Non-Stationary Additive Utility and Time Consistency

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Abstract

Within a continuous time life cycle model of consumption and savings, I study the properties of the most general class of additive intertemporal utility functionals. They are not necessarily stationary, and do not necessarily multiplicatively separate a discount factor from "per-period utility". I prove rigorously that time consistency holds if and only if the per-period felicity function is multiplicatively separable in t, the date of decision and in s, the date of consumption, or equivalently, if the Fisherian instantaneous subjective discount rate does not depend on t. The model allows to explain "anomalies in intertemporal choice" even when the agents are time consistent and various empirical regularities. On the other hand, the model allows to characterize mathematically the "effective consumption profile" of naive, time-inconsistent agents.

Code JEL: D91, E21, E71

Key words : time consistency, stationarity of preferences, life cycle theory of consumption and saving, time discounting, anomalies in intertemporal choice, optimal control, aging

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1 Introduction

For a long time, conventional wisdom has considered Samuelson (1937)'s exponential discounting model as the only additive utility model compatible with a time-consistent consumption behavior. This paper studies the most general class of utility functionals that are altogether additively time separable, not necessarily stationary and, nevertheless, imply full intertemporal rationality (time-consistent behavior). It shows that this class is broader than is generally assumed, comprising many simple cases that have never been used, and may be useful to settle some important economic problems such as, for example, the effect of aging on the intertemporal choice of consumption and savings. Thus, it hopes to clarify the notion of time consistency and, in particular, to give a better understanding of the distinction between stationarity and time consistency, which has recently been pointed out by Halevy (2015), and there concrete implications for applied economic modeling.

The notion of stationarity was introduced by Koopmans (1960, 1972), who studied choice between infinite-horizon consumption streams. If, for a given decision date, the preference order between any two consumption streams is not modified when those streams are anticipated or postponed by the same amount of time, then the preference relation fulfills stationarity. Fishburn and Rubinstein (1982) have given a definition of stationarity for choice between single dated outcomes, in which the infinity of the horizon is no longer required. Since Fishburn and Rubinstein (1982)'s settings fit well with common practice in experimental decision theory, their definition is now standard.

Stationarity is a form of weak separability that implies recursive utility (Blackorby et al., 1978). As shown in Koopmans (1960)'s seminal paper, additive separability and stationarity together imply the exponential discounting model. Thus, a first strategy for economists or psychologists dissatisfied with the exponential discounting model is to explore the more general recursive utility model of intertemporal choice that drop additivity and keep only stationarity (Epstein and Hynes, 1983, for example). But, as noted by Fishburn and Rubinstein (1982) "(...) we know of no persuasive argument for stationarity as a psychologically viable assumption." In particular, stationarity as such cannot be considered as an axiom of rationality. It is the reason why, in this article, I will take the opposite strategy to drop stationarity and keep additivity.

The notion of time consistency has been alluded by Ramsey (1928), Samuelson (1937) and Allais (1947) and was formally introduced by Strotz (1956) and clarified by Blackorby et al. (1973). To define this notion properly, it is important to distinguish (1) the *calendar date of decision (planning)* (denoted t) from (2) the *calendar date of the future act of consumption* (denoted s). The agent will be *time consistent* if, in the absence of any new information, the choice of consumption for any future calendar date is independent from the calendar date of decision. Clearly, time consistency is a criteria of rationality and as such is of great importance. Strotz's main claim was that only models that discount instantaneous utility by an exponential function of the algebraic time distance were time consistent.

Since stationarity and time consistency both give foundations to the exponential discounting model, the two notions have been progressively mixed in a sort of *conventional wisdom* that sometimes abusively sums up results which do not have the same domain of validity and have been established using different methodologies. As pointed out for example by Harrison et al. (2005) and Halevy (2015), it is now time to clearly disentangle the two notions. In particular, Halevy (2015) introduces a third notion, *time invariance*.

An intertemporal choice fulfils *time invariance* if the order of preference between two dated consumptions is not modified when the calendar date of decision and the date of both consumptions are postponed by the same amount of time. In other words, *time invariance* means that intertemporal preferences do not change with the calendar date of decision. Halevy (2015) stated the notion in the Fishburn and Rubinstein (1982) framework, but it can be easily generalized to consumption streams. The main theoretical result of Halevy (2015) is that, if intertemporal preferences fulfil any two of the following three properties : *time consistency*, *stationarity*, and *time invariance*, then they will also necessarily fulfil the third one. An immediate corollary of this proposition follows: if intertemporal preferences fulfil any one of the three properties and not any one of the two remaining, then they will also necessarily not fulfil the third one. This means that intertemporal preferences may be time consistent, but not stationary, if they are not time invariant. My aim is to study such intertemporal preferences, characterize the properties of utility functions that represent them, and draw direct implications for applied economic theory in the domain of the intertemporal choice of consumption and savings.

In this paper, I will not use the upstream, axiomatic methodology of Fishburn and Rubinstein (1982) and Halevy (2015), but instead, the downstream "choice-based" methodology which compares the solutions of dynamic programs with different decision dates. It is the methodology used originally by Strotz (1956), and now standard in behavioral macroeconomics, since the pioneering work of Laibson (1994, 1997); O'Donoghue and Rabin (1999, 2001); Gruber and Köszegi (2001); Diamond and Köszegi (2003). It is "choice based" because it not only uses a utility function that represents the preference relation, but also the budgetary constraints that the decision maker faces. As we will see, to study intertemporal choices, because of the dynamic nature of the constraint, past consumption choices will affect future choices even when preferences fulfil a coordinate independence axiom (see Wakker, 1989, for a definition). Of course, on the one hand, this methodology can be criticized as lacking of proper axiomatic foundations. But, on the other hand, I have to point out that it allows to work directly with consumption streams, a much richer object than the binary choice of dated consumption of the Fishburn and Rubinstein settings, and much closer to common practice in macro-economics or financial economics.

However, the purpose of Laibson, relying on the special case of quasi-hyperbolic discounting, was to illustrate the power of time inconsistent preferences to solve many empirical puzzles (Laibson, 1998). My goal is different. I want to characterize the most general properties of intertemporal utility functionals implying time consistency within the additive framework. I want to clarify, once and for all, the boundaries between fully rational intertemporal choice model and time inconsistent ones in this setting. Of course the question is of first importance, because beyond the boundaries of full rationality, the ways to think at welfare economics and economic policy are deeply modified introducing the possibility of *libertarian parternalism* (Thaler and Sunstein, 2003) (see, for example Bernheim and Rangel (2007) or Saint-Paul (2011) for an extended discussion).

Within a typical modern macro model, with random shocks at each period, a rational agent has to change her plan at each period to take into account the new information provided by the realisation of current shocks. That is the essence of Hall (1978) random walk result for life-cycle consumption. On the opposite, in absence of any uncertainty, only a non rational, time inconsistent, agent will change her plan over-time. A model, that introduce time inconsistency in a buffer stock model of savings, characterized by random income and borrowing constraint (see Harris and Laibson, 2001, for a rigourous resolution) may certainly be more realistic and interesting, but it prevent from using the notion of changing plans to dichotomize between time consistent and time inconsistent model. In other words, time inconsistency implies a kind of "endogenous change of choice" over time, so it is important to sterilize any "exogenous sources" that may explain changing choices. That is why it is important to work with a model with no uncertainty.

Moreover, to study time consistency, it is crucial to work with many periods. Discrete time models, with vectors of dated consumption, can rapidly become cumbersome. Continuous time allows to deal with intertemporal consumption profiles, which are just one variable functions of time, a much more tractable mathematical object. In this paper, I will develop a continuous time life-cycle model of consumption and savings, without uncertainty. I will use optimal control to solve the model, instead of the recursive approach of dynamic programming, that is better suited to deal with random shocks[1]. Optimal control allows to obtain directly closed-form solutions, that are easiest to interpret.

I define the most general possible set of intertemporally additive utility functionals, allowing the instantaneous felicity function to vary according to the consumption date and the decision date. Within this set, I search for the special functional forms which are compatible with time consistency. The main result of the paper is that any function of the form $V(c,t) = \int_t^T w(c(s), s)ds$ implies time-consistent choices. Obviously, the exponential discounting model is a special case, the only one that also fulfills stationarity. All the other cases are additive, non-stationary and, nevertheless, time consistent. Renewing with the Fisherian tradition, I define the discount factor as a marginal rate of substitution, and derive a notion of instantaneous subjective discount rate that is altogether simple and insightful, and generalizes all the known definition of the discount rate. I then have all the material to discuss additional restrictions to the intertemporal utility functional which are required to explain some empirical regularities coming from experimental economics, or from the life-cycle model of consumption and savings. In particular, I reinterpret the so called "anomalies in intertemporal choice" as behavioral requirements for the utility function, implying some restrictions on the third derivatives of the instantaneous felicity function. Finally, I fully characterize the general properties of the observable consumption behavior of a naive[2] non-time consistent agent, and illustrate them in a calibrated version of Barro (1999) continuous time model of a quasi-hyperbolic discounting consumer.

The rest of the paper is structured as follows. Section 2 describes the general model, discusses the possible form of the intertemporal utility function, provides examples of some special cases of interest, defines the generalized subjective discount rate and characterizes the general form of the solution of the maximization program. Section 3 gives the formal definition of time consistency and derives the core theorem of the paper. Section 4 reinterprets empirical findings on intertemporal discounting, within the time-consistent framework as behavioral requirements for the utility function. Part 5 characterizes the general mathematical properties of the effective consumption profile for time non-consistent agents. Finally, part 6 concludes.

[1] Each method can be more or less easy to implement, depending on the characteristics of the program to be solved. Dynamic programming works particularly well when the value function of the program is time invariant, a property that requires simultaneously the intertemporal utility functional to be stationary and the time horizon to be infinite. Because I am interested in non-stationary intertemporal utility functionals, finite horizon, and because I neutralized uncertainty except for the length of the horizon, optimal control has a clear advantage.

[2] All over this paper I use the term naive with the meaning given in the time consistency literature starting with Strotz (1956). Following Pollak (1968), "The *naive optimum path* is defined as the path which an individual would follow if, at every decision point, he believed that he could pre-commit his future behaviour".

2 The Model

Time is continuous. Let us denote the "calendar date of decision" by t. A consumer is endowed with an initial capital, $K(t)$, and a planning horizon $T > t$. At every moment, this capital brings interest at a constant rate, r, and can be used to finance consumption.

At date t, the consumer has to decide the level of consumption $c(s)$ for any further date $s \in [t, T]$. Thus s refers to the "calendar date of consumption". If the birthdate of the agent is normalized to zero, $c(s)$ can also be interpreted as the consumption at age s[3]. $\mathscr{C}_t$ will be considered the set of all functions $c : [t, T] \longrightarrow \mathbb{R}^+$ continuous and differentiable.

I assume that the consumer's intertemporal preferences at date t are represented by an additive utility functional,

$$V(c, t) = \int_t^T v(c(s), s, t) ds \tag{1}$$

with v, the "instantaneous felicity function", three times continuously differentiable in $c(s)$ and s, once continuously differentiable in t, and strictly increasing and strictly concave in $c(s)$. I also assume that $v_1(c, s, t) \to -\infty$ when $c \to 0$ (the notation v_1 meaning the partial derivative of the function v according to its first variable, here, c). $T \in \mathbb{R} + \{+\infty\}$ is the "time horizon"[4]. The set of all intertemporal utility functionals of that kind defined on $\mathscr{C}_t \times [t_0, T]$ is denoted $\mathcal{V}_t$, with t_0 the minimal possible value for t.

2.1 Examples

The specification of the utility functional (1) is as general as possible within the additive framework. It means that it includes, as special cases, all the possible continuous additive utility models of intertemporal choice. Among them, some special subclasses and specific semi-parametrical forms can be putted forward to illustrate the great versatility of the general model and connect it with some existing literature or potential future applications.

1. The **stationary exponential discounting model** provided by Samuelson (1937).

$$V(c, t) = \int_t^T e^{-\theta(s-t)} u(c(s)) ds \tag{2}$$

   It is the standard model of intertemporal choice, the workhorse of macroeconomics and financial economics. The instantaneous felicity function multiplicatively separates a discount factor, that depends only on decision and consumption dates, and an instantaneous utility function that depends only on the level of consumption and is time invariant. The discount factor is an exponential that decreases with the date of consumption at constant rate, θ, the instantaneous subjective discount rate. The exponential discounting model is often assume to be the only one that is time consistent.

[3]Although both t and s can be interpreted as age, in the rest of the paper the word age will only be used to refer to the time of consumption.

[4]Allowing the time horizon to be infinite does not change any of the results of this article as long as the improper integral of equation (1) is defined.

2. **Non-Exponential discounting models** have been proposed in the literature, for example Yaari (1964), Harvey (1986), or Harvey and Østerdal (2012), in which the discount function can be any continuous function of s. In those models, the discount rate is just the log derivative according to s of the discount factor. The most general form has been given by Burness (1976):

$$V(c,t) = \int_t^T f(s,t)u(c(s))ds \tag{3}$$

The rise of behavioral economics has emphasized those models as an answer to the empirical rejection of the exponential discounting model. Some insightfull semi-parametrical example can be shown to illustrate the interest of this approach.

2.a. **Hyperbolic discounting**

$$V(c,t) = \int_t^T \frac{s}{t} u(c(s))ds \tag{4}$$

The discount factor is equal to one when s is equal to t and decreases at a decreasing rate to reach zero as s goes to infinity. This model has been experimentally founded by Ainslie (1975), arguing that, following Strotz (1956), it provides empirical evidence of *time inconsistency*. However, I will prove in the next section that this model is fully compatible with time consistency.

2.b. **Quasi-hyperbolic discounting** has been proposed by Phelps and Pollak (1968) and increasingly used in the behavioral literature since Laibson (1997). Formally, it departs from standard exponential discounting by a supplementary weight that is added to present utility against all the future utilities. Barro (1999) proposes a continuous-time version of the model:

$$V(c,s) = \int_t^T e^{-\alpha\cdot(s-t)-\phi(s-t)} u(c(s))ds \tag{5}$$

with $\phi(s-t) = -\beta \exp[-q(s-t)] + \beta$.
The function ϕ represents the *present bias* of the agent. The instantaneous subjective rate of discount is $\alpha + \phi'(s-t)$. When the parameter q is sufficiently high the discount rate decreases rapidly to reach its long-term value α. Alternative transpositions of quasi-hyperbolic discounting in continuous time has been proposed by Harris and Laibson (2013) and Pan et al. (2015).

2.c. **Seasonal discounting.** The notion of discounting is generally associated with impatience and preference for present consumption. But the concept can be generalized to introduce the effect of mood in the consumption decision. This effect can be cyclical, for example it might be correlated with the quantity of daylight the agent is exposed to, which, as every one knows, is seasonal. In this case, she may value more utility of consumption in summer than in winter. The following model has been given by Drouhin (2009) as an example of non-exponential discounting that is time consistent:

$$V(c,t) = \int_t^T \frac{\cos[t]+\delta}{\cos[s]+\delta} u(c(s))ds (\text{with } \delta > 1) \tag{6}$$

3. **Uncertain lifetime.** A very important subclass of intertemporal choice model, are the one that introduce lifetime uncertainty into the life-cycle theory of consumption and savings. All the researches in this domain stem from the pioneering work of Yaari (1965) that proposed, and solved, a very general, non parametric, expected utility model.

   3.a. The **expected utility model of uncertain lifetime**

   $$V(c,t) = \int_t^T (1 - \Pi_t(s))e^{-\theta(s-t)}u(c(s))\,ds \tag{7}$$

   Π_t is the cumulative distribution of the age of death for an individual alive at age t (i.e. $\Pi_t(s)$ is thus "the probability for agent alive at date t to be dead at age s" and $(1 - \Pi_t(s)$ is "the probability for agent alive at date t to be alive at age s"). From equation 7, we can infer that the discount factor in this model is just the product of two sub-factors, the standard subjective factor of time discounting (here an exponential one) and the probability to be alive at age s. It can be shown easily, that in this model the discount rate is just the sum of the subjective discount rate θ and the hazard rate of the mortality process $\pi_s(s) = \pi_t(s)/(1 - \Pi_t(s)$ (with π_t the conditional probability density function of the age of death for an agent alive at date t).

   In this literature, it is the specification of the probability distribution of the age of death that will distinguished various models. For example, Blanchard (1985), assumed that, in a model with infinite maximum life duration, the hazard rate of the mortality process is the same at every age. This simplifying assumption was sufficient to allow to discuss the incidence of the finiteness of life on the effect of fiscal policy in Ramsey-like model. However, this model of "perpetual youth" (i.e. the life expectancy is the same at every age) is obviously unrealistic. That can be a problem for studying precisely the effect of demographic transition on long term growth, or to design annuity schemes or social security. For that purpose special parametrical functions for the mortality process has been designed to investigate intertemporal choice with realistic demography (Boucekkine et al., 2002; Faruqee, 2003; Sheshinski, 2007; Azomahou et al., 2009; Lau, 2009, among others).

   Whatever the supplementary assumption on the mortality process, all these expected utility model of intertemporal choice are formally equivalent with non exponential discounting models. Burness (1976) proves that they all account for the behavior of a time consistent agent.

   3.b. **The rank dependent utility model of uncertain lifetime**. The rise of behavioral economics has lead some economist to investigate the consequences of alternative models of choice (see Bommier et al., 2012, for example). Drouhin (2015) introduces Quiggin (1982)'s Rank Dependent Utility model into Yaari (1965)'s framework. Concretely, agents are supposed to subjectively transform the cumulative probability distribution of the age of death:

   $$V(c,t) = \int_t^T (1 - h(\Pi_t(s)))e^{-\theta(s-t)}u(c(s))\,ds \tag{8}$$

   Drouhin (2015) shows that time consistency is lost as a general property, holding only when the probability weighting function h is of power form.

4. **Age-varying instantaneous utility**. Beside the heavily studied problem of the discount function, the general additive utility functional of equation 1 also include the possibility of changing taste with age or decision date. If, for simplification, I drop the existence of a multiplicatively separable discount factor and the effect of the decision date, I obtain a reduced model that focus on the change of the instantaneous utility with age:

$$V(c,t) = \int_t^T u(c(s), s)\, ds \tag{9}$$

This possibility seems natural, but in practice it has not received much attention until now, and very few is known on possible realistic semi-parametric functional form that will fit the data. However, it is always possible to think at simple generalization of existing models.

4.a. **Age-varying CES**. The CRRA utility function is the most used utility function in macroeconomics. In this special functional form, the coefficient of relative risk aversion (which is also the inverse of the instantaneous elasticity of substitution) is invariant with consumption and invariant with age. If we drop this last invariance, we obtain the nice following intertemporal utility functional, that I will use in the remaining of this article as simple example of age varying instantaneous utility function.

$$V(c,t) = \int_t^T \frac{c^{1-\gamma(s)}}{1-\gamma(s)}\, ds \tag{10}$$

2.2 Time discounting and changing tastes

Preceding example illustrates the many richness that lies behind the parsimonious formulation of additive intertemporal utility functional described by Equation (1). However, it is precisely because it is parsimonious, that it will allow to abstract form specific semi-parametrical functional form and to derive general property that were hidden until now.

It is important to notice that this general formulation encompasses two important notions, *time discounting* and the possibility of *changing tastes* without necessarily multiplicatively separate them. However, going back to Fisher (1930) both discount factor and rate can easily be defined within this model.

For Irving Fisher, the discount factor is just the marginal rate of substitution between two same levels of consumption at different dates. If the constant fixed level of consumption is denoted by c and the time distance between the two dates by τ, we have:

$$e^{-\Theta(c,s,\tau,t)\tau} = \frac{v_1(c, s+\tau, t)}{v_1(c, s, t)} \tag{11}$$

Taking the limit of the Log of this expression when τ goes to 0, I get the definition of the instantaneous subjective discount rate:

$$\Theta(c,s,t) = -\lim_{\tau \to 0} \frac{\ln(v_1(c, s+\tau, t)) - \ln(v_1(c, s, t))}{\tau} = \frac{\partial}{\partial s} \ln(v_1(c, s, t))$$

Definition 1. *The* ***instantaneous subjective discount rate*** *is defined as:*

$$\Theta(c,s,t) \stackrel{def}{=} -\frac{v_{12}(c,s,t)}{v_1(c,s,t)} \tag{12}$$

The instantaneous subjective discount rate is just the rate at which the marginal felicity of a given future consumption changes with age. It is easy to verify that when the instantaneous felicity function multiplicatively separates a discount factor from the a time invariant instantaneous utility function (i.e. $v(c,s,t) = f(s,t)(u(c))$, the instantaneous subjective discount rate is the logarithmic derivative of the discount factor, proving that my definition[5] is a generalization of the usual one. It is also noteworthy that, until now, no assumption has been made about the sign of the second-order cross derivative of v. As long as the agent's horizon is finite, such an assumption is not technically required to have an optimum for the agent's program. Thus my model can explain *preference for present consumption*, when $v_{12} < 0$, as well as *preference for future consumption*, $v_{12} > 0$. However, it is important to notice that preference for present consumption is not formally characterized by a instantaneous felicity function that is decreasing with s, but by a marginal felicity of consumption that is decreasing with s. With the general utility functional characterized by equation (1), the subjective instantaneous discount rate may vary according to the level of consumption, the age, and the calendar date of decision. It will be the same for the notion of resistance toward intertemporal substitution. This point will be of first importance when discussing time consistency in the next section. Table 1 in the appendix provides the instantaneous effective discount rate for all the examples of section 2.1.

Definition 2. *The* ***rate of absolute resistance to intertemporal substitution*** *is defined as:*

$$\rho(c,s,t) \stackrel{def}{=} -\frac{v_{11}(c,s,t)}{v_1(c,s,t)} \tag{13}$$

This second definition is standard. [6] What is noteworthy is the symmetry between the definitions of *the instantaneous subjective discount rate* and *the rate of absolute resistance to intertemporal substitution*. The two rates measure the curvature of the relation between the marginal felicity of consumption and each one of its determinant taken separately. I will show that, similarly to the standard exponential discounting model, these two rates, jointly with the rate of interest, will determine the slope of the intertemporal consumption profile.

The fact that the rate of absolute resistance to intertemporal substitution can vary with the level of consumption has been discussed extensively in the literature[7]. But the possibility for this rate, for a given level of consumption, to vary according to age and decision date has generally been discarded as implying irrational behavior. Before moving on to challenge this last point, I have to discuss the possibility for consumers to change tastes, induced by the formulation of the utility functional (1).

Taking s as a variable of the "instantaneous felicity function" allows me to include the possibility of tastes evolving with age in the analysis. Obviously, this assumption may

[5]The same kind of definition is used by Gollier and Zeckhauser (2005), but they do not make the link with the Fisherian definition.

[6]$\gamma(c,s,t) = \rho(c,s,t)c$ is the inverse of the instantaneous elasticity of intertemporal substitution, that will determined the intensity of the response of present consumption to a change in the rate of interest.

[7]In the case of risk, the fact that the utility function is DARA, is generally considered as realistic (see Gollier (2001) for an extensive discussion.

imply a departure from *time invariance* and *stationarity*, because in this case postponing or anticipating two different future consumption streams can clearly alter the order of preference of these streams. In this model, the possibility that tastes for future consumption will evolve with age s is clearly anticipated by the decision maker at decision date t. Those change of taste being anticipated, it is also possible that the special functional dependance between the instantaneous felicity function and s also accounts form some sort of sophisticated behavior of the agent for dealing with them.[8]

Taking t as a parameter of the "instantaneous felicity function", in a way that can depart from the exponential discounting formulation, allows me to include in the analysis the possibility of changing tastes at different calendar dates of decision. This other possibility of changing tastes is more "drastic" than the former. Clearly, this formulation allows the decision maker, if given the possibility, to change her mind in the future and to decide for a different stream of consumption. Either because, following Strotz (1956) she does not anticipate in any way that she will be allowed to plan again at any subsequent date, or because she does not anticipate the possibility that her preferences for future consumption can change according to t. In this article, I will emphasize the second possibility, interpreting the functional dependance between the instantaneous felicity function and the decision date, when it differs from the standard exponential discounting formulation as the "naive" part of the inter-temporal utility functional . I will discuss this point further in the next section.

2.3 Intertemporal choice

The choice of the consumer at date t is the function $c_t \in \mathscr{C}_t$ solution of the program:

$$\mathcal{P}_t \quad \begin{cases} \max\limits_c \int_t^T v(c(s), s, t) ds \\ s.t. \forall s \in [t, T], \quad \dot{K}(s) = rK(s) - c(s) \\ K(t) \text{ given and } K(T) \geq 0 \end{cases}$$

Thus, $c_t(s)$ is the optimal consumption at age s, planned at date t (the value of the control variable at age s), and $K_t(s) = K(t) + \int_t^s (rK(\tau) - c_t(\tau)) d\tau$, the remaining capital at age s for an agent planning her consumption at date t (the value of the state variable at age s).

[8] As pointed out by Peleg and Yaari (1973) in the introduction, "(...) we must acknowledge the fact that, from the methodological point of view, the whole question of preferences that change over time is, at the outset, rather troublesome. An agent's preference ordering is nothing more than a summary of his choices, when confronted with dichotomous alternatives. As such, preferences are an ex-post concept, and there is a real methodological difficulty in talking today about tomorrow's preferences since tomorrow's preferences only become meaningful after tomorrow's potential choices are known". But the literature on time consistency usually depart from this general methodological statement, including Peleg and Yaari themselves in the rest of their paper. Initially, Strotz (1956) believed that, being time consistent, a sophisticated agent choice could be accounted for, ex-post, by an exponentially discounted utility model. However, relying on a model with a instantaneous utility function that is time invariant, Pollak (1968) "proves" that Strotz was wrong. In this paper, I will prove that the set of time consistent additive utility functional includes many non stationary, time varying, cases, and thus, the behavior of a sophisticated agent may well, *ex post*, be represented by one of those.

Proposition 1. *The optimal consumption profile planned at date* t (i.e. the solution of the program $\mathcal{P}_t$) *is fully described by:*

$$\dot{c}_t(s) = \frac{r - \Theta(c_t(s), s, t)}{\rho(c_t(s), s, t)} \tag{14}$$

$$\int_t^T \exp[-r(s-t)]c_t(s)ds = K(t) \tag{15}$$

Proof: The Hamiltonian of the program $\mathcal{P}_t$ is $\mathcal{H}(c, K, s, t) = v(c, s, t) + \lambda(rK - c)$. The maximum principle gives us the necessary condition for c_t and K_t to be the solution of $\mathcal{P}_t$. $\forall s \in [t, T]$:

$$\lambda(s) = v_1(c_t(s), s, t) \tag{16}$$

$$\dot{\lambda}(s) = -r\lambda(s) \tag{17}$$

$$\dot{K}_t(s) = rK_t(s) - c(s)) \tag{18}$$

$$\lambda(T)K_t(T) = 0 \tag{19}$$

Differentiating (16) according to s and substituting in (17) gives (14). The strict positive monotonicity of v according to c and condition (16) imply that $\lambda(s) > 0$ and then, taking into account the transversality condition (19) proves that $K_t(T) = 0$. Multiplying (18) by $\exp[-r(s-t)]$ and integrating by parts on the interval [t,T] gives (15).
□

Proposition 1 generalizes the standard results of the theory of consumption and savings (Yaari, 1964), making it very insightful. Equation (14) characterizes the *slope* of the optimal consumption profile and equation (15) its *level*.

The *slope* depends mainly on the psychological characteristics of the agent embodied in the intertemporal utility function. These characteristics are properly summarized by the two concepts of instantaneous subjective discount rate and rate of absolute resistance to intertemporal substitution. The key issue is the difference between the economical discount rate, r, and the subjective rate of discount. If $r > \Theta$, there is an incentive for the agent to consume more in the future. The slope will be positive. On the contrary, if $r < \Theta$, there is an incentive to consume more in the present and thus the slope will be negative. If $r = \Theta$, the intertemporal consumption profile will be flat. The intensity of the agent's response to a difference between the economic and the subjective discount rates will be determined by the rate of resistance to intertemporal substitution. In my model, the fact that both rates can vary with the consumption level and with age implies that the dynamic of consumption will be richer.

The *level* of the optimal consumption profile only depends on life-cycle wealth. In this article, it is simply the capital at decision date t.

Equation (14) describes the consumption profile planned at date t as a differential equation. At some point in the article, it will be more convenient to specify the consumption profile in an integral form. If the instantaneous rate of growth at age s of the consumption profile planned at date t is denoted $G(t, s)$, we have:

$$G(t, s) \stackrel{\text{def}}{=} \frac{r - \Theta(c_t(s), s, t)}{\rho(c_t(s), s, t)c_t(s)} \tag{20}$$

Proposition 2 (The optimal consumption profile integral form)**.**

$$c_t(s) = c_t(t) \exp\left[\int_t^s G(t, \tau)d\tau\right] \tag{21}$$

Proof: From equation (14) taken for all $\tau \in [t,T]$, we get:

$$\frac{\dot{c}_t(\tau)}{c_t(\tau)} = \frac{r - \Theta(c_t(\tau),\tau,t)}{\rho(c_t(\tau),\tau,t)c_t(\tau)} = G(t,\tau)$$

Integrating this relation between t and s, we get:

$$ln(c_t(s)) - ln(c_t(t) = \int_t^s G(t,\tau)d\tau$$

Taking the exponential, equation (21) follows directly. □

It is important to note that equation (21) is not an explicit formulation of c, but an implicit one, because both sides of the equation depend on $c_t(s)$.

Having all the necessary material now, I may turn to the main concern of this paper, time consistency.

3 Time Consistency for a Naive Decision Maker

Let us now consider a naive decision maker who is allowed to plan her consumption again at date $t' > t$. Her optimal choice $c_{t'} \in \mathscr{C}_{t'}$ is the solution of the program:

$$\mathcal{P}_{t'} \quad \begin{cases} \max\limits_{c} \int_{t'}^{T} v(c(s),s,t')ds \\ s.t. \forall s \in [t',T], \quad \dot{K}(s) = rK(s) - c(s) \\ K(t') = K_t(t') \text{ and } K(T) \geq 0 \end{cases}$$

Time consistency can now be properly defined.

Definition 3. *The agent is* ***time consistent*** *if and only if, for all possible interest rate, r, and initial value of capital, $K(t)$:*

$$\forall t' \in [t,T], \forall s \in [t',T], c_t(s) = c_{t'}(s) \equiv c(s)$$

This definition is exactly the same nature as the one given by Strotz (1956), so is the methodology. Like Strotz, within a set of utility functionals, I will seek the ones that are compatible with time consistency. The only difference is that I am searching in the set $\mathcal{V}_t$, the one described by Equation 1, apparently a much broader set than the one used by Strotz.

Within this set, what are the specificities of the utility functions compatible with time-consistent behavior?

Theorem 1. *The three following statements are equivalent:*

1. *The agent is time consistent.*
2. *The* instantaneous felicity function v *is of the form:*

$$v(c(s),s,t) = \alpha(t)w(c(s),s) + \gamma(s,t) \tag{22}$$

with w any real valued function defined on $\mathbb{R}^+ \times [t,T]$, three times continuously differentiable in c, s, and strictly concave in c; and α and γ any continuous real valued function defined on $[t,T] \times [t_0,T]$.

3. *The* instantaneous subjective discount rate *and the* rate of absolute resistance to intertemporal substitution *are both independent from t, the calendar date of decision.*

$$\Theta(c(t), s, t) = -\frac{w_{12}(c(s), s)}{w_1(c(s), s)} \text{ and } \rho(c(t), s, t) = -\frac{w_{11}(c(s), s)}{w_1(c(s), s)}$$

Proof: The solutions of the programs $\mathcal{P}_t$ and $\mathcal{P}_{t'}$ over the interval $[t', T]$ must be compared for any possible values of r and $K(t)$. Because of the strict concavity of v according to c, these solutions are unique. They both satisfy their respective differential budgetary constraint. Discounting at the market rate r and integrating these constraints over the interval $[t', T]$, taking into account the fact that $K_t(T) = K_{t'}(T) = 0$ (cf. the proof of Proposition 1), we obtain:

$$K(t') = \int_{t'}^{T} \exp[-r(s - t')]c_t(s)ds = \int_{t'}^{T} \exp[-r(s - t')]c_{t'}(s)ds \tag{23}$$

Moreover, Equation (14) of Proposition (1) implies:

$$\dot{c}_t(s) = \frac{r + \frac{v_{12}(c_t(s), s, t)}{v_1(c_t(s), s, t)}}{-\frac{v_{11}(c_t(s), s, t)}{v_1(c_t(s), s, t)}} \text{ and } \dot{c}_{t'}(s) = \frac{r + \frac{v_{12}(c_{t'}(s), s, t')}{v_1(c_{t'}(s), s, t')}}{-\frac{v_{11}(c_{t'}(s), s, t')}{v_1(c_{t'}(s), s, t')}}$$

If for all s, $c_t(s) = c_{t'}(s)$, then one must have for all s, $\dot{c}_t(s) = \dot{c}_{t'}(s)$. Put differently, it means for all age s, the derivative of the planned consumption stream according to age does not depend on the decision date t. In this case, I denote this derivative as an arbitrary function Γ of the consumption, the consumption date, and the rate of interest. Γ is of the form: $\Gamma(c, s, r) = \tilde{A}(c, s, t) + \tilde{B}(c, s, t)\, r$, with $\tilde{A}$ a real valued function and $\tilde{B}$ a strictly positive real valued function. Γ does not depend on t, implying that the partial derivative according to t of the righthand side should be null, $\tilde{A}_3(c, s, t) + r\tilde{B}_3(c, s, t) = 0$. This should be true for every r, thus in particular when $r = 0$ and $r = 1$, so we can deduce that $\tilde{A}_3(c, s, t) = 0$ and $\tilde{B}_3(c, s, t) = 0$. Thus Γ can be rewritten $\Gamma(c, s, r) = A(c, s) + B(c, s)\, r$, with A a real valued function and B a strictly positive real valued function.
Thus the time consistent utility functional should solve the following Partial Derivative Equation for every possible values of r:

$$rv_1(c, s, t) + v_{12}(c, s, t) = -\left(A(c, s) + rB(c, s)\right) v_{11}(c, s, t)$$

In particular, this is true for $r = 0$ and $r = 1$, thus:

$$\begin{cases} v_1(c, s, t) & = -B(c, s)v_{11}(c, s, t) \\ v_{12}(c, s, t) & = -A(c, s)v_{11}(c, s, t) \end{cases}$$

And because B is always strictly positive:

$$\begin{cases} \frac{\partial}{\partial c} Log\, v_1(c, s, t) & = -\frac{1}{B}(c, s)) \\ \frac{\partial}{\partial s} Log\, v_1(c, s, t) & = \frac{A}{B}(c, s) \end{cases}$$

and then:

$$\begin{cases} Log\, v_1(c, s, t) & = \int_0^c -\frac{1}{B}(k, s)dk + L(s, t) \\ & = \int_0^s \frac{A}{B}(c, \tau)d\tau + M(c, t) \end{cases}$$

With L and M two arbitrary functions. It gives:

$$\int_0^c -\frac{1}{B}(k,s)dk) - \int_0^s \frac{A}{B}(c,\tau)d\tau + L(s,t) - M(c,t) = 0$$

The two integrals does not depend on t, so we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}(L - M) = 0$$

It implies that L is of the form:

$$L(s,t) = M(c,t) + N(c,s)$$

with N an arbitrary function. Thus we have:

$$\frac{\mathrm{d}}{\mathrm{d}c}L = 0$$

And then:

$$M_1(c,t) = -N_1(c,s)$$

Which implies:

$$M_{12}(c,t) = N_{12}(c,s) = 0$$

That proves that M and N, and thus L, are necessarily additively separable:

$$\begin{cases} M(c,t) & = \gamma^1(c) + \gamma^3(t) \\ N(c,s) & = -\gamma^1(c) + \gamma^2(s) \\ L(s,t) & = \gamma^2(s) + \gamma^3(t) \end{cases}$$

With γ^i arbitrary functions.
Finally we can write:

$$v_1(c,s,t) = \exp\left[\int_0^c -\frac{1}{B}(k,s)dk + \gamma^2(s) + \gamma^3(t)\right]$$

Denoting, $w_1(c,s) = \exp\left[\int_0^c -\frac{1}{B}(k,s)dk + \gamma^2(s)\right]$ and $\alpha(t) = \exp\left[\gamma^3(t)\right]$, we get:

$$v_1(c,s,t) = \alpha(t)w_1(c,s)$$

Integrating, we get (22) which is thus necessary for c_t and $c_{t'}$ to have the same derivative over the interval $[t', T]$ for all possible values of r. Because of (23), they also have the same discounted integral on the interval $[t', T]$, they are thus equal on $[t', T]$. This proves Theorem 1.
□

Theorem 1 is the core result of this paper. The equivalence between *1.* and *2.* is certainly the most substantial part of Theorem 1. It implies that for an additive intertemporal utility functional to be time consistent, it requires only that the instantaneous felicity function be multiplicatively separable in t, the calendar date of decision, and in s, the date of consumption. It implies that all preferences that can be represented by a utility function of the kind $V(c,t) = \int_t^T w(c(s),s)ds$ are time consistent. If we substitute the instantaneous felicity function described in Equation (22) in the intertemporal utility functional, we obtain $V(c,t) = \alpha(t)\int_t^T w(c(s),s)ds + \Gamma(t,T)$. It is just a linear transformation of the former and thus represents exactly the same preferences.

The equivalence between *1.* and *3.* gives a practical way to test for time consistency. The subjective discount rate and the rate of resistance toward intertemporal substitution have to be independent from the *calendar date of decision*. But time consistency has nothing to do with the fact that these rates are invariant with the consumption level or with age. This has two important consequences. First, one can no longer consider that variations of the subjective discount rate according to the consumption level or according to the consumption date are sufficient to prove irrational behavior or anomalies in intertemporal choice as claimed by Thaler (1981). This point will be discussed further in the next section. Second, the only way to prove time inconsistency empirically on the basis of experimental measures of the subjective discount rate is to perform longitudinal experiments, as done by Halevy (2015). The discount rate and the rate of resistance to intertemporal substitution (or rate of risk aversion) must be evaluated at different decision dates t. As noted by Sayman and Onculer (2009), until very recently, there have been very few longitudinal measurements of the discount rate.

Table 1 in the appendix provides condition for time consistency for all the examples of section 2.1.

Among the set of time inconsistent preferences, the literature has focused on preference that are "present biased". According to O'Donoghue and Rabin (1999), "when considering trade-offs between two future moments, *present-biased* preferences give stronger relative weight to the earlier moment as it gets closer". Taking into account the equivalence between *1.* and *3.* in Theorem 1, a natural transposition of this statement in our general framework is the following formal definition:

Definition 4. *The agent is said to be* ***present bias*** *if:*

$$\forall t \in [t_0, T), \forall s \in [t, T), \forall c, \Theta_3(c(s), s, t) \geq 0, \textit{with } \Theta_3(c(t), t, t) > 0 \tag{24}$$

"An intertemporal utility functional can be non stationary, additive and compatible with time consistency." "Exponential discounting is not the only additive intertemporal utility functional compatible with time consistency." "Some hyperbolic discounting functions are compatible with time consistency." All these statements stem from Theorem 1. They are paradoxical because they confront some largely shared opinions[9]. In fact, these false opinions stem from a kind of fallacy of composition, that will be made clear by the following proposition. My point is that the subset of time-consistent utility functionals critically depends on the nature of the set of functionals you're searching in. Since I started from $\mathcal{V}_t$, a broader set of utility functionals than the one implicitly considered in the literature, it is no surprise that I have found a broader subset of time-consistent utility functionals.

As a corollary of Theorem 1, a subset X of $\mathcal{V}_t$ can be used. I will denote $TC(X)$ the subset of all functionals belonging to X which also implies time consistency (*i.e.* $TC(X) = \{V(c,t) \in X | \forall s \in [t,T], v(c(s),s,t) = \alpha(t)w(c(s),s) + \gamma(s,t)\}$).

Proposition 3 (Special cases)**.** *The following statements are true:*

[9] To be right, I do not claim that this opinion is universally shared. One can find in the literature many literary statements that convey some the ideas of Theorem 1. (see Becker and Mulligan, 1997; Saint-Paul, 2011, for example).

a. For $A = \{V \in \mathcal{V}_t | V(c,t) = \int_t^T f(s-t)u(c(s))ds\}$*,*
$$TC(A) = \{V \in \mathcal{V}_t | \int_t^T \exp[-\theta(s-t)]u(c(s))ds\}$$

b. For $B = \{V \in \mathcal{V}_t | V(c,t) = \int_t^T f(s,t)u(c(s))ds\}$*,*
$$TC(B) = \{V \in \mathcal{V}_t | \int_t^T \alpha(t)\beta(s)u(c(s))ds\}$$

c. For $C = \{V \in \mathcal{V}_t | V(c,t)V(c,t) = \int_t^T f(s-t)u(c(s),s)ds\}$*,*
$$TC(C) = \{V \in \mathcal{V}_t | \int_t^T \exp[-\theta(s-t)]u(c(s),s)ds\}$$

We have $A \subset B \subset \mathcal{V}_t$ and $A \subset C \subset \mathcal{V}_t$ and thus, for obvious reasons, $TC(A) \subset TC(B) \subset TC(\mathcal{V}_t)$ and $TC(A) \subset TC(C) \subset TC(\mathcal{V}_t)$, demonstrating the increasing generality of the proposition.

$TC(A)$ is the set of all the exponentially discounted intertemporal utility functionals introduced by Samuelson (1937), the only one that is conventionally assumed to be compatible with time consistency. Assuming that the instantaneous felicity function multiplicatively separates a discount function that depends on the algebraic time distance between t and s from a instantaneous utility function that is the same at all ages, then only exponential discounting is time consistent. Considering Theorem 1, the result is obvious because only exponential functions "transform sums into products". The important point to notice is that this case is very special, stemming from a rather arbitrary starting point. In light of Halevy (2015), it is easy to show that utility functionals belonging to A fulfil time invariance. Thus, it is no surprise that *time consistent* utility functionals of this kind will also fulfil *stationarity* and thus, being additive, correspond to Samuelson (1937)'s exponential discounting model.

If the separability between discounting and instantaneous utility function is preserved, but with a more general discount function, then we obtain Proposition 3-b. This result was demonstrated first by Burness (1976) (Theorem 1), but unfortunately has received very little attention until now. This case is interesting because it is intuitively close to the standard one, but it opens a broad range of time-consistent utility functionals which contradict conventional wisdom very clearly. The most striking example is the case of hyperbolic discount functions satisfying time consistency (Example 2.a. in subsection 2.1). Indeed, $\int_t^T \frac{t}{s}u(c(s))ds$ belongs to $TC(B)$. The discount factor is t/s. It decreases from 1, at date t, to 0 when s tends to infinity. The discount rate is $1/s$. It is decreasing with age and time distance, *i.e.* the agent shows decreasing impatience (Prelec, 2004). Nevertheless, an agent with such preferences is totally time consistent. It can be shown that utility functionals belonging to B are not necessarily *time invariant*, which explains why $TC(B)$ includes non-stationary intertemporal utility functionals.

The first demonstration of Proposition 3-a is generally attributed to Strotz (1956). But a careful reading shows that Strotz did not assume *time invariance* in the sense of Halevy (2015). Proposition 3-c corresponds exactly to the seminal results demonstrated by Strotz (1956). C is the set of intertemporal utility functionals that multiplicatively separate a discount factor, but with a felicity function that can vary with age. If Theorem 1 is applied directly, we can immediately deduce that the discount function has to be multiplicatively separable between s and t, implying the exponential form. But it is important to notice that instantaneous utility can always be rewritten with $u(c(s),s) = w(c(s),s)\exp[\theta s]$, with $W(c,t) = \int_t^T \exp[-\theta(s-t)]w(c(s),s)ds\}$ also belonging to C. Thus Strotz's time-consistent utility functional characterized by Proposition 3-c can be rewritten

$$V(c,t) = \exp[\theta t] \int_t^T w(c(s), s) ds$$

proving that, in this case, the exponential formulation is a mathematical artefact. In other words, $TC(C) = TC(\mathcal{V}_t)$.

Another way to shed light on this point is to note that the parameter θ of the Strotzian formulation *is not* the Fisherian instantaneous subjective discount rate. Applying equation (12), we get:

$$\Theta(c(s), s, t) = \theta - \frac{u_{12}(c(s), s)}{u_1(c(s), s)}$$

The formulation of Strotz (1956)'s main result has proved misleading, in the sense that most subsequent works on the question have focused on the exponential formulation (the mathematical artefact) and forgotten the possible variability of the instantaneous utility function with age.

4 Empirical Considerations on the Shape of the Intertemporal Utility Functional

Theorem 1 is very general and opens a broad range of possibilities to model intertemporal choice. However, one can even find that the result is "too general". In particular, more precise specifications of the intertemporal utility functional may be wanted to do comparative statics or dynamics of the model. In this section, I will discuss the shape of the utility function, considering stylized facts from the empirical literature. Two literatures can be referred to: the experimental literature in decision theory, on the one hand, and the econometrical estimation of life-cycle consumption and savings, on the other hand.

Let us start with the experimental literature. Since the work of Ainslie (1975), much evidence has been collected in the lab against the descriptive power of Samuelson (1937)'s exponential discounting model. More precisely, following Thaler (1981), Loewenstein and Prelec (1992) have proposed a typology to depart from the exponential discounting model, presented as "anomalies in intertemporal choice". In the last two decades of the twentieth century, many experimental works (see Frederick et al. (2002) for a survey) have emphasized the possibility that the subjective discount rate is decreasing with time distance (*the common difference effect*) or with the level of consumption (*the absolute magnitude effect*). These empirical findings have been one of the starting points for what is sometimes called *the behavioral economics revolution.* Even if those findings are still in debates[10], it can be interesting to consider their logical implications within the general framework of this article.

Usually, these anomalies were interpreted as stemming from time inconsistency. However, almost all of the experimental studies at the time were made with experimental protocols in which the decision date (t, with my notation) is invariant, and only the date of the future prospects (s) varies. As shown, the fact that the discount rate can vary with the date of the prospect proves absolutely nothing with regard to time consistency.

[10] More recent experimental studies with appropriate protocols and incentive schemes do not find evidence of these anomalies, for example Andreoni and Sprenger (2012); Andersen et al. (2013, 2014); Augenblick et al. (2015) but are themselves criticized for violating the weak axiom of revealed preference (Chakraborty et al., 2017).

Thus, in this section, I will interpret these anomalies as resulting from the rational choice of a time-consistent agent with an additive and non-stationary utility. Under this interpretation, I will characterize the restrictions necessary on the shape of the intertemporal utility functional to demonstrate these anomalies[11].

The results are as follows:

Proposition 4 (Behavioral restrictions for the utility function of a time-consistent agent)**.** *A time-consistent agent with* preference for present consumption*($w_{12} < 0$):*

a. demonstrates the common difference effect *if and only if*

$$-\frac{w_{122}(c(s), s)}{w_{12}(c(s), s)} > \Theta(c(s), s)$$

b. demonstrates the absolute magnitude effect *if and only if*

$$-\frac{w_{121}(c(s), s)}{w_{12}(c(s), s)} > \rho(c(s), s)$$

c. demonstrates absolute resistance to intertemporal substitution decreasing with age *if and only if*

$$-\frac{w_{112}(c(s), s)}{w_{11}(c(s), s)} > \Theta(c(s), s)$$

d. The absolute magnitude effect *is equivalent to a* resistance to intertemporal substitution decreasing with age*, $\Theta_1(c(s), s) = \rho_2(c(s), s)$.*

Proof: The *common difference effect* is characterized by $\Theta_2(c(s), s) < 0$. Using the definition of Θ given by equation (12) and elementary calculations prove *a*). Similarly, the *absolute magnitude effect* is characterized by $\Theta_1(c(s), s) < 0$, by differentiating equation (11), we obtain *b*). *c*) is obtained from $\rho_2(c(s), s) < 0$. *d*) is obtained from *b*) and *c*), and the application of Young's theorem.□

Proposition 4 gives a correspondence between the observable properties of the discount rate, which can be measured experimentally in the lab or in the field, and the properties of the utility function, that may be important for comparative statics analysis. This shows crucially that both *common difference* and *magnitude effect* depend on the sign and magnitude of some third derivative of the instantaneous utility function. More precisely, Proposition 4 i), ii) and iii) gives a condition that compares the log-derivative of a second-order derivative with the log-derivative of a first-order derivative. Because the log-derivative is a measure of the growth rate of a function, it means that, to observe *the common difference effect*, the growth rate according to s of the cross-derivative of w has to be higher than the growth rate according to s of the marginal felicity function of consumption. Identically, to observe *the absolute magnitude effect*, the growth rate according to c of the cross-derivative of w has to be higher than the growth rate according to c of the marginal felicity function of consumption.

[11]Of course, on the other hand, the fact that these anomalies can be reproduced within a generalized model of time-consistent agents does not prove that agents are necessarily time consistent. I will return to this "time-inconsistent side" of the model in the next section

What is striking is that these conditions are of the same mathematical nature as those given by Kimball (1990) to characterize the necessary conditions to observe precautionary savings. The only difference is that, in my model, the felicity function depends on two variables, and thus first-order and second-order derivatives are partial derivatives.

Theorem 1 d. may be the most remarkable part of the proposition. It stems from the internal consistency of the model. In particular, the instantaneous discount rate cannot be any function of c, t and s. It is the log-derivative according to s of the marginal instantaneous felicity of consumption, and the rate of absolute resistance to intertemporal substitution is also a log-derivative (according to c) of the same marginal instantaneous felicity of consumption).

If one believes in the absolute magnitude effect, in the context of the life-cycle model of consumption and savings, it implies that, everything else being equal, richer people will be more patient; then, one must necessarily believe that older people will be less resistent to intertemporal substitution. This can easily be illustrated with the example 4.a. of the age-varying CES in subsection 2.1. In this specific case, the instantaneous subjective rate of discount is $\Theta(c, s) = Log(c)\gamma'(s)$ and the rate of absolute resistance to intertemporal substitution is $\rho(c, s) = \gamma(s)/c$. It is easy to verify that $\Theta_1(c(s), s) = \gamma'(s)/c = \rho_2(c(s), s)$.

The problem is in direct relation with a well-known problem in finance, the problem of the evolution of risk aversion with age, since, under expected utility, parameter ρ will also characterize instantaneous absolute risk aversion.

In finance, it is common practice to consider that risk aversion increases with age. However, two famous theoretical papers, Samuelson (1969) and Merton (1969), have demonstrated that, under the standard expected utility dynamic model of portfolio choice with CRRA instantaneous utility function, the agent will have a risk aversion on wealth that is invariant with age. Reconciling the common practice with theory has since been an important research agenda (see Spaenjers and Spira (2015) for a recent survey and empirical results on the problem). Many explanations can be put forth, from assuming the convexity of absolute risk tolerance with regard to wealth (Gollier and Zeckhauser, 2002), to assuming loss aversion (Berkelaar et al., 2004). Unfortunately, the direct explanation that instantaneous absolute risk aversion can simply increase with age has not received much attention until now, even if there is some empirical evidence (Harrison et al., 2007, for example)[12]. This is probably because conventional wisdom has long wrongly associated non-stationarity with time inconsistency. Thus, in future research, the general model presented here can be a good starting point to study portfolio choices throughout the life cycle.

Regardless, it appears that assuming an absolute rate of aversion to intertemporal substitution that decreases with age is far from obvious from a life cycle perspective. The reverse assumption actually seems more natural. However, if it is the case, according to Theorem 1 d., a negative *absolute magnitude effect* must also be discarded and replaced by a positive one.

Turning now to the interpretation of the common difference effect given at the beginning of the section, one may also wonder whether it accounts for empirical regularity in life-cycle consumption and savings. The work of Thurow (1969) has established the stylized fact that the life-cycle consumption profile is hump-shaped. Following Thurow himself, many works have attributed this property to the existence of a borrowing con-

[12]In a setting focused on utility of wealth (an thus indirect utility of the one used in this paper), Nachman (1975) considered the possibility of temporally changing tastes and introduce a notion of *temporal risk aversion* on wealth.

straint and of buffer-stock savings in presence of earnings uncertainty (Carroll, 1997; Gourinchas and Parker, 2002, among others). But a more general and appealing explanation exists: uncertain lidetime (Example 3. in subsection 2.1.). Following Yaari (1965), it is well known that a primitive for the subjective discount rate is the hazard rate of the mortality process, which is typically increasing and convex with age. Bommier (2013) and Drouhin (2015) have shown that various models of life-cycle consumption and savings with uncertain lifetime can explain the hump in the life-cycle consumption profile.

Regardless, the properties of the slope of a time-consistent agent's life-cycle consumption profile can be characterized.

Proposition 5. *The slope of a time-consistent agent's optimal consumption profile fulfils:*

$$\dot{c}(s) = \frac{r - \Theta(c_t(s), s)}{\rho(c_t(s), s)} \tag{25}$$

and

$$\ddot{c}(s) = \frac{-\Theta_2(c(s), s) - 2\dot{c}(s)\rho_2(c(s), s) - \dot{c}(s)^2\rho_1(c(s), s)}{\rho(c(s), s)} \tag{26}$$

Proof: Equation (25) stems directly from equations (14) and Theorem 1. Equation (26) is just the derivative of (25) according to s.□

Considering equations (26) and (25), it is obvious that the only way to obtain a hump-shaped consumption profile, without assuming a borrowing constraint, is to have a subjective discount rate that is increasing with age. Thus, the consumption profile will be increasing for young agents, reaching a maximum at middle age, and then decreasing. More specifically, equation (26) shows that the concavity of the consumption profile in the vicinity of the maximum consumption is only possible if $\Theta_2(c(s), s) > 0$, because in this vicinity the two other terms of the numerator are approximately zero ($\dot{c}(s) \approx 0$). However, when agents are young or old, the analysis of the concavity of c is richer because, in both cases, the slope of the consumption profile interacts with the first-order derivative of $\rho(c, s)$ to determine the concavity of c. Assuming that the instantaneous felicity function is DARA ($\rho_1(c(s), s) < 0$), then the third term of the denominator will be positive, implying an effect that will attenuate the concavity of c for both young and old people. Finally, assuming that the absolute coefficient of time resistance increases with age ($\rho_2(c(s), s) > 0$)(risk aversion increases with age), then the second term of the denominator will be negative for young people (reinforcing the concavity of c) and positive for old people (attenuating the concavity of c). The demonstration of this last effect is an innovation of this article, because it will only appear in a model that does not multiplicatively separate the discount factor from a time-invariant utility function. Of course, assuming the opposite sign for the derivative of ρ, the sign of the effects will be reversed.

To conclude, it appears that the sign of the *absolute magnitude effect* and the interpretation given in this section of the *common difference effect* described by the experimental decision literature seem to contradict common stylized facts in the empirical literature on life-cycle consumption and savings. However, the theoretical model of this article, by assuming the possibility of a non-stationary intertemporal utility functional, provides a synthetical framework in which these two literatures can be connected. As such, it offers rigourous theoretical foundations for systematic empirical investigation of the effect of aging on fundamental concepts of economic theory, such as time preference and risk aversion.

5 A Naive, Time-Inconsistent Consumer's Effective Consumption Profile

With the intertemporal utility functional (1), the agent may be time inconsistent. When it is the case, the consumption path planned at date t will not be observable, because the agent will change her mind in the future. The properties of the planned consumption profile, given by proposition 1, are theoretically easy to compute. However, with the exception of the starting point $c_t(t)$, these properties may seem empirically useless because they will never be directly observed in the real world. Nevertheless, the intertemporal utility functional (1) includes a description of how the agent changes her mind. Thus, a theory of the consumption effectively consumed at each future date can be abstracted from the model.

In this section, I assume that the agent is time inconsistent, naive and characterized by a present bias. I will exploit the continuous time structure of the model to fully characterize and analyze the "effective consumption profile" of the agent, the one that is observable. I will follow an agent throughout her life cycle from the starting date[13] t_0, until her death at date T. I will suppose that the agent is continuously re-planning her consumption path over her Life Cycle. It means that at each instant $t \in [t_0, T]$, the agent solves the program $\mathcal{P}_t$ and effectively consumes $c_t(t)$, the initial value of each consumption profile planned at date t.[14]

Definition 5. *The **effective consumption profile** is defined as the function* $\bar{c}$: $[t_0, T] \longrightarrow \mathbb{R}^+$ *with:*

$$\bar{c}(t) \overset{def}{=} c_t(t) \tag{27}$$

What are the properties of the *effective consumption profile*? Relying on the general results concerning the optimal consumption profile planned at date t from Proposition 1, we have almost all the material to derive some important general properties of the *effective consumption profile.* But before doing that we have to characterize the related notion of *effective capital profile.*

Proposition 6. ***The effective capital profile**:*
fulfils the dynamic accumulation constraint.

$$\forall t \in [t_0, T], \quad \dot{K}(t) = rK(t) - \bar{c}(t) \tag{28}$$

fulfils the transversality condition $K(T) = 0$ *that is common to all programs* $\mathcal{P}_t$ *and, thus, fulfils the life-cycle budgetary constraint.*

$$\forall t \in [t_0, T], \quad \int_t^T \exp[-r(s-t)]\bar{c}(s)ds = K(t) \tag{29}$$

[13]Even if the birth date is normalized to 0, taking t_0 as a starting date is preferable for two reasons. First, agents effectively start deciding on their own consumption after their birth date. Second, for a clean exposition of problems relating to time consistency, it is always preferable to take dates as parameters.

[14]In a less general setting, Findley and Caliendo (2015) uses a concept of consumption "path actually followed", defined as the "envelope of initial values from infinitely many planned consumption paths", that is equivalent with my notion of "effective consumption profile" presented here. Pollak (1968) has been the first to consider what happens when the number of re-planning dates converge to infinity.

Proof: The differential budgetary constraint (28) is just an accounting equation. Taking equation (28) in s, and multiplying both sides by $\exp[-r(s-t)]$ and, then, integrating the obtained identity by parts on the interval [t,T] gives (29). □

As simple as it seems, Proposition 6 is in fact essential. Because the *effective capital profile* is the only notion that connects together all the program $\mathcal{P}_t$ with t varying from t_0 to T. For a time inconsistent agent, the function (c_t) is always different from the function $(\bar{c})$. But both function share the same present value on the interval $[t, T]$. That gives a mathematical structure to the problem that will allow to derive all the following propositions.

Proposition 7. *The effective consumption profile is characterized by :*

$$\bar{c}(t) = \frac{K(t)}{\int_t^T \exp\left[\int_t^s (G(t,\tau) - r)\, d\tau\right] ds} \tag{30}$$

and

$$\frac{\dot{\bar{c}}(t)}{\bar{c}(t)} = G(t,t) - \frac{\int_t^T \int_t^s \frac{\partial G(t,\tau)}{\partial t} d\tau \exp\left[\int_t^s (G(t,\tau) - r)\, d\tau\right] ds}{\int_t^T \exp\left[\int_t^s (G(t,\tau) - r)\, d\tau\right] ds} \tag{31}$$

Proof: Substituting (21) into the life-cycle budgetary constraint (15), we get:

$$K(t) = c_t(t) \int_t^T \exp\left[\int_t^s (G(t,\tau) - r)\, d\tau\right] ds \tag{32}$$

Using Definition 5, equation (30) is straightforward.

We denote,

$$I(t) = \int_t^T \exp\left[\int_t^s (G(t,\tau) - r)\, d\tau\right] ds \quad \text{with} \quad D_I = \{t; t_0 \leq t \leq T\} \tag{33}$$

Then equation (30) can be rewritten:

$$\bar{c}(t) = \frac{K(t)}{I(t)} \tag{34}$$

Taking the log-derivative of this equation, we get:

$$\frac{\dot{\bar{c}}(t)}{\bar{c}(t)} = \frac{\dot{K}(t)}{K(t)} - \frac{\dot{I}(t)}{I(t)}$$

Since for all $t \in [t_0, T]$, the effective consumption path fulfills the differential constraint (28), we have:

$$\frac{\dot{K}(t)}{K(t)} = r - \frac{\bar{c}(t)}{K(t)} = r - \frac{1}{I(t)} \tag{35}$$

Using the Leibnitz formula on $I(t)$ twice, we get:

$$\dot{I}(t) = -1 - (G(t,t) - r)I(t) + \int_t^T \int_t^s \frac{\partial G(t,\tau)}{\partial t} d\tau \exp\left[\int_t^s (G(t,\tau) - r)\, d\tau\right] ds$$

Combining these three last equations, we get equation (31). □

Equation (30) gives the value of the effective consumption at any date t. It is directly derived from the life-cycle budgetary constraints that fulfil the *planned consumption profile*. Equation (31) gives the dynamic of the *effective consumption profile*. It is particularly interesting because it allows to directly compare the slope of the *planned consumption profile* at its starting date, t, and the slope of *the effective consumption profile* at the same date, t. The sign of the difference between these two slopes is given by the sign of the integral term in equation (31). This term, which seems complicated at first, can be interpreted as a weighted average of the term $\int_t^s \frac{\partial G(t,\tau)}{\partial t} d\tau$ for each date s in $[t, T]$, the weight being the growth factor of the planned consumption profile, (at planning date t), for each date s in $[t, T]$ (i.e. $c_t(s) \exp[-r(s-t)]/c_t(t)$). Thus, when the growth rate of the planned consumption profile at any future date s is monotonic in the planning date t, because of the positivity of the exponential, the sign of the integral term will be determined, and thus the sign of the difference between the two slopes. Moreover, the integral term tends to 0, when t tends to T, we have an *horizon effect* and the two slopes are converging as the agents tends to the end. I will illustrate this general property in the continuous time version of the quasi-hyperbolic discounting model given by Barro (1999). But first, the last general comparative properties of the effective and planned consumption profiles must be specified.

Proposition 8. *For a time-inconsistent agent, the planned consumption profile (at any date t) crosses the effective consumption profile at least twice.*

Proof: Both planned and effective consumption at date t satisfy the life-cycle budgetary constraints (respectively given by equations (15) and (29)). We thus have:

$$\forall t \in [t_0, T], \qquad \int_t^T \exp[-r(s-t)](\bar{c}(s) - c_t(s))ds = 0 \tag{36}$$

By definition of the effective consumption profile, we necessarily have $c_t(t) = \bar{c}(t)$. If the agent is time inconsistent, there exists a date $s_0 \in (t, T]$ when $\bar{c}(s_0) - c_t(s_0) < 0$ or $\bar{c}(s_0) - c_t(s_0) > 0$. Then, because of equation (36), there necessarily exists a date $s_1 \in (t, T]$ when, respectively, $\bar{c}(s_1) - c_t(s_1) > 0$ or $\bar{c}(s_1) - c_t(s_1) < 0$. Since the instantaneous felicity function is strictly concave in c, c_t is necessarily continuous. Equation (30) shows that it will also be the case for the effective consumption profile. Thus, by continuity, we necessarily have a second date $\bar{s} \in (t, T]$ when $\bar{c}(\bar{s}) = c_t(\bar{s})$.
□

It is important to insist on the fact that all the propositions derived until now do not depend on the choice of special parametrical forms for the intertemporal utility functional. The model is general and admit all standard model of the literature as special cases. Relying on the power of optimal control theory and a careful derivation of intertemporal budgetary constraints and transversality conditions, those propositions accounts for effects that have not be described until now, and that without relying on unnecessary mathematical sophistication.

However, a parametric example can be taken to illustrate the results of Propositions 7 and 8. I will use the continuous time version of the quasi-hyperbolic discounting model given by (Example 2.b. in subsection 2.1.): In this special case of the model, the instantaneous felicity function multiplicatively separates the discount factor and the instantaneous utility function, which is time invariant. Using the formalism of Proposition 3, this parametric intertemporal utility functional belongs to the set B, but not to the set $TC(B)$, because the instantaneous felicity function is not multiplicatively separable in s and t. Following Definition 1, the instantaneous subjective discount rate can be written $\Theta(c, s, t) = \alpha + q\beta \exp[-q(s - t)]$. Thus, the discount rate decreases with s, from $\alpha + q\beta$ when $s = t$ to α when s goes to infinity. The parameters q characterize the speed of convergence of the instantaneous subjective discount rate to its long term value α. If q is sufficiently high, the convergence is very rapid, approximating the discrete time quasi-hyperbolic discounting model proposed by Laibson (1997). It can also be easily checked, relying on Definition 4, that the agent is *present bias* in this model.

For the purpose of simplicity and computability, I will assume that the instantaneous utility function is CRRA ($\rho(c, s, t)c_t(s) = \gamma = cst$).
With these assumptions, the growth rate of the planned consumption can be expressed:

$$\forall t \in [t_0, T], \forall s \in [t, T], \quad G(t, s) = \frac{\dot{c}_t(s)}{c_t(s)} = \frac{r - \alpha - \phi'(s - t)}{\gamma} \tag{37}$$

Contrary to the general model, the right-hand of the expression no longer depends on the level of the consumption. This implies that the planned consumption function is the solution of a linear homogenous differential equation with variable coefficient, and thus is explicitly solvable. By a simple calculation, we also obtain:

$$\forall t \in [t_0, T], \forall s \in [t, T], \quad \frac{\partial G(t, s)}{\partial t} = \frac{\phi''(s - t)}{\gamma} < 0 \tag{38}$$

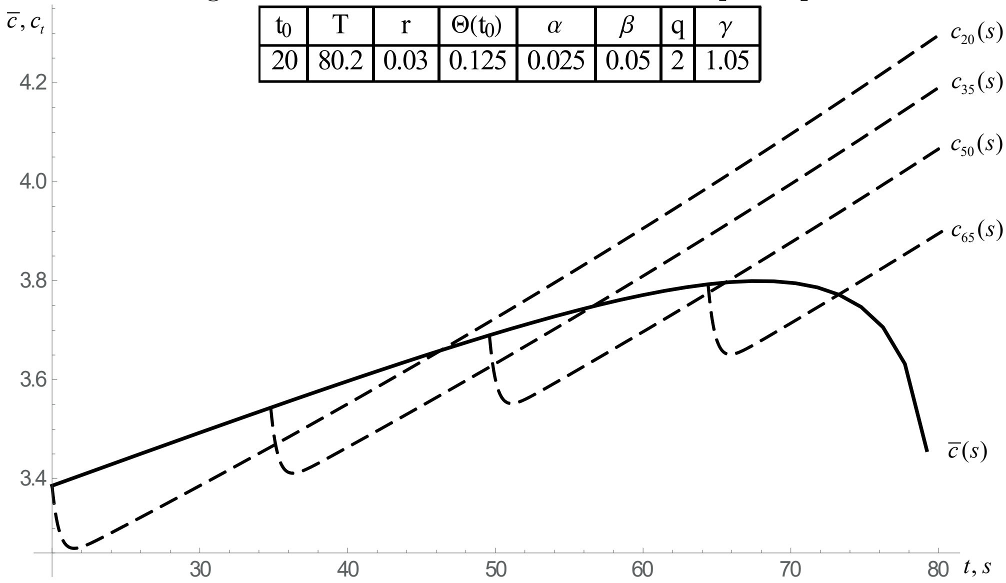
This result provides the sign of the last term of the right-hand side of equation (31), and proves, according to Proposition 7 that, in this case, the effective consumption profile will always have a starting slope that is higher than that of the planned consumption profile.

The method to compute the effective consumption profile is to numerically solve integral $I(t)$ from equation (33) for all t in (t_0, T) and differential equation (35), which is, in this parametrical example, homogenous and linear with variable coefficient, and to compute $K(t)$ for all t in (t_0, T). Then, exploiting equation (34), computing $\bar{c}(t)$ for all t in (t_0, T) is straightforward.

Figure 1 represents the effective consumption profile throughout the life cycle (with $t_0 = 20$ and $T = 80$) and the planned consumption profile at age 20, 35, 50, and 65, for special numerical values. After a "short-term effect", the planned consumption converges rapidly to a standard exponential discounting model with a CRRA utility function, characterized by a constant growth rate of the planned consumption. I choose parameters $q = 2$ to have a rapid convergence to the "long term regime". I choose γ, the coefficient of relative resistance to intertemporal substitution to be "close" to one ($\gamma = 1.05$), so that the growth rate of the planned consumption can be approximated by the difference between the interest rate (r) and the instantaneous subjective discount rate (Θ). I take a value of 3% for the interest rate. I choose a value of 2.5% for α, the "long-term" instantaneous discount rate, such that the planned consumption will be asymptotically increasing. Finally, I take a value of 5% for parameters β. Thus, the instantaneous subjective discount rate at the planning date will be 12.5% ($\Theta(t, t) = 12.5\%$), implying that the planned consumption profile will be significatively decreasing at the planning date t and immediately after.

When considering Figure 1, it is striking to note that the planned and effective consumption profiles have such completely different shapes that it seems impossible to intuitively abstract the properties of the latter when knowing the properties of the former. Hopefully, Propositions 6, 7, and 8 allow to understand what is happening. Typically,

Figure 1: Planned and effective consumption profile



t_0	T	r	$\Theta(t_0)$	α	β	q	γ
20	80.2	0.03	0.125	0.025	0.05	2	1.05

when studying intertemporal choice models, one tends to focus on the slope of the consumption profile, which is given by the first-order condition of the maximization program. Of course, this condition is very important, but to fully characterize the consumption profile, the initial value of consumption at date t must be considered. This consumption $c_t(t)$ is chosen by the agent. However, this choice is constrained by the initial level of capital. If the agent chooses "too high" a level for $c_t(t)$, the planned consumption will not be sustainable because the capital will go to zero before the terminal date T. On the contrary, with "too low" a level for $c_t(t)$, the terminal value of the capital $(K(T))$ will be strictly positive, implying that some available resources are lost for consumption. Thus, the optimal value for $c_t(t)$ is precisely the one that will allow to fulfil the transversality condition, that is, in the absence of a bequest motive, the one that drives the capital to zero exactly at the terminal date T. This is precisely what is implied by the life-cycle budgetary constraint (15).

What happens for an agent that is not time consistent, in a way characterized by Barro (1999)'s utility functional? At date t, the agent plans her consumption throughout her life-cycle. She determines the optimal slope of consumption for all ages in $[t, T]$ and the initial level of consumption $c_t(t)$. After a "small" time interval ϵ, the agent realizes that she has changed her mind. She now wants a new consumption profile $c_{t+\epsilon}$, with a lower slope for all dates in $[t+\epsilon, T]$. However, she realizes that if she starts this new consumption profile, with the previously planned consumption for date $t+\epsilon$, she will end up at date T with a strictly positive level of capital. Thus, to fulfil the transversality condition, the agent has to start from a level of consumption at date $t+\epsilon$ that is necessarily higher than the one formerly planned (i.e. $c_{t+\epsilon}(t+\epsilon) > c_t(t+\epsilon)$). This reasoning is precisely the one that is behind proposition 7 and equation (31).

Figure 1 also illustrates the fact that the difference between the slope of the planned consumption profile and the effective one is subject to the horizon effect described in proposition 7, due to the integral term in equation (31) tending to zero, when t tends to T. Thus, when the agent gets older, the slope of the effective consumption profile gets closer to the initial slope of the planned consumption $G(t,t)$. That is why the effective consumption profile is hump-shaped in our example.

Finally, Figure 1 illustrates Proposition 8. The planned consumption profile at any date t starts from the effective consumption profile, goes immediately below it, but always finishes above it.

The example is illustrative, but the method applies to any possible functional form for the intertemporal utility functional. As seen, the life-cycle budgetary constraint plays a central role in all propositions in this section, proving that "the choice-based" methodology adopted is a powerful and necessary tool to make testable predictions for the consumption behavior of non-consistent agents.

6 Conclusion

By using the most general possible additive utility representation of intertemporal preferences, encompassing the possibility that the instantaneous felicity function varies with age and decision date, I have established three important and original sets of results:

1. I have shown, contrary to what is conventionally assumed, that exponential discounting is not the only additive intertemporal utility functional that is compatible with time consistency. In practice, I have demonstrated rigourously that any intertemporal utility functional of the form $V(c,t) = \int_t^T w(c(s), s)ds$ implies a time-consistent consumption behavior.

2. This implies that the possibility that the discount rate can vary with age or with the level of consumption cannot be rejected as implying irrational behavior (time-inconsistent behavior). There is a rising interest in econometrics on this question (see Attanasio, 1999, for a survey). The results of this article provide a rigourous theoretical framework for doing so. In particular, the fact that both the instantaneous subjective rate of discount and coefficient of absolute resistance to intertemporal substitution (risk aversion) are log-derivatives of the marginal utility of consumption should be taken into account when exploring the effect of aging on intertemporal consumption behavior.

3. Finally, when using the model to describe a naive, time-inconsistent agent's observable behavior, the effective consumption profile and its general mathematical properties can be characterized.

In line with Halevy (2015), but with a different methodology, I hope to have contributed to the clarification of the distinction between time consistency and stationarity. Halevy has done it, upstream, through rigourous axiomatic work, sustained by experimental evidence. In this article, I have done it, downstream, starting from a very general class of additive intertemporal utility functionals and solving dynamic optimization programs. Of course, the two approaches are complementary and should be investigated in parallel. The advantage of the "dynamic optimization methodology" is that, being closest to the standard practice of economists, it allows to discuss functional forms (various forms of separability, sign of high-order derivatives, etc.), not only on the basis of axioms, but also taking into account empirical regularities outside the lab.

For too long, it has been considered that time consistency and stationarity were two sides of the same coin. In an aging economy, it is crucial to have a better understanding of how preferences evolve throughout the life cycle. For that purpose, stationarity must be discarded and alternative models explored. As such, this article is a first step in an applied research agenda, concerning not only consumption and savings, but all the related fields in which intertemporal choice plays a central role: labor, health, education, finance, and many others.

Appendix

$$V(c,t) = \int_t^T v(c(s),s,t)ds$$

Table 1: Summary of instantaneous subjective discount rates and time consistency for Examples of subsection 2.1.

	$v(c,s,t)$	Θ	TC	Reference
1. Exponetial discounting	$e^{-\theta(s-t)}u(c)$	θ	Yes	Samuelson (1937)
2. Non Exponential discounting	$f(s,t)u(c)$	$-f_1(s,t)/f(s,t)$	iif $f(s,t)=\alpha(s)\beta(t)$	Burness (1976)
a. Hyperbolic discounting	$\frac{t}{s}u(c)d$	$1/s$	Yes	Ainslie (1975); Drouhin (2009)
b. Quasi-hyperbolic discounting	$e^{-\alpha\cdot(s-t)-\phi(s-t)}u(c)$	$\alpha+\phi'(s-t)$	No, if $\phi''\neq 0$	Barro (1999)
c. Seasonal discounting	$e^{-\theta(s-t)}\frac{\cos[t]+\delta}{\cos[s]+\delta}u(c)$	$\theta+\frac{sin(s)}{\delta+cos(s)}$	Yes	Drouhin (2009)
3. Uncertain lifetime				
a. Expected Utility Model	$e^{-\theta(s-t)}(1-\Pi_t(s))u(c)$	$\theta+\pi_s(s)$	Yes	Yaari (1965); Burness (1976)
b. RDU Model	$e^{-\theta(s-t)}(1-h(\Pi_t(s)))u(c)$	$\theta+\pi_t(s)\frac{h'(\Pi_t(s)))}{1-h(\Pi_t(s))}$	iif $h(x)=1-(1-x)^\alpha$	Drouhin (2015)
4. Age-varying inst. utility	$v(c,s,t)$	$-\frac{u_{12}(c,s,t)}{u_1(c,s,t)}$	iif $v(c,s,t)=\alpha(t)w(c,s)+\gamma(s,t)$	Theorem 1
a. Age varying CES	$\frac{c^{1-\gamma(s)}}{1-\gamma(s)}$	$Log(c)\gamma'(s)$	yes	

N.B. Notations are given in the text.

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