{O}_i(q) = q_i$ and $\mathbb{G}_i(q) = q_i + 1$. Hence by (40), $\mu^B_q$ is monotonic w.r.t. $i$ (i.e. $\mu^B_q(S \cup \{i\}) \geq \mu^B_q(S)$). Thus $U_v^\wedge$ is monotone w.r.t. each attribute $X_i$, see (39).

Consider now $i \in A^-(q)$. Then $\mathbb{O}_i(q) = q_i + 1$ and $\mathbb{G}_i(q) = q_i$. Hence $\mu_q^B$ is anti-monotonic w.r.t. $i$ (i.e. $\mu_q^B(S \cup \{i\}) \leq \mu_q^B(S)$). As $\phi_i^B$ is anti-monotone w.r.t. $x_i$ in $[x_i^{q_i}, x_i^{q_i+1}]$ (see (41)), we conclude that $U_v^\wedge$ is monotone w.r.t. each attribute $X_i$. ■

The next lemma shows that Property **Stability of interpolation regarding additivity** is fulfilled.

**Lemma 4.** *If a p-ary capacity $v$ on $N$ takes the form $v(A) = \sum_{S \in \mathcal{S}} v_S(A \cap S)$, where $v_S$ is a $p_S$-ary capacity on $S$, then for every $x \in X$*

$$U_v^\wedge(x) = \sum_{S \in \mathcal{S}} U_{v_S}^\wedge(x_S).$$

**Proof :** Clear as the Choquet integral is linear in the capacity. ■

Finally $U_v^\wedge$ satisfies some compensation property.

**Lemma 5.** *For every $q \in \mathcal{Q}_p(N)$ and every $\lambda \in (0,1)$, we have*

$$U_v^\wedge\Big(\mathbb{O}(q) + (\mathbb{G}(q) - \mathbb{O}(q))\,\lambda\Big) = U_v^\wedge(\mathbb{O}(q)) + \lambda\,\Big(U_v^\wedge(\mathbb{G}(q)) - U_v^\wedge(\mathbb{O}(q))\Big).$$

**Proof :** Let $q \in \mathcal{Q}_p(N)$ and $x = \mathbb{O}(q) + (\mathbb{G}(q) - \mathbb{O}(q))\,\lambda$. By definition of $\mathbb{O}$ and $\mathbb{G}$, we have $\phi_1^B(x) = \cdots = \phi_n^B(x) = \lambda$. Hence

$$U_v^\wedge(x) = C_{\mu^B_{q(x)}}(\lambda, \dots, \lambda) + v(\mathbb{O}(q)) = (v(\mathbb{G}(q)) - v(\mathbb{O}(q)))\,\lambda + v(\mathbb{O}(q))$$

We conclude as $v(\mathbb{O}(q)) = U_v^\wedge(\mathbb{O}(q))$ and $v(\mathbb{G}(q)) = U_v^\wedge(\mathbb{G}(q))$ (see Lemma 2). ■

*4.4. Link between the GAI model and p-ary capacities*

We are now in a position to compare $U^\Pi$ (see (28)) and $U_v^\wedge$ (see (39)).

A GAI model is characterized by a decomposition relation (23). $U^{\Pi}$ is defined from the knowledge of $u_S^D(q_S)$ for all $S \in \mathcal{S}$ and all $q_S \in D_S$. Note that the function $U^D : D \to \mathbb{R}$ defined by

$$U^D(q) = \sum_{S \in \mathcal{S}} u_S^D(q_S)$$

corresponds to a $p$-ary capacity $v$, as already observed in Section 2.3: For every $q \in \mathcal{Q}_p(N)$

$$U^D(a_1^{q_1}, \ldots, a_n^{q_n}) = v(q).$$

According to (26) and Lemma 2 (Property **Interpolation**), $U^{\Pi}$ defined from $U^D$ and $U_v^{\wedge}$ defined from $v$ are identical on $D$: for all $x \in D$

$$U^{\Pi}(x) = U^D(x) = v(q(x)) = U_v^{\wedge}(x).$$

Hence $U^{\Pi}$ and $U_v^{\wedge}$ return the same value on $D$. The only difference between $U^{\Pi}$ and $U_v^{\wedge}$ is that $U^{\Pi}$ is the multi-linear extension of the values $\{U^D(x)\,,\ x \in D\}$ (see (27)), whereas $U_v^{\wedge}$ is the Lovász extension of the values $\{U^D(x)\,,\ x \in D\}$ (see (37)).

## 5. Conclusion

We have proposed a comparison between the Choquet integral and the GAI model. To make this comparison possible, the Choquet integral is taken w.r.t. a $p$-ary capacity, where $p$ is a vector depicting the number of reference elements that are picked-up from each attribute. When the attributes are discrete, there is a clear correspondence between the Choquet integral and the GAI model as they both collapse to a $p$-ary capacity.

As a result, we have focused on the situation where attributes are continuous in this paper. These two models are based on interpolation from values assigned to pre-defined points. The Choquet integral and the GAI model coincide on the grid formed of the reference elements or levels. This restriction corresponds to a $p$-ary capacity. These two models differ then only by the interpolation method that is used.

To obtain this comparison, we extended the GAI model by introducing interpolation and also extended the Choquet integral w.r.t. a $p$-ary capacity to allow more complex monotonicity conditions.

Concerning the GAI model, we introduced a multilinear extension over a discrete set of values for the attributes. This interpolation satisfies three properties: continuity, stability regarding additivity, and an extension property saying that the interpolated model gives back the discrete model on the grid.

The usual model based on a $k$-ary capacity (where $k$ is a scalar) is decomposable, demarcating the marginal utility functions and the aggregation function – namely a Choquet integral. We propose an expression of the $p$-ary Choquet integral which is not decomposable. Instead of having marginal utility functions, the values of attributes modulo the grid of the reference levels are directly used in the interpolation à la Lóvasz. This modulo operation amounts to identifying the cell in the grid where this point belongs to. The restriction of the $p$-ary capacity to this cell returns a usual capacity. The marginal utility functions are then replaced by the distance of $x$ to the cell. Depending on the local monotonicity of the restricted capacity, the interpolation is performed in the same spirit as for a bi-capacity.

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