A Plaidoyer for the Play LeveL

Shahid Rahman, Zoe Mc Conaughey, Ansten Klev, Nicolás Clerbout

To cite this version:

Shahid Rahman, Zoe Mc Conaughey, Ansten Klev, Nicolás Clerbout. A Plaidoyer for the Play LeveL. 2015. halshs-01226153v2

HAL Id: halshs-01226153
https://shs.hal.science/halshs-01226153v2

Preprint submitted on 13 Feb 2018

IMMANENT REASONING OR EQUALITY IN ACTION

A PLAIDOYER FOR THE PLAY LEVEL

Shahid Rahman,[1] Zoe M^{c}Conaughey,[2] Ansten Klev,[3] Nicolas Clerbout[4]

---

[1] Univ. Lille, CNRS, UMR 8163 - STL - Savoirs Textes Langage, F-59000 Lille, France, ADA-MESHS (NPdC)

[2] Univ. Lille, CNRS, UMR 8163 - STL - Savoirs Textes Langage, F-59000 Lille, France
and Université du Québec à Montréal

[3] Academy of Sciences of the Czech Republic

[4] Instituto de Filosofía, Universidad de Valparaíso. The results of the present work have been developed in the frame of the Project: Fondecyt Regular N° 1141260.

To all of the present and past members of
the Dialogicians' team of Lille and beyond

TABLE OF CONTENTS

IV. ADVANCED DIALOGUES: PLAY LEVEL 64

V. ADVANCED DIALOGUES: STRATEGY LEVEL 74

PREFACE

Prof. Göran Sundholm of Leiden University inspired the group of logicians who nowadays develop their work in Lille and Valparaíso to undertake a fundamental review of the dialogical conception of logic by linking it to Constructive Type Logic. One of Sundholm's insights was that inference can be understood as involving an *implicit interlocutor*. This led to several investigations whose purpose was to explore the consequences of joining winning strategies to the proof-theoretical conception of meaning: while introduction rules lay down the conditions under which a winning strategy for the Proponent can be built, the elimination rules lay down precisely those elements of the Opponent's assertions that the Proponent has the right to use for building a winning strategy. The pragmatic and ethical features of obligations and rights naturally brings forth the dialogical interpretation of natural deduction.

During the 2012 Visiting Professorship of Prof. Sundholm in Lille, the logic group of Lille started probing possible ways of implementing Per Martin-Löf's Constructive Type Theory (CTT)[5] in the dialogical perspective. The first publication in particular on the subject—Aarne Ranta's (1988) paper—was read and discussed during Sundholm's seminar. These discussions strongly suggested that the game-theoretical conception of quantifiers, which marshalls interdependent moves, provides a natural link between CTT and dialogical logic. This idea triggered several publications by the group of Lille in collaboration with Nicolas Clerbout and Juan Redmond at the University of Valparaíso, including that of the (2015) book by Clerbout & Rahman providing a systematic development of this way of linking CTT and the dialogical conception of logic.

However the (Clerbout & Rahman, 2015) book was written from the CTT perspective on dialogical logic, rather than the other way round. The present book, *Immanent Reasoning or Equality in Action*, should provide the other perspective in the dialogue between the dialogical framework and Constructive Type Theory.

In order to develop a dialogical perspective on the links between CTT and dialogical logic we will follow three complementary paths:

A. One of the chief ideas animating our study is that we believe Sundholm's (1997)[6] notion of *epistemic assumption* is closely linked to the *Copy-Cat rule* or *Socratic rule* that distinguishes the dialogical framework from any other game-theoretical approach; this link is established through the dialogical understanding of definitional equality.

B. We will join—with some nuances linked to point **C** below—Martin-Löf's (2017a; 2017b) suggestions that the new insights provided by the dialogical framework mainly amount to the following three interconnected points:
   B.1. the introduction of *rules of interaction* rather than of inference rules;
   B.2. the challenge to what Kuno Lorenz (2010a, p. 71) calls the *semantization of pragmatics*: deontic features are formalized with the help of specific propositional operators (and indexes) upon which the truth-value of the resulting proposition is made dependent;
   B.3. the central role of the notion of *execution* in the rules of interaction: executions are responses to questions of *knowing how*.

---

[5] For an overview see for instance (Nordström, Petersson, & Smith, 1990; 2000), (Primiero, 2008), (Thompson, 1991).

[6] See also (Sundholm, 1998; 2012; 2013b).

C. As indicated by the subtitle, "A Plaidoyer for the Play Level", we will stress the importance of the play level over the strategy level: this binds the point of execution with that of equality.

In relation to A and B.3, the present book can indeed be read as furthering Sundholm's own extension to inference of Austin's remark (1946, p. 171) on assertion acts; Sundholm (2013a, p. 17) did indeed produce this forceful formulation:

> *When I say* therefore, *I give others my authority for asserting the conclusion, given theirs for asserting the premisses.*

In recent lectures, Per Martin-Löf used the dialogical perspective with epistemic assumptions in order to escape a form of circle threatening the explanation of the notions of inference and demonstration. A demonstration may indeed be explained as a chain of (immediate) inferences starting from no premisses at all. That an inference

$$\frac{J_1 \cdots J_n}{J}$$

is valid means that one can make the conclusion (judgement J) evident on the assumption that $J_1, \dots, J_n$ are known. Thus the notion of epistemic assumption appears when explaining what a valid inference is. According to this explanation however, we cannot take 'known' in the sense of *demonstrated*, or else we would be explaining the notion of inference in terms of demonstration when demonstration has been explained in terms of inference. Hence the threatening circle. In this regard Martin-Löf suggests taking 'known' here in the sense of *asserted*, which yields epistemic assumptions as judgements others have made, judgements whose responsibility others have already assumed. An inference being valid would accordingly mean that, given others have assumed responsibility for the premisses, I can assume responsibility for the conclusion:

A. Martin-Löf's circularity problem

> *The circularity problem is this: if you define a demonstration to be a chain of immediate inferences, then you are defining demonstration in terms of inference. Now we are considering an immediate inference and we are trying to give a proper explanation of that; but, if that begins by saying: Assume that $J_1, \dots, J_n$ have been demonstrated—then you are clearly in trouble, because you are about to explain demonstration in terms of the notion of immediate inference, hence when you are giving an account of the notion of immediate inference, the notion of demonstration is not yet at your disposal. So, to say: Assume that $J_1, \dots, J_n$ have already been demonstrated, makes you accusable of trying to explain things in a circle. The solution to this circularity problem, it seems to me now, comes naturally out of this dialogical analysis.* […]
>
> *The solution is that the premisses here should not be assumed to be known in the qualified sense, that is, to be demonstrated, but we should simply assume that they have been asserted, which is to say that others have taken responsibility for them, and then the question for me is whether I can take responsibility for the conclusion. So, the assumption is merely that they have been asserted, not that they have been demonstrated. That seems to me to be the appropriate definition of epistemic assumption in Sundholm's sense.*[7]

The present study makes a further step, namely that of relating judgemental equality with the rule known in the literature as the *Copy-Cat rule*, or *Formal rule*, or, as more aptly called now by (Marion & Rückert, 2015), the *Socratic rule*.[8] We hold it as one of our main tenets that this relation provides both a simpler and a more direct way to implement the Constructive Type Theoretical approach within the dialogical framework. Such a reconsideration of the Socratic rule roughly amounts to the following:

1. When Proponent **P** makes a move bringing forward a local reason—say *b*—to defend an elementary proposition *A*, this move can be challenged by the

---

[7] Transcription by Ansten Klev of Martin-Löf's talk in May 2015 (Martin-Löf, 2015).

[8] See for instance below, section III.2.2 or section VII.2.1.

Opponent **O**. That is, given **P** $b{:}A$, the antagonist may play the attack **O** $? = \mathrm{b}$.

2. To respond to such a challenge from **O**, **P** must bring forward a definitional equality expliciting that the local reason chosen by **P** copies precisely the reason **O** chose when stating A. In short, this equality expresses at the object-language level the fact that **P**'s defence move rests on the authority **O** has previously asserted when producing her local reason.

More generally, according to this view a definitional equality established by **P** and brought forward while defending the proposition A expresses the equality between a local reason (introduced by **O**) on the one hand, and the instruction on the other hand used for building a local reason brought forward by **O** when stating A. A definitional equality can therefore be read as a computation rule indicating how to compute the instructions **O** brought forward during a play.[9]

From the dialogical perspective though, providing local reasons must be distinguished from providing equalities: while providing an explicit local reason b is a way of answering a *why* question, such as "why does A hold?", providing an equality is more a way of answering a *how* question, such as "how do you show that b accomplishes the explicative task?". Equalities thus express how to execute or carry out the actions encoded by the local reasons.

Let us recall that from the strategic point of view, **O**'s moves correspond to elimination rules (including the selector-functions deployed by these rules) of demonstrations. Thus, the dialogical rules prescribing how to introduce a definitional equality correspond—at the strategy level—to the definitional equality rules for CTT as applied to the selector-functions involved in the elimination rules.

We are in this fashion extending the dialogical interpretation of Sundholm's *epistemic assumption* to the rules that set up the definitional equality of a type. Actually, Sundholm (2017) himself suggests in his section 4 this extension when he points out that if some object, say a, is granted by a suitable *epistemic assumption* to be a proof-object of C, then it *executes to a canonical proof of* C. In other words, on the grounds of the epistemic assumption we know that a must be equal to a canonical element of C.

Notice however that from the dialogical perspective equalities grounded on the sole authority of the Opponent (i.e. on epistemic assumptions) are a trade-mark of what we call *formal dialogues*.

Yet the dialogical perspective also includes *material dialogues*, where the Opponent must carry out some process *specific* to the proposition at stake before the Proponent can answer to the *how*-question with a suitable equality. In other words, though equalities of material dialogues are the result of the application of the Socratic rule, they are not "merely" grounded on epistemic assumptions.

In relation to **B1** and **B2,** the Oslo and Stockholm lectures of Martin-Löf (2017a; 2017b) contain challenging and deep insights in dialogical logic, and the understanding of *defences as duties* and *challenges as rights* is indeed at the core of the deontic feature underlying the dialogical framework. More precisely, these two rules *Req1* and *Req2*:

$$(Req\ 1)\,\frac{\vdash C}{?\vdash_{may}\, C}$$

and

---

[9] These elements are formalized in the Socratic rule for immanent reasoning, section VII.2.1.

$$(Req\ 2)\,\frac{\vdash C \qquad ?\vdash C}{\vdash_{must}\ C'}$$

both condense the particle rules of meaning and bring to the fore the normative feature of those rules. What is more, Martin-Löf points out rightly that they should not be called *rules of inference* but *rules of interaction.*

Still, a dialogician might wish to draw further distinctions to the divide between play level rules and those of the strategy level, such as distinctions between players, or the distinction in terms of choice as to how to defend or challenge moves: it is such a distribution of choices that distinguishes the meaning for instance of the conjunction and of the disjunction; the meaning of a disjunction binds the right to state a disjunction with the *defender's duty to choose* a component of the disjunction to defend, but the meaning of the conjunction binds the right to challenge it with the *challenger's duty to choose* the side to be requested.[10]

On our view, point **C** is at the core of the innovations of the dialogical framework and our point of departure from Ranta's (1988) seminal paper: he proposes to identify proof-objects with winning-strategies, so that we have canonical and non-canonical winning-strategies. Winning strategies are however not primitive in the dialogical framework, but are constituted by some finite sequence of legal moves (that is, a sequence of moves which observes the game rules) called plays. The notion of plays is what grounds meaning within the dialogical framework, and this notion also leads to the notion of proposition: in the standard presentation of dialogical logic a proposition is defined as a *dialogue-definite expression*, that is, an expression *A* such that there is an individual play about *A* that can be said to be lost or won after a finite number of steps, following some given rules of dialogical interaction.[11]

As discussed in chapter III and section XI.1, the rock-bottom of the dialogical approach to CTT is the play level notion of dialogue definiteness of the proposition. Thus to paraphrase (Lorenz, 2001, p. 258): *for an expression to count as a proposition A* there must exist an individual play about the statement **X** ! *A*, in the course of which **X** is committed to bring forward a local reason to back that proposition, play which must reach a final position with either win or loss after a finite number of moves according to definite particle and structural rules.

Though performing the interaction schemata defining a play is in this sense a crucial aspect of the dialogical framework, it must be stressed that the actualization of a play (performing it) *does not* require winning the play. Immanent reasoning thus conceives *performance* as *putting dialogue definiteness into action*.

In a nutshell, we call our dialogues involving rational argumentation *dialogues for immanent reasoning* precisely because the *reasons* backing a statement, now *explicit denizens* of the object-language of plays, *are internal* to the development of the dialogical interaction itself: the emergence of concepts are not only games of giving and asking for reasons (games involving *why*-questions), they are also games including moves establishing *how is it that the reason brought forward accomplishes the explicative task.* Immanent reasoning is thus a dialogical framework for games of *why* and *how*.

---

[10] In the conclusion we enrich Martin-Löf's (2017a; 2017b) rules Req1 and Req2 with players and with choice-options.

[11] See for instance (Lorenz, 2001, p. 258): "[…] *for an entity to be a proposition there must exist an individual play, such that this entity occupies the initial position, and the play reaches a final position with either win or loss after a finite number of moves according to definite rules."* See also below chapter III and conclusion XI.1.

Acknowledgments:

Many thanks to Mark van Atten (Paris1), Giuliano Bacigalupo (Zürich), Charles Zacharie Bowao (Univ. Ma Ngouabi de Brazzaville, Congo), Christian Berner (Lille3), Michel Crubellier (Lille3), Pierre Cardascia (Lille3), Oumar Dia (Dakar), Marcel Nguimbi (Univ. Marien Ngouabi de Brazzaville, Congo), Adjoua Bernadette Dango (Lille3 / Alassane Ouattara de Bouaké, Côte d'Ivoire), Steephen Rossy Eckoubili (Lille3), Johan Georg Granström (Zürich), Gerhard Heinzmann (Nancy2), Muhammad Iqbal (Lille3), Matthieu Fontaine (Lille3), Radmila Jovanovic (Belgrad), Hanna Karpenko (Lille3), Laurent Keiff (Lille3), Sébastien Magnier (Saint Dennis, Réunion), Clément Lion (Lille3), Gildas Nzokou (Libreville), Juan Redmond (Valparaíso), Fachrur Rozie (Lille3), Helge Rückert (Mannheim), Mohammad Shafiei (Paris1), and Hassan Tahiri (Lisbon), for rich interchanges, suggestions on the main claims of the present study and proof-reading of specific sections of the text.

Special thanks to Kuno Lorenz (Univ. Saarland) and Mathieu Marion (Montreal), for many suggestions and inspired discussions.

Special thank also to Gildas Nzokou (Libreville), Steephen Rossy Eckoubili (Lille) and Clément Lion (Lille), who are working out through a visiting professorship of Gildas Nzokou in Lille a Pre-Graduate Textbook in French on dialogical logic and CTT. In the present book, Eckoubili is the author of the sections with exercises for dialogical logic.

We are also very thankful to our host institutions which have fostered our researches in the framework of specific research-programs: the results of the present work have been developed in the frameworks of

- the transversal research axis *Argumentation* (UMR 8163: STL),
- the research project ADA at the MESHS-Nord-pas-de-Calais and
- the research projects: ANR-SÊMAINÔ (UMR 8163: STL) and
- Fondecyt Regular N° 1141260 (Chile).

Shahid Rahman,[12] Zoe M^{c}Conaughey,[13]
Ansten Klev,[14] Nicolas Clerbout[15]

---

[12] Univ. Lille, CNRS, UMR 8163 - STL - Savoirs Textes Langage, F-59000 Lille, France, ADA-MESHS (NPdC)

[13] Univ. Lille, CNRS, UMR 8163 - STL - Savoirs Textes Langage, F-59000 Lille, France
and Université du Québec à Montréal

[14] Academy of Sciences of the Czech Republic

[15] Instituto de Filosofía, Universidad de Valparaíso. The results of the present work have been developed in the frame of the Project: Fondecyt Regular N° 1141260.

I. INTRODUCTION: SOME BRIEF HISTORICAL AND PHILOSOPHICAL REMARKS

The present volume develops a new way of linking Constructive Type Theory (CTT) with dialogical logic by following these three complementary paths, as mentionned in the preface:

A. The path observing that Sundholm's (1997)[16] notion of *epistemic assumption* is closely linked to the *Copy-cat* and *Socratic rules*[17] and that it provides the dialogical conception of definitional equality;
B. the path joining (in principle) Martin-Löf in his (2017a; 2017b) suggestions, according to which the new insights provided by the dialogical framework mainly amount to the following three interconnected points:
    B.1. the introduction of *rules of interaction* rather than of rules of inference;
    B.2. the challenge to the *semantization of pragmatics* and the claim of the deontic nature of logic;[18]
    B.3. the central role of the notion of *execution* in the rules of interaction: executions are responses to questions of *knowing how*.
C. The path stressing the importance of the *play level* and the associated notion of *dialogue-definiteness*.

Before displaying some of the conceptual background behind the project we will first make some brief historical remarks concerning the *dialogical turn* that took Lorenzen (1955) from his *Operative Logik* to the inception of *dialogical logic*.

I.1 The dialogical turn and the operative justification of intuitionistic logic

The origins of a deontic nature of logic in its dialogical conception can be traced back to Paul Lorenzen's 1958 endeavour of overcoming difficulties specific to his (1955) *Einführung in die Operative Logik und Mathematik* (1955), which lead him to turn the normative perspectives of the operative logic into the dialogical framework. We will here closely follow Schroeder-Heister's thorough (2008) paper on the subject.[19] It should be noted that these difficulties are reminiscent of Martin-Löf's circularity puzzle mentioned in the Preface and which motivated his dialogical interpretation of the notion of epistemic assumption (see p. 7).

---

[16] See also (Sundholm, 1998; 2012; 2013a).

[17] See section III.2.2 for the Copy-cat rule, and section VII.2.1 for the Socratic rule.

[18] In fact, as opposed to Martin-Löf's understanding of dialogical logic, Lorenz's dialogical constructivism does not only reject the *semantization of pragmatics* in which deontic features are formalized using specific propositional operators and indexes upon which depends the truth-value of the resulting proposition, but it also rejects the *pragmatization of semantics* in which a propositional kernel is complemented by moods yielding assertions, questions, commands, and so on. According to dialogical constructivism, pragmatic and semantic features are produced within one and the same act. See (Lorenzen, 1969), (Kamlah & Lorenzen, 1972), (Lorenzen & Schwemmer, 1975). It is precisely this tenet on the dual nature of actions in both their *significative* and *communicative* role, thoroughly worked out by Lorenz (2010a, pp. 71-80), that leads to this central claim that logic is part of ethics—see section XI.5 for further details.

[19] See also Lorenz's (2001) study of the origins of the dialogical approach to logic.

I.1.1 Admissibility in operative logic

In the context of the operative justification of intuitionistic logic, the operative meaning of an elementary proposition is understood as a proof of its derivability in relation to some given calculus. Calculus is here understood as a general term close to the *formal systems* of Curry (1951) which include basic expressions, and rules for producing complex expressions out of basic ones. More precisely, as Schroeder-Heister puts it:

> *Lorenzen starts with elementary calculi (OL, §1) which permit to generate words (strings of signs) over an arbitrary (finite) alphabet. The elements of the alphabet are called* atoms, *the words are called* sentences *("Aussagen"). A calculus K is specified by giving certain initial formulas ("Anfänge") A and rules* $A_1, \dots, A_n \rightarrow A$.[20]

Instead of starting with the functor-argument structure common in logic, Lorenzen starts here with an arbitrary word-structure, where expressions in K are just strings of atoms and variables, allowing his notion of calculus to be particularly general.

In such a framework, logic is introduced as a system of proof procedures for asserting the *admissibility* of rules:[21]

> *A rule R is called admissible in a calculus* K*, if its addition to the primitive rules of* K*—resulting in an extended calculus* $K + R$*—does not enlarge the set of derivable sentences. If* $\vdash_K A$ *denotes the derivability of* A *in* K*, then R is admissible in K if*
>
> $$\vdash_{K+R} A \text{ implies } \vdash_K A$$
>
> *for every sentence A.*

Thus, implication is explained in terms of admissibility. But how is admissibility to be explained?

I.1.2 Implication and admissibility: another circle?

Since implication is explained by the notion of admissibility, admissibility cannot be explained by the notion of implication. In fact, Lorenzen (1955) in his chapter 3 invests admissibility with an operative meaning through the notion of an *elimination procedure*, stating that R is admissible in K if every application of R can be eliminated from every derivation in $K + R$.The above implication ($\vdash_{K+R} A$ implies $\vdash_K A$) reduces to a form of elimination procedure, for the derivation in $K + R$ can be brought down to a derivation which no longer uses R: the R rule can be disposed of. Schroeder-Heister thus concludes:

> *According to Lorenzen, this is the sort of insight (evidence) on which constructive logic and mathematics is based. It goes beyond the insight that something is derivable in K, but is still something which has a "definite" meaning. (Schröder-Heister, 2008, p. 217)*

This approach thus goes beyond the formalistic focus on derivability: what provides meaning is *the further understanding gained through the notion of admissibility*. In this respect, according to Schröder-Heister:

> *Lorenzen's theory of implication is based on the idea that an implicational sentence* $A \rightarrow B$ *expresses the admissibility of the rule* $A \rightarrow B$*, so the assertion of an implication is justified if this implication, when read as a rule, is admissible. In this sense an implication expresses a meta-statement about a calculus. This has a clear meaning as long as there is no iteration of the implication sign. (Schröder-Heister, 2008, p. 222)*

---

[20] (Schröder-Heister, Lorenzen's Operative Justification of Intuitionistic Logic, 2008). All the following quotations of this section, if not otherwise specified, will come from this same source.

[21] Nowadays, the notion of *admissibility* is a fundamental concept of proof-theory; Schröder-Heister (2008, p. 218) pointed out that Lorenzen was the one to have coined this term.

It is precisely to deal with iterated implications that Lorenzen develops the idea of finitely iterated meta-calculi. Schröder-Heister (2008, p. 235) points out that the operative approach has its own means to draw the distinction between *direct* and *indirect inferences*, a distinction which triggered Martin-Löf's puzzle quoted in our preface (see p. 7). In this sense, the implication $A \to B$ can be asserted as either

(i) *a direct derivation* in a meta-calculus MK, based on a demonstration of the admissibility in K of the rule $A \to B$, or as

(ii) *an indirect derivation* by means of a formal derivation in MK using axioms and rules already shown to be valid.

In the context of operative logic, *direct knowledge* or *canonical inference* of the implication $A \to B$ is obtained by the demonstration of the admissibility in K of the rule $A \to B$, and *indirect knowledge* or *non-canonical inference* results from the derivation of $A \to B$ by means of rules already established as admissible.

I.1.3 **From admissibility to dialogue-definiteness**

There is however a high price to pay for this way out of the circularity problem, as knowledge cannot be characterized in the required way showing that the reasoner actually masters the meaning of an implication. Schröder-Heister (2008, p. 236) indeed pointed out that in the Gentzen-style introduction rule for implication, the conclusion prescribes that there is a derivation of the consequent from the antecedent, *independently of the validity of the hypothetical derivation* itself. Indeed, the meaning explanation of the implication is based on the idea that from the assumption of a derivation of the antecedent a method can be found that transforms the derivation in one of the consequent.

This undesired consequence on knowledge motivated Lorenzen to move to the dialogical framework in which the *play level* takes care of all the issues on meaning and *strategies* are associated to validity features: in this context, a proof of admissibility amounts to showing that some specific sequence of plays yields a winning strategy.

Now, if dialogues are to be conceived as mediators of meaning, these dialogues must be games actually playable by human beings: it must be the case *that we can actually perform them*—see our chapter XI.1.[22] These games must therefore be finite, though this does not excluded that there might be a (potentially) infinite number of them. In fact it is the notion of *dialogue-definiteness* that provides both, the basis for implementing the requirement of human-playable games, and the notion of proposition. Under such a background a proposition is defined as a dialogue-definite expression, that is, an expression *A* such that there is an individual play about *A*, that can be said to be lost or won after a finite number of steps, following some given rules of dialogical interaction.

Notice however that the notion of dialogue-definiteness is not bound to knowing how *to win*—this is rather a feature that characterizes winning strategies; to master meaning of an implication, within the dialogical framework, amounts rather to *know how to* develop an *actual play for it*. In this context it is worth mentioning that during the Stockholm and Oslo talks on dialogical logic, Martin-Löf (2017a; 2017b) points out that one of the hallmarks of the dialogical approach is the notion of *execution*, which—as mentioned in the preface—is close to the requirement *of bringing forward a suitable equality while performing an actual play*. Indeed from the dialogical point of view (see I.2.2), an equality statement comes out as an answer to a question on the local reason *b* of the form *how*: *How do you show the efficiency of b as providing a reason for A*? In this sense the *how*-question presupposes that *b* has been brought-forward as an

---

[22] See also (Marion, 2006, p. 231).

answer to a *why* question: *Why does A hold*? Thus, equalities express the way how to execute or carry out the actions encoded by the local reason; however, the actualization of a play-schema *does not require the ability of knowing how to win a play*. Thus, while execution, or *performance*, is indeed important—see our point **B**.3 above, the backbone of the framework lies in the *dialogue-definiteness notion of a play*.

Perhaps a way of formulating the distinction we are aiming at is to stress the difference between *ability* and *knowing how*. In this context, one might speak of *ability* in the sense of the *ability to win*—in a way not far from Peregrin's (2014, pp. 228-229) notion of *tactics*—, but *ability* has strategy level underpinnings rather than play level ones. The fundamental notion in this dialogical perspective is therefore that of *knowing how to do develop a play* for some proposition C, rather than that of having the ability to develop a *winning* play for C.

This is how the problematic case of operative logic is overcome by a turn where the actions that were understood as operations within the framework of *operative logic* are now understood as *dialogical interaction*. In other words, *the dialogical approach turned monological operations into dialogical interactions* (see section XI.5).[23]

Content and Interaction

Another important issue in the passage from the operative to the dialogical framework is that while the operative framework allowed quite naturally to deal with mathematical content, the dialogical framework appears to be restricted to the meaning of logical constants. This has been the subject of many criticisms, old (Hintikka, 1973, pp. 77-82), and new (Trafford, 2017, pp. 86-88); see chapter XI, in particular section XI.4. There have nonetheless been attempts to compensate this gap by introducing in the dialogical framework definitions conceived as operation rules—see (Piecha & Schröder-Heister, 2011) and (Piecha, 2012). However, these attempts have rather been received as highly programmatic.

It is actually quite fair to say that the notion of *material dialogues*—that is dialogues containing rules for expressions other than logical constants—seems to be underdeveloped in respect to *formal dialogues* (restricted mostly to logical constants) which have gathered much more attention. It is also true that a similar kind of criticism has also been raised against inferentialist approaches to meaning, and *operative logic* and dialogical logic, inspired by these inferentialist approaches, seemed to inherit this problem. However, let us stress that the fathers of dialogical logic where aware of the need of a contentual (*material* was the chosen term) basis from the beginning, and they tackled the issue with different devices. Lorenz (1970) in particular dedicated to this issue very thorough and deep studies, most of them collected in (Lorenz, 2010a; 2010b).

One of the widely acknowledged achievements of Constructive Type Theory rests in its ability to furnish the means to develop a language in which mathematical content can be introduced with the same kind of inferential rules displayed by systems of natural deduction. This virtue of CTT motivated us to explore the possibilities of enriching the language of the dialogical framework with the means of CTT. However, though chapter X deals to some extent with mathematical content and contains some brief remarks on empirical content, we shall here content ourselves with the more modest task of setting the basis for future, more thorough, developments on the issue.

The Ancient Greek Roots of the Dialogical Turn and its Renaissance

Before turning to the links between CTT and the dialogical framework, let us point out that Lorenzen's *dialogical turn* did not come out of the blue: Lorenzen was an

---

[23] Winning strategies in the first writings of Lorenzen and Lorenz (1978) were formulated in the form of sequent-calculus; thus the demonstration of "admissibility" amounts in this context to show that the sequence of plays determined by the local and structural rules for the logical constants yield those of the sequent calculus.

admirer of Ernst Kapp’s (1942) perspective on the dialectical origins of logic, and had a frequent and lively interaction with the philologist Kurt von Fritz. In fact, Lorenzen had a thorough and intimate knowledge of Ancient Greek mathematics and logic, even before he gathered the chair in Kiel in 1956, where he continued these kinds of studies then in contact with Oskar Becker (who influenced Lorenzen’s appointment). In this context it might also well be that the inception of operative logic had a dialectical background that finally found its explicit expression in his *Logik und Agon* (1958).[24]

A striking witness of the Ancient Greek roots of the passage from the operative to the dialogical framework is *Ein formales Modell der Syllogistik des Aristoteles* (1964) by Lorenzen's student Kurt Ebbinghaus, where, after developing a remarkable proof-theoretical reconstruction of Aristotle's syllogistic in the style of *Operative Logik*, he discusses the advantages of a dialogical approach to Aristotle's notion of quantification—see (Ebbinghaus, 1964, pp. 57-58).[25]

In this context it is worth mentioning that nowadays history of mathematics is experiencing a revival in the studies linking the development of deductive proof in Ancient Greek mathematics with the dialogical practices of those days. Some of the most thorough studies on the subject are the ones of G. E. R. Lloyd (1996) and Reviel Netz (1999; 2005; 2009) who stress the importance of debates and oral dialogues for the emergence of classical mathematics in Ancient Greece. It seems like from the very start of mathematics the notion of proof was associated with the endeavour of explaining *why* the putative statement is true. Explaining *why* something is the case requires conceiving this explanation as directed towards a *stubborn interlocutor*, a point which does not hold only for the notion of proof in Ancient Greek mathematics—see (Fischer, 1989, p. 50).

Let us end this more historical section with the remark that the normative approach underlying the Dialogical Constructivism Program of Erlangen that emerged from the dialogical turn coupled explaining *why a putative statement is true* with the task of explaining *what the statement is good for*: according to the Erlangen-Programme the general notion of explaining is always conceived as *explaining to an audience what the purposes of an specific action (that give rise to the claim) are*—see (Lorenzen, 1969), (Kamlah & Lorenzen, 1972), (Lorenzen & Schwemmer, 1975). [26]

The general epistemological lesson behind Lorenzen's bold proposal of a dialogical turn might be put in the following words: the dialogical turn is an invitation to think of actions involving scientific enquiry as interaction. It took a while until the scientific community picked up Lorenzen's gauntlet, but as the most recent studies and projects in history and philosophy of logic, mathematics, foundations of computer sciences, linguistics, and epistemology point out, the time seems ripe now for the development of such a perspective.[27]

I.2 Linking Dialogues and Constructive Type Theory

I.2.1 Equality and the Socratic Rule

One of the main tenets of the present study is that a direct way to implement the Constructive Type Theoretical (CTT) approach within the dialogical framework is to focus on the CTT notion of judgemental equality.

---

[24] Kuno Lorenz conveyed this information to S. Rahman by a personal email.

[25] See (Crubellier, Marion, McConaughey, & Rahman, 2018) and (Rahman & Lion, 2018).

[26] For a brief presentation of the philosophical tenets of Dialogical Constructivism see section XI.7.

[27] See, among others, (Fischer, 1989), (Sellars, 1991), (Brandom, 1997), (Girard, 1999), (Heinzmann, 2006), (Ginzburg, 2012), (Lecomte, 2011), (Lecomte & Quatrini, 2010), (Paseau, 2011), (Peregrin, 2014), Duthil Novaes (2015).

In CTT, every category needs to be associated with a criterion of identity (see chapter II, written by Ansten Klev). More precisely, there are two basic forms of categorical judgement in CTT:

i) $a : C$
ii) $a = b : C$

The first is read "a is an object of the category C", and the second, the judgemental or definitional equality, is read "a and b are identical objects of the category C". We thus require that any category C occurring in a judgement of CTT be associated with a

- *criterion of application*, which tells us what a C is; the fact that a meets this criterion is precisely what is expressed in $a : C$; and a
- *criterion of identity*, which tells us what it is for a and b to be identical C's; the fact that a and b together meet this criterion is precisely what is expressed in $a = b : C$.

In the dialogical framework, on the other hand, equality involves plays in which players explicitly expose in the object-language[28] the reasons they have for stating judgements. More precisely, as mentioned in the preface, definitional equality is implemented at the play level by means of the Socratic rule.

The Socratic rule is one of the most salient characteristics of dialogical logic. As discussed by (Marion & Rückert, 2015), it can be traced back to Aristotle's reconstruction of the Platonic dialectics. A purely argumentative point of view can be defined within dialectics as refraining from calling on some authority beyond what has actually been brought forward during the current argumentative interaction (following the suitable rules determined by the game). Thus, when an elementary statement is challenged, the challenge can be answered only by invoking the challenger's own concessions (or his own constructions). In such a context, the Socratic rule can be understood in the following way, when a player plays an elementary statement:[29]

> *"My reasons for stating this proposition you are now challenging are exactly the same as the ones you brought forward when you yourself stated that very same proposition."*

In this fashion the Socratic rule provides for equality, but through interaction: equality is built within an argumentative play by copying exactly the same reasons for a proposition as what the other player has already provided. Statements of definitional equality have thus emerged in a dialogical perspective, in particular reflexivity statements such as

$$p = p : A$$

which express the fact that if the Opponent states the elementary proposition A, then the Proponent can do the same, that is, play the same move and do it on the same grounds which provide the meaning and justification of A, namely p.

In order to introduce in the object-language of the dialogical framework (dialogues for immanent reasoning) definitional identities at the play level, we must extend the language of a dialogical game with statements of the form

$$p : A$$

where on the right-hand side of the colon is the proposition A, and on the left-hand side is the *local reason* brought forward to back the proposition *during a play* (see chapter VI). The local reason is therefore *local* if the force of the statement is limited to the level of plays. But when the assertion $p : A$ is backed by a *winning strategy*, the

---

[28] This is the main feature of dialogues for immanent reasoning, the dialogical framework which incorporates features of CTT. For a presentation of this framework, see chapters VI-VII. The Socratic rule is the equivalent in immanent reasoning of the Copy-Cat rule in the standard dialogical framework. For a presentation of the standard framework, see chapters III-V.

[29] See (Rahman, Clerbout, & Keiff, 2009) and (Rahman & Keiff, 2010).

judgement asserted draws its justification precisely from that strategy, thus endowing p with the status of a *strategic reason* (see section VII.7).

Thus *reasons* backing a statement are manifest at the object-language level, and are internal to the development of a play, which is why we have named this dialogical framework incorporating CTT features making these reasons explicit, *dialogues for immanent reasoning*.

The notion of *reason* (local and strategic) shows how we link the dialogical framework to CTT, but also how we can preserve the flexibility of the dialogical framework and bring out its full potential, ranging from material dialogues (at the play level) to the equivalent of the CTT demonstrations (at the strategy level) and all that which comes in-between. Immanent reasoning and equality in action are in this sense not exclusively at one level, but are embedded in the whole framework through the constitutive role of the Socratic rule.

I.2.2 Local Reasons and Content: The Socratic Rule within *Material Dialogues*

Local reasons are fundamental to dialogues for immanent reasoning as they also contribute to *material dialogues* for elementary propositions (see chapter X). Informally, the idea is that if the Proponent is entitled to his statement on the elementary proposition A, it is because he is ready to defend A by giving a *reason* in favor of that statement. The Proponent can find such a local reason backing A in a process governed by the Socratic rule which spells out the precise forms of the local reason required by the *content* of A. The appropriate local reason will thus be governed by the Socratic rule (which ensures, by preventing the Proponent to provide his own grounds for what he says, that the grounds for stating an elementary proposition are taken from the play itself, that is, it ensures the reasons are immanent to the play), but this rule will have to be adapted to each individual content brought forward, bringing us to *material dialogues*, and should be contrasted with the development of *formal dialogues* in which the Socratic rule allows the Proponent to replicate an elementary proposition *A* stated by the Opponent, but also to replicate the local reasons that the Opponent brought forward when stating A, and this independently of checking what these local reasons are: in purely formal dialogues, if the Opponent states for instance that 2 is odd, the Proponent can state this too on the sole grounds that the Opponent herself stated it and provided some reason for it, whatever this reason be.

But in material dialogues, if the Proponent asserts for example: “1 is an uneven number,” the Opponent would be entitled to request of the Proponent a natural number n such that $1 = 2.n + 1$. The local reason in this case would be $1 = 2.0 + 1$, which is a reason specific to that particular statement the Opponent challenged.

In these dialogues, the Socratic rule determines the canonical elements and the definitions (the definitional equalities) specific to each of the elementary expressions in a play. This yields *material truth*.

Stating the *material truth* of a proposition requires exhibiting a local reason *specific* to the content of that proposition.

The origins of the normative approach to meaning can be found in this aspect of the dialogical framework: *meaning as use* should be understood as *the use is spelled out by a rule of dialogical interaction* which applies to the meaning of the logical constants, but also to the meaning of the elementary propositions. Strictly speaking, the meaning of each elementary expression requires a specific rule that determines its proper content and distinguishes it from other elementary propositions.[30] Material dialogues in this

---

[30] Jaroslav Peregrin (2014, pp. 3-5) calls the notion of *use* understood as *following a rule* “role”. *Role* distinguish linguistic uses from other uses such as using a hammer.

perspective are not only a matter of putting back normativity in logic, they also deal with the important matter of elaborating a contentful language.

I.2.3 **Dialogues for Immanent Reasoning as *Games of Why and How***

The present study aims at showing that, if we follow Lorenzen's and Lorenz's advice of looking at mathematical operations as interaction, then definitional equality can be considered as exposing the dialogical intertwining of *entitlements* and *duties*. In this perspective, the standard monological presentation of these rules for both definitional and predicative equality implicitly encodes an underlying process, a process in which the Proponent "copies" some of the Opponent's choices, thus providing its dialogical and normative roots.

We shall in this fashion rally to some extent to Robert Brandom's insight[31] that conceptual meaning is entirely constituted by the way judgements are inserted into *games of giving and asking for reasons*, the touchstone of inferential pragmatism. Our task now lies in describing, in the context of these *games of giving and asking for reasons*, the moves on the ontological level grounding statements of equality (definitional equality), and on the propositional level grounding statements of identity (the dyadic-predicate of standard first-order logic). This is necessary in order to have games of *Why* and *How*.[32]

The emergence of concepts, we claim, are not only games of giving and asking for reasons (games involving *Why*-questions), they are also games that include moves establishing *how it is that the reason brought forward accomplishes the explicative task.* Immanent reasoning are dialogical games of *Why* and *How*.

We call our dialogues involving rational argumentation *dialogues for immanent reasoning* precisely because *reasons* backing a statement, now *explicit* denizens of the object-language of plays, are *internal* to the development of the dialogical interaction itself.

I.3 A Basic Overview of the Book

One of the important lessons of the Constructive Type Theory approach to meaning is that equality is at the center of a constructivist project of types. Indeed, it has been stressed that the constructivist parallel to Quine's (1969, p. 23) notorious "*no entity without identity*" is

- No entity without a type
- No type without a criterion of identity

---

[31] See for instance (Brandom, 1994). To some extent only, for it seems like Brandom starts from the strategy level rather than from the play level as we do.

32 As discussed in section X.5, Brandom's approach only has the propositional level (i.e. his framework does not include the ontological level of the *local reasons* relevant fort the backing of the proposition involved in the judgement), perhaps because he fears that such a move would amount to incorporating into the framework an *authority* which would be *external* to the games that determine concepts. As far as we understand it, this is a serious limitation of Brandom's approach since it fails to distinguish between the notations, or written forms, concerning the ontological level, and those concerning the propositional level: the present book, we hope, shows how to make the ontological level immanent to the dialogical process of reasoning. This suggests that the dialogical approach to CTT offers a way to integrate within one epistemological framework the two conflicting readings of Willfried Sellars' (1991, pp. 129-194) notion of *space of reasons* brought forward by John McDowell (2009, pp. 221-238) on the one side, who insists in distinguishing world-direct thought and knowledge gathered by inference, and by Robert Brandom (1997) on the other side, who interprets Sellars' work in a more radical anti-empiricist manner. The point is not only that we can deploy the CTT-distinction between reason as a premise and reason as the piece of evidence justifying a proposition, but it is also that the dialogical framework allows distinguishing between the objective justification (strategy) level targeted by Brandom (1997, p. 129) and the subjective (play) level stressed by McDowell—see also (Rahman, 2017).

Definitional equality is central to the constitution of a type. Moreover, in the context of logic, definitional equality makes the coordination of analytic and synthetic steps explicit. So, if we are looking at ways of linking the normativity of dialogical logic with the normativity of CTT, it becomes apparent that we should try to provide an answer to the question of how the criterion of identity of a type is manifested in the dialogical framework—this is what the book is about.

The purpose of the book is therefore rather technical, though it has deep philosophical roots in what argumentation and reasoning are. Perhaps one way to condense our philosophical perspective on identity is that it has been developed in the general epistemological framework according to which

> *argumentation is, all in all, nothing more—but nothing less—than a collaborative enquiry into the ways of constructing the symmetries grounding rationality within inquisitive interaction.*[33] *By building these symmetries we provide meaning to our actions, a meaning deployed in our actions' internal coordination with the actions of others (interaction).*

In order to allow readers to follow the technical aspects of this book, we have divided it into progressive sections, with examples and exercises.

The next section (chapter II) provides an elementary introduction to Constructive Type Theory (CTT), original in its equal emphasis on basic ideas and finer technical details.

The third chapter introduces essential notions for the dialogical framework and provides a basic and step by step approach to dialogues.

The fourth and fifth chapters aim at more advanced readers, who are either already familiar with the dialogical framework or well versed into logic; they respectively deal with the two fundamental levels characteristic of dialogues: the play level and the strategy level.

The reader should by then be fully equiped for the following sections, which are the core of this book and deal precisely with the problem at issue, that is, with immanent reasoning and equality in action. Thus the sixth chapter introduces local reasons in the dialogical framework, a crucial step for immanent reasoning; the seventh chapter presents again the local reasons but from a more technical side, deals with the strategy level in dialogues for immanent reasoning, and introduces a key notion: strategic objects, the dialogical counterpart to CTT proof-objects.

The eighth chapter illustrates some of the imports of the constructive perspective in general and of dialogues for immanent reasoing in particular through the case of the Axiom of Choice.

The ninth chapter provides an algorithm allowing to go from dialogical strategies to CTT demonstrations, and reversewise.

The tenth chapter touches on the less studied material dialogues, that is, dialogues with rules for local and global meaning that are not restricted to the rules of logical constants. We study the case of propositional identity, the set **Bool**, Boolean operations, and finite sets.

The final chapter will be our conclusion, which contains some philosophical remarks on dialogical constructivism and suggests some responses to standard objections to the dialogical framework.

---

[33] For more details on symmetry in the dialogical framework, see section IV.3.

II. A BRIEF INTRODUCTION TO CONSTRUCTIVE TYPE THEORY

By ***Ansten Klev***

Martin-Löf's Constructive Type Theory (CTT) is a formal language developed in order to reason constructively about mathematics. It is thus a formal language conceived primarily as a tool to reason *with* rather than a formal language conceived primarily as a mathematical system to reason *about*. Constructive Type Theory is therefore much closer in spirit to Frege's ideography and to the language of Russell and Whitehead's *Principia Mathematica* than to the majority of logical systems ("logics") studied by contemporary logicians. Since CTT is designed as a language to reason with, much attention is paid to the explanation of basic concepts. This is perhaps the main reason why the style of presentation of CTT differs somewhat from the style of presentation typically found in, for instance, ordinary logic textbooks. For those new to the system it might be useful to approach an introduction, such as the one given below, more as a language course than as a course in mathematics.

II.1 Judgements and categories

Statements made in Constructive Type Theory are called *judgements*. *Judgement* is thus a technical term, chosen because of its long pedigree in the history logic. (cf. e.g. (Martin-Löf, 1996; 2011) and (Sundholm, 2009)). Judgement thus understood is a logical notion and not, as it is commonly understood in contemporary philosophy, a psychological notion. As in traditional logic, a judgement may be categorical or hypothetical. Categorical judgements are conceptually prior to hypothetical judgements, hence we must begin by explaining them.

II.1.1 Forms of categorical judgement

There are two basic forms of categorical judgement in CTT:

$$a : \mathcal{C}$$

$$a = b : \mathcal{C}$$

The first is read "a is an object of the category $\mathcal{C}$" and the second is read "a and b are identical objects of the category $\mathcal{C}$". Ordinary grammatical analysis of $a : \mathcal{C}$ yields a as subject, $\mathcal{C}$ as predicate, and the colon as copula. We thus call the predicate $\mathcal{C}$ in $a : \mathcal{C}$ a *category*. This use of the term 'category' is in accordance with one of the original meanings of the Greek *katēgoria*, namely as predicate. It is also in accordance with a common use of the term 'category' in current philosophy.[34] We require, namely, that any category $\mathcal{C}$ occurring in a judgement of CTT be associated with

- a *criterion of application*, which tells us what a $\mathcal{C}$ is; that a meets this criterion is precisely what is expressed in $a : \mathcal{C}$;
- a *criterion of identity*, which tells us what it is for a and b to be identical $\mathcal{C}$s; that a and b together meet this criterion is precisely what is expressed in $a = b : \mathcal{C}$.

What the categories of CTT are will be explained below.

In CTT any object belongs to a category. The theory recognizes something as an object only if it can appear in a judgement of the form $a : \mathcal{C}$ or $a = b : \mathcal{C}$. Since

---

[34] See, in particular, the definition of category given by Dummett (1973, pp. 75-76), which has been taken over by Hale and Wright (2001) for instance.

associated with any category there is a criterion of identity, we can recover Quine's (1969, p. 23) precept of "no entity without identity" as

no object without category +
no category without a criterion of identity.

Thus we derive Quine's precept from two of the fundamental principles of CTT. We shall have more to say later about the treatment of identity in CTT.

Neither semantically nor syntactically does $a : C$ agree with the basic form of statement in predicate logic:

$$F(a)$$

In $F(a)$ a function F is applied to an argument a (in general there may be more than one argument). The judgement $a : C$, by contrast, does not have function–argument form. In fact, the '$a : C$'-form of judgement is closer to the 'S is P'-form of traditional syllogistic logic than to the function-argument form of modern, Fregean logic. Since we have required that the predicate C be associated with criteria of application and identity, the judgement $a : C$ can only be compared with a special case of the 'S is P'-form, for no such requirement is in general laid on the predicate P in a judgement of Aristotle's syllogistics—it can be any general term.

To understand the restriction that P be associated with criteria of application and identity, in terms of traditional logic, we may invoke Aristotle's doctrine of predicables from the *Topics*.[35] A predicable may be thought of as a certain relation between the S and the P in an 'S is P'-judgement. Aristotle distinguishes four predicables: genus, definition, *idion* or *proprium*, and accident. That P is a genus of S means that P reveals a *what*, or a *what-it-is*, of the subject S; a genus of S may thus be proposed in answer to the question of what S is. The class of judgements of Aristotelian syllogistics to which judgements of the form $a : C$ may be compared is the class of judgements whose predicate is a genus of the subject. Provided the judgement $a : C$ is correct, the category C is namely an answer to the question of what a is; we may thus think of C as the genus of a. Aristotle's other predicables will not concern us here.

Being a natural number is in a clear sense a *what* of 7. The number 7 is also a prime number; but being prime is not a *what* of 7 in the sense that being a natural number is, even though 7 is necessarily, and perhaps even essentially, a prime number. Following Almog (1991) we may say that being prime is one of the *how*s of 7. This difference between the *what* and the *how* of a thing captures quite well the difference in semantics between a judgement $a : C$ of CTT and a sentence $F(a)$ of predicate logic. In the predicate-logical language of arithmetic we do not express the fact that 7 is a number by means of a sentence of the form $F(a)$. That the individual terms of the language of arithmetic denote numbers is rather a feature of the interpretation of the language that we may express in the metalanguage.[36] We do, however, say in the language of arithmetic that 7 is prime by means of a sentence of the form $F(a)$, for instance as $\mathbf{Pr}(7)$. It is therefore natural to suggest that by means of the form of statement $F(a)$ we express a *how*, but not the *what*, of the object a. The opposite holds for the form of statement $a : C$—by means of this we express the *what*, but not the *how*, of the object a. Thus, in CTT we do say that 7 is a number by means of a judgement, namely as $7 : \mathbb{N}$, where $\mathbb{N}$ is the category of natural numbers; but we do not say that 7 is prime by means of a similar judgement such as $7 : \mathbf{Pr}$. Precisely how we express in CTT that 7 is prime will become

---

[35] (Barnes, 1984), (Crubellier, 2008).

[36] Compare Carnap's treatment of what he calls *Allwörter* ('universal words' in the English translation) in §§ 76, 77 of *Logische Syntax der Sprache,* (Carnap, 1934).

clear only later; it will then be seen that we express the primeness of 7 by a judgement of the form

$$p : \mathbf{Pr}(7)$$

where $\mathbf{Pr}(7)$ is a *proposition* and p is a *proof* of this proposition. The proposition $\mathbf{Pr}(7)$ has function-argument form, just as the atomic sentences of ordinary predicate logic.

II.1.2 **Categories**

The forms of judgement $a : \mathcal{C}$ and $a = b : \mathcal{C}$ are only schematic forms. The specific forms of categorical judgement employed in CTT are obtained from these schematic forms by specifying the categories of the theory. There is then a choice to be made, namely between what may be called a higher-order and a lower-order presentation of the theory. The higher-order presentation results in a somewhat conceptually cleaner theory, but the lower-order presentation is preferable for pedagogical purposes, both because it requires less machinery and because it is the style of presentation found in the standard references of Martin-Löf (1975b; 1982; 1984) and Nordström et al. (1990, pp. ch. 4-16). We shall therefore follow this style of presentation. The categories are then the following. There is a category **set** of sets in the sense of Martin-Löf; and for any **set** A, A itself is a category. We therefore have the following four forms of categorical judgement:

$$A : \mathbf{set}$$
$$A = B : \mathbf{set}$$

and for any **set** A,

$$a : A$$
$$a = b : A$$

In the higher-order presentation the categories are **type** and α, for any **type** α. The higher-order presentation in a sense subsumes the lower-order presentation, since we have there, firstly, as an axiom **set** : **type**, hence **set** itself is a category; and secondly, there is a rule to the effect that if A : **set**, then A : **type**, hence also any **set** A will be a category. The higher-order presentation can be found in Nordström et al. (1990, pp. ch. 19-20; 2000).

We have so far only given names to our categories. To justify calling **set** as well as any **set** A a category we must specify the criteria of application and identity of **set** and of A, for any **set** A. Thus we have to explain four things: what a **set** is, what identical **set**s are, what an element of a **set** A is, and what identical elements of a **set** A are. By giving these explanations we also explain the four forms of categorical judgement A : **set**, $A = B : \mathbf{set}$, $a : A$, and $a = b : A$. Our explanations follow those given by Martin-Löf (1984, pp. 7-10).

We explain the form of judgement A : **set** as follows. A **set** A is defined by saying what a *canonical* element of A is and what equal canonical elements of A are. (Instead of ‘canonical element’ one can also say ‘element of canonical form’.) What the canonical elements are, as well as what equal canonical elements are, of a **set** A is determined by the so-called introduction rules associated with A. For instance, the introduction rules associated with the set of natural numbers ℕ are as follows.

$$0 : \mathbb{N} \quad 0 = 0 : \mathbb{N} \quad \frac{n : \mathbb{N}}{\mathbf{s}(n) : \mathbb{N}} \qquad \frac{n = m : \mathbb{N}}{\mathbf{s}(n) = \mathbf{s}(m) : \mathbb{N}}$$

By virtue of these rules 0 is a canonical element of ℕ, as is $\mathbf{s}(n)$ provided n is a ℕ, which does not have to be canonical. Moreover, 0 is the same canonical element of ℕ as

0, and $\mathbf{s}(n)$ is the same canonical element of $\mathbb{N}$ as $\mathbf{s}(m)$ provided $n = m : \mathbb{N}$. It is required that the specification of what equal canonical elements of a **set** A are renders this relation reflexive, symmetric, and transitive.

The form of judgement $A = B : \mathbf{set}$ means that from a's being a canonical element of A we may infer that a is also a canonical element of B, and *vice versa*; and that from a and b's being identical canonical elements of A we may infer that they are also identical canonical elements of B, and *vice versa*.

Thus we have given the criteria of application and identity for the category **set**.

Suppose that A is a **set**. Then we know how the canonical elements of A are formed as well as how equal canonical elements of A are formed. The judgement $a : A$ means that a is a programme which, when executed, evaluates to a canonical element of A. For instance, once one has introduced the addition function, +, and the definitions $1 = \mathbf{s}(0) : \mathbb{N}$ and $2 = \mathbf{s}(1) : \mathbb{N}$, one can see that $2 + 2$ is an element of $\mathbb{N}$, since it evaluates to $\mathbf{s}(2 + 1)$, which is of canonical form. A canonical element of a **set** A evaluates to itself; hence, any canonical element of A is an element of A.

The judgement $a = b : A$ presupposes the judgements $a : A$ and $b : A$. Hence, if we can make the judgement $a = b : A$, then we know that both a and b evaluate to canonical objects of A. The judgement $a = b : A$ means that a and b evaluate to equal canonical elements of A. The value of a canonical element a of a **set** A is taken be a itself. Hence, if b evaluates to a, then we have $a = b : A$.

Thus we have given the criteria of application and identity for the category A, for any **set** A.

A note on terminology is here in order. 'Set' is the term used by Martin-Löf from (Martin-Löf, 1984) onwards for what in earlier writings of his were called types.[37] A set in the sense of Martin-Löf is a very different thing from a set in the sense of ordinary axiomatic set theory. In the latter sense a set is typically conceived of as an object belonging to the cumulative hierarchy V. It is, however, this hierarchy V itself rather than any individual object belonging to V that should be regarded as a set in the sense of Martin-Löf. A set in the sense of Martin-Löf is in effect a domain of individuals, and V is precisely a domain of individuals. That was certainly the idea of Zermelo in his paper on models of set theory (Zermelo, 1930): he there speaks of such models as *Mengenbereiche*, domains of sets. And Aczel (1978) has defined a set in the sense of Martin-Löf that is "a type theoretic reformulation of the classical conception of the cumulative hierarchy of types" (Aczel, 1978, p. 61). It is in order to mark this difference in conception that we denote a set in the sense of Martin-Löf with boldface type, thus writing '**set**'.[38]

II.1.3 General rules of judgemental equality

Recall that when defining a **set** A, it is required that the relation of being equal canonical elements then specified be reflexive, symmetric, and transitive. From the explanation of the form of judgement $a = b : A$, it is then easy to see that the relation of the so-called judgemental identity, namely the relation expressed to hold between a and b by means of the judgement $a = b : A$, is also reflexive, symmetric, and transitive. Thus the following three rules are justified.

$$\frac{a : A}{\qquad} \qquad\qquad \frac{a = b : A}{\qquad} \qquad\qquad \frac{a = b : A \qquad b = c : A}{\qquad}$$

---

[37] This older terminology is retained for instance in *Homotopy Type Theory* (The Univalent Foundations Program, 2013); what is there called a set (The Univalent Foundations Program, 2013, p. Definition 3.1.1) is only a special case of a set in Martin-Löf's sense, namely a set over which every identity proposition has at most one proof.

[38] For a further discussion of the difference between Martin-Löf's notion and other notions of set, see (Granström, 2011, pp. 53-63) and (Klev, 2014a, pp. 138-140).

$$\frac{}{a = a : A} \qquad \frac{}{b = a : A} \qquad \frac{}{a = c : A}$$

The explanation of the form of judgement $A = B : \mathbf{set}$ justifies the same rules at the level of sets.

$$\frac{A : \mathbf{set}}{A = A : \mathbf{set}} \qquad \frac{A = B : \mathbf{set}}{B = A : \mathbf{set}} \qquad \frac{A = B : \mathbf{set} \qquad B = C : \mathbf{set}}{A = C : \mathbf{set}}$$

They also justify the following two important rules.

$$\frac{a : A \qquad A = B : \mathbf{set}}{a : B} \qquad \frac{a = b : A \qquad A = B : \mathbf{set}}{a = b : B}$$

II.1.4 **Propositions**

The notion of proposition has already been alluded to above; and it is reasonable to expect that a system of logic should give some account of this notion. In CTT there is a category **prop** of propositions. The reason this category was not explicitly introduced above is that it is identified in CTT with the category **set**. Thus we have

$$\mathbf{prop} = \mathbf{set}$$

The identification of these two categories[39] is the manner in which the so-called Curry–Howard isomorphism (Howard, 1980) is implemented in CTT. This "isomorphism" is one of the fundamental principles on which the theory rests.

When regarding A as a proposition, the elements of A are thought of as the proofs of A. Thus *proof* is employed as a technical term for elements of propositions. A proposition is, accordingly, identified with the **set** of its proofs. That a proposition is true means that it is inhabited.

By the identification of **set** and **prop** the meaning explanation of the four basic forms of categorical judgement carries over to the explanation of the similar forms

$$\begin{gathered} A : \mathbf{prop} \\ A = B : \mathbf{prop} \\ a : A \\ a = b : A \end{gathered}$$

To define a **prop** one must lay down what are the canonical proofs of A and what are identical canonical proofs of A. That the propositions A and B are identical means that from a's being a canonical proof of A we may infer that it is also a canonical proof of B, and *vice versa*; and that from a and b's being identical canonical proofs of A we may infer that they are also identical canonical proofs of B, and *vice versa*. Thus, by the identification of **set** and **prop** we get for free a criterion of identity for propositions.

That a is a proof of A means that a is a method which, when executed, evaluates to a canonical proof of A. That a and b are identical proofs of A means that a and b evaluate to identical canonical proofs of A. Thus we have provided a criterion of identity for proofs.

Let us illustrate the concept of a canonical proof in the case of conjunction. A canonical proof of $A \land B$ is a proof that ends in an application of $\land$-introduction

---

[39] In the higher-order presentation this identification can be made in the language itself, namely as the judgement $\mathbf{prop} = \mathbf{set} : \mathbf{type}$.

$$\dfrac{\begin{matrix}\mathcal{D}_1 & \mathcal{D}_2 \\ A & B\end{matrix}}{A \wedge B}$$

where $\mathcal{D}_1$ is a proof of A and $\mathcal{D}_2$ a proof of B. An example of a non-canonical proof is therefore

$$\dfrac{\begin{matrix}\mathcal{D}_1 & \mathcal{D}_2 \\ C \supset A \wedge B & C\end{matrix}}{A \wedge B}$$

where $\mathcal{D}_1$ is a proof of $C \supset A \wedge B$ and $\mathcal{D}_2$ a proof of C.

The proofs occurring in the above illustration are in tree form. Proofs in the technical sense of CTT are not given in tree form, but rather as the subjects a of judgements of the form $a : A$, where A is a **prop**. Proofs in this sense are in effect terms in a certain rich typed lambda-calculus and they are often called *proof-objects* (this term was introduced by (Diller & Troelstra, 1984)).

We may introduce a new form of judgement '$A\ true$' governed by the following rule of inference

$$\dfrac{a : A}{A\ true}$$

Thus, provided we have found a proof a of A, we may infer A true. The conclusion $A\ true$ can be seen as suppressing the proof a of A displayed in $a : A$.

II.1.5 Forms of hypothetical judgement

One of the characteristic features of Constructive Type Theory is that it recognizes hypothetical judgements as a form of statement distinct from the assertion of the truth of an implicational proposition $A \supset B$. In fact, hypothetical judgements are fundamental to the theory. It is, for instance, hypothetical judgements that give rise to the various dependency structures in CTT, by virtue of which it is a dependent type theory.

Assume A : **set**. Then we have the following four forms of hypothetical judgement with one assumption.

$$x : A \vdash B : \mathbf{set}$$
$$x : A \vdash B = C : \mathbf{set}$$
$$x : A \vdash b : B$$
$$x : A \vdash b = c : B$$

We have used the turnstile symbol, ⊢, to separate the antecedent, or assumption, of the judgement from the consequent. In (Martin-Löf, 1984) the notation used is

$$B : \mathbf{set}\ (x : A)$$

for what we here write $x : A \vdash B : \mathbf{set}$. We read this judgement as "B is a set under the assumption $x : A$". Similar remarks apply to the other three forms of hypothetical judgement. Let us consider the more precise meaning explanations of these forms of judgement.

A judgement of the form $x : A \vdash B : \mathbf{set}$ means that

$B[a/x] : \textbf{set}$ whenever $a : A$, and
$B[a/x] = B[a'/x] : \textbf{set}$ whenever $a = a' : A$.

Here '$B[a/x]$' signifies the result of substituting 'a' for 'x' in 'B'. Thus we may think of B as a function from A into **set**; or using a different terminology, B may be thought of as a family of **set**s over A. We are assuming that x is the only free variable in B and that A contains no free variables, hence that the judgement $A : \textbf{set}$ holds categorically, that is, under no assumptions. It follows that $B[a/x]$ is a closed term, hence that $B[a/x] : \textbf{set}$ holds categorically; by the explanation given of the form of categorical judgement $A : \textbf{set}$ we therefore know the meaning of $B[a/x] : \textbf{set}$. Thus we see that the meaning of a hypothetical judgement is explained in terms of the meaning of categorical judgements. It holds in general that the meaning explanation of hypothetical judgements is thus reduced to the meaning explanation of categorical judgements.

The explanation of the form of judgement $x : A \vdash B : \textbf{set}$ justifies the following two rules.

$$\frac{a : A \qquad x : A \vdash B : \textbf{set}}{B[a/x] : \textbf{set}} \qquad\qquad \frac{a = a' : A \qquad x : A \vdash B : \textbf{set}}{B[a/x] = B[a'/x] : \textbf{set}}$$

Note that by the second rule here, substitution into **set**s is extensional with respect to judgemental identity. That is to say, if we think of $x : A \vdash B : \textbf{set}$ as expressing that B is a set-valued function (a family of sets), then B has the expected property that for identical arguments $a = a' : A$ we get identical values $B[a/x] = B[a'/x] : \textbf{set}$.

We note that the notion of substitution is here understood only informally and that the notation $B[a/x]$ belongs to the metalanguage. The notion of substitution can be made precise, and a notation for substitution introduced into the language of CTT itself; but it would take us too far afield to get into the details of that (cf. (Martin-Löf, 1992) and (Tasistro, 1993)).

A judgement of the form $x : A \vdash B = C : \textbf{set}$ means that

$B[a/x] = C[a/x] : \textbf{set}$ whenever $a : A$.

Hence, in this case we may think of B and C as identical families of **set**s over A. The explanation justifies the following rule.

$$\frac{a : A \qquad x : A \vdash B = C : \textbf{set}}{B[a/x] = C[a/x] : \textbf{set}}$$

A judgement of the form $x : A \vdash b : B$ means that

$b[a/x] : B[a/x]$ whenever $a : A$, and
$b[a/x] = b[a'/x] : B[a/x]$ whenever $a = a' : A$.

Here we are presupposing $x : A \vdash B : \textbf{set}$, hence we know that $B[a/x] : \textbf{set}$ whenever $a : A$, and therefore we also know the meaning of $b[a/x] : B[a/x]$ and $b[a/x] = b[a'/x] : B[a/x]$ whenever $a : A$ and $a = a' : A$. The judgement $x : A \vdash b : B$ can be understood as saying that b is a function from A into the family B; that is to say, b is a function that for any $a : A$ yields an element $b[a/x]$ of the set $B[a/x]$. The explanation justifies the following two rules.

$$\frac{a : A \qquad x : A \vdash b : B}{b[a/x] : B[a/x]} \qquad\qquad \frac{a = a' : A \qquad x : A \vdash b : B}{b[a/x] = b[a'/x] : B[a/x]}$$

Note that by the second rule here, substitution into elements of **set**s is extensional with respect to judgemental identity. That is to say, if we think of $x : A \vdash b : B$ as expressing that b is a function, then b has the expected property that for identical arguments $a = a' : A$ we get identical values $b[a/x] = b[a'/x] : B[a/x]$.

A judgement of the form $x : A \vdash b = c : B$ means that

$$b[a/x] = c[a/x] : B[a/x] \text{ whenever } a : A.$$

Thus, in this case, b and c are identical functions into the family B. The explanation justifies the following rule.

$$\frac{a : A \qquad x : A \vdash b = c : B}{b[a/x] = c[a/x] : B[a/x]}$$

II.1.6 Assumptions and other speech acts

The notions of proposition, categorical judgement, and hypothetical judgement can be seen all of them to be presupposed by what is arguably the most natural interpretation of natural deduction derivations (Sundholm, 2006). Consider the following natural deduction proof sketch:

$$\cfrac{\cfrac{\begin{matrix} A \\ \mathcal{D}_1 \\ B \end{matrix}}{A \supset B} \qquad \begin{matrix} \mathcal{D}_2 \\ A \end{matrix}}{B}$$

Here $\mathcal{D}_1$ is a proof of B from A, and $\mathcal{D}_2$ is a closed proof of A. Let us regard this natural deduction proof sketch as a representation of an actual mathematical demonstration and let us consider which speech acts the individual formulae here then represent.

- The topmost A represents an assumption, namely the assumption that the proposition A is true.
- The formula A that is the conclusion of $\mathcal{D}_2$ is the conclusion of a closed proof; this formula therefore represents the categorical judgement, or assertion, that A is true; the same considerations apply to $A \supset B$ and to the final conclusion B.
- The B that is the conclusion of $\mathcal{D}_1$ represents neither an assumption nor a categorical assertion; it rather represents a hypothetical judgement, namely the judgement that B is true on the hypothesis that A is true.
- The formula A occurring as a subformula in $A \supset B$ represents neither an assumption nor a categorical assumption nor a hypothetical judgement. It rather represents a proposition that is a part of a more complex proposition $A \supset B$, which in the given proof is asserted categorically to be true.

Thus we see that in order to make the semantics of natural deduction derivations explicit we should employ a notation that is able to distinguish not only propositions from judgements, but also categorical judgements from hypothetical judgements, and perhaps also assumptions from all of these. Assumptions can, however, be subsumed under hypothetical judgements, since we may regard the assumption of some categorical judgement J as the assertion of J on the hypothesis that J. In particular, the assumption of $a : A$ and the assumption that the proposition A is true may be analyzed as respectively:

$$a : A \vdash a : A \quad \text{and} \qquad A\ true\ \vdash\ A\ true$$

In CTT one can therefore make the semantics of the above natural deduction proof sketch explicit as follows

$$\cfrac{\cfrac{\begin{array}{c} A\ true\ \vdash\ A\ true \\ \mathcal{D}_1 \\ A\ true\ \vdash\ B\ true \end{array}}{A \supset B\ true} \qquad \begin{array}{c} \mathcal{D}_2 \\ A\ true \end{array}}{B\ true}$$

From the meaning explanation of hypothetical judgements it is clear that the following rule is justified.

$$\frac{A : \mathbf{set}}{x : A \vdash x : A}$$

Nordström et al. (1990, p. 37) call this the rule of assumption, since it in effect allows us to introduce assumptions.

II.1.7 **Hypothetical judgements with more than one assumption**

The forms of hypothetical judgement where the number of hypotheses is $n > 1$ are explained by induction on n. We consider the case of $n = 2$ for illustration. We assume that $A_1 : \mathbf{set}$ and $x : A_1 \vdash A_2 : \mathbf{set}$. Thus A_1 is a **set** categorically, while A_2 is a family of **set**s over A_1. The four forms of judgement to be considered are the following.

$$\begin{array}{l} x : A_1, x_2 : A_2 \vdash B : \mathbf{set} \\ x : A_1, x_2 : A_2 \vdash B = C : \mathbf{set} \\ x : A_1, x_2 : A_2 \vdash b : B \\ x : A_1, x_2 : A_2 \vdash b = c : B \end{array}$$

The first of these judgements means that $B[a_1/x_1,\ a_2/x_2] : \mathbf{set}$ whenever $a_1 : A_1$ and $a_2 : A_2[a_1/x_1]$ and that $B[a_1/x_1,\ a_2/x_2] = B[a_1'/x_1,\ a_2'/x_2] : \mathbf{set}$ whenever $a_1 = a_1' : A_1$ and $a_2 = a_2' : A_2\ [a_1/x_1]$. Note that A_2 here in general may be a family of **set**s over A_1. Which member of the family the second argument a_2 is taken from depends on the first argument a_1. Thus B is a family of **set**s over A_1 and A_2, where A_2 itself may be a family of **set**s over A_1.

The meaning of the third judgement is that $b[a_1/x_1, a_2/x_2] : B[a_1/x_1, a_2/x_2]$ whenever $a_1 : A_1$ and $a_2 : A_2[a_1/x_1]$, and that $b[a_1/x_1, a_2/x_2] = b[a_1'/x_1, a_2'/x_2] : B[a_1/x_1, a_2/x_2]$ whenever $a_1 = a_1' : A_1$ and $a_2 = a_2' : A_2\ [a_1/x_1]$. Thus b is a binary function whose first argument is an element of A_1; if this element is a_1, then the second argument is an element of $A_2[a_1/x_1]$; if the second argument is a_2, then the value $b[a_1/x_1, a_2/x_2]$ is an element of $B[a_1/x_1, a_2/x_2]$. Here one sees the complex dependency structures that can be expressed in CTT.

It should be clear how the explanation of the second and fourth forms of judgement above, as well as the explanation for arbitrary n, should go.

Let J be any categorical judgement, that is, a judgement of one of the forms $B : \mathbf{set}$, $B = C : \mathbf{set}$, $b : B$, $b = b' : B$. In a hypothetical judgement

$$x_1 : A_1, \ldots, x_n :\ A_n \vdash J$$

we call the sequence of hypotheses $x_1 : A_1, \ldots, x_n : A_n$ a *context*. A judgement of the form

$$x_1 : A_1, \ldots, x_n : A_n \vdash B : \mathbf{set}$$

may thus be expressed by saying that B is a **set** in the context $x_1 : A_1, \ldots, x_n : A_n$. Let Γ be a context. From the meaning explanation of hypothetical judgements one sees that rules of the following kind are justified.

$$\frac{\Gamma \vdash J \qquad \Gamma \vdash B : \mathbf{set}}{\Gamma, y{:}\, B \vdash J}$$

These rules may be called *rules of weakening*, in accordance with the terminology used in sequent calculus.

With the general hypothetical form of judgement explained we may introduce a notion of category in a wider sense, in effect what is called a category in (Martin-Löf, 1984, p. 21-23). Let us write the four general forms of judgement in the style of Martin-Löf, namely as follows.

$$\begin{aligned} B &: \mathbf{set}\ (x_1 : A_1, \ldots, x_n : A_n) \\ B = C &: \mathbf{set}\ (x_1 : A_1, \ldots, x_n : A_n) \\ b &: B\ (x_1 : A_1, \ldots, x_n : A_n) \\ b = c &: B\ (x_1 : A_1, \ldots, x_n : A_n) \end{aligned}$$

In a grammatical analysis of the first of these it is natural to view not only **set** but everything that is to the right of the colon, namely

$$\mathbf{set}\ (x_1 : A_1, \ldots, x_n : A_n)$$

as the predicate. The relation between the notions of predicate and category thus suggests that we may regard this as a category. Indeed, this may be regarded as the category of families of **set**s in n variables ranging over the **set**s or families of **set**s $A_1, \ldots, A_n$, among which there may be dependency relations as explained for the case of $n = 2$ above. Likewise we may regard

$$B\ (x_1 : A_1, \ldots, x_n : A_n)$$

as a category. It is the category of n-ary functions from $A_1, \ldots, A_n$ into the family B (again keeping dependency relations in mind).

Thus we may extend the notion of category to include not only **set** and A for any **set** A, but also n-ary families of **set**s and n-ary functions into a **set** A. Note that these are indeed categories in the present sense since they are associated with criteria of application and identity, namely through the explanation of the general forms of hypothetical judgement.

II.2 Rules

So far we have only the frame of a language, namely an explanation of its basic forms of statement as well as explanations of the basic notions of **set**, proposition, element of a **set**, and proof of a proposition. The frame is filled by the introduction of symbols signifying **set**s, operations for forming **set**s, and operations for forming elements of **set**s. These symbols are not explained one by one, but rather in groups. The meaning of the symbols in a given group is determined by rules of four kinds:

- Formation rules
- Introduction rules
- Elimination rules
- Equality, or computation, rules

The inclusion of formation rules in the language itself is a distinctive feature of CTT. The introduction and elimination rules are like those of Gentzen (1933), though generalized to the syntax of CTT so as also to cover the construction of proof objects. The equality rules correspond to the reduction rules of Prawitz (1965). The best way of getting a grip on these notions is by looking at concrete examples, which we now proceed to do.

In the following we shall in most cases write $A[b,c]$ and $a[b,c]$, etc., instead of $A[b/x, c/y]$ and $a[b/x, c/y]$, etc. That is, for ease of readability we shall usually not mention the variables for which b, c, etc. are substituted in A, a, etc. Which variables are replaced will usually be clear from the context. Although variables are not mentioned, square brackets will still stand for substitution and not for function application.

II.2.1 Cartesian product of a family of sets

Given a **set** A and a family B of **set**s over A we can form the product of B over A. That is the content of the Π-formation rule:

$$(\Pi\text{-form}) \qquad \frac{A : \mathbf{set} \qquad x : A \vdash B : \mathbf{set}}{(\Pi x : A)B : \mathbf{set}}$$

This rule lays down when we may judge that $(\Pi x : A)B$ is a **set**. There is a second Π-formation rule that lays down when we may judge that two sets of the form $(\Pi x : A)B$ are identical:

$$\frac{A = A' : \mathbf{set} \qquad x : A \vdash B = B' : \mathbf{set}}{(\Pi x : A)B = (\Pi x : A')B' : \mathbf{set}}$$

All formation, introduction, and elimination rules are paired with identity rules of this kind, but we shall state these rules explicitly only in the present case of Π.

The conclusion of Π-formation says that $(\Pi x : A)B$ is a **set**. Since we have the right to judge that C is a **set** only if we can say what the canonical elements of C are, as well as what equal canonical elements of C are, we see that the rule of Π-formation requires justification.

The required justification is provided by the Π-introduction rules:

$$(\Pi\text{-intro}) \qquad \frac{x : A \vdash b : B}{\lambda x.b : (\Pi x : A)B} \qquad\qquad \frac{x : A \vdash b = b' : B}{\lambda x.b = \lambda x.b' : (\Pi x : A)B}$$

According to this rule a canonical element of $(\Pi x : A)B$ has the form $\lambda x.b$, where $b[a] : B[a]$ whenever $a : A$. Note that such a b is of a category different from the category of $\lambda x.b$. Namely, b is of category $B(x : A)$ whereas $\lambda x.b$ is of category $(\Pi x : A)B$. It was noted above that we may regard such a b as a function from A into the family B. We may think of $\lambda x.b$ as an individual that codes this function. The λ-operator is thus similar to Frege's course-of-values operator (cf. e.g. (Frege G. , 1893, p. § 9)) which, given a function $f(x)$, yields an individual $\acute{\alpha}f(\alpha)$. Note, however, that $\lambda x.b$

belongs to a separate $\mathbf{set}$ $(\Pi x : A)B$ and not to the domain A of the function b; whence we cannot make sense of applying the function b to $\lambda x.b$, hence a contradiction along the lines of Russell's Paradox cannot be derived.

The role of the elements of $(\Pi x : A)B$ as codes of functions is made clear by the Π-elimination rule:

$$(\Pi\text{-elim}) \quad \frac{c : (\Pi x : A)B \qquad a : A}{\mathbf{ap}(c, a) : B[a]} \qquad\qquad \frac{c = c' : (\Pi x : A)B \quad a = a' : A}{\mathbf{ap}(c, a) = \mathbf{ap}(c', a') : B[a]}$$

The conclusion of this rule asserts that $\mathbf{ap}(c, a)$ is an element of the set $B[a]$. Since we have the right to judge that c is an element of a set C only if we can specify how to compute c to a canonical element of C, we see that the rule of Π-elimination requires justification.

The required justification is provided by the rule of Π-equality, which specifies how $\mathbf{ap}(c, a)$ is computed in the case where c is of canonical form, namely $\lambda x.b$.

$$(\Pi\text{-eq}) \qquad \frac{x : A \vdash b : B \qquad a : A}{\mathbf{ap}(\lambda x.b, a) = b[a] : B[a]}$$

We can now justify Π-elimination as follows. By the assumption $c : (\Pi x : A)B$ we know how to evaluate c to canonical form $\lambda x.b$, where $x : A \vdash b : B$; thus we have $c = \lambda x.b : (\Pi x : A)B$. But then also $\mathbf{ap}(c, a) = \mathbf{ap}(\lambda x.b, a) : B[a]$, so $\mathbf{ap}(c, a) = b[a] : B[a]$, whence the value of $\mathbf{ap}(c, a)$ is equal to the value of $b[a]$; by the assumption $x : A \vdash b : B$ we know how to find this value.

From the Π-equality rule we see that **ap** is an application operator; as such it is similar to the function $x^\frown y$, satisfying the equation $\Delta^\frown \acute{\alpha} f(\alpha) = f(\Delta)$, defined by Frege (1893, p. § 34).

We have now seen that the Π-introduction rules enable us to justify the Π-formation rule and that the Π-equality rule enables us to justify the Π-elimination rule. These relations of justification hold in general and not only in the case of Π.

The advantage of the higher-order presentation of CTT is most readily seen when we ask about the categories of Π, λ, and $\mathbf{ap}$. Intuitively we may think of Π as a certain higher-order function that takes a $\mathbf{set}$ A and a family of $\mathbf{set}$s B over A and yields a $\mathbf{set}$ $(\Pi x : A)B$. But we have no means of naming the category of such a function in the language frame introduced here. In the higher-order presentation such a name is easily constructed; indeed we then express the category assignment of Π by means of the judgement $\Pi : (X : \mathbf{set})\big((X)\mathbf{set}\big)\mathbf{set}$. Similar remarks apply to λ and $\mathbf{ap}$, and in fact to all of the various symbols that we are now in the process of introducing into the language (apart from the constant sets $\mathbb{N}_n$ and $\mathbb{N}$ to be introduced below—these are of category $\mathbf{set}$).

II.2.2 The logical interpretation of the Cartesian product

Recall that $\mathbf{prop} = \mathbf{set}$. Hence we may regard a family B of $\mathbf{set}$s over a $\mathbf{set}$ A as a family of propositions over A. A family of propositions over A is a function from A into the category of propositions; it is thus a propositional function.

Let us consider B as a propositional function over A and $(\Pi x : A)B$ as a proposition, and let us ask what a canonical proof of this proposition looks like. Such a canonical proof has the form $\lambda x.b$, where $x : A \vdash b : B$, and is in effect a code of the function b. This function b takes an element a of A and yields a proof $b[a]$ of the proposition $B[a]$. Keeping in mind the Brouwer–Heyting–Kolmogorov interpretation of the logical connectives (cf. e.g. (Troelstra & van Dalen, 1988, pp. 9-10)), we see thus that $(\Pi x : A)B$, when regarded as a proposition, is the proposition $(\forall x : A)B$, which intuitively says that all elements of A have the property B. Note that this proposition is not written $\forall x B$ as in ordinary predicate logic; rather, the domain of quantification, A, is explicitly mentioned.

On the understanding of Π as $\forall$, we can recover the rule of $\forall$-introduction from the rule of Π-introduction by employing the form of judgement '$C\ true$' as follows.

$$\frac{x : A \ \vdash \ B\ true}{(\forall x : \ A)B\ true}$$

That is to say, if $B[a]$ is true whenever $a : A$, then $(\forall x : A)B$ is true. Let us also consider the version of $\forall$-introduction where the proof objects have not been suppressed:

$$\frac{x : A \ \vdash \ b : B}{\lambda x. b : (\forall x : A)B}$$

Here we should think of b as an open proof of B, a proof depending on a parameter $x : A$. For instance, A may be the natural numbers, $\mathbb{N}$, and B may be the propositional function that for any element n of $\mathbb{N}$ yields the proposition that n is either even or odd; b is then a proof of the proposition that x is either even or odd, where x is a generic or arbitrary natural number. By binding x we get a proof $\lambda x. b$ of $(\forall x : A)B$ where x is no longer free; if x is the only free variable in b, then $\lambda x. b$ is a closed proof of $(\forall x : A)B$.

Since the domain of quantification is explicitly mentioned in $(\forall x : A)B$, it also has to be mentioned in the $\forall$-elimination rule:

$$\frac{(\forall x : A)B\ true \qquad a : A}{B[a] true}$$

Making the proof-objects explicit yields the following $\forall$-elimination rule.

$$\frac{c : (\forall x : A)B \qquad a : A}{\mathbf{ap}(c, a) : B[a]}$$

The rule says that if c is a proof of $(\forall x : A)B$ and $a : \ A$, then $\mathbf{ap}(c, a)$ is a proof of $B[a]$. The Π-equality rule can now be seen to correspond to the $\forall$-reduction of Prawitz (1965, p. 37) at the level of proof-objects. We shall illustrate this in the case of $\supset$, to which we now turn.

Suppose $B : \mathbf{set}$. Then, by weakening, $x : A \vdash B : \mathbf{set}$ holds. In this case an element of $(\Pi x : A)B$ codes a function from the **set** A to the **set** B. Since x is not free in B in this case, we may write $A \ \rightarrow \ B$ instead of $(\Pi x : A)B$, thereby also indicating that this is the function space from A to B. Regarding both A and B as propositions, and again keeping in mind the Brouwer–Heyting–Kolmogorov interpretation of the logical connectives, it is clear that $A \ \rightarrow \ B$ can be interpreted as the implication $A \supset B$.

The Π-introduction and elimination rules become $\supset$-introduction and elimination in this case. A canonical proof-object of $A \supset B$ has the form $\lambda x. b$, where b is an open proof from A to B. Given a proof of $c : A \supset B$ and a proof $a : A$, then $\mathbf{ap}(c, a)$ is a proof of B.

The Π-equality rule yields the following rule of $\supset$-equality.

$$\frac{x : A \vdash b : B \qquad a : A}{\mathbf{ap}(\lambda x. b, a) = b[a] : B}$$

Here a is a proof of A; b is an open proof of B from A; $\lambda x. b$ is a proof of $A \supset B$ obtained by extending b with one application of $\supset$-introduction; $\mathbf{ap}(\lambda x. b, a)$ is the proof of B got by applying $\supset$-elimination to $\lambda x.b$ and a; and $b[a]$ is a proof of B got from b by supplying it in the suitable sense with the proof a of A. The $\supset$-equality rule says that $\mathbf{ap}(\lambda x.b, a)$ and $b[a]$ are equal proofs of B. Using the standard notation of natural deduction this equality can be expressed as follows (where we write $\mathcal{D}_1$ instead of b and $\mathcal{D}_2$ instead of a).

$$\dfrac{\dfrac{\begin{matrix}A\\ \mathcal{D}_1\\ B\end{matrix}}{A\supset B} \qquad \begin{matrix}\mathcal{D}_2\\ A\end{matrix}}{B} \qquad = \qquad \begin{matrix}\mathcal{D}_2\\ A\\ \mathcal{D}_1\\ B\end{matrix}$$

By replacing '$=$' here with a sign for Prawitz's reduction relation, one sees that what is displayed here is just the rule of $\supset$-reduction. Thus the rule of $\supset$-equality can be read as saying that a proof containing a "detour" like that in the proof on the left hand side above is identical to the proof got by deleting this detour by means of a $\supset$-reduction.

II.2.3 **Disjoint union of a family of sets**

Given a $\mathbf{set}$ A and a family B of $\mathbf{set}$s over A we can form the disjoint union of the family B. That is the content of Σ-formation:

$$(\Sigma\text{-form}) \qquad \frac{A : \mathbf{set} \qquad x : A \vdash B : \mathbf{set}}{(\Sigma x : A)B : \mathbf{set}}$$

According to the rule of Σ-introduction, the canonical elements of $(\Sigma x : A)B$ are pairs:

$$(\Sigma\text{-intro}) \qquad \frac{a : A \qquad b : B[a]}{\langle a, b\rangle : (\Sigma x : A)B}$$

Assume $A : \mathbf{set}$, $x : A \vdash B : \mathbf{set}$. Then we may form $(\Sigma x : A)B :$ $\mathbf{set}$. Assume further that C is a family of sets over $(\Sigma x : A)B$, that is, assume $z : (\Sigma x : A)B \vdash C : \mathbf{set}$. The rule of Σ-elimination is as follows:

$$(\Sigma\text{-elim}) \qquad \frac{c : (\Sigma x : A)B \qquad x : A, y : B \vdash d : C[\langle x, y\rangle]}{\mathbf{E}(c, xy.d) : C[c]}$$

We may think of the binary function d as a unary function on the canonical elements of $(\Sigma x : A)B$—it takes $\langle a, b\rangle$, where $a : A$ and $b : B[a]$, and yields an element $d[a, b]$ of $C[\langle a, b\rangle]$. The Σ-elimination rule provides us with a function $c \mapsto \mathbf{E}(c, xy.d)$ defined for all elements c (not only canonical ones) of $(\Sigma x : A)B$.

Two clarificatory remarks pertaining to Σ-elimination are in order here. The first remark concerns the premiss $x : A, y : B \vdash d : C[\langle x, y\rangle]$. By the preliminary assumption $z : (\Sigma x : A)B \vdash C :$ $\mathbf{set}$, the variable z, ranging over $(\Sigma x : A)B$, occurs (or, is allowed to occur) in C. Since $x : A$, $y : B \vdash \langle x, y\rangle : (\Sigma x : A)B$ holds by Σ-introduction, the substitution of $\langle x, y\rangle$ for z in C in the context $x : A$, $y : B$ makes sense. The second remark concerns the conclusion $\mathbf{E}(c, xy.d) : C[c]$. The operation $\mathbf{E}$ is variable-binding: it binds the free variables x and y in d. This is symbolized by prefixing d with x and y inside $\mathbf{E}(-,-)$.[40]

---

[40] In the higher-order presentation there is only one variable-binding operation, namely abstraction, by means of which higher-order functions are formed. The **E** above is then a higher-order function whose second argument is itself a higher-order function $d : (x : A)(y : B)\, C(\langle x, y\rangle)$. To make the notation for

The Σ-equality rule tells us how to compute $\mathbf{E}(c, xy.d)$ when c is in canonical form.

(Σ-eq)
$$\frac{a : A \qquad b : B[a] \qquad x : A, y : B \vdash d : C[\langle x, y\rangle]}{\mathbf{E}(\langle a, b\rangle, xy.d) = d[a, b] : C\langle a, b\rangle}$$

The conclusion of Σ-elimination introduces a non-canonical element $\mathbf{E}(c, xy.d)$ in $C[c]$. To justify this rule we have to explain how to evaluate this non-canonical element to canonical form. This is done by reference to the Σ-equality rule. First evaluate $c : (\Sigma x : A)B$ to get a pair $\langle a, b\rangle$, where $a : A$ and $b : B[a]$. We have

$$\mathbf{E}(c, xy.d) = \mathbf{E}(\langle a, b\rangle, xy.d) = d[a, b] : C[\langle a, b\rangle]$$

by Σ-equality. By the premiss $x : A,\ y : B \vdash d : C[\langle x, y\rangle]$ we know how to compute $d[a, b]$ to obtain a canonical element of $C[\langle a, b\rangle]$; since $C[c] = C[\langle a, b\rangle] : \mathbf{set}$, this will also be a canonical element of $C[c]$.

By means of **E** we can define projection operations, which justifies our speaking of the canonical elements of $(\Sigma x : A)B$ as pairs. For the first projection we put $C = A$ and $d = x$ in the rule of Σ-elimination, thereby obtaining:

$$\frac{c : (\Sigma x : A)B \qquad x : A, y : B \vdash x : A}{\mathbf{E}(c, xy.x) : A}$$

By Σ-equality we have in this case:

$$\mathbf{E}(\langle a, b\rangle, xy.x) = x[a/x, b/y] = a : A$$

We may therefore define the first projection $\mathbf{fst}$ as follows.

$$c : (\Sigma x : A)B \vdash \mathbf{fst}(c) = \mathbf{E}(c, xy.x)$$

For the second projection we put $C = B[\mathbf{fst}(z)]$ and $d = y$ in the rule of Σ-elimination:

$$\frac{c : (\Sigma x : A)B \qquad x : A, y : B \vdash y : B[\mathbf{fst}(\langle x, y\rangle)]}{\mathbf{E}(c, xy.y) : B[\mathbf{fst}(c)]}$$

The second premiss here is valid since $x : A,\, y : B \vdash B[\mathbf{fst}(\langle x, y\rangle)] = B[x] = B : \mathbf{set}$ holds. By Σ-equality we have

$$\mathbf{E}(\langle a, b\rangle, xy.y) = y[a/x, b/y] = b : B[\mathbf{fst}(\langle a, b\rangle)]$$

But $\mathbf{fst}(\langle a, b\rangle) = a : A$, hence

$$B[\mathbf{fst}(\langle a, b\rangle)] = B[a] : \mathbf{set}$$

We therefore define the second projection by

$$c : (\Sigma x : A)B \vdash \mathbf{snd}(c) = \mathbf{E}(c, xy.y)$$

The following four rules are then justified

λ in the Π-introduction rule accord with the notation here used for **E** we should write $\lambda(x.b)$ instead of $\lambda x.b$. The latter, being more familiar, will be preferred here.

$$\frac{c : (\Sigma x : A)B}{\mathbf{fst}(c) : A} \qquad \frac{a : A \qquad b : B[a]}{\mathbf{fst}(\langle a, b\rangle) = a : A}$$

$$\frac{c : (\Sigma x : A)B}{\mathbf{snd}(c) : B[\mathbf{fst}(c)]} \qquad \frac{a : A \qquad b : B[a]}{\mathbf{snd}(\langle a, b\rangle) = b : B[a]}$$

II.2.4 The logical interpretation of the disjoint union of a family of sets

If we regard B as a propositional function over A, then $(\Sigma x : A)B$ can be regarded as the existentially quantified proposition $(\exists x : A)B$. A canonical proof of $(\exists x : A)B$ is a pair $\langle a, b\rangle$ where $a : A$ and $b : B[a]$; that is to say, a is a witness and b is a proof that a indeed has the property B. When suppressing proof objects and employing the form of judgement $D\ true$, the rule of Σ-elimination becomes ∃-elimination:

$$\frac{(\exists x : A)B\ true \qquad x : A, B\ true\ \vdash\ C\ true}{C\ true}$$

In ordinary natural deduction the assumption $x : A$ in the second premiss is usually not made explicit.

If $B : \mathbf{set}$ holds categorically, then the rules for Σ yield rules for ordinary Cartesian product. On the logical interpretation, the Cartesian product becomes conjunction. Indeed the Σ-formation and introduction rules then become:

$$\frac{A : \mathbf{prop} \qquad B : \mathbf{prop}}{A \wedge B : \mathbf{prop}} \qquad \frac{a : A \qquad b : B}{\langle a, b\rangle : A \wedge B}$$

The Σ-elimination rule, with and without proof-objects, becomes:

$$\frac{A \wedge B\ true \qquad A\ true, B\ true\ \vdash\ C\ true}{C\ true}$$

$$\frac{c : A \wedge B \qquad x : A, y : B \vdash d : C[\langle a, b\rangle]}{\mathbf{E}(c, xy.d) : C[c]}$$

This is a generalization of the ordinary rules of ∧-elimination also found in Schroeder-Heister (1984, p. 1294). The ordinary rules are obtained as a special case by letting C be A or B. We remark that in the higher-order presentation a generalized elimination rule in this sense can also be given for Π Nordström et al. (Nordström, Petersson, & Smith, 1990, pp. 51-52); using this generalized elimination rule instead of the rule of Π-elimination presented above in fact yields a strictly stronger theory, as shown by Garner (2009).

II.2.5 Disjoint union of two sets

Given two sets we may form their disjoint union. That is content of the rule of +-formation.

$$(+\text{-form}) \qquad \frac{A : \mathbf{set} \qquad B : \mathbf{set}}{A + B : \mathbf{set}}$$

A canonical element of $A + B$ is an element of A or an element of B together with the information that it comes from A or B respectively. Thus there are two rules of $+$-introduction:

(+-intro)
$$\frac{a : A}{\mathbf{i}(a) : A + B} \qquad \frac{b : B}{\mathbf{j}(b) : A + B}$$

Assume A : **set**, B : **set**, and $z : A + B \vdash C$: **set**. The rule of $+$-elimination is:

(+-elim)
$$\frac{c : A + B \qquad x : A \vdash d : C[\mathbf{i}(x)] \quad y : B \vdash e : C[\mathbf{j}(y)]}{\mathbf{D}(c, x.d, y.e) : \ C[c]}$$

The rule can be glossed as follows. Assume that C is a family of **set**s over $A + B$ and that we are given a function d which takes an $a : A$ to an element $d[a]$ of $C[\mathbf{i}(a)]$ and a function e which takes a $b : B$ to an element $e[b]$ of $C[\mathbf{j}(b)]$. Then $C[c]$ is inhabited for any $c : \ C$, namely by $\mathbf{D}(c, x.d, y.e)$. How to compute $\mathbf{D}(c, x.d, y.e)$ is determined by the $+$-equality rules. Since there are two $+$-introduction rules, there are also two $+$-equality rules.

(+-eq)
$$\frac{a : A \qquad x : A \vdash d : C[\mathbf{i}(x)] \quad y : B \vdash e : C[\mathbf{j}(y)]}{\mathbf{D}(\mathbf{i}(a), x.d, y.e) \ = \ d[a] : \ C[\mathbf{i}(a)]}$$

$$\frac{b : B \qquad x : A \vdash d : C[\mathbf{i}(x)] \quad y : B \vdash e : C[\mathbf{j}(y)]}{\mathbf{D}(\mathbf{j}(b), x.d, y.e) \ = \ e[b] : C[\mathbf{j}(b)]}$$

In the logical interpretation $+$ becomes disjunction $\vee$.

II.2.6 **Finite sets**

We have introduced operations for constructing **set**s from other **set**s or families of **set**s; but so far we have no basic **set** to start from. We shall now provide a scheme of rules which, when specified for any particular natural number n, gives us a **set** with n canonical elements.

In the following n is a generic natural number. There is a set $\mathbb{N}_n$:

($\mathbb{N}_n$-form) $\qquad \mathbb{N}_n$: **set**

The set $\mathbb{N}$ has n canonical elements, each introduced by its own introduction rule; thus $\mathbb{N}_n$ has n introduction rules:

($\mathbb{N}_n$-intro) $\qquad m_1 : \mathbb{N}_n, \dots, m_n : \mathbb{N}_n$

The $\mathbb{N}_n$-elimination rule can be seen as a principle of proof by n cases. We assume that C is a family of **set**s over $\mathbb{N}_n$; that is, we assume $z : \mathbb{N}_n \vdash C$: **set**.

($\mathbb{N}_n$-elim)
$$\frac{m : \mathbb{N}_n \qquad c_1 : C[m_1] \qquad \dots \quad c_n : C[m_n]}{\mathbf{case}_n(m, c_1, \dots, c_n) : C[m]}$$

Thus, assume that for each canonical $m_k : \mathbb{N}_n$ we have an element $c_k : C[m_k]$. The $\mathbb{N}_n$-elimination rule allows us to infer that for any $m : \mathbb{N}_n$ there is an element, namely $\mathbf{case}_n(m, c_1, \dots, c_n)$, in $C[m]$. How to compute $\mathbf{case}_n(m, c_1, \dots, c_n)$ to canonical form is determined by the $\mathbb{N}_n$-equality rules. Since there are n $\mathbb{N}_n$-introduction rules, there are also n $\mathbb{N}_n$-equality rules, one for each introduction rule. We state the rule for a generic $k \leq n$.

$$(\mathbb{N}_n\text{-eq}) \qquad \frac{c_1 : C[m_1] \qquad \dots \qquad c_n : C[m_n]}{\mathbf{case}_n(m_k, c_1, \dots, c_n) = c_k : C[m_k]}$$

On the basis of this rule we may explain how to compute $\mathbf{case}_n(m, c_1, \dots, c_n) : C[c]$. Evaluate $m : \mathbb{N}_n$, thereby obtaining a canonical element $m_k : \mathbb{N}_n$. Since $m = m_k : \mathbb{N}_n$, we have both $\mathbf{case}_n(m, c_1, \dots, c_n) = \text{c }\mathbf{ase}_n(m_k, c_1, \dots, c_n) : C[m_k]$ and $C[m_k] = C[m] : \mathbf{set}$. Therefore, $\mathbf{case}_n(m, c_1, \dots, c_n) = c_k : C[m]$, by $\mathbb{N}_n$-equality. Hence the value of $\mathbf{case}_n(m, c_1, \dots, c_n)$ equals the value of c_k, which we know how to compute by the premiss $c_k : C[m_k]$.

We may give $\mathbb{N}_2$ the name **bool**; the canonical elements of $\mathbb{N}_2$ the names **t** and **f**; and the expression $\mathbf{case}_2(m, c_1, c_2)$ may be written **if** m **then** c_1 **else** c_2. Let C be a family of **set**s over **bool**; that is, assume $z : \mathbf{bool} \vdash C : \mathbf{set}$. We have the following two rules of **bool**-elimination, which we here state as one rule with two conclusions:

$$\frac{c : C[\mathbf{t}] \qquad\qquad d : C[\mathbf{f}]}{\begin{array}{c}\mathbf{if\ t\ then}\ c\ \mathbf{else}\ d = c : C[\mathbf{t}] \\ \mathbf{if\ f\ then}\ c\ \mathbf{else}\ d = d : C[\mathbf{f}]\end{array}}$$

Familiar Boolean functions can be defined from $\mathbf{t}, \mathbf{f}$, and **if** m **then** c_1 **else** c_2 as follows.

$$a : \mathbf{bool}, b : \mathbf{bool} \vdash a\ \mathbf{and}\ b = \mathbf{if}\ a\ \mathbf{then}\ b\ \mathbf{else\ f} : \mathbf{bool}$$

$$a : \mathbf{bool}, b : \mathbf{bool} \vdash a\ \mathbf{or}\ b = \mathbf{if}\ a\ \mathbf{then\ t\ else}\ b : \mathbf{bool}$$

$$a : \mathbf{bool} \vdash \mathbf{not}\ a = \mathbf{if}\ a\ \mathbf{then\ f\ else\ t} : \mathbf{bool}$$

The **set** $\mathbb{N}_0$ has 0, that is no, introduction rules; but it does have an elimination rule:

$$\frac{m : \mathbb{N}_0}{\mathbf{case}_0(m) : C[m]}$$

Thus, in particular, if we are given $C : \mathbf{set}$ and $m : \mathbb{N}_0$, then we may infer that C is inhabited, namely by $\mathbf{case}_0(m)$. Since there is no $\mathbb{N}_0$-introduction rule, neither is there a $\mathbb{N}_0$-equality rule. The justification of $\mathbb{N}_0$-elimination is therefore different in character from the justification of the other elimination rules. A rule of inference is justified if we can make the conclusion of the rule evident on the assumption that its premisses are known (cf. eg. (Sundholm, 2012)). Since $\mathbb{N}_0$ has no introduction rule and therefore no canonical elements, the premiss $m : \mathbb{N}_0$ of $\mathbb{N}_0$-elimination cannot be known. The rule of $\mathbb{N}_0$-elimination is therefore vacuously justified, for the assumption that its premiss is known cannot be fulfilled.

In the logical interpretation $\mathbb{N}_0$ becomes absurdity $\bot$. In CTT absurdity is thus the proposition that by definition has no introduction rule. The rule of $\bot$-elimination is the principle of *ex falso quodlibet*:

$$\frac{\bot\ true}{C\ true}$$

We define the negation of a proposition A to be the proposition $A \supset \bot$:[41]

---

[41] In the higher-order presentation $\neg$ may be defined in the empty context, namely as follows.

$$\neg = [A]A \supset \bot : (\mathbf{prop})\mathbf{prop}$$

$$A : \mathbf{prop} \vdash \neg A = A \supset \bot$$

The following rules are then derivable.

$$\frac{A : \mathbf{prop}}{\neg A : \mathbf{prop}} \qquad \frac{x : A \vdash b : \bot}{\lambda x.b : \neg A} \qquad \frac{b : \neg A \qquad a : A}{\mathbf{ap}(b, a) : C}$$

II.2.7 **The natural numbers**

We shall not need the natural numbers in later chapters, but it may nevertheless be useful to see that the primitive notions of arithmetic can be introduced by Gentzen–Prawitz style rules.

(ℕ-form) $\quad \mathbb{N} : \mathbf{set}$

The canonical elements of ℕ are 0 and $\boldsymbol{s}(n)$, where n is any ℕ, not necessarily of canonical form:

(ℕ-intro) $\quad 0 : \mathbb{N} \qquad \dfrac{n : \mathbb{N}}{\boldsymbol{s}(n) : \mathbb{N}}$

The rule of ℕ-elimination is simultaneously a principle of definition by recursion and proof by mathematical induction. Assume that C is a family of **set**s over ℕ; that is, assume $z : \mathbb{N} \vdash C : \mathbf{set}$.

(ℕ-elim) $\quad \dfrac{n : \mathbb{N} \qquad d : C[0] \qquad x : \mathbb{N}, y : C[x] \vdash e : C[\boldsymbol{s}(x)]}{\mathbf{R}(n, d, xy.e) : C[n]}$

We are given an element $d : C[0]$ together with a function e that takes $k : \mathbb{N}$ and $c : C[k]$ and yields $e[k, c] : C[\boldsymbol{s}(k)]$. The rule tells us that for an arbitrary $n : \mathbb{N}$ the **set** $C[n]$ is then inhabited, namely by $\mathbf{R}(n, d, xy.e)$.

To see that ℕ-elimination encapsulates the ordinary principle of proof by induction over the natural numbers, assume that C is a propositional function over ℕ. Then d is a proof of the base case $C[0]$ and e is a proof of the induction step; that is, e is an open proof from $C[k]$ to $C[\boldsymbol{s}(k)]$. The conclusion of ℕ-elimination says that $C[n]$ is inhabited, that is, true, for an arbitrary $n : \mathbb{N}$.

The rule of ℕ-equality tells us how to compute $\mathbf{R}(n, d, xy.e)$ when n is of canonical form. Since there are two ℕ-introduction rules, there are also two ℕ-equality rules, which we here state more simply without the premisses

(ℕ-eq)
$$\mathbf{R}(0, d, xy.e) = d : C[0]$$
$$\mathbf{R}(\boldsymbol{s}(n), d, xy.e) = e[n, \mathbf{R}(n, d, xy.e)] : C[\boldsymbol{s}(n)]$$

This gives in particular:

$$\mathbf{R}(\boldsymbol{s}(0), d, xy.e) = e[0, \mathbf{R}(0, d, xy.e)] = e[0, d] : C[\boldsymbol{s}(0)]$$
$$\mathbf{R}(\boldsymbol{s}(\boldsymbol{s}(0)), d, xy.e) = e[\boldsymbol{s}(0), \mathbf{R}(\boldsymbol{s}(0), d, xy.e)] = e[\boldsymbol{s}(0), e[0, d]] : C[\boldsymbol{s}(\boldsymbol{s}(0))]$$
$$\mathbf{R}\left(\boldsymbol{s}\left(\boldsymbol{s}(\boldsymbol{s}(0))\right), d, xy.e\right) = e\left[\boldsymbol{s}(\boldsymbol{s}(0)), e[\boldsymbol{s}(0), e[0, d]]\right] : C\left[\boldsymbol{s}\left(\boldsymbol{s}(\boldsymbol{s}(0))\right)\right]$$

Thus, ¬ is a function which takes a **prop** A as argument and yields a **prop** $\neg A$ as value. Similar considerations apply to the definitions of **fst** and **snd** above as well as to the definitions given below pertaining to ℕ.

It should be clear from these few computations that **R** provides a means for defining functions by recursion.

In general, to compute $\mathbf{R}(m, d, xy.e)$ first evaluate m to canonical form. If the value is 0, output d; if the value is $\boldsymbol{s}(n)$, continue by computing $e[n, \mathbf{R}(n, d, xy.e)]$.

By letting C be $\mathbb{N}$ and e be $\boldsymbol{s}(y)$ in $\mathbb{N}$-elimination, one can infer $a : \mathbb{N},\ b : \mathbb{N} \vdash \mathbf{R}\big(b,\ a,\ xy.\boldsymbol{s}(y)\big) : \mathbb{N}$. The reader may check that we here have the definiens of the addition function, in other words, that addition can be defined as follows.

$$a : \mathbb{N}, b : \mathbb{N} \vdash a + b = \mathbf{R}\big(b, a, xy.\boldsymbol{s}(y)\big) : \mathbb{N}$$

To define multiplication we let C be $\mathbb{N}$**,** d be 0, and e be $y + a$.

$$a : \mathbb{N}, b : \mathbb{N} \vdash a \times b = \mathbf{R}\big(b, 0, xy.(y + a)\big) : \mathbb{N}$$

II.2.8 **Propositional identity**

In the language developed so far we can express identities by means of judgements $a = b : C$. Judgements cannot, however, be operated on by the propositional operators, that is, for instance by conjunction or universal quantification. Since we want to be able to operate on identity statements with the propositional operations, it is clear that we need identity propositions. Thus, for each $\mathbf{set}$ A we wish to introduce a binary propositional function $x =_A y$ of identity over A. Instead of $x =_A y$, we shall usually write $\mathbf{Id}(A, x, y)$. The rule of $\mathbf{Id}$-formation states that given a $\mathbf{set}$ A and two elements a, b of A, there is a proposition $\mathbf{Id}(A, a, b)$.

($\mathbf{Id}$-form) $$\frac{A : \mathbf{set} \qquad a : A \qquad\qquad b : A}{\mathbf{Id}(A, a, b) : \mathbf{prop}}$$

Note that by this rule there is no identity proposition between a and b unless a and b belong to the same $\mathbf{set}$. Hence, assuming that Julius Caesar and the number 7 do not belong to the same $\mathbf{set}$, there is no proposition to the effect that Julius Caesar and 7 are identical.

It is only with the introduction of $\mathbf{Id}$ that we are able to define propositional functions in our language. Namely, given a $\mathbf{set}$ A we now have a propositional function $\mathbf{Id}(A, x, y)$ over A, by means of which other propositional functions can be defined. For instance, we may now define $x \leq y$ over $\mathbb{N}$ by

$$x : \mathbb{N}, y : \mathbb{N} \vdash x \leq y = (\exists z : \mathbb{N})\mathbf{Id}(\mathbb{N}, x + z, y) : \mathbf{prop}$$

Notice the use of judgemental identity $A = B : \mathbf{prop}$ here: the definition in effect identifies two propositions in the context $x : \mathbb{N}, y : \mathbb{N}$. It may be useful to see how one may define the propositional function $\mathbf{Pr}(x)$, saying that x is a prime number, in CTT. In ordinary predicate logic with restricted quantifiers we may use a definition such as the following (for the purposes of this presentation let us assume that 0 and 1 are prime numbers).

$$\mathbf{Pr}(n) \equiv \forall x, y \leq n\big(x \times y = n \supset\ (x = 1 \vee x = n)\big)$$

In CTT restricted quantification may be defined as quantification over $(\exists z : \mathbb{N})z \leq n$. Such quantification makes sense, since $(\exists z : \mathbb{N})z \leq n$ is a set for any $n : \mathbb{N}$. An element of this set is a pair $\langle k, p\rangle$, where k is a $\mathbb{N}$ and p is a proof of $k \leq n$. We may define **Pr** as follows.

$$n : \mathbb{N} \vdash \mathbf{Pr}(n) = (\forall x, y : (\exists z : \mathbb{N}) z \leq n)$$
$$\big((\mathbf{fst}(x) \times \mathbf{fst}(y) =_{\mathbb{N}} n) \supset (\mathbf{fst}(x) =_{\mathbb{N}} 1 \vee \mathbf{fst}(x) =_{\mathbb{N}} n)\big) : \mathbf{prop}$$

Here we have used the notation $x =_{\mathbb{N}} y$ instead of the official $\mathbf{Id}(\mathbb{N}, x, y)$, and we have contracted the two quantifiers by writing $(\forall x, y : (\exists z : \mathbb{N}) z \leq n)$.

To justify the rule of **Id**-formation we have to specify what is a canonical proof of $\mathbf{Id}(A, a, b)$. What, for instance, should be a canonical proof of the proposition $\mathbf{Id}(\mathbb{N}, 0, 0)$? The Brouwer–Heyting–Kolmogorov explanation will not help us in answering this question, since it is silent about identity propositions. Since we want $\mathbf{Id}(\mathbb{N}, x, y)$ to be the relation of identity over $\mathbb{N}$ and since a proposition is here taken to be true if it is inhabited as a **set**, it is clear that we simply have to introduce a proof of $\mathbf{Id}(\mathbb{N}, 0, 0)$ by stipulation; we call this proof $\mathbf{refl}(\mathbb{N}, 0)$. The rule of **Id**-introduction is as follows.

(**Id**-intro)
$$\frac{a : A}{\mathbf{refl}(A, a) : \mathbf{Id}(A, a, a)}$$

Thus, provided $a : A$, we stipulate that there is a proof $\mathbf{refl}(A, a) : \mathbf{Id}(A, a, a)$. We here emphasize the aspect of stipulation, but in fact, all introduction rules are purely stipulatory in nature. An introduction rule stipulates that the canonical elements of the **set** under consideration look such and such. The **Id**-introduction rule is in this regard no different from other introduction rules.

The **Id**-introduction rule *is* different from other introduction rules in that it does not immediately yield an answer to the question of what is a canonical element of $\mathbf{Id}(A,\ a,\ b)$, that is, of a **set** of the form introduced by **Id**-formation. It yields an answer only to the question of what is a canonical element of $\mathbf{Id}(A, a, a)$. Since no introduction rule has $\mathbf{Id}(A, a, b)$ as the predicate C of its conclusion $c : C$, it is clear that the only way in which we can come to judge that c is a canonical element of $\mathbf{Id}(A, a, b)$ is on the basis of an identity judgement of the form $C = \mathbf{Id}(A, a, b) : \mathbf{prop}$, where $c : C$ is the conclusion of an application of an introduction rule. It is, moreover, clear that any such C must have the form $\mathbf{Id}(A', a', a')$, where $A = A' : \mathbf{set}$, $a = a' : A$ and $b = a' : A$. A canonical element of $\mathbf{Id}(A, a, b)$ is therefore of the form $\mathbf{refl}(A', a')$ where $A = A' : \mathbf{set}$, $a = a' : A$ and $b = a' : A$.

Martin-Löf (1971), in a paper concerned with natural deduction rather than Type Theory, provided a general scheme of introduction and elimination rules as well as reduction procedures for so-called inductively defined predicates. The rules provided above for $\mathbb{N}_n$ and $\mathbb{N}$ follow this scheme, although they are adapted to the syntax of CTT. Also the identity predicate of ordinary predicate logic is covered by this rule scheme (Martin-Löf, 1971, p. 190). The ordinary binary identity predicate, which Martin-Löf designates by E, is the predicate that has the introduction rule

$$Exx$$

with no premisses. It should be clear how the rule of **Id**-introduction above is an adaptation of this rule to the syntax of CTT. Martin-Löf's scheme yields the following elimination rule for E:

$$\frac{Etu \qquad C[z/x, z/y]}{C[t/x, u/y]}$$

Here C is any formula of the language, and $C[z/x, z/y]$ is the result of substituting the variable z for both x and y in C. If we assume that x and y are all and only the free variables in C, then we may think of C as defining a binary relation over the underlying domain. That we can prove $C[z/x, z/y]$ means that the relation defined by C is reflexive. The E-elimination rule allows us to infer that the relation defined by C is true of t and u provided we have a derivation of Etu. Thus the rule says in effect that E is the smallest reflexive relation over the underlying domain.

The rule of **Id**-elimination generalizes the E-elimination rule to the syntax of CTT; it generalizes the E-elimination rule also in allowing the relation C occurring in the minor premiss to include as argument a proof-object of the identity proposition being eliminated. Assume $A : \mathbf{set}$, $a : A$, $b : A$, and $x : A, y : A, u : \mathbf{Id}(A, x, y) \vdash C : \mathbf{set}$.

(**Id**-elim)

$$\frac{p : \mathbf{Id}(A, a, b) \quad z : A \vdash d : C[z, z, \mathbf{refl}(A, z)]}{\mathbf{J}(p, z.d) : C[a, b, p]}$$

Thus, if we have a proof p of the proposition $\mathbf{Id}(A, a, b)$ and a function d taking any a' of A to a proof that the ternary relation C holds of the triple a', a', $\mathbf{refl}(A, a')$; then **Id**-elimination allows us to infer that there is a proof $\mathbf{J}(p, z.d)$ of $C[a, b, p]$.

The rule of **Id**-equality tells us how to compute $\mathbf{J}(p, z.d)$ when p is of canonical form, $\mathbf{refl}(A, a)$.

(**Id**-eq) $$\mathbf{J}(\mathbf{refl}(A, a), z.d) = d[a] : C[a, a, \mathbf{refl}(A, a)]$$

The rule of **Id**-elimination can now be justified as follows. To evaluate $\mathbf{J}(p, z.d)$ first evaluate p to get a canonical element of $\mathbf{Id}(A, a, b)$. As noted above, such a canonical element has the form $\mathbf{refl}(A', a')$, where $A = A' : \mathbf{set}$, $a = a' : A$ and $b = a' : A$. Hence,

$$\mathbf{J}(p, z.d) = \mathbf{J}(\mathbf{refl}(A', a'), z.d) = d[a'] : C[a', a', \mathbf{refl}(A', a')].$$

By the assumption $z : A \vdash d : C[z, z, \mathbf{refl}(z)]$ we know how to evaluate $d[a']$ to canonical form. It remains then only to see that $C[a', a', \mathbf{refl}(A', a')] = C[a, b, p] : \mathbf{set}$, but this follows from the judgemental identities

$$A = A' : \mathbf{set};\ a = a' : A;\ b = a' : A;\ p = \mathbf{refl}(A', a') : \mathbf{Id}(A, a, b)$$

together with the extensionality of substitution into **set**s with respect to judgemental identity.

In many applications of **Id**-elimination the family C in its minor premiss does not depend on the set $\mathbf{Id}(A, a, b)$ of its major premiss. Thus, C will then be a **set** already in the context $x : A, y : A$; that is, C will then be a binary relation over A. What is required then for an application of **Id**-elimination is that we have a function d witnessing that C is a reflexive relation over A; more precisely, that $d[a]$ is a proof of $C[a, a]$ for any $a : A$. We shall now demonstrate how **Id**-elimination is used in practice by showing that the relation $\mathbf{Id}(A, x, y)$ is symmetric and transitive, and by showing that if $F[a]$ and $\mathbf{Id}(A,\ a,\ b)$ are true, then so is $F[b]$, that is, by showing the indiscernibility of elements related by $\mathbf{Id}(A, x, y)$. The main task in each case is to find a suitable C and a suitable function d taking $a : A$ and yielding a proof of $C[a, a]$.

For symmetry let C be $\mathbf{Id}(A, y, x)$. It is clear that $z : A \vdash \mathbf{refl}(A, z) : \mathbf{Id}(A, z, z)$, so we let d be $\mathbf{refl}(A, z)$. If we insert these data into **Id**-elimination we get:

$$\frac{p :\ \mathbf{Id}(A, a, b) \qquad z : A \vdash \mathbf{refl}(A, z) : \mathbf{Id}(A, z, z)}{\mathbf{J}\big(p, z.\mathbf{refl}(A, z)\big) : \mathbf{Id}(A, b, a)}$$

Hence, from a proof $p :\ \mathbf{Id}(A, a, b)$ we get a proof $\mathbf{J}\big(p, z.\mathbf{refl}(A, z)\big) : \mathbf{Id}(A, b, a)$.

Assuming that p here is a closed term, one can argue, on the basis of the explanation of what a canonical proof of an identity proposition is, that p is identical with the proof $\mathbf{J}\big(p,\ z.\mathbf{refl}(A,z)\big)$. Namely, the judgement $p : \mathbf{Id}(A,a,b)$ means that p, since it is a closed term, evaluates to a proof of the form $\mathbf{refl}(A',a')$, where $A = A' : \mathbf{set}$, $a = a' : A$, and $b = a' : A$; hence, $p = \mathbf{refl}(A',a') : \mathbf{Id}(A,a,b)$. Therefore,

$$\mathbf{J}\big(p, z.\mathbf{refl}(A,z)\big) = \mathbf{J}\big(\mathbf{refl}(A',a'), z.\mathbf{refl}(A,z)\big) = \mathbf{refl}(A,a') : \mathbf{Id}(A,b,a)$$

Since $A = A' : \mathbf{set}$, we get:

$$p = \mathbf{refl}(A',a') = \mathbf{refl}(A,a') = \mathbf{J}\big(p, z.\mathbf{refl}(A,z)\big) : \mathbf{Id}(A,b,a)$$

It should be emphasized that this reasoning i) presupposes that p is closed and ii) is not a computation in the theory itself; the computation in any particular case depends on what the given proof-object $p : \mathbf{Id}(A,a,b)$ looks like.

For transitivity we let C be $\mathbf{Id}(A,y,c) \supset \mathbf{Id}(A,x,c)$ for an arbitrary $c : A$. We have $\lambda u.u : \mathbf{Id}(A,z,c) \supset \mathbf{Id}(A,z,c)$, so we let d be $\lambda u.u$. Inserting these data into **Id**-elimination yields:

$$\frac{p : \mathbf{Id}(A,a,b) \qquad z : A \vdash \lambda u.u : \mathbf{Id}(A,z,c) \supset \mathbf{Id}(A,z,c)}{\mathbf{J}(p,\lambda u.u) : \mathbf{Id}(A,b,c) \supset \mathbf{Id}(A,a,c)}$$

Hence, from a proof $p : \mathbf{Id}(A,a,b)$ and a proof $q : \mathbf{Id}(A,b,c)$, we get a proof $\mathbf{ap}(\mathbf{J}(p,\ \lambda u.u),q) : \mathbf{Id}(A,a,c)$. Note that $\lambda u.u$ does not depend on $z : A$, hence no variable gets bound by this application of **Id**-elimination. Assuming that p is closed, we can argue as above that $p = \mathbf{refl}(A',a') : \mathbf{Id}(A,a,b)$, whence by **Id**-equality and Π-equality we get:

$$\mathbf{ap}(\mathbf{J}(p,\lambda u.u),q) = \mathbf{ap}(\mathbf{J}(\mathbf{refl}(A',a'),\lambda u.u),q) = \mathbf{ap}(\lambda u.u,q) = q : \mathbf{Id}(A,a,c)$$

Again it must be emphasized that this argument is not the same as an actual computation in the theory.

For the indiscernibility of elements a,b for which $\mathbf{Id}(A,a,b)$ is true, let F be a propositional function over A, that is $x : A \vdash F : \mathbf{prop}$. Let C be $F[x] \supset F[y]$. Again we have $\lambda u.u : F[z] \supset F[z]$, hence **Id**-elimination yields:

$$\frac{p : \mathbf{Id}(A,a,b) \qquad z : A \vdash \lambda u.u : F[z] \supset F[z]}{\mathbf{J}(p,\lambda u.u) : F[a] \supset F[b]}$$

Hence, from a proof $p : \mathbf{Id}(A,a,b)$ and a proof $q : F[a]$, we get a proof

$$\mathbf{ap}(\mathbf{J}(p,\lambda u.u),q) : F[b].$$

As in the case of transitivity above we can argue that $\mathbf{ap}(\mathbf{J}(p,\lambda u.u),q) = q : F[b]$ on the assumption that p is closed.

The second **Id**-formation rule, the rule that governs when two sets of the form $\mathbf{Id}(A,\ a,\ b)$ are identical, is as follows.

$$\frac{A = A' : \mathbf{set} \qquad a = a' : A \qquad b = b' : A}{\mathbf{Id}(A,a,b) = \mathbf{Id}(A',a',b') : \mathbf{set}}$$

Employing this rule together with the general rules governing judgemental identity, one sees that the following rule is derivable:

$$\frac{a = b : A}{\mathbf{refl}(A, a) : \mathbf{Id}(A, a, b)}$$

The rule of identity elimination employed in (Martin-Löf, 1984) lays down that one can also go the other way:

$$\frac{p : \mathbf{Id}(A, a, b)}{a = b : A}$$

In the literature this rule is sometimes called extensional identity elimination, whereas the rule of $\mathbf{Id}$-elimination is called intensional. Subsequent metamathematical work has shown that this extensional rule has several undesirable consequences, perhaps the strongest of which is that in the presence of this rule judgements of the form $a = b : A$ become undecidable in general (Hofmann, 1995, p. Theorem 3.2.1). From the normalization theorem of Martin-Löf (1975b) for the system employing the intensional $\mathbf{Id}$-elimination rule it follows that judgements of the form $a = b : A$ are decidable in this system. For this and other reasons,[42] most presentations of CTT prefer the intensional system.

It follows from Martin-Löf's normalization theorem that if $p : \mathbf{Id}(A, a, b)$ is demonstrable in the empty context, then so is $a = b : A$; hence if one can construct a closed proof p of $\mathbf{Id}(A, a, b)$, then the judgemental identity $a = b : A$ is demonstrable (Martin-Löf, 1975b, p. Theorem 3.14). In a non-empty context, however, one can in general not infer $a = b : A$ from $p : \mathbf{Id}(A, a, b)$. For instance, using Σ-elimination one can prove

$$z : A \times B \vdash \mathbf{E}\big(z, xy.\mathbf{refl}(A \times B, \langle x, y \rangle)\big) \ : \mathbf{Id}(A \times B, z, \langle \mathbf{fst}(z), \mathbf{snd}(z) \rangle)$$

But there is no way of demonstrating the corresponding judgemental identity

$$z : A \times B \vdash z = \langle \mathbf{fst}(z), \mathbf{snd}(z) \rangle : A \times B$$

since z and $\langle \mathbf{fst}(z), \mathbf{snd}(z) \rangle$ are different normal forms in the context $z : A \times B$ (this example is taken from (Martin-Löf, 1975a, pp. 103-104)). We see therefore that judgemental identity $a = b : A$ and propositional identity $\mathbf{Id}(A, a, b)$ do not only have different logical form, but also that they differ in logical strength. These are therefore two essentially different notions of identity.

---

[42] In Homotopy Type Theory the main reason to prefer the intensional $\mathbf{Id}$-elimination rule is that it does not entail

$$\big(\forall p, q : \mathbf{Id}(A, a, b)\big)\mathbf{Id}(\mathbf{Id}(A, a, b), p, q) \ true$$

for arbitrary A and $a, b : A$, a result that was established by (Hofmann & Streicher, 1998).

Exercises

The word ‘proof’ has been reserved for proof-objects. Derivations in Constructive Type Theory are often called *demonstrations*. Most exercises below ask the reader to demonstrate A *true* for a given proposition A. It is then intended that a demonstration be given whose conclusion has the form $a : A$.

1. Let $A : \mathbf{prop}$, $B : \mathbf{prop}$. Demonstrate the following judgements.

a) $A \supset (B \supset A)$ *true*
b) $(A \wedge B) \supset (A \vee B)$ *true*
c) $\neg(A \vee B) \supset \neg A$ *true*

2. Let D : set, $x : D \vdash A : \mathbf{prop}$, and $x : D \vdash B : \mathbf{prop}$. That is to say, A and B are propositional functions over D. Demonstrate the following judgements.

a) $(\exists x : D)(A \wedge B) \supset ((\exists x : D)A \wedge (\exists x : D)B)$ *true*
b) $(\forall x : D)(A \wedge B) \supset ((\forall x : D)A \wedge (\forall x : D)B)$ *true*
c) $(\exists x : D)(A \vee B) \supset ((\exists x : D)A \vee (\exists x : D)B)$ *true*

3. Let D : set and $x : D, y : D \vdash R : \mathbf{prop}$. That is to say, R is a binary relation over D. Demonstrate

$(\exists x : D)(\forall y : D)R[x, y] \supset (\forall y : D)(\exists x : D)R[x, y]$ *true*

(Note that $R[x, y] \equiv R[x/x, y/y] \equiv R$.)

4. In its class-theoretic form, the syllogism of Barbara may be formulated as follows.

$$\frac{\text{All } A\text{'s are } B. \quad \text{All } B\text{'s are } C.}{\therefore \text{ All } A\text{'s are } C.}$$

Formalize Barbara in CTT. Assume that you have been given proof-objects of the premisses; construct a proof-object of the conclusion.

5. Formalize in CTT the following class-theoretical reasoning. “Everything is an A; whatever is an A is a B; hence, everything is a B.”

Given a proof-object of the two premisses, construct a proof-object of the conclusion.

6. Let $A : \mathbf{set}$, $B : \mathbf{set}$, and assume $c : A \to B$. Demonstrate

$\mathbf{Id}(A, a, a') \supset \mathbf{Id}(B, \mathbf{ap}(c, a), \mathbf{ap}(c, a'))$ *true*

7. Let $A : \mathbf{set}$, $B : \mathbf{set}$. Assume $p : \mathbf{Id}(A, a, a')$ and $q : \mathbf{Id}(B, b, b')$. Demonstrate

$\mathbf{Id}(A \times B, <a, b>, <a', b'>)$ *true*

8. Demonstrate
a) $(\forall x : \mathbf{bool})(\mathbf{Id}(\mathbf{bool}, x, \mathbf{f}) \vee \mathbf{Id}(\mathbf{bool}, x, \mathbf{f}))$ *true*
b) $(\forall x : \mathbb{N})(\mathbf{Id}(\mathbb{N}, x, 0) \vee (\exists y : \mathbb{N})\mathbf{Id}(\mathbb{N}, x, \mathbf{s}(y)))$ *true*

Solutions

1. In the solutions to these exercises we will include all the details. Each demonstration begins with the judgements A : **prop** and B : **prop**. These judgements could also be included in the demonstration as hypotheses, namely by placing them to the left of $\vdash$. Here we rather choose to regard these judgements as given. We may think of an interlocutor who has taken responsibility for these judgements, A : **prop** and B : **prop**, and consider it to be our task to make evident each of $A \supset (B \supset A)$ *true*, $(A \wedge B) \supset (A \vee B)$ *true*, and $\neg(A \vee B) \supset \neg A$ *true*.[43]

a)

$$\dfrac{\dfrac{\dfrac{\dfrac{A : \mathbf{prop}}{x : A \vdash x : A} \qquad \dfrac{B : \mathbf{prop}}{x : A \vdash B : \mathbf{prop}}}{x : A, y : B \vdash x : A}}{x : A \vdash \lambda y.x : B \supset A}}{\lambda x.\lambda y.x : A \supset (B \supset A)}$$

b)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A : \mathbf{prop} \qquad B : \mathbf{prop}}{A \wedge B : \mathbf{prop}}}{x : A \wedge B \vdash x : A \wedge B}}{x : A \wedge B \vdash \mathbf{fst}(x) : A}}{x : A \wedge B \vdash \mathbf{i}(\mathbf{fst}(x)) : A \vee B}}{\lambda x.\mathbf{i}(\mathbf{fst}(x)) : (A \wedge B) \supset (A \vee B)}$$

There is a similar demonstration having as conclusion

$$\lambda x.\mathbf{j}(\mathbf{snd}(x)) : (A \wedge B) \supset (A \vee B).$$

c)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A : \mathbf{prop} \qquad B : \mathbf{prop}}{A \vee B : \mathbf{prop}}}{\neg(A \vee B) : \mathbf{prop}}}{y : \neg(A \vee B) \vdash y : \neg(A \vee B)} \qquad \dfrac{\dfrac{A : \mathbf{prop}}{x : A \vdash x : A}}{x : A \vdash \mathbf{i}(x) : A \vee B}}{y : \neg(A \vee B), x : A \vdash \mathbf{ap}(y, \mathbf{i}(x)) : \bot}}{y : \neg(A \vee B) \vdash \lambda x.\mathbf{ap}(y, \mathbf{i}(x)) : \neg A}$$

---

[43] This dialogical view on demonstration has been developed in some recent lectures of Per Martin-Löf. This proposal, as communicated by Prof. Sundholm to the group of dialogicians in Lille, is one of the main motivations for the research documented in the following chapters of this book.

$$\frac{\qquad\qquad}{\lambda y.\lambda x.\mathbf{ap}(y, \mathbf{i}(x)) : \neg(A \vee B) \supset \neg A}$$

2. Here we shall use the natural deduction style of presenting demonstrations. That means that we shall not exhibit the hypotheses, but take these to be implicitly understood. The reader may want to supply the missing hypotheses.

   a) To save space we shall write **p** instead of **fst** and **q** instead of **snd.**

$$\dfrac{\dfrac{\dfrac{y : (\exists x : D)(A \wedge B)}{\mathbf{p}(y) : D} \qquad \dfrac{\dfrac{y : (\exists x : D)(A \wedge B)}{\mathbf{q}(y) : (A \wedge B)[\mathbf{p}(y)]}}{\mathbf{p}(\mathbf{q}(y)) : A[\mathbf{p}(y)]}}{<\mathbf{p}(y), \mathbf{p}(\mathbf{q}(y))> : (\exists x : D)A} \qquad \dfrac{\dfrac{y : (\exists x : D)(A \wedge B)}{\mathbf{p}(y) : D} \qquad \dfrac{\dfrac{y : (\exists x\!: D)(A \wedge B)}{\mathbf{q}(y)\!: A \wedge B)[\mathbf{p}(y)]}}{\mathbf{q}(\mathbf{q}(y)) : B[\mathbf{p}(y)]}}{<\mathbf{p}(y), \mathbf{q}(\mathbf{q}(y))> : (\exists x : D)B}}{<<\mathbf{p}(y), \mathbf{p}(\mathbf{q}(y))>, <\mathbf{p}(y), \mathbf{q}(\mathbf{q}(y))>> : (\exists x : D)A \wedge (\exists x : D)B}$$

   b)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{z : (\forall x : D)(A \wedge B) \qquad x : D}{\mathbf{ap}(z, x) : A \wedge B}}{\mathbf{fst}(\mathbf{ap}(z, x)) : A}}{\lambda x.\mathbf{fst}(\mathbf{ap}(z, x)) : (\forall x : D)A} \qquad \dfrac{\dfrac{\dfrac{z : (\forall x : D)(A \wedge B) \qquad x : D}{\mathbf{ap}(z, x) : A \wedge B}}{\mathbf{snd}(\mathbf{ap}(z, x)) : B}}{\lambda x.\mathbf{snd}(\mathbf{ap}(z, x)) : (\forall x : D)B}}{<\lambda x.\mathbf{fst}(\mathbf{ap}(z, x)), \lambda x.\mathbf{snd}(\mathbf{ap}(z, x))> : (\forall x : D)A \wedge (\forall x : D)B}}{\lambda z.<\lambda x.\mathbf{fst}(\mathbf{ap}(z, x)), \lambda x.\mathbf{snd}(\mathbf{ap}(z, x))> : (\forall x : D)(A \wedge B) \supset ((\forall x : D)A \wedge (\forall x : D)B)}$$

   c) In the solution to this exercise we use both Σ-elimination and +-elimination. To save space we define **Dis** = (∃*x* : *D*)*A* ∨ (∃*x* : *D*)*B* : **prop**.

$$\dfrac{\dfrac{z : (\exists x : D)(A \vee B) \qquad \dfrac{y : A \vee B \qquad \dfrac{\dfrac{x : D \qquad v : A}{<x, v> : (\exists x : D)A}}{\mathbf{i}(<x, v>) : \mathbf{Dis}} \qquad \dfrac{\dfrac{x : D \qquad w : B}{<x, w> : (\exists x : D)B}}{\mathbf{j}(<x, w>) : \mathbf{Dis}}}{\mathbf{D}(y, v.\mathbf{i}(<x, v>), w.\mathbf{j}(<x, w>)) : \mathbf{Dis}}}{\mathbf{E}(z, xy.\mathbf{D}(y, v.\mathbf{i}(<x, v>), w.\mathbf{j}(<x, w>))) : \mathbf{Dis}}}{\lambda z.\mathbf{E}(z, xy.\mathbf{D}(y, v.\mathbf{i}(<x, v>), w.\mathbf{j}(<x, w>))) : (\exists x : D)(A \vee B) \supset \mathbf{Dis}}$$

Note that since *A* and *B* are propositional functions over *D*, the judgement *v* : *A* is made in the context *x* : *D*, *v* : *A*; the judgement *w* : *B* is made in the context *x* : *D*, *w* : *B*; and the judgement *y* : *A* ∨ *B* is made in the context *x* : *D*, *y* : *A* ∨ *B*. The variables *w* and *v* get bound by the application of **D**, whereas *x* and *y* get bound by the application of **E**.

3. Again we use the natural deduction style of presenting demonstrations.

$$\dfrac{\dfrac{\dfrac{\dfrac{z : (\exists x : D)(\forall y : D)R[x, y]}{\mathbf{fst}(z) : D} \qquad \dfrac{\dfrac{z : (\exists x : D)(\forall y : D)R[x, y]}{\mathbf{snd}(z) : (\forall y : D)R[\mathbf{fst}(z), y]} \qquad y : D}{\mathbf{ap}(\mathbf{snd}(z), y) : R[\mathbf{fst}(z), y)]}}{<\mathbf{fst}(z), \mathbf{ap}(\mathbf{snd}(z), y)> : (\exists x : D)R[x, y]}}{\lambda y.<\mathbf{fst}(z), \mathbf{ap}(\mathbf{snd}(z), y)> : (\forall y : D)(\exists x : D)R[x, y]}}{\lambda z.\lambda y.<\mathbf{fst}(z), \mathbf{ap}(\mathbf{snd}(z), y)> : (\exists x : D)(\forall y : D)R[x, y] \supset (\forall y : D)(\exists x : D)R[x, y]}$$

4. The important observation is that the classes A, B and C are defined over a universe of discourse. The universe of discourse is made explicit in CTT. Thus we let $D : \mathbf{set}$ and we let A, B and C be propositional functions over D, that is, we assume $x : D \vdash A : \mathbf{prop}$, $x : D \vdash B : \mathbf{prop}$, and $x : D \vdash C : \mathbf{prop}$. That all A's are B is *not* formalized as $(\forall x : D)(A \supset B)$. This proposition says that all D's have property of being "B–if–A". The formalization is rather $(\forall z : (\exists x : D)A)B[\mathbf{fst}(z)]$. A **set** of the form $(\exists x : D)A$ may be understood as formalizing the idea of "the D's such that A" or "the D's that are A" (Ranta, 1995, p. 61-64). The proposition $(\forall z : (\exists x : D)A)B[\mathbf{fst}(z)]$ can therefore be understood as expressing that all the D's that are A are B. Since B is a propositional function over D, $B[\mathbf{fst}(z)]$ is a family over $(\exists x : D)A$ in the variable z. The formalization, then, is as follows.

$$\dfrac{\begin{array}{c}(\forall z : (\exists x : D)A)B[\mathbf{fst}(z)] \ \mathit{true} \\ (\forall z : (\exists x : D)B)C[\mathbf{fst}(z)] \ \mathit{true}\end{array}}{(\forall z : (\exists x : D)A)C[\mathbf{fst}(z)] \ \mathit{true}}$$

Assume now that we are given

$$p : (\forall z : (\exists x : D)A)B[\mathbf{fst}(z)]$$
$$q : (\forall z : (\exists x : D)B)C[\mathbf{fst}(z)]$$

To save space we define

$$\mathbf{P} = (\forall z : (\exists x : D)A)B[\mathbf{fst}(z)] : \mathbf{prop}$$
$$\mathbf{Q} = (\forall z : (\exists x : D)B)C[\mathbf{fst}(z)] : \mathbf{prop}$$

We construct a proof of $(\forall z : (\exists x : D)A)C[\mathbf{fst}(z)]$ as follows.

$$\dfrac{\dfrac{q : \mathbf{Q} \qquad \dfrac{\dfrac{z : (\exists x : D)A}{\mathbf{fst}(z) : D} \qquad \dfrac{p : \mathbf{P} \quad z : (\exists x : D)A}{\mathbf{ap}(p, z) : B[\mathbf{fst}(z)]}}{<\mathbf{fst}(z), \mathbf{ap}(p, z)> : (\exists x : D)B}}{\mathbf{ap}(q, <\mathbf{fst}(z), \mathbf{ap}(p, z)>) : C[\mathbf{fst}(z)]}}{\lambda z.\mathbf{ap}(q, <\mathbf{fst}(z), \mathbf{ap}(p, z)>) : (\forall z : (\exists x : D)A)C[\mathbf{fst}(z)]}$$

5. Again we must be careful to remember the universe of discourse. Hence, when we say that everything is an A, we mean that everything in the universe of discourse is an A. We have the following formalization. We assume $D : \mathbf{set}$, $x : D \vdash A : \mathbf{set}$, and $x : D \vdash B : \mathbf{set}$.

$$\begin{array}{l} (\forall x : D)A \text{ true} \\ (\forall z : (\exists x : D)A)B[\mathbf{fst}(z)] \text{ true} \\ \hline \therefore\ (\forall x : D)B\ \textit{true} \end{array}$$

Now assume

$$\begin{array}{l} q : (\forall x : D)A \\ p : (\forall z : (\exists x : D)A)B[\mathbf{fst}(z)] \end{array}$$

We construct a proof of $(\forall x : D)B$ as follows. We use $\mathbf{P}$ as in the previous exercise.

$$\dfrac{\dfrac{\dfrac{p : \mathbf{P} \qquad \dfrac{x : D \qquad \dfrac{q : (\forall x : D)A \qquad x : D}{\mathbf{ap}(q, x) : A}}{<x, \mathbf{ap}(q, x)> : (\exists x : D)A}}{\mathbf{ap}(p, <x, \mathbf{ap}(q, x)>) : B[\mathbf{fst}(<x, \mathbf{ap}(q, x)>)]} \qquad \dfrac{\mathbf{fst}(<x, \mathbf{ap}(q, x)>) = x : D}{B[\mathbf{fst}(<x, \mathbf{ap}(q, x)>)] = B : \mathbf{set}}}{\mathbf{ap}(p, <x, \mathbf{ap}(q, x)>) : B}}{\lambda x.\mathbf{ap}(p, <x, \mathbf{ap}(q, x)>) : (\forall x : D)B}$$

Note here the use of the extensionality of substitution into **set**s as well as the use of the principle that we can infer $a : B$ from $a : A$ and $A = B : \mathbf{set}$. We rely on the syntactic identity $B[x] \equiv B$.

6. We apply **Id**-elimination.

$$\dfrac{\dfrac{p : \mathbf{Id}(A, a, a') \qquad \dfrac{x : A \vdash \mathbf{ap}(c, x) = \mathbf{ap}(c, x) : B}{x : A \vdash \mathbf{refl}(B, \mathbf{ap}(c, x)) : \mathbf{Id}(B, \mathbf{ap}(c, x), \mathbf{ap}(c, x))}}{\mathbf{J}(p, x.\mathbf{refl}(B, \mathbf{ap}(c, x))) : \mathbf{Id}(B, \mathbf{ap}(c, a), \mathbf{ap}(c, a'))}}{\lambda p.\mathbf{J}(p, x) : \mathbf{Id}(A, a, a') \supset \mathbf{Id}(B, \mathbf{ap}(c, a), \mathbf{ap}(c, a'))}$$

An alternative demonstration is the following.

$$\dfrac{q : \mathbf{Id}(A, a, a') \qquad \dfrac{\dfrac{\dfrac{x : A \vdash \mathbf{ap}(c, x) = \mathbf{ap}(c, x) : B}{x : A \vdash \mathbf{refl}(B, \mathbf{ap}(c, x)) : \mathbf{Id}(B, \mathbf{ap}(c, x), \mathbf{ap}(c, x))}}{x : A, p : \mathbf{Id}(A, x, x) \vdash \mathbf{refl}(B, \mathbf{ap}(c, x)) : \mathbf{Id}(B, \mathbf{ap}(c, x), \mathbf{ap}(c, x))}}{x : A \vdash \lambda p.\mathbf{refl}(B, \mathbf{ap}(c, x)) : \mathbf{Id}(A, x, x) \supset \mathbf{Id}(B, \mathbf{ap}(c, x), \mathbf{ap}(c, x))}}{\mathbf{J}(q, x.\lambda p.\mathbf{refl}(B, \mathbf{ap}(c, x))) : \mathbf{Id}(A, a, a') \supset \mathbf{Id}(B, \mathbf{ap}(c, a), \mathbf{ap}(c, a'))}$$

7. We apply **Id**-elimination twice.

$$\dfrac{q : \mathbf{Id}(B, b, b') \qquad \dfrac{p : \mathbf{Id}(A, a, a') \quad x : A, y : B \vdash \mathbf{refl}(A \times B, <x, y>) : \mathbf{Id}(A \times B, <x, y>, <x, y>)}{y : B \vdash \mathbf{J}(p, x.\mathbf{refl}(<x, y>)) : \mathbf{Id}(A \times B, <a, y>, <a', y>)}}{\mathbf{J}(q, y.\mathbf{J}(p, x.\mathbf{refl}(<x, y>))) : \mathbf{Id}(A \times B, <a, b>, <a', b'>)}$$

8. a) We apply **bool**-elimination with $\mathbf{Id}(\mathbf{bool}, x, \mathbf{t}) \vee \mathbf{Id}(\mathbf{bool}, x, \mathbf{f})$ as our C. Hence we need a proof-object of $\mathbf{Id}(\mathbf{bool}, \mathbf{t}, \mathbf{t}) \vee \mathbf{Id}(\mathbf{bool}, \mathbf{t}, \mathbf{f})$ and a proof-object of $\mathbf{Id}(\mathbf{bool}, \mathbf{f}, \mathbf{t}) \vee \mathbf{Id}(\mathbf{bool}, \mathbf{f}, \mathbf{f})$. These are constructed as follows.

$$\dfrac{\dfrac{\mathbf{t} : \mathbf{bool}}{\mathbf{refl}(\mathbf{t}) : \mathbf{Id}(\mathbf{bool}, \mathbf{t}, \mathbf{t})}}{\mathbf{i}(\mathbf{refl}\ (\mathbf{t})) : \mathbf{Id}(\mathbf{bool}, \mathbf{t}, \mathbf{t}) \vee \mathbf{Id}(\mathbf{bool}, \mathbf{t}, \mathbf{f})} \qquad \dfrac{\dfrac{\mathbf{f} : \mathbf{bool}}{\mathbf{refl}\ (\mathbf{f}) : \mathbf{Id}(\mathbf{bool}, \mathbf{f}, \mathbf{f})}}{\mathbf{j}(\mathbf{refl}\ (\mathbf{f})) : \mathbf{Id}(\mathbf{bool}, \mathbf{f}, \mathbf{t}) \vee \mathbf{Id}(\mathbf{bool}, \mathbf{f}, \mathbf{f})}$$

Continuing the demonstration, we write the three premisses of **bool**-elimination below each other.

$$\dfrac{\dfrac{\begin{array}{c} x : \mathbf{bool} \\ \mathbf{i}(\mathbf{refl}\ (\mathbf{t})) : \mathbf{Id}(\mathbf{bool}, \mathbf{t}, \mathbf{t}) \vee \mathbf{Id}(\mathbf{bool}, \mathbf{t}, \mathbf{f}) \\ \mathbf{j}(\mathbf{refl}\ (\mathbf{f})) : \mathbf{Id}(\mathbf{bool}, \mathbf{f}, \mathbf{t}) \vee \mathbf{Id}(\mathbf{bool}, \mathbf{f}, \mathbf{f}) \end{array}}{\mathbf{if}\ x\ \mathbf{then}\ \mathbf{i}(\mathbf{refl}\ (\mathbf{t}))\ \mathbf{else}\ \mathbf{j}(\mathbf{refl}\ (\mathbf{f})) : \mathbf{Id}(\mathbf{bool}, x, \mathbf{t}) \vee \mathbf{Id}(\mathbf{bool}, x, \mathbf{f})}}{\lambda \mathrm{x}.\mathbf{if}\ \mathrm{x}\ \mathbf{then}\ \mathbf{i(refl(t))}\ \mathbf{else}\ \mathbf{j}(\mathbf{refl}(\mathbf{f})) : (\forall x : \mathbf{bool})(\mathbf{Id}(\mathbf{bool}, x, \mathbf{t}) \vee \mathbf{Id}(\mathbf{bool}, x, \mathbf{f}))}$$

b) In this solution, we sometimes omit parentheses and write, for instance, $\mathbf{s}x$ instead of $\mathbf{s}(x)$. We aim to apply $\mathbb{N}$-elimination with $\mathbf{Id}(\mathbb{N}, x, 0) \vee (\exists y : \mathbb{N})\mathbf{Id}(\mathbb{N}, x, \mathbf{s}y)$ as our C. Hence we need a proof-object d of $\mathbf{Id}(\mathbb{N}, 0, 0) \vee (\exists y : \mathbb{N})\mathbf{Id}(\mathbb{N}, 0, \mathbf{s}y)$ and a function e which, given a proof-object

$$w : \mathbf{Id}(\mathbb{N}, x, 0) \vee (\exists y : \mathbb{N})\mathbf{Id}(\mathbb{N}, x, y)$$

yields a proof-object

$$e[w] : \mathbf{Id}(\mathbb{N}, \mathbf{s}x, 0) \vee (\exists y : \mathbb{N})\mathbf{Id}(\mathbb{N}, \mathbf{s}x, y)$$

The proof-object d is easily constructed.

$$\dfrac{\dfrac{0 : \mathbb{N}}{\mathbf{refl}(0) : \mathbf{Id}(\mathbb{N}, 0, 0)}}{\mathbf{i}(\mathbf{refl}(0)) : \mathbf{Id}(\mathbb{N}, 0, 0) \vee (\exists y : \mathbb{N})\mathbf{Id}(\mathbb{N}, 0, \mathbf{s}y)}$$

To find the function e requires more work. Since we are given

$$w : \mathbf{Id}(\mathbb{N}, x, 0) \vee (\exists y : \mathbb{N})\mathbf{Id}(\mathbb{N}, x, y)$$

it is natural to try to construct $e[w]$ by means of $\vee$-elimination. Assume, therefore, first that we are given $q : \mathbf{Id}(\mathbb{N}, x, 0)$.

$$\dfrac{\dfrac{\dfrac{q : \mathbf{Id}(\mathbb{N}, x, 0) \qquad \mathbf{refl}(\mathbf{s}z) : \mathbf{Id}(\mathbb{N}, \mathbf{s}z, \mathbf{s}z)}{\mathbf{J}(q, z.\mathbf{refl}(\mathbf{s}z)) : \mathbf{Id}(\mathbb{N}, \mathbf{s}x, \mathrm{s}0)}}{<0, \mathbf{J}(q, z.\mathbf{refl}(\mathbf{s}z))> : (\exists \mathrm{y} : \mathbb{N})\mathbf{Id}(\mathbb{N}, \mathbf{s}x, \mathbf{s}y)}}{\mathbf{j}<0, \mathbf{J}(q, z.\mathbf{refl}(\mathbf{s}z))> : \mathbf{Id}(\mathbb{N}, \mathbf{s}x, 0) \vee (\exists y : \mathbb{N})\mathbf{Id}(\mathbb{N}, \mathbf{s}x, \mathbf{s}y)}$$

Next assume that we are given $p : (\exists y : \mathbb{N})\mathbf{Id}(\mathbb{N}, x, \mathbf{s}y)$.

$$p : (\exists y : \mathbb{N})\mathbf{Id}(\mathbb{N}, x, \mathbf{s}y)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\mathbf{snd}p : \mathbf{Id}(\mathbb{N}, x, \mathbf{s}(\mathbf{fst}p))} \qquad \mathbf{refl}(\mathbf{s}z) : \mathbf{Id}(\mathbb{N}, \mathbf{s}z, \mathbf{s}z)}{\mathbf{J}(\mathbf{snd}p, z.\mathbf{refl}(\mathbf{s}z)) : \mathbf{Id}(\mathbb{N}, \mathbf{s}x, \mathbf{ss}(\mathbf{fst}p))}}{<\mathbf{s}(\mathbf{fst}p), \mathbf{J}(\mathbf{snd}p, z.\mathbf{refl}(\mathbf{s}z))> : (\exists y : \mathbb{N})\mathbf{Id}(\mathbb{N}, \mathbf{s}x, \mathbf{s}y)}}{\mathbf{j}<\mathbf{s}(\mathbf{fst}p), \mathrm{J}(\mathbf{snd}p, z.\mathbf{refl}(\mathbf{s}z))> : \mathbf{Id}(\mathbb{N}, \mathbf{s}x, 0) \vee (\exists y : \mathbb{N})\mathbf{Id}(\mathbb{N}, \mathbf{s}x, \mathbf{s}\mathrm{y})}$$

With $w : \mathbf{Id}(\mathbb{N}, x, 0) \vee (\exists y : \mathbb{N})\mathbf{Id}(\mathbb{N}, x, \mathbf{s}y)$ as major premiss, $\vee$-elimination then yields

$$\begin{array}{l}\mathbf{D}(\mathrm{w}, q.\mathbf{j}<0, \mathbf{J}(q, z.\mathbf{refl}(\mathbf{s}z))>, p.\mathbf{j}<\mathbf{s}(\mathbf{fst}p), \mathbf{J}(\mathbf{snd}p, z.\mathbf{refl}(\mathbf{s}z))>) : \\ \mathbf{Id}(\mathbb{N}, \mathbf{s}x, 0) \vee (\exists y : \mathbb{N})\mathbf{Id}(\mathbb{N}, \mathbf{s}x, \mathbf{s}y)\end{array}$$

With the assumption $x : \mathbb{N}$ as major premiss, $\mathbb{N}$-elimination yields

$$\begin{array}{l}\mathbf{R}(x, \mathbf{i}(\mathbf{refl}0), w.\mathbf{D}(w, q.\mathbf{j}<0, \mathbf{J}(q, z.\mathbf{refl}(\mathbf{s}z))>, p.\mathbf{j}<\mathbf{s}(\mathbf{fst}p), \mathbf{J}(\mathbf{snd}p, z.\mathbf{refl}(\mathbf{s}z))>)) : \\ \mathbf{Id}(\mathbb{N}, x, 0) \vee (\exists y : \mathbb{N})\mathbf{Id}(\mathbb{N}, x, \mathbf{s}y)\end{array}$$

Note that w has become bound here, so only x is free. A proof-object of

$$(\forall x : \mathbb{N})(\mathbf{Id}(\mathbb{N}, x, 0) \vee (\exists y : \mathbb{N})\mathbf{Id}(\mathbb{N}, x, \mathbf{s}y))$$

is then constructed by means of Π-introduction.

III. BASIC NOTIONS FOR DIALOGICAL LOGIC

The dialogical approach to logic is not a specific logical system; it is rather a general framework having a rule-based approach to meaning (instead of a truth-functional or a model-theoretical approach) which allows different logics to be developed, combined and compared within it. The main philosophical idea behind this framework is that meaning and rationality are constituted by argumentative interaction between epistemic subjects; it has proved particularly fruitful in history of philosophy and logic. We shall here provide a brief overview of dialogues in a more intuitive approach than what is found in the rest of the book in order to give a feeling of what the dialogical framework can do and what it is aiming at.

III.1 The general framework

Dialogues and interaction

As hinted by its name, this framework studies dialogues; but it also takes the form of dialogues. In a dialogue, two parties (players) argue on a thesis (a certain statement that is the subject of the whole argument) and follow certain fixed rules in their argument. The player who states the thesis is the Proponent, called **P**, and his rival, the player who challenges the thesis, is the Opponent, called **O**. By convention, we refer to **P** as he and to **O** as she. In challenging the Proponent's thesis, the Opponent is requiring of the Proponent that he defends his statement.

The interaction between the two players **P** and **O** is spelled out by challenges and defences, implementing Robert Brandom's take on meaning as a game of giving and asking for reasons (see the introduction, section I.2). Actions in a dialogue are called moves; they are often understood as speech-acts involving declarative utterances (*statements*) and interrogative utterances (*requests*).[44] The rules for dialogues thus never deal with expressions isolated from the act of uttering them.

The rules in the dialogical framework are divided into two kinds of rules: particle rules, and structural rules.

Particle rules

Particle rules (*Partikelregeln*), or rules for logical constants, determine the legal moves in a play and regulate interaction by establishing the relevant moves constituting *challenges*: moves that are an appropriate attack to a previous move (a statement) and thus require that the challenged player play the appropriate defence to the attack. If the challenged player defends his statement, he has answered the challenge.

Particle rules determine how reasons are asked for and are given for each kind of statement, thus providing the meaning of that statement. In other words, the appropriate attacks and defences—that is, the appropriate ways of asking for and giving reasons—for

---

[44] Litterature pertaining to the dialogical framework also uses the terms posits and assertions to designate what we will here call statements, that is, the act of stating a proposition within a game of giving and asking for reasons; the meaning of a statement is defined by an appropriate challenge and defence, or, in other words, how reasons for this statement can be requested, what constitutes reasons for this statement and how these reasons can be provided.

each statement (or move) gives the meaning of these statements: a conjunction, a disjunction, or a universal quantification, for instance, receive their meaning through the appropriate interaction in a dialogical game, spelled out by the particle rules.

The particle rules provide the meaning of the different logical connectives, which they provide in a dynamic way through appropriate challenges and answers. This feature of dialogues is fundamental for immanent reasoning: the meaning of the moves in a dialogue does not lie in some external semantic, but is immanent to the dialogue itself,[45] that is, in the specific and appropriate way the players interact; we here join the Wittgensteinian conception of meaning as use. The particle rules are spelled out in an anonymous way, that is, without mentioning if it is **P** or **O** who is attacking or defending: the rules are the same for the two players; the meaning of the connectives is therefore independent of who uses them.[46]

An essential aspect of the meaning of logical constants in the dialogical framework pertains to the actions, such as choices, the particle rules associate to the use of such constants. In this regard, since the interaction constitutes the meaning, all the actions involved in the constitution of the meaning of an expression should be made explicit; — otherwise part of the meaning would be left implicit. These essential aspects must therefore be part of the object language in order to figure explicitly. This is the main reason for importing in the dialogical framework many Constructive Type Theory features (and will be presented in chapters VI-VII). The roots of this kind of perspective can be found in Wittgenstein's *Unhintergehbarkeit der Sprache*:[47] language games are supposed to show this internal feature of meaning.

Structural rules

Structural rules (*Rahmenregeln*) on the other hand determine the general course of a dialogue game, such as how a game is initiated, how to play it, how it ends, and so on. The point of these rules is not so much to spell out the meaning of the logical constants by specifying how to act in an appropriate way—this is the role of the particle rules—; it is rather to specify according to what structure interactions will take place. It is one thing to determine the meaning of the logical constants as a set of appropriate challenges and defences, it is another to define whose turn it is to play and when a player is allowed to play a move. One could thus have the same local meaning and change a structural rule, saying for instance that one of the players is allowed to play two moves at a time instead of simply one: this would considerably change the game without changing the local meaning of what is said.

One of the most important structural rules for the present study on immanent reasoning is the Copy-cat rule (or Socratic rule when introducing CTT features in the

---

[45] Göran Sundholm (1997; 2001) voiced some criticism against metalogical frameworks for meaning: standard model-theoretic semantics convert semantics in a formal metamathematical object for which the syntax is linked to the meaning by attributing truth values to each sign that is uninterpreted (formula). The language thus does not express any content but is rather conceived as a system of signs *speaking of* the world, provided that a metalogical adequation between the signs and the world has been defined. For more on this issue, see section VI.

[46] In this sense, the particle rules are said to be *symmetric*, see section IV.3. This is imperative to preserve the dialogical framework from connectives as Prior's (1960) *tonk*. See (Redmond & Rahman, 2016).

[47] This is a Wittgensteinian principle that Hintikka explicitely adopted. The reasons for linking the dialogical framework to CTT, allowing a greater explicitation of the meaning in the object-language, are thus analogous to Hintikka's vindication for the fecundity of game-theoretic semantics (GTS) in the epistemic framework for logic, semantics, and the foundations of mathematics.

dialogical context). This rule is not anonymous, it is a restriction on the moves the Proponent is allowed to play: the Proponent is allowed to assert an elementary judgement only if the Opponent has already asserted it. So the Opponent is not concerned by the same exact rules as the Proponent.

The Copy-cat rule accounts for analyticity: the Proponent, who brings forward the thesis, will have to defend it without bringing any element of his own in the play: his defence of the thesis will have to rely only on what the Opponent has conceded, and everything the Opponent concedes comes only from the meaning of the thesis. The Opponent will be challenging the thesis, and challenging and defending the subsequent moves made by the Proponent in reaction to her initial challenge of the thesis; but all these challenges and defences are made according to the particle rules. So everything the Opponent will concede during a play stems from an application of the particle rules starting with the thesis. The only elements whose meaning is left unspecified, in formal plays, are the elementary statements (specifying their meaning is the point of material plays, see chapter X). The Copy-cat rule makes sure that the Proponent is not bringing in any elementary statement to back his thesis that the Opponent might not agree with: the Proponent can only back his thesis with elementary statements that the Opponent herself has already conceded.

III.2 The rules at the play level

III.2.1 Particle rules

Here are the particle rules for propositional logic in a standard framework. We use anonymous players, **X** and **Y**, so that the meaning of the logical constants stays player-independent. The exclamation mark after the player indicates that the player states the following proposition (utterance), the question mark indicates that the player is making a request.

Conjunction : X ! $A \wedge B$

The meaning of logical constants are not defined in the dialogical framework by their truth-values, but by the appropriate challenges and defences: a conjunction is a statement that is challenged by asking one of the two conjuncts, the choice lying with the challenger (**Y**). The challenge is answered by providing the conjunct asked, so the defender (**X**) does not have the choice in a conjunction:

- if the challenger asks for the left conjunct (**Y** ? $L^\wedge$) then the defender must provide the left conjunct (**X** ! A) to defend his initial conjunction;
- if the challenger asks for the left conjunct (**Y** ? $R^\wedge$) then the defender must provide the left conjunct (**X** ! B) to defend his initial conjunction.

Table 1: particle rules for conjunction

	Move	Challenge	Defence
Conjunction	**X** ! $A \wedge B$	**Y** ? $L^\wedge$ or **Y** ? $R^\wedge$	**X** ! A (respectively) **X** ! B

Properties of conjunction:

- the challenge is a request;
- the challenger has the choice;
- the defender must provide the conjunct requested.

Disjunction: X ! $A \vee B$

A disjunction is a statement that is challenged by asking one of the two disjuncts, but the challenger (**Y**) does not have the choice. The challenge is answered by providing either of the two disjuncts, the choice lying with the defender (**X**).

Table 2: particle rules for disjunction

	Move	Challenge	Defence
Disjunction	**X** ! $A \vee B$	**Y** $?_{\vee}$	**X** ! A or **X** ! B

Properties of disjunction:

- the challenge is a request;
- the defender has the choice;
- the defender provides the disjunct he wants.

Implication: X ! $A \supset B$

An implication is a statement that is challenged by stating the antecedent. The challenge is answered by stating the consequent. So to challenge an implication stated by the other player **X** ! $A \supset B$, the challenger must state the antecedent **Y** ! A; the defender must then state the consequent **X** ! B.

Notice that by challenging an implication, the challenger himself is making a statement, which can be challenged if it is not elementary. Implication in this sense can be considered as distributing the burden of proof.

Table 3: particle rules for implication

	Move	Challenge	Defence
Implication	**X** ! $A \supset B$	**Y** ! A	**X** ! B

Properties of implication:

- the challenge is the stating of the antecedent;
- the defence is the stating of the consequent;
- there is no choice involved.

Negation: X! $\neg A$

A negation is a statement that is challenged by stating the negated proposition (**Y** ! A). It cannot be answered.

In the table presentation of dialogues, to distinguish the cells that are empty because the defence is still pending from the cells that will remain empty because a challenge on a negation cannot be defended, we insert the sign "**—**" in the cell for the defence of negation.

Table 4: particle rules for negation

	Move	Challenge	Defence
Negation	**X** ! $\neg A$	**Y**! A	**—**

Properties of negation:

- the challenge is the stating of the negated proposition;
- there is no defence;

- there is no choice involved.

Summing up the particle rules for propositional classical logic

Table 5: particle rules

	Conjunction	**Disjunction**	**Implication**	**Negation**
Move	**X** ! $A \wedge B$	**X** ! $A \vee B$	**X** ! $A \supset B$	**X** ! $\neg A$
Challenge	**Y** ? $L^{\wedge}$ or **Y** ? $R^{\wedge}$	**Y** ?$_{\vee}$	**Y** ! A	**Y** ! A
Defence	**X** ! A (resp.) **X** ! B	**X** ! A or **X** ! B	**X** ! B	—

III.2.2 Structural rules

Here are the structural rules for standard classical logic in the dialogical framework. They define how a play unravels, that is how a statement (the thesis) can generate a certain dialogue. There are four basic structural rules (SR) specifying: 1. how to start; 2. how to play; 3. how to preserve formality (Copy-cat rule); 4. how to win.

Each of these rules can be modified, producing variants in the dialogical framework: adding for instance a condition on 3—called "last duty first"—will yield intuitionistic logics. Some variants will be introduced in the next section for more advanced dialogues.

SR 1: starting the play

A play starts with a player stating a proposition called the *thesis*; that player becomes the Proponent (**P**) and the move is labelled move 0.

The other player—the Opponent (**O**)—chooses a *repetition rank* determining how many times she is allowed to challenge or defend any move in a play. It is usually enough for **O** to choose a repetition rank of 1 ($m \coloneqq 1$), it is move 1.

P then chooses a repetition rank: 2 is usually enough ($n \coloneqq 2$); it is move 2.

SR 2: how to play

Each player in turn plays one move: once the repetition ranks have been chosen, each move is either a challenge on a previous statement or a defence of a previous challenge.

SR 3: Copy-cat rule

P cannot play an elementary statement if **O** has not stated it previously.

SR 4: winning rule

The play ends when it is a player's turn to make a move but that player has no available move left. That player loses, the other player wins.

III.3 Building a dialogue: step-by-step instructions

Setting up the game

In order to build a dialogue, first start with a large table with two columns, one for the Opponent (on the left) and one for the Proponent (on the right).

O	**P**

Add a column:

- on the *outer side* of your two columns (A cells); this column will specify the number of the move played;
- on the *inner side* of your two columns (B cells); this column will specify the number of the move challenged, if applicable.

O			**P**		
A		B	B		A
A		B	B		A

Write the *thesis* as **P**'s first move, that is move 0 (see the structural rule for initializing the play, SR0).

O			**P**		
				$!\,(A \wedge B) \supset A$	0

Each player (**O** first, then **P**) now choose their repetition rank. For convenience, we usually choose rank 1 for **O** and rank 2 for **P**.[48] Repetition ranks determine how many times a player is allowed to attack or to defend each and every move.

O			**P**		
				$!\,(A \wedge B) \supset A$	0
1	$m := 1$			$n := 2$	2

Playing the game

- Each player *in turn* will play a move,[49] that is, he will either *attack* a move made by the other player, or *defend* against an attack.
    - If the player attacks: write the attack *on a new line*, and specify the number of the attacked move in the inner column.[50]

---

[48] These ranks are enough for propositional logic: **P** can attack the two sides of a conjunction and defend the two sides of a disjunction. If the players are playing at their best (no mistakes), then 1 is enough for **O**: if she has a move allowing her to win, she will choose it straightaway.

[49] Since the players will play alternately, all of **O**'s moves will be *uneven* numbers, whereas all of **P**'s moves will be *even* numbers. There are no exceptions.

[50] Expressions are not listed by following the order of the moves, but by writing an attack on a new line and the defence on the same line as the corresponding attack, thus showing when a challenge is answered.

Elementary statements cannot be attacked.[51]

- If the player defends: write the defence *on the same line as the attack*.
  An attack on a negation cannot be defended, though the player can attack or defend some other move, if there are any available.

Ending the game

A play stops when one of the two players *has no move left* (attack or defence).

In order to be sure that *every* possible move has been played, ask yourself, for the player whose turn it is to play:

- if there are any non-elementary statements played by the adversary that has not been attacked (or that has been attacked only once if the repetition rank is 2).
  If there are, then the player can attack it (except if there is a restriction by the Copy-cat rule).
- If there are any attacks left undefended (or defended only once if the repetition rank is 2). You can see this easily by the empty case in the player's column. Remember: attacks on negations cannot be defended, so cells with "— " are left empty.

If it is a player's turn and everything that could be attacked has already been attacked, and everything that could be defended has already been defended, then, and then only, has the play ended and that player has lost.

III.4 Commented construction of a play: $(A \wedge B) \supset A$

The thesis to be tested will be $(A \wedge B) \supset A$.

0. The play starts with the Proponent (**P**) stating the thesis. It is move 0.

O			**P**		
				$! \, (A \wedge B) \supset A$	0

1. It is now the Opponent's (**O**'s) turn to play. She must choose her repetition rank, that is, the number of times she can attack or defend any move. We will in general use a repetition rank of 1 ($m \coloneqq 1$) for **O**, and of 2 ($n \coloneqq 2$) for **P**.

O			**P**		
				$! \, (A \wedge B) \supset A$	0
1	$m \coloneqq 1$				

2. Now it is **P**'s turn to choose his repetition rank.

O			**P**		
				$! \, (A \wedge B) \supset A$	0
1	$m \coloneqq 1$			$n \coloneqq 2$	2

---

[51] This would bring us into material plays. See the introduction, section I.2 and chapter **Erreur ! Source du renvoi introuvable.**.

3. It is **O**’s turn, and she has only one availaible move: attacking the thesis (move 0). Since it is an implication, she must state the antecedent ($A \wedge B$).

O			P		
				$!\,(A \wedge B) \supset A$	0
1	$m := 1$			$n := 2$	2
3	$!\,A \wedge B$	0			

4. **P** has two available moves: either he defends his thesis against **O**’s challenge (move 3), or he attacks **O**’s move 3.

   O’s challenge being on an implication, the defence must be to state the consequent, here A. But A is an elementary statement, so **P** is not allowed to state A if **O** has not played it beforehand—according to the Copy-cat (structural) rule. So **P** cannot defend his thesis (move 0) as long as he has not found a way of making **O** assert A.

   P must therefore attack. His move is written on a new line (each new challenge is written on a new line, so that the corresponding defence can be written on the same line: at a glance one should see if challenges are left unanswered).

   P has to challenge **O**’s move 3 which is a conjunction. So **P** has the choice as to which of the conjuncts to ask for. Since **P** wants to be able to defend his thesis, he chooses to ask for the first conjunct.

O			P		
				$!\,(A \wedge B) \supset A$	0
1	$m := 1$			$n := 2$	2
3	$!\,A \wedge B$	0			
			3	$?\,L^{\wedge}$	4

5. It is now **O**’s turn to play, and she has only one available move: she must defend her conjunction (move 3) from **P**’s challenge (move 4).

O			P		
				$!\,(A \wedge B) \supset A$	0
1	$m := 1$			$n := 2$	2
3	$!\,A \wedge B$	0			
5	$!\,A$		3	$?\,L^{\wedge}$	4

6. Since **O** has stated A, **P** is now allowed to state A: he can defend his thesis (move 0) against **O**’s still unanswered challenge (move 3).

O			P		
				$!\,(A \wedge B) \supset A$	0
1	$m := 1$			$n := 2$	2

3	$!\, A \wedge B$	0		$!\, A$	6
5	$!\, A$		3	$?\, L^{\wedge}$	4

7. It is now **O**’s turn, but she has no available move: every **P**-challenge has been answered and every non-elementary statement of **P** has been challenged. Therefore the play is over and **O** has lost.

O			**P**		
				$!\, (A \wedge B) \supset A$	0
1	$m \coloneqq 1$			$n \coloneqq 2$	2
3	$!\, A \wedge B$	0		$!\, A$	6
5	$!\, A$		3	$?\, L^{\wedge}$	4

P wins.

Exercises

Build a play for the following theses:

1. $(A \vee B) \supset (B \vee A)$
2. $A \vee \neg A$
3. $\neg\neg A \supset A$
4. $\neg\neg(A \vee \neg A)$

Solutions

The solution for 1 is given in the next section (III.5) as the first step in building a winning strategy.

The other solutions are given as examples at the end of the next chapter (IV). See the following passages:

- p. 71 for the third excluded: $A \vee \neg A$;
- p. 72 for the double negation elimination: $\neg\neg A \supset A$;
- p. 73 for the double negation of the third excluded: $\neg\neg(A \vee \neg A)$.

The examples in the next section compare the intuitionistic (structural) rules and the classical ones; in this section we have only presented the classical (structural) rules.

III.5 Approaching the strategy level

Up to now we have only looked at individual plays; that is, we have stayed at the play level, which is a distinctive feature of the dialogical framework. The strategy level on the other hand converges with other logical frameworks, for a winning strategy can be turned into a demonstration in a non-dialogical framework (and reversely): such an algorithm will be provided in chapter IX.

The strategy level allows to compare different plays on the same thesis. A winning strategy is always defined in relation to a specific player, either **O** or **P**, though it must be noted that *strategies* are not actually carried out by the players, they are only a *perspective* on the possibles plays for a given thesis. Usually (and by default), we consider **P**-winning strategies. A **P**-winning strategy determines if **P** has a way to win a play regardless of **O**’s choices during the play: whatever be **O**’s choice, **P** will be able to find a way to win. **P**-strategies are therefore built on **O**'s choices: each possible choice of **O** must be taken into account and dealt with in order to determine if **P** is able to win in all the different cases stemming from **O**'s possible choices.

A **P**-strategy is constructed first like a normal play, which must end with **P** winning (otherwise we will not be constructing a **P**-winning strategy). Then we proceed from the last move up to the first move and we stop when we come across a choice made by **O**. At that point, we branch the play: on the first branch we leave the initial play and on the second branch we do as if **O** had chosen her other option *at that point*, that is with exactly the same previous moves of the play. This play also needs to be won by **P** in order to continue building the **P**-strategy. We then proceed again by going up the moves until we find another **O**-choice and make another branching, up to the thesis. Once all of **O**'s choices have been dealt with, we can determine whether or not **P** has a winning strategy for this thesis: if somewhere **O** won a play, then **O** has a way of playing that will allow her to win; but if for each and every choices of **O**, **P** has a way to play allowing him to win, he has a winning strategy. Winning strategies are analogous in chess to saying "checkmate in x moves": whatever the opponent will play, the other player has a way to win in maximum x moves.

With the particle rules given in this didactic section, **O** has a choice only when:

- Challenging a conjunction and
- Defending a disjunction.

Building a P-winning strategy step-by-step

In order to illustrate how to build a winning strategy, let us consider the thesis **P** $!(A \vee B) \supset (A \vee B)$.

1. First build a play for this thesis.

O			**P**		
				$!(A \vee B) \supset (A \vee B)$	0
1	$m := 1$			$n := 2$	2
3	$!A \vee B$	0		$!A \vee B$	4
5	$?_{\vee}$	4		$!A$	8
7	$!A$		3	$?_{\vee}$	6

P wins.

2. Proceed backwards from the last move up and stop at the first **O** choice encountered.

O			**P**		
				$!(A \vee B) \supset (A \vee B)$	0
1	$m := 1$			$n := 2$	2
3	$!A \vee B$	0		$!A \vee B$	4
5	$?_{\vee}$	4		$!A$	8
7	$!A$		3	$?_{\vee}$	6

P wins.

Here it is move 7, in which **O** had a choice because she had to defend a disjunction. The other option that was available to her was to choose the other disjunct, **O** $!B$. We need to branch the plays so that both options are considered. We therefore copy the play up to move 7 and leave this move blank; we draw two branches and copy the first play on one branch and continue the other play with move 7 being the other option, that is **O** $!B$.

O			P		
				$!\,(A \vee B) \supset (A \vee B)$	0
1	$m \coloneqq 1$			$n \coloneqq 2$	2
3	$!\,A \vee B$	0		$!\,A \vee B$	4
5	$?_{\vee}$	4			
7			3	$?_{\vee}$	6

Play 1: left option

O			P		
				$!\,(A \vee B) \supset (A \vee B)$	0
1	$m \coloneqq 1$			$n \coloneqq 2$	2
3	$!\,A \vee B$	0		$!\,A \vee B$	4
5	$?_{\vee}$	4		$!\,A$	8
7	$!\ A$		3	$?_{\vee}$	6

P wins

Play 2: right option

O			P		
				$!\,(A \vee B) \supset (A \vee B)$	0
1	$m \coloneqq 1$			$n \coloneqq 2$	2
3	$!\,A \vee B$	0		$!\,A \vee B$	4
5	$?_{\vee}$	4		$!\ B$	8
7	$!\ B$		3	$?_{\vee}$	6

P wins.

3. Proceed backwards: we fall on the thesis without any other **O**-choice; so we have considered all the relevant cases for a **P** strategy. Since **P** can win in every case, **P** has a winning strategy for $(A \vee B) \supset (A \vee B)$.

Exercise

Build a **P**-winning strategy for the following thesis, with the repetitions ranks **O** $!m \coloneqq 1$ and **P** $!\ m \coloneqq 2$:

1. $\big((A \vee B) \wedge \neg A\big) \supset B$

Solution

The branching is here triggered by **O**'s repetition rank $m \coloneqq 1$: she cannot defend twice against the same attack (move 8), so her two options yield two different plays (play 2 and play 3) in which she chooses each time one of the two options (right option in play 2 and left option in play 3).

Play 1: $\big((A \vee B) \wedge \neg A\big) \supset B$

O			P		
				$!\ \big((A \vee B) \wedge \neg A\big) \supset B$	0
1	$m \coloneqq 1$			$n \coloneqq 2$	2

3	$!\ (A \vee B) \wedge \neg A$	0			
5	$!\neg A$		3	$?\,R^{\wedge}$	4
7	$!\,A\ \vee B$		3	$?\ \mathrm{L}^{\wedge}$	6
			7	$?_{\vee}$	8

Play 2: right option

O			**P**		
				$!\ ((A \vee B) \wedge \neg A) \supset B$	0
1	$m := 1$			$n := 2$	2
3	$!\ (A \vee B) \wedge \neg A$	0		$!\,B$	10
5	$!\ \neg A$		3	$?\,R^{\wedge}$	4
7	$!\ A \vee B$		3	$?\ L^{\wedge}$	6
9	$!\,B$		7	$?_{\vee}$	8

P wins (intuitionistic rules, see section IV.4.2, p. 69)

Play 3: left option

O			**P**		
				$!\ ((A \vee B) \wedge \neg A) \supset B$	0
1	$m := 1$			$n := 2$	2
3	$!\ (A \vee B) \wedge \neg A$	0			
5	$!\ \neg A$		3	$?\,R^{\wedge}$	4
7	$!\ A \vee B$		3	$?\ L^{\wedge}$	6
9	$!\,A$		7	$?_{\vee}$	8
			5	$!\,A$	10

P wins (intuitionistic rules)

III.6 Rounding up some key notions

Considering what has been said up to now, the following points are essential to the dialogical approach:

1. The distinction between *local* meaning (particle rules for logical constants) and *global* meaning (structural rules determining how to play).
2. The player independence of local meaning.
3. The distinction between the play level (local winning, or winning of a play) and the strategic level (existence of a winning strategy).
4. A notion of demonstration that amounts to building a winning strategy.
5. The distinction between *material* dialogues, *formal* dialogues that include a rule allowing Copy-cat moves, and dialogues combining both.

III.7 Further reading

Textbook presentations

For a textbook presentation of the dialogical framework, see (Clerbout, 2014b), (Redmond & Fontaine, 2011) or (Rückert, 2011a).

Logical studies in dialogic

- The main original papers on the dialogical approach to logic are collected in (Lorenzen & Lorenz, 1978).
- For a historical overview of the transition from operative logic to dialogical logic see (Lorenz, 2001).
- For a presentation about the initial role of the dialogical framework as a foundation for intuitionistic logic, see (Felscher, 1985);
- for its bearings with argumentation theory and everyday dialogues see (Krabbe, 1982; 1985; 2006).
- Other papers have been collected more recently in (Lorenz, 2010a; 2010b).
- An account of developments since (Rahman, 1993) can be found in (Rahman & Keiff, 2005; 2010), (Keiff, 2007; 2009), (Beirlaen & Fontaine, 2016) and (Cardascia, 2016).
- For the underlying metalogic see (Clerbout, 2014a; 2014b; 2014c).

On the use of the dialogical framework in epistemology, philosophy and history of ideas

The dialogical framework has proven a powerful tool in logic, but it also has further ambitions: the fact that meaning in entirely provided by the interaction that makes a dialogue (even, in a large part, the material aspect of meaning; see chapter 1) and that this dialogical framework allows for multiple levels of consideration (the play level and the strategy level are the most important ones) makes this framework a very flexible and adaptable tool useful in history of philosophy, deontics, epistemology, and many other fields of inquiry that are not primarily logical. Here are a few studies showing the range of possible applications of the dialogical framework; many more are currently under progress, exploring the gamut of immanent reasoning.

- For the key role of the dialogical framework in linking dialectics, games, and logic, see (Rahman & Tulenheimo, 2009), (Rahman & Keiff, 2010) and (Marion & Rückert, 2015).
- (Clerbout, Gorisse, & Rahman, 2011) studied Jain Logic in the dialogical framework.
- (Popek, 2012) develops a dialogical reconstruction of medieval *obligationes*.

For other books see

- (Redmond, 2010) and (Rahman & Redmond, 2015) on fiction and the dialogical framework;
- (Fontaine, 2013) on intentionality, fiction, and dialogues;
- (Magnier, 2013) on dynamic epistemic logic and legal reasoning in a dialogical framework;
- and (Nzokou, 2013) on the dialogical framework and non-monotonic reasoning in legal debates within oral traditions.

IV. ADVANCED DIALOGUES: PLAY LEVEL

This chapter will provide a more technical approach to the standard (non-CTT) dialogical framework at the play level. The next chapter (V) will do the same at the strategy level. It will then be possible to introduce local reasons in the dialogues and thus start making it explicit that dialogues are games of giving and asking for reasons;[52] in this sense, the elements contributing to the meaning as use will appear in the object language. The link to equality in action will then be spelled out in the following two chapters (VI-VII), based on what will be presented in the next two chapters (IV-V).

In the previous chapter (III), we have introduced less formally the standard dialogical framework and have given step-by-step instructions as how to proceed in building a dialogue. We shall here be presenting the same material, but in a more technical way. First (section IV.1), we will provide some preliminary notions. Then (section IV.2) we will deal with local meaning (provided by the particle rules) with a more extended language than the propositional logic presented in the previous chapter. Third (section IV.3), we will expound briefly on the distinction between harmony at the global and the strategy level on the one hand, which defines dynamic identity, and a kind of harmony at the local level on the other hand, which we call symmetry and is player-independent, a feature essential to the particle rules. Then (section IV.4) we will provide the structural rules and the global meaning these rules convey, and provide different variants for games in this framework, giving in particular the rules for intuitionistic logic (as opposed to classical logic). Last (section IV.5), we will carry out three examples of plays and insist on the difference between classical and intuitionistic logic.

IV.1 Preliminary notions

The language

Let $\mathbb{L}$ be a first-order language built as usual upon the propositional connectives, the quantifiers, a denumerable set of individual variables, a denumerable set of individual constants and a denumerable set of predicate symbols (each with a fixed arity).

We extend the language $\mathbb{L}$ with two labels **O** and **P**, standing for the players of the game, and the two symbols ‘!’ and ‘?’ standing respectively for statements and requests. When the identity of the player does not matter, we use the variables **X** or **Y** (with **X** ≠ **Y**).[53]

Plays

A *play* is a legal sequence of moves, that is, a sequence of moves which observes the game rules. Particle rules are not the only rules which must be observed in this respect: the second kind of rules, the *structural rules*, are the rules providing the precise conditions under which a given sequence is a play.

---

[52] This idea will remain implicit until then, appearing only when we stress that the particle rules as provide the meaning of logical constants through appropriate challenges and defences

[53] This aspect (player independence) is fundamental for the symmetry of the rules. See section IV.3.

Dialogical games

The *dialogical game* for a statement is the set of all plays from a given *thesis* (initial statement, see below the Starting rule, SR0).

A move in a play

A move $\mathbb{M}$ is an expression of the form '**X**-*e*', where *e* is either

- of the form '! A' (read: *the player* ***X*** *states* A), for some proposition A of $\mathbb{L}$; we say it is an elementary statement, or
- of one of the forms specified by the particle rules (see below).

Challenges and defences

The words 'attack' and 'defence' are convenient to name certain moves according to their relation to other moves which can be defined in the following way.

- Let σ be a sequence of moves. The function ρ_σ assigns a position to each move in σ, starting with 0.
- The function $\mathbb{F}_\sigma$ assigns a pair $[m, Z]$ to certain moves $\mathbb{M}$ in σ, where m denotes a position smaller than $\rho_\sigma(\mathbb{M})$ and Z is either A or D, standing respectively for 'attack' and 'defence'. That is, the function $\mathbb{F}_\sigma$ keeps track of the relations of attack and defence as they are given by the particle rules.

Let us point that at the local level (the level of the particle rules), this terminology should be bereft of any strategic undertone.

Terminological note: challenge, attack and defence

The standard terminology uses the terms *challenge,* or *attack,* and *defence* (sometimes *answer* in respect of challenges). We shall here make a (subtle) distinction between challenge and attack: a challenge is initiated by an attack and needs this attack to be defended against in order to be answered to. So a challenge requires a defence to be settled, whereas an attack is simply the move that opens the challenge. For instance, using the particle rules exposed below, an attack on an implication will be simply to state the antecedent, and challenging an implication will be to attack it and thus demanding that the player who stated the implication defends her posit by positing the consequent, knowing that the challenger stated the antecedent. As one can see, the difference between challenge and attack is slim, and they may oftentimes be taken as synonymous.

IV.2 Local meaning of logical constants

Particle rules:

In the dialogical framework, the particle rules state the *local semantics*: only challenges and the corresponding defences for a given logical constant are at stake here, that is, we only take the main logical constant of the proposition into account.

Particle rules provide a decontextualized description of how the game can proceed locally: they specify the way a statement can be challenged and defended according to its main logical constant. In this way the particle rules govern the local level of meaning.

The table for the propositional connectives has already been presented in the previous chapter (see p. 53) and we simply refer the reader to that section as an explanation of the following table summing up these particle rules.

Table 6: Particle rules for dialogical games: propositional connectives

	Conjunction	**Disjunction**	**Implication**	**Negation**
Move	$\boldsymbol{X} ! A \wedge B$	$\boldsymbol{X} ! A \vee B$	$\boldsymbol{X} ! A \supset B$	$\boldsymbol{X} ! \neg A$
Challenge	$\boldsymbol{Y} ? L^{\wedge}$ or $\boldsymbol{Y} ?\ R^{\wedge}$	$\boldsymbol{Y} ?_{\vee}$	$\boldsymbol{Y} ! A$	$\boldsymbol{Y} ! A$
Defence	$\boldsymbol{X} ! A$ (resp.) $\boldsymbol{X} ! B$	$\boldsymbol{X} ! A$ or $\boldsymbol{X} ! B$	$\boldsymbol{X} ! B$	—

The particle rules for quantifiers has not been introduced, so we will be commenting these rules briefly.

The rules for universal quantification are similar to those for conjunction: stating a universally quantified proposition means that the challenger may choose any individual constant a_i and request of the utterer to make his statement by instantiating every free occurrence of x with a_i. That is, the challenger chooses which proposition he wants the utterer to state.

Properties of universal quantification:

- the challenge is a request;
- the challenger has the choice;
- the defender must state the requested proposition.

The rules for existential quantification are similar to those for disjunction: it is the defender who chooses the proposition he wants to state in response to the challenge.

Properties of existential quantification:

- the challenge is a request;
- the defender has the choice;
- the defender chooses which proposition to state.

Table 7: Particle rules for dialogical games: quantifiers

	Universal quantification	**Existential quantification**
Move	$\boldsymbol{X} ! \forall x B(x)$	$\boldsymbol{X} ! \exists x B(x)$
Challenge	$\boldsymbol{Y} ? [x/a_i]$	$\boldsymbol{X} ?_{\exists}$
Defence	$\boldsymbol{X} ! B(x/a_i)$	$\boldsymbol{X} ! B(x/a_i)$ with $1 \leq i \leq n$

Summing up

Table 8: Summing up the properties of the particle rules

	Nature of the challenge	**Who has the choice**	**Step to defend**
Conjunction	Request	Challenger	State the requested proposition
Universal quantification	Request	Challenger	State the requested proposition
Disjunction	Request	Defender	State the desired proposition
Existential quantification	Request	Defender	State the desired proposition
Implication	Statement	No choice	State the consequent
Negation	Statement	No choice	None

Table 9: Summing up the particle rules

	Conjunction	**Disjunction**	**Implication**	**Negation**	**Universal quantification**	**Existential quantification**
Move	$\boldsymbol{X} ! A \wedge B$	$\boldsymbol{X} ! A \vee B$	$\boldsymbol{X} ! A \supset B$	$\boldsymbol{X} ! \neg A$	$\boldsymbol{X} ! \forall x B(x)$	$\boldsymbol{X} ! \exists x B(x)$
Challenge	$\boldsymbol{Y} ? L^{\wedge}$ or $\boldsymbol{Y} ?\ R^{\wedge}$	$\boldsymbol{Y} ?_{\vee}$	$\boldsymbol{Y} ! A$	$\boldsymbol{Y} ! A$	$\boldsymbol{Y} ? [x/a_i]$	$\boldsymbol{X} ?_{\exists}$
Defence	$\boldsymbol{X} ! A$ (resp.) $\boldsymbol{X} ! B$	$\boldsymbol{X} ! A$ or $\boldsymbol{X} ! B$	$\boldsymbol{X} ! B$	—	$\boldsymbol{X} ! B(x/a_i)$	$\boldsymbol{X} ! B(x/a_i)$ with $1 \leq i \leq n$

IV.3 Symmetry and harmony

In providing the properties of the particle rules, a central feature we have distinguished is who has the choice: is it the challenger or the defender? The meaning of the logical constants is largely determined by who has the choice in the interaction. But notice that in formulating the particle rules, the players' identities are not specified: we do not use **O** and **P** but we use **X** and **Y** instead, thus only specifying who is the challenger and who is the defender *for this particular statement*. That is, we simply provide the appropriate challenge and defence for certain logical constants and determine in this way who has the choice: we provide their meaning in terms of interaction within a dialogue (a game of giving and asking for reasons).

It would not be reasonable to base a game-theoretical approach to the meaning of logical constants in which the meaning differs according to which players utters it: this approach would make interaction senseless, for each player would be meaning something different when uttering the same thing. Equality in action is precisely based on the possibility for a player to say *the same thing* as the other player, and by that to be meaning also the same thing. Equality in action is in this regard the idea that a statement made by a player can be made by another player in a game of giving and asking for reasons (a dialogue) with the exact same meaning as the statement made by the first player, that is with the same particle rules for challenging and defending it. It is thus the interaction based on player-independent rules that allows two different players to be speaking of the same thing: equality between different statements emerges from the interaction itself.

Since the rules for the logical constants are independent of the player's identities—the rules are exactly the same for the two players—we say that these rules are symmetric. This feature captures one of the strengths of the dialogical approach to meaning: the dialogical approach is in this way immune to a wide range of trivializing connectives such as Prior's *tonk*.[54]

Symmetry, or player-independence in the particle rules, must be contrasted with the dialogical rendering of harmony, which concerns the structural rules and the strategy level, not the particle rules. The structural rules, which will be introduced in the next section, are not player independent: the first rule (SR0) specifies who the players are (Proponent or Opponent) according to who plays the first move, that is who states the thesis; that player will be the Proponent. But the rule that matters most in regard to immanent reasoning is the Copy-cat rule (or Socratic rule in a CTT framework); this rule puts a restriction on the Proponent's moves, while those of the Opponent are left unrestricted: which the Proponent cannot play an elementary statement that has not been previously stated by the Opponent.

The purpose of this restriction is to insure that the thesis will be grounded only on what the Opponent has conceded, and thereby secure a form of analyticity that we call immanent reasoning: the Proponent has to ground his thesis on what the Opponent brings forward in the course of their interaction (the dialogue), an interaction that is initiated by the Proponent stating a thesis, which the Opponent challenges with the ensuing series of challenges and defences defined by the particle rules and constituting the dialogue. Thus the Opponent will not bring forward anything that does not stem from the meaning (defined by the particle rules) of the thesis, and the Proponent will not bring any elementary proposition into the game that he cannot justify within that very same dialogue by refering to the Opponent's own statements ("I am entitled to state this because you have stated it yourself").

Symmetry and harmony are two essential aspects of the dialogical framework and are the principles for immanent reasoning.

Dialogical harmony thus coordinates a player-independent level (the local meaning) and a player-dependent level (the global meaning and the strategy level). This aspect contrasts with the Constructive Type Theory notion of harmony which belongs to proof-theory and stays only at the level of strategies.[55] Immanent reasoning and equality in action emerge from taking the specific aspects of the three levels (local, global and strategic) into account and considering how they intertwine to build these complex and dynamic frameworks that are dialogues.

IV.4 Global meaning:

The global meaning—as opposed to the local meaning defined by the particle rules—is defined by means of *structural rules* which specify the general way plays

---

[54] See (Rahman, Clerbout, & Keiff, 2009) and (Redmond & Rahman, 2016).

[55] See (Rahman & Redmond, 2016) and (Rahman, Redmond, & Clerbout, 2017).

unravel by specifying who starts in a play, what moves are allowed and in which order, when a play ends and who wins it. For a more informal presentation of the structural rules determining the global meaning in the dialogical framework, see chapter III.2.2, p.55.

IV.4.1**Preliminary terminology**

Terminal plays:

A play is called *terminal* when it cannot be extended by further moves in compliance with the rules.

X-terminal plays:

A play is ***X**-terminal* when the play is terminal and the last move in the play is an **X**-move.

IV.4.2**The structural rules**

SR0 (Starting rule)

Any dialogue starts with the Opponent stating initial concessions (if any) and the Proponent stating the thesis (labelled move 0). After that, each player chooses in turn a positive integer called the *repetition rank* which determines the upper boundary for the number of attacks and of defences each player can make in reaction to each move during the play.

*Example: if the repetition rank of **O** is $m \coloneqq 1$, then **O** may attack or defend against* at most *once each move of **P**. If **P**'s repetition rank is $n \coloneqq 2$, then **P** may attack or defend against* at most *twice each move of **O**.*

SR1: Development rule

The Development rule depends on what kind of logic is chosen: if the game uses classical logic, then it is SR1c that should be used; but if intuitionistic logic is used, then SR1i must be used.

SR1c (Classical Development rule)

Players move alternately. Once the repetition ranks have been chosen, each move is either attacking or defending a move made by the other player, in accordance with the particle rules.

SR1i (Intuitionisitic Development rule)

Players move alternately. Once the repetition ranks have been chosen, each move is either attacking or defending a move made by the other player, in accordance with the particle rules.

Players can respond only to the *last non-answered* challenge of the other player.

Note: This last clause is known as the Last Duty First *condition, and makes dialogical games suitable for intuitionistic logic (hence this rule's name).*

SR2 (Copy-cat rule)

P may not play an elementary statement unless **O** has stated it first.

Elementary propositions cannot be challenged.

Note: The formulation of this rule has a downside: the thesis of a dialogical game cannot be an elementary statement. For a special version of the Copy-cat rule allowing plays on elementary statements, see below (section 0) where we link this rule to equality.

SR3 (Winning rule)

Player **X** wins the play ζ only if it is **X**-terminal.

Linking the Copy-cat rule (SR2) and equality

The Copy-cat rule[56] is one of the most salient characteristics of dialogical logic. As discussed by (Marion & Rückert, 2015), it can be traced back to Aristotle's reconstruction of the Platonic dialectics. A purely argumentative point of view can be defined within dialectics as refraining from calling on some authority beyond what has actually been brought forward during the current argumentative interaction, the ultimate authority being the fact that the other person has said it, any other consideration being set aside for the time of the dialectical exchange (in this argumentative perspective). Thus, when an elementary statement is challenged, the challenge can be answered only by invoking the challenger's own concessions. In such a context, the Copy-cat rule can be understood in the following way, when a player plays an elementary statement:

> *my grounds for stating the proposition you are challenging are exactly the same as the ones you brought forward when you yourself stated that very same proposition.*[57]

In this regard, elementary statements actually can be challenged (as opposed to the SR2 formulation above), the answer then being of the form "but you have said it yourself". A special formulation of the Copy-cat rule SR2 addresses this problem.

Special Copy-cat rule

O's elementary statements cannot be challenged. However, **O** can challenge an elementary statement played by **P**. The challenge and corresponding defence is determined by the following table. Notice that this (structural) rule is not player-independent and uses the names of the players.

Table 10: SR2 special Copy-cat rule

	Move	**Challenge**	**Defence**
Special Copy-cat rule (Structural rule 2)	**P** $!A$ For elementary A	**O** $?_A$	**P** $!sic(n)$ **P** indicates that **O** stated A at move n

The Copy-cat rule, and even more in this special formulation, introduces an asymmetry between the two players (the Proponent's moves are restricted in a way the Opponent's are not).

---

[56] In previous literature on dialogical logic this rule has been called the *Formal rule*. Since here we will distinguish different formulations of this rule that yield different kind of dialogues we will use the term *Copy-cat rule* when we speak of the rule in standard contexts (such as in the present section)—contexts in which the constitution of the elementary propositions involved in a play is not rendered explicit. When we use the rule in a dialogical framework for CTT, as in the next chapter, we speak of the *Socratic rule*. However, we will continue to use the expression Copy-cat *move* in order to characterize moves of **P** that copies moves of **O**.

[57] See (Rahman, Clerbout, & Keiff, 2009) and (Rahman & Keiff, 2010).

IV.5 Examples of plays

These examples should allow the reader to fully understand the rules given above and their implications, especially the difference between SR1c (classical Development rule) and SR1i (intuitionistic Development rule). For an introduction to the table presentation of dialogues, see chapter III.3, p.56. In the next chapter (V), strategies will be introduced, which allow to compare different plays (with different choice sequences of the players) and build the best possible way of playing for one of the players.

First example, the third excluded: $A \lor \neg A$

The third excluded (*tertium non datur*) is a principle stating that a proposition either is (A) or is not ($\neg A$), without any third possible option. This principle is much discussed in philosophy and logic,[58] it is a valid principle in classical logic, but is not accepted in intuitionistic logic. If this principle is accepted, the principle of non-contradiction ($\neg(A \land \neg A)$) follows, but the reverse is not the case (and intuitionistic logic accepts the principle of non-contradiction but not the principle of third excluded). We will here give a play according to the classical (structural) rules, and then a play according to the intuitionistic (structural) rules.

Play 4: the third excluded—classical rules

O			**P**		
				$! A \lor \neg A$	0
1	$n := 1$			$m := 2$	2
3	$?_\lor$	0		$! \neg A$	4
5	$! A$	2		—	
$\{3\}$	$\{?_\lor\}$	$\{0\}$		$! A$	6

P wins (classical rules).

*Note : the curly brackets are inserted to stress the fact that **O** is not actually making a move, but that **P** is using his repetition rank of 2 in order to defend twice **O**'s challenge (move 3). We repeat that challenge (in brackets) in order to know where **P**'s move 6 comes from.*

Notice that **P** would not have won without a repetition rank higher than 1: he would not have been allowed to answer twice to **O**'s challenge (move 3), and thus use her own assertion of A (move 5) triggered by **P**'s first defence to **O**'s challenge (move 3). This example is a good illustration for the Copy-cat rule and for the use of repetition ranks.

Notice also that **P**'s move 6 is an answer to the challenge of move 3, that is a challenge preceeding the last unanswered challenge, which is move 5. This challenge of move 5 will never be answered, because an attack on the negation cannot be defended. So **P** wins because the classical rules for dialogues *do not* restrict **P**'s answers only to the last unanswered challenge. This fact is the key to understand the outcome of the next play, which uses the intuitionistic rules.

---

[58] For an overview of the philosophical positions behind such principles, see for instance (Read, 1995, pp. 222-224), or see (Dummett, 1978) for seminal papers.

Play 5: the third excluded—intuitionistic rules

O			**P**		
				$! A \vee \neg A$	0
1	$n := 1$			$m := 2$	2
3	$?_{\vee}$	0		$! \neg A$	4
5	$! A$	4		—	

O wins (intuitionistic rules).

Second example, the double negation elimination: $\neg\neg A \supset A$

The elimination of double negation is another example of a principle accepted in classical logic but rejected in intuitionistic logic. This principle is at the core of classical mathematics, for it is what is used in indirect proofs (concluding A from the demonstration that the negation of A leads to a contradiction, that is from the fact that $\neg\neg A$ holds). This principle is closely linked to the principle of excluded middle. Once again, we give a play with classical rules (**P** wins) and a play with intuitionistic rules (**P** loses).

Play 6: the elimination of double negation—classical rules

O			**P**		
				$! \neg\neg A \supset A$	0
1	$m := 1$			$n := 2$	2
3	$! \neg\neg A$	0		$! A$	6
	—		3	$! \neg A$	4
5	$! A$	4		—	

P wins (classical rules).

Notice that as for the third excluded, **P** wins here because he does not have to answer only to the last unanswered challenge (which is move 5) but answers a previous challenge (his move 6 is an answer to the challenge of move 3). This move is forbidden by the intuitionistic (structural) rules ("Last Duty First") illustrated in the next play: **P** should play his move 6, but is not allowed to; it is his turn and he cannot play, so he loses.

Play 7: the elimination of double negation—intuitionistic rules

O			**P**		
				$! \neg\neg A \supset A$	0
1	$m := 1$			$n := 2$	2
3	$! \neg\neg A$	0			
	—		3	$! \neg A$	4
5	$! A$	4		—	

O wins (intuitionistic rules).

It should be clear from these two examples that the intuitionistic rules for dialogues only concern the *structural* rules, namely *when* (in what conditions) a move (challenge or defence) is allowed, but not the *particle* rules which determine *how* to challenge a move or *how* to answer a challenge.

The intuitionistic rules are only a *restriction* imposed on the classical rules, so if **P** wins a play according to the intuitionistic rules, *a fortiori* he should win according to the classical rules.

Third example, the double negation of the third excluded: $\neg\neg(A \vee \neg A)$

This example is a combination of the previous two. But whereas the principle of third excluded and the principle of double negation elimination are not intuitionistic principles (**P** loses), the double negation of the third excluded $\neg\neg(A \vee \neg A)$ does actually hold with intuitionistic rules. This clearly shows that, for intuitionistic logic, an expression is not equivalent to its double negation (the elimination of the double negation of the third excluded would not yield the third excluded, which would contradict the first example).

Play 8: double negation of third excluded—intuitionistic rules

O			P		
				$! \neg\neg(A \vee \neg A)$	0
1	$m := 1$			$n := 2$	2
3	$! \neg(A \vee \neg A)$	0		—	
	—		3	$! A \vee \neg A$	4
5	$?_{\vee}$	4		$! \neg A$	6
7	$! A$	6		—	
	—		3	$! A \vee \neg A$	8
9	$?_{\vee}$	8		$! A$	10

P wins (intuitionistic rules).

In this play, **P** also uses his repetition rank of 2 (move 4 and move 8), but this time to challenge move 3 (instead of defending a move). As opposed to the previous examples, he does not need to defend a move preceeding the last unanswered challenge, so this play in winnable by **P** in the intuitionistic and in the classical contexts.

V. ADVANCED DIALOGUES: STRATEGY LEVEL

The strategy standpoint is but a generalisation of the procedure which is implemented at the play level; it is a systematic exposition of all the relevant variants of a game—the relevancy of the variants being determined from the viewpoint of one of the two players. For a more intuitive approach of strategies and a step-by-step introduction of strategies as branching tables, see section III.5, p. 59. Such trees with branching tables are a good didactic approach to strategies, for the rules in building the tree are the same as those for building the plays: we simply use an algorithm yielding all the relevant plays for a player, keeping the table presentation we use for plays. The link from plays to strategies is thus clearly apparent. This method however is rather cumbersome and becomes unmanageable as soon as we deal with games involving more than two choices, the generated trees taking too much space. We will here present strategies from another perspective, that of extensive forms of dialogical games (more precisely from their core; see below, section V.3) rather than the table presentation; the extensive form presentation has this advantage over the table presentation that strategies can be linked more straightforwardly to demonstrations, which will be useful in chapter IX. This link is crucial to the logical framework of dialogues, for the dialogical notion of validity is secured through the notion of a *winning strategy* for the Proponent. Many metalogical results in the dialogical framework are obtained by leaving the level of rules and plays to move to the level of strategies; winning strategies for a player are one of these metalogical results.

We will here first (section V.1) provide some definitions pertaining to the strategy level along with three important results from existing litterature on this level; then (section V.2) we will introduce what the *core* of a strategy is and the assumptions on the plays we need in order to extract such a core from the extensive form of a strategy; we will then (section V.3) be able to provide the procedure for building a heuristical presentation of the core, that is a presentation of the strategy explicitly preserving the link with the play level; finally (section V.4) we will provide the procedure for building a tree-shaped presentation of the core, which takes up less space than the heuristical (table) presentation and makes it easier to establish links between the dialogical framework and other logical frameworks.

V.1 Preliminary notions

V.1.1 Definitions

Extensive form of a dialogical game:

The *extensive form* $\mathbb{E}(\phi)$ of the dialogical game $\mathbb{D}(\phi)$ is simply its tree presentation, also called the game-tree. Nodes are labelled with moves so that the root is labelled with the thesis and paths in $\mathbb{E}(\phi)$ are linear representations of plays and maximal paths represent terminal plays in $\mathbb{D}(\phi)$.

That is, the extensive form of a dialogical game is an infinitely generated tree in which each branch is of a finite length.

Strategy

A *strategy* for player **X** in $\mathbb{D}(\phi)$ is a function which assigns an **X**-move M to every non terminal play ζ having a **Y**-move as last member, such that extending ζ with M results in a play.

X-winning strategy (or X-strategy)

An ***X**-strategy* is *winning* if playing according to it leads to **X**-terminal plays no matter how **Y** moves.

That is, a winning strategy for player **X** defines the situation in which, for any move choice made by player **Y**, **X** has at least one possible move at his disposal allowing him to win.

Extensive form of an X-strategy

Let s*x* be a strategy of player **X** in *D(ϕ)* of extensive form E*(ϕ)*. The extensive form of s*x* is the fragment S*x* of E*(ϕ)* such that:

1. The root of E*(ϕ)* is the root of S*x*,
2. For any node *t* which is associated with an **X**-move in E*(ϕ)*, any immediate successor of *t* in E*(ϕ)* is an immediate successor of *t* in S*x*,
3. For any node *t* which is associated with a **Y**-move in E*(ϕ)*, if *t* has at least one immediate successor in E*(ϕ)* then *t* has exactly one immediate successor in S*x*, namely the one labelled with the **X**-move prescribed by s*x*.

Validity

A proposition is *valid* in a certain dialogical system if and only if **P** has a winning strategy for it.

V.1.2 Some results from existing literature on the strategy level

The following three results—extracted from existing literature on the subject—establish the correspondance between the dialogical framework and other frameworks involving classical and intuitionistic logics. We will be using them in order to facilitate the building of dialogical demonstrations: the procedure presented in the next section (V.2) will use results justified in this literature: for details and the proofs, see in particular (Clerbout, 2014a; 2014b) and (Felscher, 1985).

P-winning strategies and leaves

Let w be a **P**-winning strategy in $\mathbb{D}(\phi)$. Then every leaf in the extensive form W_ϕ of w is labelled with an elementary **P**-sentence.

Determinacy

There is an **X**-winning strategy in $\mathbb{D}(\phi)$ if and only if there is no **Y**-winning strategy in $\mathbb{D}(\phi)$.

Soundness and Completeness of Tableaux

Consider first-order tableaux and first-order dialogical games. There is a tableau proof for φ if and only if there is a ***P***-winning strategy in $\mathbb{D}(\phi)$.

The fact that the existence of a ***P***-winning strategy coincides with validity (*there is a **P**-winning strategy in $\mathbb{D}(\phi)$ if and only if φ is valid)* follows from the soundness and completeness of the tableau method with respect to model-theoretical semantics.

This third metalogical result for the standard dialogical framework will be taken here for granted (the proof is given for instance in (Clerbout, 2014a)), but we will provide in chapter IX the algorithm transforming a **P**-winning strategy in the framework for dialogues for immanent reasoning into CTT-demonstrations and reversewise.

V.2 Developping a dialogical demonstration: reaching the core of a strategy

Extensive forms of strategies have some redundant information—like the different order of moves—that can be set aside by extracting from the strategy its *core*.[59]

The core of a strategy will allow us to build translation algorithms from the dialogical framework to other logical frameworks; chapter IX will thus use the core of strategies in dialogues for immanent reasoning (the dialogical framework incorporating CTT features presented in chapters VI-VII) in order to build a translation procedure from winning strategies to CTT-demonstrations.

A dialogical game is the set of plays starting with the same thesis (see section IV.1, p. 65). Bear in mind that the extensive form of a game is simply its tree presentation. Since a game takes into account every possible choice for each player, its extensive form has an infinite number of branches (each of a finite length, because of the repetition ranks): there will be a branch for each possible combination of repetition ranks (**O** choosing 1 and **P** choosing 1 to n yields n branches with each combination, then if **O** chooses 2, P will have again the choice from 1 to n, etc.), there will be a branch for each possible choice of instantiation of the quantifiers, there will be different branches for different orders of moves, etc. In all these branches, many are not relevant, either because they do not add any information, or because the players did not play optimally (one of the greatest strengths of the dialogical framework, at least for its philosophical value, is that it allows players to play in a non-optimal fashion).

Extensive forms of strategies do not sift through these informations and thus keep a lot of pointless information. To obtain the core of the extensive form of a strategy, that is, to obtain only the relevant information for building a strategy for a player, we need some kind of heuristic procedure allowing us to deal with such bountiful arborescences. We will be using the three metalogical results provided in the previous section (V.1.2), pertaining to 1) **P**-winning strategies and leaves, 2) determinacy, and 3) the link with validity.

V.2.1 Features essential to the procedure yielding the core of a strategy

Here are a series of assumptions, definitions, and rules, that will allow us to speak about the plays in the following procedures—the heuristical one (V.3), and the graphic tree-shaped one (V.4)—linking the play level and the strategy level of the dialogical framework. Since our goal is to build a procedure allowing us to arrive at the *strategy* level through an arborescence of *plays*, we need to be able to speak *about* the plays, for instance saying which player had which choices in a play, what other options were available, and so on. We will name these plays $\wp_1$, $\wp_2$, etc. in order to differenciate them from each other; we classify them with respect to the last decision taken (see p. 78).

Our goal in this chapter is to find the games won by the Proponent; all the relevant

---

[59] See (Rahman & Clerbout, 2015).

plays constituting the game therefore need to be **P**-terminal. But we are more specifically looking for a **P**-strategy, so only **O**-choices will matter: they are the ones that will show if **P** can win or not, whatever **O** chooses; that is, even if **O** plays optimally. Think for instance of chess-mate in *x* moves: what matters are the possible movements for **O**, and **P** adapts his moves according to the **O**-move actually played.

Assumption on repetition ranks

We assume that the number of repetition rank for **O** is 1.

*The point is that, since **O** is not restricted by the Copy-cat rule in any of its forms (including the Socratic rule), she can always choose the move that is the best for her own interests, and even if she makes a bad choice in a play, she can correct it in a new play. Thus, rank 1 is sufficient.*

We choose a repetition rank for **P** allowing him to win the first play (but not necessarily every possible play). Once the first play is won by **P**, the procedure will allow him to choose another repetition rank for a new play.

*Finding the right repetition rank for **P** that will always allow him to win is one of the objectives of developing demonstrations.*

Assumption on move preferences for O

When **O** has to choose an individual constant she will always choose a new one.

*The reason is pretty straight-forward: it is the best possible choice for **O**. Since **P** is restricted by the Copy-cat rule, he needs to rely on **O**'s choices in order to apply a Copy-cat move. In such a context, the only way to (try to) block the use of the Copy-cat move is to always choose a new constant, whenever a choice is to be made. Thus, this assumption is in **O**'s best interest.*

When **O** can challenge a move where **P** has several defensive options, **O** will launch the challenge before carrying out other moves.

*The reason is similar to the previous one: it is better for **O** to force **P** to make his choice as soon as possible.*

O-Decisions

We say that **O** *makes a decision* in $\wp_{\mathbf{n}}$ in the following cases:

(i) When she *challenges a conjunction or an existential*: she must choose which side to ask for.

(ii) When she *defends a disjunction*: she must choose one of the sides of the disjunction.

(iii) When she has *an implication to defend*: she must choose either to defend it (state the consequent) or to counterattack, that is to launch a challenge on **P**'s attack of the implication (where he stated the antecedent of the initial implication).

(iv) When she *defends an existential*: she chooses a (new) constant.

(v) When she *challenges a universal*: she chooses a (new) constant.

Assumption on O-decisions

The following procedure only takes into consideration decisions of type i) challenge a conjunction or existential, ii) defend a disjunction, and iii) defend an implication: decisions of type iv) defend an existential, and v) challenge a universal, are not really options since our assumptions stipulate that **O** will always choose a new constant and challenge if she can.

Table 11: **O**-decisions in a play

O has a choice and takes a decision when	
O challenges a…	**O defends a…**
Conjunction Existential	Disjunction Implication

Choices: using up the available options

We say that a decision has *used up* the available options if and only if this decision chooses an option when the other option has previously been chosen.

We say that a decision has *not used up* the available options when, because of **O**'s repetition rank 1, she chooses one of two available options, the other option not having been previously chosen: this second option, which remains unchosen, is said *unused*.

Definition of dependent moves

We say that a move *M depends on* the move *M'* if there is a chain of applications of (particle) rules leading from *M'* to *M*.[60]

Left and right decisions

We speak of a *left-decision* when **O**

- decides to defend the left side of a disjunction, or
- decides to challenge the left side of a conjunction, or
- decides to counterattack instead of defending an implication.

We speak of a *right-decision* when **O**

- decides to defend the right side of a disjunction, or
- decides to challenge the right side of a conjunction, or
- decides to defend her implication (instead of counterattacking).

Last unused decision

The *last decision* is the last decision taken by **O** in a play *m*—proceeding bottom up in the order of the moves—such that this decision *does not use up* the available options.

Labeling decisions: conjunctions and disjunctions

We label the moves for which a decision has been taken. For a disjunction or a conjunction, we simply add a label on the right hand side of the move:

---

[60] Dependent moves are important for building subplays triggered by **P** challenging an **O**-implication: the moves depending on **O** counterattacking must be separated from the moves depending on **O** defending her implication.

- $[\delta_n , \ldots]$ when the left decision has been taken but the the right option is still unused; or
- $[\ldots , \delta_n]$ when the right decision has been taken but the left option is still unused.
- $[\delta_m , \delta_n]$ when both options have been chosen (the decision is thus used up): the left decision has been taken in play m and the right decision in play n.

The label $[\delta_3 , \ldots]$ *for instance spells out that the left decision has been taken in play 3, but that the right option has not at that point been chosen before. The label* $[\delta_3 , \delta_5]$ *indicates that the left decision has been taken in play 3 and the right in play 5, so that from play 5 and on, this decision is considered used up.*

Labeling decisions: implications and subplays (branching rule)

When **O** makes a decision for an *implication* in the play $\wp_{\mathbf{n}}$, she opens two new *subplays* (and not plays) $\wp_{\mathbf{n}.L}$ and $\wp_{\mathbf{n}.R}$, one after the other, such that

- $\wp_{\mathbf{n}.L}$ indicates the subplay where **O** decides to counterattack (left decision), and
- $\wp_{\mathbf{n}.R}$ indicates the subplay where **O** decides to defend (right decision).

We say that the available choices for an implication have been used up if both of the subplays have been opened.

The idea of dividing the play into two subplays is to avoid giving **P** an unfair advantage: he cannot use what is given in the development of the left subplay within the development of the right one; **P** must be able to win without using the information of the other subplay. These are not full plays—contrary to the plays opened by using up a decision-option concerning conjunction or disjunction—, but these subplays are only a division of the play: it is as if there were no splitting, but we do it in order to check the moves **P** uses.

It must be noted that the procedure prescribes to start with the subplay involving the counterattack (see the second move preference for **O**, p. 77); but once the counterattack on the antecedent has been launched, the repetition rank 1 has been used up. Thus, in the second subplay involving the defence, a challenge to the antecedent of the implication is no longer available. This shows that the two subplays are only a *graphical device* to present both options within the same (main) play. This is possible because **O** is not restricted by the Copy-cat rule and so, contrary to **P**, never needs the information provided in the other subplay in order to win. This branching rule simply divides the two decision-options of **O** in two subplays.

The opening of two subplays is called the *Branching rule* in (Rahman & Keiff, 2005, pp. 273-275). This rule has a strategic motivation, not a play level one. The reason is that we are after the development of a demonstration (this is the point of this chapter and the procedure for obtaining the core of a strategy). It is also possible to develop a notation where both responses are in the same play, counterattack as well as defence, but this comes with a heavy notation. Moreover, in the present study we have repetition ranks and thus we can use them: the rank 1 of **O** blocks from the branch involving the defence of the implication the problematic second counterattack upon **P**'s-challenge.[61] This shows that the subplays are really two options in the same (main) play.

---

[61] For a discussion of this problem see section 1.6 in (Rahman, Clerbout, & Keiff, 2009).

Starting a subplay

Each subplay starts with the **O**-move immediately succeeding the **P**-move challenging the **O**-implication.

So if the implication was challenged in move n of play $\wp_n$*, then both* $\wp_{n.L}$ *and* $\wp_{n.R}$ *start with move n+1.*

The first move of the play $\wp_{n.R}$ is the defence of the challenge. Graphically, dots are inserted in the upperplay in the space of the defence, and in the subplay in the space of the challenge.

Moving from subplays to upperplays

O's moves in a subplay may allow **P** to make a move in the upperplay. In such a case, the move in the upperplay depending on a move in a subplay will be indexed with its origin (e.g. 12 [$\wp_{n.R}$]).

Play 9: Illustration of subplays and upperplays

$\mathcal{P}_1$

O			**P**		
				$!((A \supset B) \supset A) \supset A$	0
1	$m \coloneqq 1$			$n \coloneqq 2$	2
3	$!(A \supset B) \supset A$	0		$!A$ $!A$	6 [$\mathcal{P}_{1L}$] 6 [$\mathcal{P}_{1R}$]
	...		3	$!A \supset B$	4

$\mathcal{P}_{1L}$

5	$!A$	4			

P wins

$\mathcal{P}_{1R}$

5	$!A$			...	

P wins

P- and O-terminal plays

The play $\wp_n$ is ***P**-terminal* if and only if *every* path starting with the thesis, continuing with $\wp_{n.L}$, $\wp_{n.R}$ and all further subplays, are **P**-terminal.

The play $\wp_n$ is ***O**-terminal* if and only if *one* of the paths is **O**-terminal.

V.3 Heuristical presentation of the core (succession of plays)

Assume we have a play $\wp_n$ won by **P** where **O** played according to the assumptions mentioned above (pertaining to repetition ranks, move preferences and **O**-decisions). The following procedure allows us to build a collection of the relevant plays for a **P**-strategy, taking into account all the **O**-decisions. The *core* of a **P**-winning strategy for the thesis is the collection of plays thus generated: $\wp_1$, $\wp_2$, $\wp_3$, ….

Note: the *final repetition rank* for **P** is the highest repetition number chosen in $\wp_i$.

V.3.1 **Procedure**

Step 0: starting with P-terminal $\wp_1$

The process starts with a **P**-terminal play $\wp_1$ for which **O**'s repetition rank is 1, **P**'s rank is high enough to allow him to win the first play. If you cannot find one, then stop the process: **P** does not have a winning strategy.

Step 1: unused decisions?

If there is no (remaining) unused decision to be taken by **O** in $\wp_n$ then go to step 4. Otherwise go to step 2.

Step 2: using up decisions

Proceeding bottom up in the flow of moves, take in $\wp_n$ **O**'s *last unused decision*:[62]

- if it concerns a conjunction or a disjunction and has not been labeled yet, label it $[\delta_n, \dots]$ or $[\dots, \delta_n]$ respectively for a left or right decision;
- if it concerns a conjunction or disjunction which has already been labeled, open a new play by applying one of the following substeps 2.1 or 2.2, as the case may be;
- if it concerns an implication, open two subplays following substep 2.3.

*Note: when a new play is opened **P** may change his repetition rank.*

Step 2.1: the decision concerns a challenge on a conjunction or an existential

If δ_n concerns the challenge of a conjunction or existential, then open a new play $\wp_{m=n+1}$ in which the move uses the other decision-option:

- **P** may change his repetition rank;
- if the decision in $\wp_n$ was a right-decision, take now the left-decision and reversewise to end up with both sides of the conjunction being challenged in the two different plays, $\wp_m$ and $\wp_n$.
- Label the decision in $\wp_m$ as $[\delta_m, \delta_n]$ or $[\delta_n, \delta_m]$, respectively, if it is a left decision or a right decision, to indicate that it used up both of the available decision-options.
- The new play then proceeds as if the $\wp_n$ challenge of the conjunction or existential had not taken place.[63]
- The $\wp_n$ moves previous to δ_n are imported into the new play.
- If the new play is **O**-terminal, go to step 3; otherwise, go to step 1.

---

[62] The bottom up procedure can be understood through two aspects:

- from the viewpoint of **O**: she is trying to win a play but with the minimal cost; so if she can win by changing a minimum amount of moves, she will choose that option. The last decision reveals this economy: if **O** changed the first decision, then all of the play would have to be replayed; whereas simply changing the last decision allows to keep all the rest. If this last decision was not the faulty decision bringing her to lose, then she goes to the decision before last, etc. up to the very first decision.
- From the viewpoint of the logician: the bottom up procedure allows us to build the core of the strategy: it reveals what is important in the strategies.

[63] The idea is that **O** has just lost the previous play with the left- or right-decision she made; so she is now looking for what has gone wrong and tries to change her way of playing in order to win. Her last decision was the left- or right-decision for the conjunction; so a way of seeing if she can win is to start the game again with exactly the same moves up to this decision, choose the other decision this time, and play with that. This is the whole idea of the core: allowing **O** to go through all of her possible decisions, and if in this rational way of proceeding that takes all of her opportunities into account, **P** still wins, then **P** has a winning strategy. The core is all of these relevant plays, bringing the strategy to its minimal aspect.

Step 2.2: the decision concerns a defence of a disjunction

If δ_{n} concerns a defence of a disjunction, then open a new play $\wp_{\mathrm{m}=\mathbf{n+1}}$ in which the move uses the other decision-option:

- **P** may change his repetition rank;
- if the decision in $\wp_{\mathrm{n}}$ was a right decision, take now the left decision and reversewise to end up with both sides of the disjunction being defended in the two different plays, $\wp_{\mathrm{m}}$ and $\wp_{\mathrm{n}}$.
- Label the decision in $\wp_{\mathrm{m}}$ as [δ_{m}, δ_{n}] or [δ_{n}, δ_{m}], respectively, if it is a left decision or a right decision.
- The new play then proceeds as if the $\wp_{\mathrm{n}}$ defence of the disjunction had not taken place.
- The $\wp_{\mathbf{n}}$ moves previous to δ_n are imported into the new play.
- If the new play is **O**-terminal, go to step 3; otherwise, go to step 1.

Step 2.3: the decision concerns an implication

Recall that the two decision-options for **O** when one of her implications is challenged is either to counterattack (challenging the antecedent posited by **P**), or to defend the implication (providing the consequent), and that the branching here is not the starting of two new plays, as for the conjunction, existential, and disjunction, but only two subplays.

- If **O** *counterattacks*, start a new subplay $\wp_{\mathbf{n}.\boldsymbol{L}}$ in which figure all the moves depending on **O** counterattacking (and not on **O** defending her implication).
  If the development of the subplay yields that $\wp_{\mathbf{n}}$ is **O**-terminal, go to step 3; otherwise go to step 1.

- If **O** *defends* an implication, start a new subplay $\wp_{\mathbf{n}.\boldsymbol{R}}$ in which figure all the moves depending on **O** defending her implication (and not on the counterattack).
  If the development of the subplay yields that $\wp_{\mathbf{n}}$ is **O**-terminal, go to step 3; otherwise go to step 1.

Step 3: O-terminal plays

If there is no (remaining) unused decision to be taken by **O** in play $\wp_{\mathbf{m}}$ and $\wp_{\mathbf{m}}$ is **O**-terminal, then stop the process and start again at Step 0 with another play $\wp'_{\mathbf{0}}$ won by **P**—if you can find any; if you cannot find any other play won by **P**, it simply means he lost and **O** has a way of winning.

Step 4: stopping the process

If there is no (remaining) unused decision to be taken by **O** in play $\wp_{\mathbf{m}}$ and $\wp_{\mathbf{m}}$ is **P**-terminal, then stop the process. **P** has a winning strategy.

Two examples of the heuristical procedure

To illustrate this procedure, let us take two examples:

1. the first on the thesis $\big((A \vee B) \wedge \neg A\big) \supset B$ for which a strategy has already been provided in the solution of the exercise of section III.5, p. 61;

2. the second on the thesis $\Big(\big((A \lor (B \land C)) \land \neg A\big) \supset (B \land C)\Big) \land \big((D \lor E) \supset (D \lor E)\big)$,[64] which has many **O**-decisions to be processed.

Note: the labels are added after the play has been carried out, they do not belong to the play but are labels to navigate between the different plays.

First example: $\big((A \lor B) \land \neg A\big) \supset B$

We start the procedure with step 0: we build a **P**-terminal play $\mathcal{P}_1$ with **O**'s repetition rank being 1:

$\mathcal{P}_1$

	O			**P**	
				$!\ \big((A \lor B) \land \neg A\big) \supset B$	0
1	$m := 1$			$n := 2$	2
3	$!\ (A \lor B) \land \neg A$	0			
5	$!\ \neg A$		3	$?\ R^{\land}$	4
7	$!\ A \lor B$		3	$?\ L^{\land}$	6
9	$!\ A$		7	$?_{\lor}$	8
			5	$!\ A$	10

P wins (intuitionistic rules)

Now that we have our **P**-terminal play $\mathcal{P}_1$, we can proceed to step 1, finding unused **O**-decisions. Recall the **O**-decisions given in section V.2.1 p. 78:

O has a choice and takes a decision when	
O challenges a…	**O defends a…**
Conjunction Existential	Disjunction Implication

In $\mathcal{P}_1$ **O** takes a decision move 9, when she defends a disjunction. So we proceed to step 2 of the procedure: going bottom up in the flow of the moves (from move 10 to move 0) we stop on the last unused decision—which, going bottom up is the first decision for which the two decision-options have not already been taken—, and so we stop move 9. This decision concerns a disjunction and has not been labelled yet, so we insert a label, that of the left-decision because **O** provided the left disjunct in her defence.

$\mathcal{P}_1$

	O			**P**	
				$!\ \big((A \lor B) \land \neg A\big) \supset B$	0
1	$m := 1$			$n := 2$	2
3	$!\ (A \lor B) \land \neg A$	0			
5	$!\ \neg A$		3	$?\ R^{\land}$	4

---

[64] Notice that the structure of this thesis is the following one, using schematic letters: $\Big(\big((X \lor Y) \land \neg X\big) \supset Y\Big) \land (Z \supset Z)$.

7	$! A \vee B$		3	$?\ L^{\wedge}$	6
9	$! A\ [\delta_1, \dots]$		7	$?_{\vee}$	8
			5	$! A$	10

P wins (intuitionistic rules)

We go to step 1: there is an unused decision, so we go to step 2: going bottom up, we stop again on move 9 since that decision has still not been used up (the right decision-option has not been used). Since there is already a label on this decision, we open a new play $\boldsymbol{\mathcal{P}_2}$ by following step 2.2. Since we open a new play, **P** may choose to change his repetition rank. Let us assume he does not. In this play $\boldsymbol{\mathcal{P}_2}$, **O** will be taking the left decision-option, so every move up to move 9 remains identical, move 9 is the defence of the right disjunct, which we label $[\delta_1, \delta_2]$ to show both decision-options have been used, and we proceed with the play $\boldsymbol{\mathcal{P}_2}$ from there on.

$\boldsymbol{\mathcal{P}_2}$

O			**P**		
				$!\ \big((A \vee B) \wedge \neg A\big) \supset B$	0
1	$m \coloneqq 1$			$n \coloneqq 2$	2
3	$!\ (A \vee B) \wedge \neg A$	0		$! B$	10
5	$!\ \neg A$		3	$?\ R^{\wedge}$	4
7	$! A \vee B$		3	$?\ L^{\wedge}$	6
9	$! B\ [\delta_1, \delta_2]$		7	$?_{\vee}$	8

P wins (intuitionistic rules)

Since the play $\boldsymbol{\mathcal{P}_2}$ is not **O**-terminal, we do not go to step 3 but go to step 1 again, and then to step 2: we scan in $\boldsymbol{\mathcal{P}_2}$ bottom up for the last unused decision, and since there are none left, we proceed to step 4. There is no unused decision left in $\boldsymbol{\mathcal{P}_2}$ and the play is **P**-terminal, so we stop the process. **P** has a winning strategy.

The core for the thesis $\big((A \vee B) \wedge \neg A\big) \supset B$ is the collection of plays $\wp_1$ and $\wp_2$; this result demonstrates the validity of the thesis, based on the third melalogical result provided in section V.1.2, p. 75 on the soundness and completeness of tableaux.

Second example: $\Big(\big(\big(\boldsymbol{A} \vee (\boldsymbol{B} \wedge \boldsymbol{C})\big) \wedge \neg \boldsymbol{A}\big) \supset (\boldsymbol{B} \wedge \boldsymbol{C})\Big) \wedge \big((\boldsymbol{D} \vee \boldsymbol{E}) \supset (\boldsymbol{D} \vee \boldsymbol{E})\big)$

$\boldsymbol{\mathcal{P}_1}$

O			**P**		
				$!\Big(\big(\big(A \vee (B \wedge C)\big) \wedge \neg A\big) \supset (B \wedge C)\Big) \wedge \big((D \vee E) \supset (D \vee E)\big)$	0
1	$m \coloneqq 1$			$n \coloneqq 2$	2
3	$?\ L^{\wedge}$	0		$!\ \big(\big(A \vee (B \wedge C)\big) \wedge \neg A\big) \supset (B \wedge C)$	4
5	$!\big(A \vee (B \wedge C)\big) \wedge \neg A$	4			
7	$! A \vee (B \wedge C)$		5	$?\ L^{\wedge}$	6
9	$! A\ [\boldsymbol{\delta_1}, \dots]$		7	$?_{\vee}$	8

11	$!\neg A$		5	$?R^{\wedge}$	10
			11	A	12

P wins.

The last decision (move 9) has not been used up; it is a left-decision, so we label it **[δ_1, …].**

⬇

O can now open a new play $\wp_2$ in which she defends the other available option: the right disjunct:

$\mathcal{P}_2$

	O			P	
				$!\Big(\big(\big(A \vee (B \wedge C)\big) \wedge \neg A\big) \supset (B \wedge C)\Big)$ $\wedge \big((D \vee E) \supset (D \vee E)\big)$	0
1	$m \coloneqq 1$			$n \coloneqq 2$	2
3	$?L^{\wedge}$	0		$!\ \big(\big(A \vee (B \wedge C)\big) \wedge \neg A\big) \supset (B \wedge C)$	4
5	$!\big(A \vee (B \wedge C)\big)$ $\wedge \neg A$	4			
7	$!\,A \vee (B \wedge C)$		5	$?L^{\wedge}$	6
9	$!\,\mathrm{B} \wedge \mathrm{C}\ [\delta_1, \boldsymbol{\delta_2}]$		7	$?_{\vee}$	8
11	$?\,\mathrm{L}^{\wedge}\ [\boldsymbol{\delta_2}, \dots]$	10		$!\,B$	14
13	$!\,B$		9	$?L^{\wedge}$	12

P wins.

The available decision-options in $\wp_2$ pertaining to the disjunction have both been used up: we now have the label [δ_1, δ_2] showing this fact.

⬇

The last decision [δ_2, ...] (move 11) has not been used up so **O** can now open a new play $\wp_3$ in which she challenges the right-hand side of the conjunction: at move 11 in $\wp_3$, **O** does not play $?L^{\wedge}$ but $?R^{\wedge}$. We therefore delete all the moves after move 11; we keep exactly the same moves up to move 11, and we change move 11 to the other decision-option: $?R^{\wedge}$. We then proceed as if the other play had not taken place:

$\mathcal{P}_3$

	O			P	
				$!\Big(\big(\big(A \vee (B \wedge C)\big) \wedge \neg A\big) \supset (B \wedge C)\Big)$ $\wedge \big((D \vee E) \supset (D \vee E)\big)$	0
1	$m \coloneqq 1$			$n \coloneqq 2$	2
3	$?L^{\wedge}\ [\boldsymbol{\delta_3}, \dots]$	0		$!\ \big(\big(A \vee (B \wedge C)\big) \wedge \neg A\big) \supset (B \wedge C)$	4

5	$!\big(A \vee (B \wedge C)\big) \wedge \neg A$	4		$!\,B \wedge C$	10
7	$!\,A \vee (B \wedge C)$		5	$?\,L^{\wedge}$	6
9	$!\,\mathrm{B} \wedge \mathrm{C}\ [\delta_1, \boldsymbol{\delta_2}]$		7	$?_{\vee}$	8
11	$?\,\mathrm{R}^{\wedge}\ [\delta_2, \boldsymbol{\delta_3}]$	10		$!\,C$	14
13	$!\,C$		9	$?\,R^{\wedge}$	12

P wins.

Both of the available challenges of the conjunction have been used up in $\wp_3$: we label this move $[\delta_2, \delta_3]$ to show this fact.

⇓

O has one more unused decision left, namely move 3. So we add the label **$[\delta_3, \ldots]$** to that move and start a new play $\wp_4$ with **O** choosing the other option, that is, attacking the left-hand side of the conjunction in the thesis:

$\boldsymbol{\mathcal{P}_4}$

	O			P	
				$!\Big(\big((A \vee (B \wedge C)) \wedge \neg A\big) \supset (B \wedge C)\Big) \wedge \big((D \vee E) \supset (D \vee E)\big)$	0
1	$m := 1$			$n := 2$	2
3	$?\,R^{\wedge}\ [\delta_3, \boldsymbol{\delta_4}]$	0		$!\,(D \vee E) \supset (D \vee E)$	4
5	$!\,D \vee E$	4		$!\,D \vee E$	6
7	$?_{\vee}$	6		$!\,D$	10
9	$!\,D\ [\boldsymbol{\delta_4}, \ldots]$		5	$?_{\vee}$	8

P wins.

The decision-options in move 3 have all been used up in $\wp_4$.

⇓

Repeating step 2, a new unused decision appears: move 9, which is labelled accordingly $[\delta_4, \ldots]$; we start a new play $\wp_5$ with **O** defending the right-hand side of the disjunction:

$\boldsymbol{\mathcal{P}_5}$

	O			P	
				$!\Big(\big((A \vee (B \wedge C)) \wedge \neg A\big) \supset (B \wedge C)\Big) \wedge \big((D \vee E) \supset (D \vee E)\big)$	0
1	$m := 1$			$n := 2$	2
3	$?\,R^{\wedge}\ [\delta_3, \delta_4]$	0		$!\,(D \vee E) \supset (D \vee E)$	4
5	$!\,D \vee E$	4		$!\,D \vee E$	6
7	$?_{\vee}$	6		$!\,E$	10
9	$!\,E\ [\delta_4, \boldsymbol{\delta_5}]$		5	$?_{\vee}$	8

P wins.

Every possible option for **O** has now been tried out, we arrive at step 4, the procedure stops and defeat for **O** must be acknowledged: each of the plays are won by **P**.

The core for the thesis $\Big(\big((A \vee (B \wedge C)) \wedge \neg A\big) \supset (B \wedge C)\Big) \wedge \big((D \vee E) \supset (D \vee E)\big)$ is the collection of plays $\wp_1$, $\wp_2$, $\wp_3$, $\wp_4$ and $\wp_5$; this result demonstrates the validity of the thesis; to obtain a tree representation of it, see the next section.

V.4 Graphic presentation of the core (tree-shaped)

The heuristic procedure provided above shows plainly how the play level is linked to the strategy level; there is a continuity between the two. The downside of that method is that it becomes cumbersome as soon as there a more than a couple of **O**-choices to take into account; we rather use, for the strategy level, a tree presentation, which is much easier to manage and provides the added benefit that metalogical correspondences with sequent calculus and CTT-demonstrations are simplified, making the validity of dialogical demonstrations more straightforward to see (cf. chapter IX).

The graphic presentation without the heuristic procedure

The graphic tree presentation of the core corresponds to a tree in which the nodes are the players' moves displayed in vertical sequences of dialogical **P**-steps and **O**-steps; **O**-decisions trigger a branching in the tree yielding the different plays of the core.

Here is a simple way to sketch the procedure for the development of such a tree:

1. The root of the tree is the thesis.
2. The next step is either **O** challenging the thesis or **O** positing the required initial concessions.
3. The tree develops as a vertical sequence of dialogical **P**- and **O**-steps, until the first **O**-decision occurs.
4. When the first **O**-decision occurs, split the tree in two branches and explore one of them.
5. If the branch ends with an **O**-move, then **O** won and the procedure terminates.
6. Otherwise, start exploring the second main branch and so on until the end.

Note: As opposed to the heuristic method presented above, the graphic (tree-shaped) presentation of the core is built top to bottom, that is from the first decision of the first play and not from the last decision of the first play.

Clearly, the whole demonstration—or **P**-winning strategy for the given thesis—could be developped directly in such a tree from, without going through the table presentation of the heuristic procedure. But the table presentation highlights the dialogical background from which the strategy emerges, and shows the continuity between the two levels.

It is important to stress this continuity because the play level is what differentiates the dialogical framework from proof-theory; but without the heuristic procedure, there seems to be a cleft in the framework between strategies and plays: there is one method for plays (table dialogues) and one method for strategies (trees), but nothing showing how the two levels unite. The two perspectives (plays and strategies) could thus be considered separate, and have often been, proof-theoricians feeling concerned only with the dialogical demonstrations of the strategy level and not with the play level. We however believe the two levels are closely linked, and that the framework's strength and fruitfulness stems from its capacity to move seamlessly from one perspective to another. The heuristic procedure is a way of showing this link, which will be further stressed in section VII.7 through the notion of strategic reasons, and in chapter X; harmony and symmetry mentionned above (IV.3, p. 67) is another one.

We will therefore provide a procedure for building a graphic presentation of the core, but assuming that the heuristic procedure has been carried out and that the strategy is thus at hand.

V.4.1 **Building the graphic (tree-shaped) presentation of the core**

We assume we actually have carried out the heuristic procedure for finding the core of a **P**-strategy, and have found one. We will thus be using the same assumptions as for the heuristic procedure concerning **O**'s repetition rank, her move preferences and decisions (see V.2.1 p. 76). Here are the instructions for transforming the core of a **P**-strategy from the heuristic table presentation to the graphical tree presentation; thus simply providing a new exposition of the core.

Dissociating the core from the plays it stems from

Take the collection of plays constituting the core that the heuristic procedure has yielded; proceed in the following way, from the *first* play to the last, and in each play from *top* to bottom, in order to erase the table presentation of the plays but without deleting any information; this is simply a new *exposition* of the core, we need to be able to go back to the other exposition and not lose important information along the way. Repetition ranks are made implicit: they do not appear in the tree but still regulate the players' moves.

i. Start the tree with the number of the move for the thesis (that is 0);
ii. Insert to the right of the number of the move the name of the player (**P**);
iii. Insert to the right of the player:
    a. "! proposition" if the move is a statement and the proposition stated, or
    b. "? request" if the move is a request and the matter of the request;
iv. Proceed from there on, following the order of the moves, by writing each player's move one under the other according to the same notation as in steps i-iii:

    [number of move][**P/O**][! proposition/? request]

v. Insert to the right of each proposition the following indication:
    a. If the move is a challenge, add [?n], where n is the number of the move being challenged.
    b. If the move is a defence, add [!n], where n is the number of the move launching the challenge to which this move is a response.

    Note: this is much like erasing the lines of the dialogue table and writing the moves one under the other.

vi. When you come across an **O**-decision, insert between the move and the indication of it being a challenge or a defence the indication of the **O**-decisions provided by the heuristic procedure, and branch the tree:
    a. In the left branch of the tree, continue down the play implementing **O**'s left decision-option;

b. in the right branch of the tree, continue down the play implementing **O**'s right decision-option.

Two examples of the graphic presentation of the core

In order to illustrate the procedure for building a graphic (tree-shaped) presentation of the core of the extensive form of a **P**-winning strategy, but also to show plainly the similarities with and the differences from the heuristic procedure, we will take the same two examples as in the previous section (V.3, see p. 83 for the first example and p. 84 for the second) and build a graphic presentation of their core.

First example: $\boldsymbol{((A \vee B) \wedge \neg A) \supset B}$

We start by writing the number of the move of the thesis (0), the name of the player (**P**), the force of the utterance (statement: !) and the proposition stated, that is $((A \vee B) \wedge \neg A) \supset B)$. This is the root of the tree:

$0\ \boldsymbol{P}!\ ((A \vee B) \wedge \neg A) \supset B$

We then proceed down the first play of the core, $\wp_1$, and write each move one under the other, recording the challenges and defences, until reaching an **O**-decision:

$0\ \boldsymbol{P}!\ ((A \vee B) \wedge \neg A) \supset B$
$1\ \boldsymbol{O}!\ m \coloneqq 1$
$2\ \boldsymbol{P}!\ n \coloneqq 2$
$3\ \boldsymbol{O}\ !\ (A \vee B) \wedge \neg A\ [?\,0]$
$4\ \boldsymbol{P}?\ R^{\wedge}[?\,3]$
$5\ \boldsymbol{O}\ !\ \neg A\ [!\,4]$
$6\ \boldsymbol{P}\ ?\ L^{\wedge}[?\,3]$
$7\ \boldsymbol{O}!\ A \vee B\ [!\,6]$
$8\ \boldsymbol{P}\ ?_{\vee}\ [?\,7]$

We have reached the move 9 which requires **O** to make a decision. So we branch the tree and carry on each branch, with the left branch being $\wp_1$ from move 9 down, and the right branch being $\wp_2$ from move 9 down:

$0\ \boldsymbol{P}!\ ((A \vee B) \wedge \neg A) \supset B$
$1\ \boldsymbol{O}!\ m \coloneqq 1$
$2\ \boldsymbol{P}!\ n \coloneqq 2$
$3\ \boldsymbol{O}\ !\ (A \vee B) \wedge \neg A\ [?\,0]$
$4\ \boldsymbol{P}?\ R^{\wedge}[?\,3]$
$5\ \boldsymbol{O}\ !\ \neg A\ [!\,4]$
$6\ \boldsymbol{P}\ ?\ L^{\wedge}[?\,3]$
$7\ \boldsymbol{O}!\ A \vee B\ [!\,6]$
$8\ \boldsymbol{P}\ ?_{\vee}\ [?\,7]$

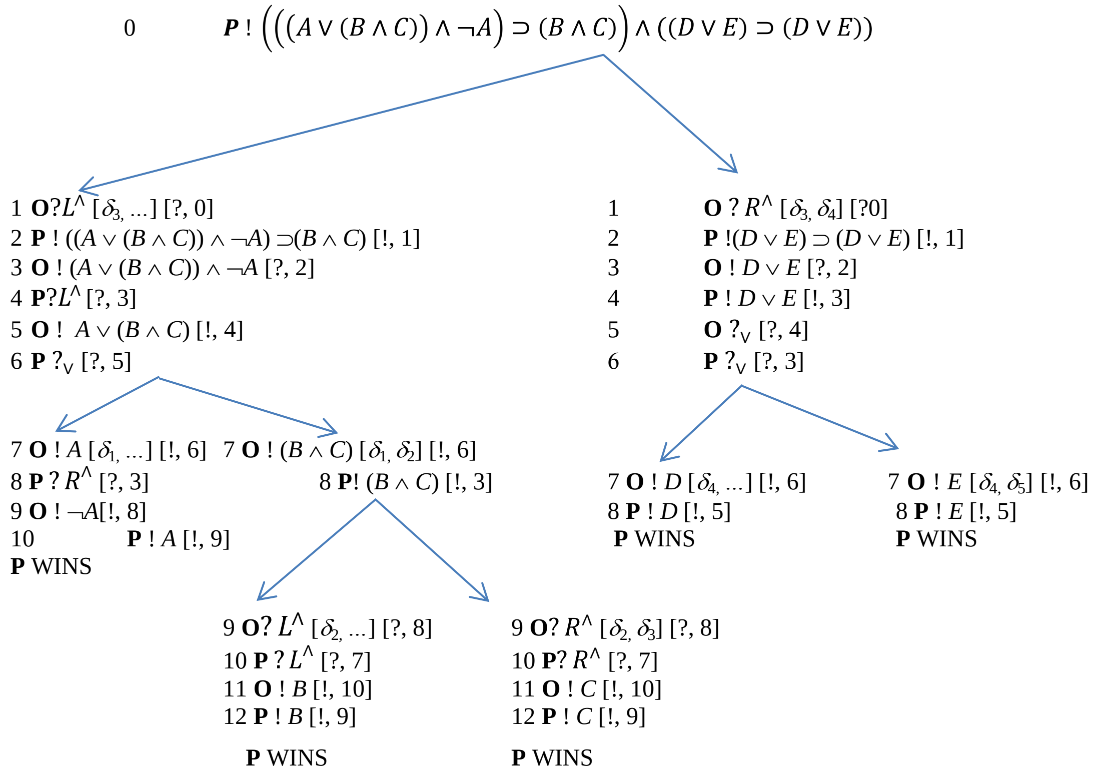
$9\ \boldsymbol{O}!\ A\ [\delta_1, \dots][!\,8]$	$9\ \boldsymbol{O}\ !\ B\ [\delta_1, \delta_2][!\,8]$
$10\ \boldsymbol{P}\ !\ \ A\ [?\,5]$	$10\ \boldsymbol{P}\ !\ B\ [!\,3]$

P wins **P** wins

We have thus transformed the heuristic presentation of the core into its graphic tree presentation that keeps the information of the play level and yet provides a synoptic view of all the relevant plays for a **P**-winning strategy.

Second example: $\left(\left((\boldsymbol{A} \vee (\boldsymbol{B} \wedge \boldsymbol{C})) \wedge \neg \boldsymbol{A}\right) \supset (\boldsymbol{B} \wedge \boldsymbol{C})\right) \wedge \left((\boldsymbol{D} \vee \boldsymbol{E}) \supset (\boldsymbol{D} \vee \boldsymbol{E})\right)$

Applying the same procedure to the heuristic presentation of the core for this second thesis, we obtain the following tree presentation of the core. Since there are five plays in the core ($\wp_1$- $\wp_5$), there will be four branchings in the tree:



0 **P** ! $\left(\left((A \vee (B \wedge C)) \wedge \neg A\right) \supset (B \wedge C)\right) \wedge ((D \vee E) \supset (D \vee E))$

1 **O**?$L^{\wedge}$ [$\delta_{3,}$ …] [?, 0]
2 **P** ! $((A \vee (B \wedge C)) \wedge \neg A) \supset (B \wedge C)$ [!, 1]
3 **O** ! $(A \vee (B \wedge C)) \wedge \neg A$ [?, 2]
4 **P**?$L^{\wedge}$ [?, 3]
5 **O** ! $A \vee (B \wedge C)$ [!, 4]
6 **P** ?$_{\vee}$ [?, 5]

7 **O** ! A [$\delta_{1,}$ …] [!, 6]
8 **P** ? $R^{\wedge}$ [?, 3]
9 **O** ! $\neg A$[!, 8]
10 **P** ! A [!, 9]
P WINS

7 **O** ! $(B \wedge C)$ [$\delta_{1,}$ δ_2] [!, 6]
8 **P**! $(B \wedge C)$ [!, 3]

9 **O**? $L^{\wedge}$ [$\delta_{2,}$ …] [?, 8]
10 **P** ? $L^{\wedge}$ [?, 7]
11 **O** ! B [!, 10]
12 **P** ! B [!, 9]

P WINS

9 **O**? $R^{\wedge}$ [$\delta_{2,}$ δ_3] [?, 8]
10 **P**? $R^{\wedge}$ [?, 7]
11 **O** ! C [!, 10]
12 **P** ! C [!, 9]

P WINS

1 **O** ? $R^{\wedge}$ [$\delta_{3,}$ δ_4] [?0]
2 **P** !$(D \vee E) \supset (D \vee E)$ [!, 1]
3 **O** ! $D \vee E$ [?, 2]
4 **P** ! $D \vee E$ [!, 3]
5 **O** ?$_{\vee}$ [?, 4]
6 **P** ?$_{\vee}$ [?, 3]

7 **O** ! D [$\delta_{4,}$ …] [!, 6]
8 **P** ! D [!, 5]
P WINS

7 **O** ! E [$\delta_{4,}$ δ_5] [!, 6]
8 **P** ! E [!, 5]
P WINS

VI. LOCAL REASONS AND DIALOGUES FOR IMMANENT REASONING

In this chapter we will provide the logical framework of dialogues for immanent reasoning, the dialogical framework incorporating features of Constructive Type Theory and making explicit the players' reasons for asserting a proposition. We will therefore be using the material provided in chapters II-V on CTT and on the standard dialogical framework and assume the reader is familiar enough with it. The framework of dialogues for immanent reasoning takes a further step in the task of making explicit the dynamic foundations of reasoning, based on equality in action: reasons adducing statements are introduced in the object-language, and the Proponent's task in formal dialgoues is to force the Opponent to provide herself the reasons the Proponent needs in order to justify his thesis; once the Opponent has produced her reasons, the Proponent uses equality rules (the Socratic rule for instance) to copy these reasons and use them to his own ends. In this respect, CTT proof-objects are adapted to the dialogical framework, yielding two new elements in our framework: local reasons, the backbone of dialogues for immanent reasoning at the play level which will be introduced in this chapter and the next (VI-VII); and strategic reasons, justifications at the strategy level corresponding to proof-objects which will be the object of chapter IX. Having local reasons in the dialogical framework provides a structure in which the reasons given and asked for actually appear in the object-language, and the Proponent can then (locally) justify his statements by explicitly copying the Opponent's reasons for his own statements.

This chapter will first present the motivation behind dialogues for immanent reasoning (a dialogical framework) by providing some philosophical considerations in relation to the modern developments of logic (section VI.1), thus extending the enquiries initiated in the introduction. A second section (VI.2) will present a limit of the current study, that is the link between local reasons and material truth, in order to better grasp the potency of local reasons, but also to better understand the scope of formal plays through this insight into what they are not; more on material dialogues can be found in the last chapter (X). A third section (VI.3) will introduce progressively the rules defining local reasons in formal plays: formation rules (VI.3.1), the synthesis rules (VI.3.2), and the analysis rules (VI.3.3). A fourth section (VI.4) will expatiate on the link between local reasons and equality before providing an example with step-by-step explanations in order to grasp the meaning of the rules presented and how they intertwine in an actual play.

This chapter is to be read in connection with the following two chapters (VII-VII.7): chapter VII provides the rules for dialogues for immanent reasoning without much explanations, considering that this chapter VI is enough to follow on local reasons; it thus provides the local and global meanings for immanent reasoning, as well as the rules for the strategy level using local reasons; chapter VII.7 develops extensively the case of the Axiom of Choice and how the framework of dialogues for immanent reasoning is capable of dealing with it. The presentation of this framework will nonetheless be incomplete until strategic reasons are introduced in chapter IX, wrapping up in a notion the specificities of this framework, and yet still dovetailing other logical frameworks.

VI.1 Introductory remarks on the choice of CTT

Recent developments in dialogical logic show that the Constructive Type Theory approach to meaning is very natural to the game-theoretical approaches in which (standard) metalogical features are explicitly displayed at the object language-level.[65] This vindicates, albeit in quite a different fashion, Hintikka's plea for the fruitfulness of game-theoretical semantics in the context of epistemic approaches to logic, semantics, and the foundations of mathematics.[66]

From the dialogical point of view, the actions—such as choices—that the particle rules associate with the use of logical constants are crucial elements of their full-fledged (local) meaning: if meaning is conceived as constituted during interaction, then all of the actions involved in the constitution of the meaning of an expression should be made explicit; that is, they should all be part of the object-language.

This perspective roots itself in Wittgenstein's remark according to which one cannot position oneself outside language in order to determine the meaning of something and how it is *linked to syntax*; in other words, language is unavoidable: this is his *Unhintergehbarkeit der Sprache*, one of Wittgenstein's tenets that Hintikka explicitly rejects.[67] According to this perspective of Wittgensteins, language-games are supposed to accomplish the task of studying language from a perspective that acknowledges its *internalized feature*. This is what underlies the approach to meaning and syntax of the dialogical framework in which all the speech-acts that are relevant for rendering the meaning and the "formation" of an expression are made explicit. In this respect, the metalogical perspective which is so crucial for model-theoretic conceptions of meaning does not provide a way out. It is in such a context that Lorenz writes:

> *Also propositions of the metalanguage require the understanding of propositions, […] and thus cannot in a sensible way have this same understanding as their proper object. The thesis that a property of a propositional sentence must always be internal, therefore amounts to articulating the insight that in propositions about a propositional sentence this same propositional sentence does not express a meaningful proposition anymore, since in this case it is not the propositional sentence that is asserted but something about it.*
>
> *Thus, if the original assertion (i.e., the proposition of the ground-level) should not be abrogated, then this same proposition should not be the object of a metaproposition […].*[68]
>
> *While originally the semantics developed by the picture theory of language aimed at determining unambiguously the rules of "logical syntax" (i.e. the logical form of linguistic expressions) and thus to justify them […]—now language use itself, without the mediation of theoretic constructions, merely via "language games", should be sufficient to introduce the talk about "meanings" in such a way that they supplement the syntactic rules for the use of ordinary language expressions (superficial grammar) with semantic rules that capture the understanding of these expressions (deep grammar).*[69]

Similar criticism to the metalogical approach to meaning has been raised by Göran Sundholm (1997; 2001) who points out that the standard model-theoretical semantic turns semantics into a meta-mathematical formal object in which syntax is linked to meaning by the assignation of truth values to uninterpreted strings of signs (formulae). Language does not express *content* anymore, but it is rather conceived as a system of signs that speak *about* the world—provided a suitable metalogical link between the signs and the world has been fixed. Moreover, Sundholm (2016) shows that the cases of quantifier-dependences motivating Hintikka's IF-logic can be rendered in the CTT framework. What we will here add to Sundholm's observation is that even the interactive features of

---

[65] See for instance (Clerbout & Rahman, 2015) (Rahman & Clerbout, 2013; 2015), (Dango, 2014; 2015; 2016), (Jovanovic, 2013; 2015), (Rahman, Clerbout, & McConaughey, 2014) (Rahman, Clerbout, & Jovanovic, 2015).

[66] Cf. (Hintikka, 1973; 1983; 1996b).

[67] Hintikka (1997) shares this rejection with all those who endorse model-theoretical approaches to meaning.

[68] (Lorenz, 1970, p. 75), translated from the German by Shahid Rahman.

[69] (Lorenz, 1970, p. 109), translated from the German by Shahid Rahman.

these dependences can be given a CTT formulation, provided the latter is developed within a dialogical setting.

Ranta (1988) was the first to link game-theoretical approaches with CTT. Ranta took Hintikka's (1973) Game-Theoretical Semantics (GTS) as a case study, though his point does not depend on that particular framework: in game-based approaches, a proposition is a set of winning strategies for the player stating the proposition.[70] In game-based approaches, the notion of truth is at the level of such winning strategies. Ranta's idea should therefore in principle allow us to apply, safely and directly, instances of game-based methods taken from CTT to the pragmatist approach of the dialogical framework.

From the perspective of a general game-theoretical approach to meaning however, reducing a proposition to a set of winning strategies is quite unsatisfactory. This is particularly clear in the dialogical approach in which different levels of meaning are carefully distinguished: there is indeed the level of strategies, but there is also the level of plays in the analysis of meaning which can be further analysed into local, global and material levels. The constitutive role of the play level for developing a meaning explanation has been stressed by Kuno Lorenz in his (2001) paper:

> *Fully spelled out it means that for an entity to be a proposition there must exist a dialogue game associated with this entity, i.e., the proposition A, such that an individual play of the game where A occupies the initial position, i.e., a dialogue D(A) about A, reaches a final position with either win or loss after a finite number of moves according to definite rules: the dialogue game is defined as a finitary open two-person zero-sum game. Thus, propositions will in general be dialogue-definite, and only in special cases be either proof-definite or refutation-definite or even both which implies their being value-definite.*
>
> *Within this game-theoretic framework […] truth of A is defined as existence of a winning strategy for A in a dialogue game about A; falsehood of A respectively as existence of a winning strategy against A.*[71]

Given the distinction between the play level and the strategy level, and deploying within the dialogical framework the CTT-explicitation program, it seems natural to distinguish between *local* reasons and *strategic* reasons: only the latter correspond to the notion of *proof-object* in CTT and to the notion of *strategic-object* of Ranta. In order to develop such a project we enrich the language of the dialogical framework with statements of the form “$p : A$”. In such expressions, what stands on the left-hand side of the colon (here p) is what we call a *local reason*; what stands on the right-hand side of the colon (here A) is a proposition (or set).[72] Strategic reasons will be introduced in chapter IX.

The *local* meaning of such statements results from the rules describing how to compose (*synthesis*) within a play the suitable local reasons for the proposition *A* and how to separate (*analysis*) a complex local reason into the elements required by the composition rules for *A*. The synthesis and analysis processes of *A* are built on the formation rules for *A*.

VI.2 Local reasons and material truth

The most basic contribution of a local reason is its contribution to a material dialogue involving an elementary proposition. Informally, we can say that if the

---

[70] That player can be called Player 1, Myself or Proponent.

[71] (Lorenz, 2001, p. 258).

[72] See (Rahman, Redmond, & Clerbout, 2017), (Clerbout & Rahman, 2015) (Rahman & Clerbout, 2013; 2015).

Proponent **P** states the elementary proposition A, it is because **P** claims that he can bring forward a reason in defence of his statement. It is the Socratic rule that determines the precise form of that local reason, $specific\ to\ A$.[73] Our study focuses on formal—not material—dialogues, but we will still provide some basic elements on material truth in regard to local reasons so as to render in a clearer fashion the limits of our study and its philosophical background, the meaning of formal plays by contrast with what they are not, and the further work that can be carried out from this presentation of dialogues for immanent reasoning.

Approaching material truth

Assume the Proponent states that 1 is an odd number:

$$\mathbf{P}\ !\ 1\ is\ an\ odd\ number$$

the Opponent can then express the following demand, asking **P** for reasons for his statement:

$$\mathbf{O}\ !\ find\ a\ natural\ number\ n\ such\ that\ 1 = 2.n + 1$$

Because of the restriction the Socratic rule imposes on **P**, he can defend his statement by choosing "0", provided that **O** has already endorsed the statement "0 is a natural number" ($0: \mathbb{N}$). This produces *material truth*.

Material truth can then be described in the following way: the statement that a given proposition is materially true requires displaying a local reason *specific to that very proposition*.

Material truth and local reasons

A local reason adduced in defence of a proposition thus prefigures a material dialogue displaying the specific content of that proposition. This constitutes the bottom of the normative approach to meaning of the dialogical framework: *use* (dialogical interaction) is to be understood as *use prescribed by a rule of dialogical interaction*. This applies not only to the meaning of logical constants, but also to the meaning of elementary propositions. This is what Jaroslav Peregrin (2014, pp. 2-3) calls the *role* of a linguistic statement: according to this terminology, and if we place his suggestion in our dialogical setting, we can say that the meaning of an elementary proposition amounts to its *role* in that form of interaction that the Socratic rule for a material dialogue prescribes for that specific proposition.[74] It follows from such a perspective that material dialogues are important not only for the general question of the normativity of logic, but also for the elaboration of a language with content.

Material dialogues and formal dialogues

Summing up, what distinguishes formal dialogues from material dialogues resides in the following:

- The formulation of the Socratic rule of a formal dialogue prescribes a form of interaction based only on the meaning of the logical constant(s) involved, irrespective of the meaning of the elementary propositions in the scope of that constant.

---

[73] Recall that the Socratic rule does not prohibit the Opponent **O** from challenging an elementary proposition of **P**; the rule only restricts **P**'s authorized moves.

[74] In the last chapter (0) we come back to the relation between normativity and material dialogues.

- The choice of the local reason for the elementary propositions involved is left to the authority of the Opponent.

In other words, in a formal dialogue the Socratic rule is *not* specific to any elementary proposition in particular, but it is *general*; definitions that distinguish one proposition from another are introduced during the game according to the local meaning of the logical constant involved: formal dialogues are the purest kind of immanent reasoning.

The synthesis and analysis of local reasons for a proposition *A* are determined by the actions prescribed by the Socratic rule specific to the kind of play in which *A* has been stated:

- If the play is *material*, the Socratic rule will describe a kind of action specific to the formation of *A*.
- If the play is *formal*, as assumed in the main body of our study, the Socratic rule will allow **O** to bring forward the relevant local reasons during the development of the play.

The point is that in formal dialogues, when the Opponent challenges the thesis, the thesis is assumed to be well-formed up to the logical constants, so the formation of the elementary statements is displayed during the development of the dialogue and left to the authority of **O**. So the formation rule for elementary statements does not really take place at the level of local meaning but at level of global meaning.

Our study here focuses on *formal plays* of immanent reasoning. A thorough development of material dialogues will be left for future work, though in the last chapter of the book (chapter X) we will provide some insights as to their structure and discuss how to "internalize" empirical content within a game.

Since the local reasons for the elementary statements are left to **O**'s authority, what we now need is to describe the process of *synthesis* and *analysis* for local reasons of the logical constants. However, before starting to enrich the language of the standard dialogical framework with local reasons for logical constants let us discuss how to implement a dialogical notion of *formation rules*. The formation rules together with the synthesis and analysis rules settle the local meaning of dialogues for immanent reasoning.

VI.3 The local meaning of local reasons

Here is an introduction of the formation rules (section VI.3.1), the synthesis rules (section VI.3.2), and the analysis rules (section VI.3.3) for local reasons. But we first need to make a clarification on statements and add a piece of notation to the framework:

Statements in dialogues for immanent reasoning

Dialogues are games of giving and asking for reasons; yet in the standard dialogical framework, the reasons for each statement are left implicit and do not appear in the notation of the stament: we have statements of the form $\mathbf{X} ! A$ for instance where A is an elementary proposition. The framework of dialogues for immanent reasoning allows to have explicitly the reason for making a statement, statements then have the form $\mathbf{X}\, a : A$ for instance where a is the (local) reason **X** has for stating the proposition A. But even in dialogues for immanent reasoning, all reasons are not always provided, and sometimes statements have only implicit reasons for bringing the proposition forward, taking then the same form as in the standard dialogical framework: $\mathbf{X} ! A$. Notice that when (local) reasons are not explicit, an exclamation mark is added before the proposition: the statement then has an implicit reason for being made.

A statement is thus both a proposition and its local reason, but this reason may be left implicit, requiring then the use of the exclamation mark.

Adding concessions

In the context of the dialogical conception of CTT we also have statements of the form

$$\mathbf{X} \; ! \; \pi(x_1, \ldots, x_n) \; [x_i : A_i]$$

where "π" stands for some statement in which $(x_1, \ldots, x_n)$ ocurs, and where $[x_i : A_i]$ stands for some condition under which the statement $\pi(x_1, \ldots, x_n)$ has been brought forward. Thus, the statement reads:

X states that $\pi(x_1, \ldots, x_n)$ under the condition that the antagonist concedes $x_i : A_i$.

We call *required concessions* the statements of the form $[x_i : A_i]$ that condition a claim. When the statement is challenged, the antagonist is accepting, through his own challenge, to bring such concessions forward. The concessions of the thesis, if any, are called *initial concessions*. Initial concessions can include formation statements such as A : *prop, B* : *prop,* for the thesis, $A \supset B$: *prop*.

VI.3.1Formation rules for local reasons: an informal overview

It is presupposed in standard dialogical systems that the players use well-formed formulas (*wff*). The well formation can be checked at will, but only with the usual meta reasoning by which one checks that the formula does indeed observe the definition of a *wff*. We want to enrich our CTT-based dialogical framework by allowing players themselves to first enquire on the formation of the components of a statement within a play. We thus start with dialogical rules explaining the formation of statements involving logical constants (the formation of *elementary* propositions is governed by the Socratic rule, see the discussion above on material truth). In this way, the well formation of the thesis can be examined by the Opponent before running the actual dialogue: as soon as she challenges it, she is *de facto* accepting the thesis to be well formed (the most obvious case being the challenge of the implication, where she has to state the antecedent and thus explicitly endorse it). The Opponent can ask for the formation of the thesis before launching her first challenge; defending the formation of his thesis might for instance bring the Proponent to state that the thesis is a proposition, provided, say, that *A is a* ***set*** is conceded; the Opponent might then concede that *A is a* ***set***, but only after the constitution of *A* has been established, though if this were the case, we would be considering the constitution of an elementary statement, which is a material consideration, not a formal one.

These rules for the *formation* of statements with logical constants are also particle rules which are added to the set of particle rules determining the local meaning of logical constants (called synthesis and analysis of local reasons in the framework of dialogues for immanent reasoning).

These considerations yield the following condensed presentation of the logical constants (plus *falsum*), in which "$\mathcal{K}$" in $A\mathcal{K}B$ " expresses a connective, and "$\mathcal{Q}$" in "$(\mathcal{Q}x : A)\; B(x)$ " expresses a quantifier.

Table 12: Formation rules, condensed presentation

	Connective	**Quantifier**	**Falsum**
Move	**X** $A\mathfrak{K}B$: *prop*	**X** $(\mathfrak{Q}x : A)\ B(x)$: *prop*	**X** $\bot$: *prop*
Challenge	**Y** $?_{F\mathfrak{K}\,1}$ and/or **Y** $?_{F\mathfrak{K}2}$	**Y** $?_{F\mathfrak{Q}1}$ and/or **Y** $?_{F\mathfrak{Q}2}$	—
Defence	**X** A : *prop* (resp.) **X** B : *prop*	**X** A : *set* (resp.) **X** $B(x)$: *prop* $(x : A)$	—

Because of the *no entity without type* principle, it seems at first glance that we should specify the type of these actions during a dialogue by adding the type "*formation-request*". But as it turns out, we should not: an expression such as "$?_F$: *formation-request*" is a judgement that some action $?_F$ is a formation-request, which should not be confused with the actual act of requesting. We also consider that the force symbol $?_F$ makes the type explicit.

VI.3.2Synthesis of local reasons

The synthesis rules of local reasons determine how to produce a local reason for a statement; they include rules of interaction indicating how to produce the local reason that is required by the proposition (or set) in play, that is, they indicate what kind of dialogic action—what kind of move—must be carried out, by whom (challenger or defender), and what reason must be brought forward.

Implication

For instance, the synthesis rule of a local reason for the implication $A \supset B$ stated by player **X** indicates:

i. that the challenger **Y** must state the antecedent (while providing a local reason for it): **Y** $p_1 : A$.[75]
ii. that the defender **X** must respond to the challenge by stating the consequent (with its corresponding local reason): **X** $p_2 : B$.

In other words, the rules for the synthesis of a local reason for implication are as follows:

Table 13: Synthesis of a local reason for implication

	Implication
Move	**X** ! $A \supset B$
Challenge	**Y** $p_1 : A$
Defence	**X** $p_2 : B$

Notice that the initial statement (**X** ! $A \supset B$) *does not* display a local reason for the claim the the implication holds: player **X** simply states that he has some reason supporting the claim. We express such kind of move by adding an *exclamation mark*

---

[75] This notation is a variant of the one used by (Keiff, 2004).

before the proposition. The further dialogical actions indicate the moves required for producing a local reason in defence of the initial claim.

Conjunction

The synthesis rule for the conjunction is straightforward:

Table 14: synthesis of a local reason for conjunction

	Conjunction
Move	**X** ! $A \wedge B$
Challenge	**Y** ? $L^{\wedge}$ or **Y** ? $R^{\wedge}$
Defence	**X** $p_1 : A$ (resp.) **X** $p_2 : B$

Disjunction

For disjunction, as we know from the standard rules, it is the defender who will choose which side he wishes to defend: the challenge consists in requesting of the defender that he chooses which side he will be defending:

Table 15: Synthesis of a local reason for disjunction

	Disjunction
Move	**X** ! $A \vee B$
Challenge	**Y** $?_{\vee}$
Defence	**X** $p_1 : A$ or **X** $p_2 : B$

The general structure for the synthesis of local reasons

More generally, the rules for the synthesis of a local reason for a constant $\mathcal{K}$ is determined by the following triplet:

Table 16: general structure for the synthesis of a local reason for a constant

	A constant $\mathcal{K}$	**Implication**	**Conjunction**	**Disjunction**
Move	**X** ! $\varphi[\mathcal{K}]$ ***X** claims that* ϕ	**X** ! $A \supset B$	**X** ! $A \wedge B$	**X** ! $A \vee B$
Challenge	***Y** asks for the reason backing such a claim*	**Y** $p_1 : A$	**Y** ? $L^{\wedge}$ or **Y** ? $R^{\wedge}$	**Y** $?_{\vee}$
Defence	**X** $p : \varphi[\mathcal{K}]$ ***X** states the local reason p for $\varphi[\mathcal{K}]$ according to the rules for the synthesis of local reasons prescribed for* $\mathcal{K}$	**X** $p_2 : B$	**X** $p_1 : A$ (resp.) **X** $p_2 : B$	**X** $p_1 : A$ or **X** $p_2 : B$

VI.3.3Analysis of local reasons

Apart from the rules for the synthesis of local reasons, we need rules that indicate how to parse a complex local reason into its elements: this is the *analysis* of local reasons. In order to deal with the complexity of these local reasons and formulate general rules for the analysis of local reasons (at the play level), we introduce certain operators that we call *instructions*, such as $L^{\vee}(p)$ or $R^{\wedge}(p)$.

Approaching the analysis rules for local reasons

Let us introduce these instructions and the analysis of local reasons with an example: player **X** states the implication $(A \wedge B) \supset A$. According to the rule for the synthesis of local reasons for an implication, we obtain the following:

Move	**X** ! $(A \wedge B) \supset B$
Challenge	**Y** $p_1 : A \wedge B$

Recall that the synthesis rule prescribes that **X** must now provide a local reason for the consequent; but instead of defending his implication (with **X** $p_2 : B$ for instance), **X** can choose to parse the reason p_1 provided by **Y** in order to force **Y** to provide a local reason for the right-hand side of the conjunction that **X** will then be able to copy; in other words, **X** can force **Y** to provide the local reason for B out of the local reason p_1 for the antecedent $A \wedge B$ of the initial implication. The analysis rules prescribe how to carry out such a parsing of the statement by using *instructions*. The rule for the analysis of a local reason for the conjunction $p_1 : A \wedge B$ will thus indicate that its defence includes expressions such as

- the left instruction for the conjunction, written $L^{\wedge}(p_1)$, and
- the right instruction for the conjunction, written $R^{\wedge}(p_1)$.

These instructions can be informally understood as carrying out the following step: for the defence of the conjunction $p_1 : A \wedge B$ separate the local reason p_1 in its left (or right) component so that this component can be adduced in defence of the left (or right) side of the conjunction.

Here is a play with local reasons for the thesis $(A \wedge B) \supset B$ using instructions:

O			**P**		
				$! \ (A \wedge B) \supset B$	0
1	$m := 1$			$n := 2$	2
3	$p_1 : A \wedge B$	0		$R^{\wedge}(p_1) : B$	6
5	$R^{\wedge}(p_1) : B$		3	$?R^{\wedge}$	4

P wins.

In this play, **P** uses the analysis of local reasons for conjunction in order to force **O** to state $R^{\wedge}(p_1) : B$, that is to provide a local reason[76] for the elementary statement B; **P** can then copy that local reason in order to back his statement B, the consequent of his initial implication. With these local reasons, we explicitly have in the object-language the reasons that are given and asked for and which constitute the essence of an argumentative dialogue.

---

[76] Speaking of local reasons is a little premature at this stage, since only instructions are provided and not actual local reasons; but the purpose is here to give the general idea of local reasons, and instructions are meant to be resolved into proper local reasons, which requires only an extra step.

The general structure for the analysis rules of local reasons

	Move	Challenge	Defence
Conjunction	**X** $p: A \wedge B$	**Y** ? $L^{\wedge}$ or **Y** ? $R^{\wedge}$	**X** $L^{\wedge}(p)^{X}: A$ (resp.) **X** $R^{\wedge}(p)^{X}: B$
Disjunction	**X** $p: A \vee B$	**Y** $?_{\vee}$	**X** $L^{\vee}(p)^{X}: A$ or **X** $R^{\vee}(p)^{X}: B$
Implication	**X** $p: A \supset B$	**Y** $L^{\supset}(p)^{Y}: A$	**X** $R^{\supset}(p)^{X}: B$

The superscripts with the player label indicate which player is entitled to decide how to resolve the instruction, that is, to decide which local reason to bring forward when carrying out the instruction.

Interaction procedures embedded in instructions

Carrying out the prescriptions indicated by instructions require the following three interaction-procedures:

1. *Resolution of instructions*: this procedure determines how to carry out the instructions prescribed by the rules of analysis and thus provide an actual local reason.
2. *Substitution of instructions*: this procedure ensures the following; once a given instruction has been carried out through the choice of a local reason, say *b*, then every time the same instruction occurs, it will always be substituted by the same local reason *b*.
3. *Application of the Socratic rule*: the Socratic rule prescribes how to constitute equalities out of the resolution and substitution of instructions, linking synthesis and analysis together.

Let us discuss how these rules interact and how they lead to the main thesis of this study, namely that immanent reasoning is equality in action.

VI.4 From Reasons to Equality

As we have already discussed to some extent one of the most salient features of dialogical logic is the so-called, *Socratic rule* (or Copy-cat rule in the standard—that is, non-CTT—context), establishing that the Proponent can play an elementary proposition only if the Opponent has played it previously.

The Socratic rule is a characteristic feature of the dialogical approach: other game-based approaches do not have it. With this rule the dialogical framework comes with an internal account of elementary propositions: an account in terms of interaction only, without depending on metalogical meaning explanations for the non-logical vocabulary. More prominently, this means that the dialogical account does not rely—contrary to Hintikka's GTS games—on the model-theoretical approach to meaning for elementary propositions.

The rule has a clear Platonist and Aristotelian origin and sets the terms for what it is to carry out a *formal argument*: see for instance Plato's *Gorgias* (472b-c). We can sum up the underlying idea with the following statement:

there is no better grounding of an assertion within an argument than indicating that it has been already conceded by the Opponent or that it follows from these concessions.[77]

What should be stressed here are the following two points:

1. formality is understood as a kind of *interaction*; and
2. formal reasoning *should not* be understood here as devoid of content and reduced to purely syntactic moves.

Both points are important in order to understand the criticism often raised against formal reasoning in general, and in logic in particular. It is only quite late in the history of philosophy that formal reasoning has been reduced to syntactic manipulation—presumably the first explicit occurrence of the syntactic view of logic is Leibniz's "pensée aveugle" (though Leibniz's notion was not a reductive one). Plato and Aristotle's notion of formal reasoning is neither "static" nor "empty of meaning".[78] In the Ancient Greek tradition logic emerged from an approach of assertions in which meaning and justification result from what has been brought forward during argumentative interaction. According to this view, dialogical interaction is constitutive of meaning.

Some former interpretations of standard dialogical logic did understand formal plays in a purely syntactic manner. The reason for this is that the standard version of the framework is not equipped to express meaning at the object-language level: there is no way of asking and giving reasons for elementary propositions. As a consequence, the standard formulation simply relies on a syntactic understanding of *Copy-cat moves,* that is, moves entitling **P** to copy the elementary propositions brought forward by **O**, regardless of its content.

The dialogical approach to CTT (dialogues for immanent reasoning) however provides a fine-grain study of the contentual aspects involved in formal plays, much finer than the one provided by the standard dialogical framework. In dialogues for immanent reasoning which we are now presenting, a statement is constituted both by a proposition and by the (local) reason brought forward in defence of the claim that the proposition holds. In formal plays not only is the Proponent allowed to copy an elementary proposition stated by the Opponent, as in the standard framework, but he is also allowed to adduce in defence of that proposition the *same* local reason brought forward by the Opponent when she defended that same proposition. Thus immanent reasoning and equality in action are intimately linked. In other words, a formal play displays the *roots of the content* of an elementary proposition, and *not* a syntactic manipulation of that proposition.

Statements of definitional equality emerge precisely at this point. In particular reflexivity statements such as

$$p = p : A$$

express from the dialogical point of view the fact that if **O** states the elementary proposition A, then **P** can do the same, that is, play the same move and do it on the same grounds which provide the meaning and justification of A, namely p.

---

[77] Recent work (Crubellier, 2014, pp. 11-40) and (Crubellier, Marion, McConaughey, & Rahman, 2018) claim that this rule is central to the interpretation of dialectic as the core of Aristotle's logic. Neither Ian Lukasiewicz's (1957) famous reconstruction of Aristotle's syllogistic, nor the Natural deduction approach of Kurt Ebbinghaus (1964) and John Corcoran (1974) deploy this rule, but Marion and Rückert (2015) showed that this rule displays Aristotle's view on universal quantification.

[78] (Smith R. , 1982; Smith R. , 1989), (Crubellier, 2014).

These remarks provide an insight only on simple forms of equality and barely touch upon the finer-grain distinctions discussed above; we will be moving to these by means of a concrete example in which we show, rather informally, how the combination of the processes of analysis, synthesis, and resolution of instructions lead to equality statements.

Example
Assume that the Proponent brings forward the thesis $(A \wedge B) \supset (B \wedge A)$:

O			P		
				$!\,(A \wedge B) \supset (B \wedge A)$	0

Both players then choose their repetition ranks:

O			P		
				$!\,(A \wedge B) \supset (B \wedge A)$	0
1	$m := 1$			$n := 2$	2

O must now challenge the implication if she accepts to enter into the discussion. The rule for the synthesis of a local reason for implication (provided above) stipulates that in order to challenge the thesis, **O** must state the antecedent *and provide a local reason for it*:

Synthesis of a local reason for conjunction

O			P		
				$!\,(A \wedge B) \supset (B \wedge A)$	0
1	$m := 1$			$n := 2$	2
3	$p : A \wedge B$	0			

According to the same synthesis-rule **P** must now state the consequent, which he is allowed to do because the consequent is not elementary:

O			P		
				$!\,(A \wedge B) \supset (B \wedge A)$	0
1	$m := 1$			$n := 2$	2
3	$p : A \wedge B$	0		$q : B \wedge A$	4

The Opponent launches her challenge asking for the left component of the local reason *q* provided by **P**, an application of the rule for the *analysis* of a local reason for a conjunction described above.

Analysis of a local reason for conjunction

O			P		
				$!\,(A \wedge B) \supset (B \wedge A)$	0
1	$m := 1$			$n := 2$	2
3	$p : A \wedge B$	0		$q : B \wedge A$	4
5	$?\,L^{\wedge}$	4			

Assume that **P** responds immediately to this challenge:

O	P

				$! (A \wedge B) \supset (B \wedge A)$	0
1	$m := 1$			$n := 2$	2
3	$p : A \wedge B$	0		$q : B \wedge A$	4
5	$? L^\wedge$	4		$L^\wedge(q) : B$	6

O will now ask for the *resolution of the instruction*:

O			P		
				$! (A \wedge B) \supset (B \wedge A)$	0
1	$m := 1$			$n := 2$	2
3	$p : A \wedge B$	0		$q : B \wedge A$	4
5	$? L^\wedge$	4		$L^\wedge(q) : B$	6
7	$? \ldots / L^\wedge(q)$	6			

Resolution of an instruction

In this move 7, **O** is asking **P** to carry out the instruction $L^\wedge(q)$ by bringing forward the local reason of his choice. The act of choosing such a reason and replacing the instruction for it is called *resolving the instruction*.

In this case, resolving the instruction will lead **P** to bring forward an *elementary statement*—that is, a statement in which *both* the local reason and the proposition are elementary, which falls under the restriction of the Socratic rule. The idea for **P** then is to postpone his answer to the challenge launched with move 7 and to force **O** to choose a local reason first so as to copy it in his answer to the challenge. This yields a further application of the *analysis rule* for the conjunction:

O			P		
				$! (A \wedge B) \supset (B \wedge A)$	0
1	$m := 1$			$n := 2$	2
3	$p : A \wedge B$	0		$q : B \wedge A$	4
5	$? L^\wedge$	4		$L^\wedge(q) : B$	6
7	$? \ldots / L^\wedge(q)$	6		$b : B$	12
9	$R^\wedge(p) : B$		3	$? R^\wedge$	8
11	$b : B$		9	$? \ldots / R^\wedge(p)$	10

O responds according to the *analysis rule*

P launches his challenge asking for the right side of the concession move 3

O responds to the challenge by choosing the local reason *b*

P asks **O** to resolve the instruction by providing a local reason

P wins.

Move 11 thus provides **P** with the information he needed: he can then copy **O**'s choice to answer the challenge she launched at move 7.

Note: It should be clear that a similar end will come about if ***O*** *starts by challenging the right component of the conjunction statement, instead of challenging the left component.*

Analysis of the example

Let us now go deeper in the analysis of the example and make explicit what happened during the play:

When **O** resolves $R^{\wedge}(p)$ with the local reason b (for instance) and **P** resolves the instruction $L^{\wedge}(q)$ with the same local reason, then **P** is not only stating $b : B$ but he is doing this by choosing b as local reason for B, that is, by choosing *exactly the same* local reason as **O** for the resolution of $R^{\wedge}(p)$.

Let us assume that **O** can ask **P** to make his choice for a given local reason explicit. **P** would then answer that his choice for his local reason depends on **O**'s own choice: he simply copied what **O** considered to be a local reason for B, that is $R^{\wedge}(p)^{O} = b{:}B$. The application of the Socratic rule yields in this respect definitional equality. This rule prescribes the following response to a challenge on an elementary local reason:

> When **O** challenges an elementary statement of **P** such as $b : B$, **P** must be able to bring forward a definitional equality such as **P** $R^{\wedge}(p) = b : B$.

Which reads:

> **P** grounds his choice of the local reason b for the proposition B in **O**'s resolution of the instruction $R^{\wedge}(p)$. At the very end **P**'s choice is the *same local reason* brought forward by **O** for the same proposition B.

In other words, the definitional equality $R^{\wedge}(p)^{O} = b{:}B$ that provides content to B makes it explicit at the object-language level that an application of the Socratic rule has been initiated and achieved by means of dialogical interaction.

The development of a dialogue determined by immanent reasoning thus includes four distinct stages:

1. applying the rules of synthesis to the thesis;
2. applying the rules of analysis;
3. launching the Resolution and Substitution of instructions;
4. applying the Socratic rule.

We can then add a fifth stage, which will be presented in chapter IX:

5. Producing the strategic reason.

While the first two steps involve local meaning, step 3 concerns global meaning and step 4 requires describing how to produce a winning strategy. Now that the general idea of local reasons has been provided, we will present in the next chapter all the rules together, according to their level of meaning.

VII. THE DIALOGICAL ROOTS OF EQUALITY: DIALOGUES FOR IMMANENT REASONING

In this chapter we will spell out all the relevant rules of dialogues for immanent reasoning, that is, the dialogical framework incorporating features of Constructive Type Theory—a dialogical framework making the players' reasons for asserting a proposition explicit. The rules can be divided, just as in the standard framework, into rules determining local meaning and rules determining global meaning. These include:

1. Concerning *local meaning* (section VII.1):
    a. formation rules (p. 105);
    b. rules for the synthesis of local reasons (p. 108); and
    c. rules for the analysis of local reasons (p. 109).
2. Concerning *global meaning*, we have the following (structural) rules (section VII.2):
    a. rules for the resolution of instructions (p. 112);
    b. rules for the substitution of instructions (p. 113);
    c. equality rules determined by the application of the Socratic rules (p. 113); and
    d. rules for the transmission of equality (p. 115).

We will be presenting these rules in this order in the next two sections, along with the adaptation of the other structural rules to dialogues for immanent reasoning in the second section. The following section (VII.4) provides a series of exercises for the play level.

A fourth section (VII.5) will deal with the strategic level of dialogues for immanent reasoning, and will be followed with solved exercises (VII.6).

A final section (VII.7) will introduce *strategic reasons*, a fundamental notion for dialogues for immanent reasoning, and will be accompanied with two examples showing how to build these reasons of a special kind.

VII.1 Local meaning in dialogues for immanent reasoning

VII.1.1 The formation rules

Formation rules for logical constants and falsum

The formation rules for *logical constants* and for *falsum* are given in the following table. Notice that a statement '$\bot$: **prop**' cannot be challenged; this is the dialogical account for falsum '$\bot$' being by definition a proposition.

Table 17: Formation rules

	Move	Challenge	Defence
Conjunction	**X** $A \wedge B : \boldsymbol{prop}$	**Y** $?F_{\wedge 1}$ or **Y** $?F_{\wedge 2}$	**X** $A : \boldsymbol{prop}$ (resp.) **X** $B : \boldsymbol{prop}$
Disjunction	**X** $A \vee B : \boldsymbol{prop}$	**Y** $?F_{\vee 1}$ or **Y** $?F_{\vee 2}$	**X** $A : \boldsymbol{prop}$ (resp.) **X** $B : \boldsymbol{prop}$

Implication	**X** $A \supset B\colon \boldsymbol{prop}$	**Y** ? $F_{\supset 1}$ or **Y** ? $F_{\supset 2}$	**X** $A\colon \boldsymbol{prop}$ (resp.) **X** $B\colon \boldsymbol{prop}$
Universal quantification	**X** $(\forall x\colon A)B(x)\colon \boldsymbol{prop}$	**Y** ? $F_{\forall 1}$ or **Y** ? $F_{\forall 2}$	**X** $A\colon \boldsymbol{set}$ (resp.) **X** $B(x)\colon \boldsymbol{prop}[x\colon A]$
Existential quantification	**X** $(\exists x\colon A)B(x)\colon \boldsymbol{prop}$	**Y** ? $F_{\exists 1}$ or **Y** ? $F_{\exists 2}$	**X** $A\colon \boldsymbol{set}$ (resp.) **X** $B(x)\colon \boldsymbol{prop}[x\colon A]$
Subset separation	**X** $\{x : A \mid B(x)\}\colon \boldsymbol{prop}$	**Y** ? F_1 or **Y** ? F_2	**X** $A\colon \boldsymbol{set}$ (resp.) **X** $B(x)\colon \boldsymbol{prop}[x\colon A]$
Falsum	**X** $\bot\colon \boldsymbol{prop}$	—	—

The substitution rule within dependent statements

The following rule is not really a formation-rule but is very useful while applying formation rules where one statement is dependent upon the other such as $B(x)\colon \boldsymbol{prop}[x\colon A]$.[79]

Table 18: Substitution rule within dependent statements (subst-D)

	Move	**Challenge**	**Defence**
Subst-D	**X** $\pi(x_1, \dots, x_n)[x_i\colon A_i]$	**Y** $\tau_1\colon A_1, \dots, \tau_n\colon A_n$	**X** $\pi(\tau_1, \dots, \tau_n)$

In the formulation of this rule, "π" is a statement and "τ_i" is a local reason of the form either $a_i\colon A_i$ or $x_i : A_i$.

A particular case of the application of Subst-D is when the challenger simply chooses the same local reasons as those occurring in the concession of the initial statement. This is particularly useful in the case of formation plays:

Example of a formation-play

Here is an example of a formation play with some explanation. The standard development rules are enough to understand the following plays (see the rules provided in chapter III or IV).

In this example, the Opponent provides initial concession before the Proponent states his thesis. Thus the Proponent's thesis is

$$(\forall x : A)(B(x) \supset C(x)) : \mathbf{prop}$$

given these three provisos that appear as initial concessions by the Opponent:

A : **set**,
$B(x)$: **prop** $[x : A]$
and $C(x)$: **prop** $[x : A]$,

---

[79] This rule is an expression at the level of plays of the rule for the substitution of variables in a hypothetical judgement. See (Martin-Löf, 1984, pp. 9-11).

This yields the following play:

Play 10: formation-play with initial concessions: first decision-option of **O**

O			**P**		
0.1	A : **set**				
0.2	$B(x)$: **prop** $[x : A]$				
0.3	$C(x)$: **prop** $[x : A]$			$(\forall x : A)\ B(x) \supset C(x)$: **prop**	0
1	$m := 1$			$n := 2$	2
3	$?\, F_{\forall 1}$	0		A : **set**	4

P wins.

Explanation:

- 0.1 to 0.3: **O** concedes that *A* is a **set** and that *B*(*x*) and *C*(*x*) are propositions provided *x* is an element of *A*.
- Move 0: **P** states that the main sentence, universally quantified, is a proposition (under the concessions made by **O**).
- Moves 1 and 2: the players choose their repetition ranks.
- Move 3: **O** challenges the thesis by asking the left-hand part as specified by the formation rule for universal quantification.
- Move 4: **P** responds by stating that *A* is a **set**. This has already been granted with the concession 0.1 so even if **O** were to challenge this statement the Proponent could refer to her initial concession.

This dialogue obviously does not cover all the aspects related to the formation of

$(\forall x : A)\ B(x) \supset C(x)$: **prop**.

Notice however that the formation rules allow an alternative move for the Opponent's move 3,[80] so that **P** has another possible course of action, dealt with in the following play.

Play 11: formation-play with initial concessions: second decision-option of **O**

O			**P**		
0.1	A : **set**				
0.2	$B(x)$: **prop** $[x : A]$				
0.3	$C(x)$: **prop** $[x : A]$			$(\forall x : A)\ B(x) \supset C(x)$: **prop**	0
1	$m := 1$			$n := 2$	2
3	$?\, F_{\forall 2}$	0		$B(x) \supset C(x)$: **prop** $[x : A]$	4
5	$x : A$	4		$B(x) \supset C(x)$: **prop**	6
7	$?\, F_{\supset 1}$	6		$B(x)$: **prop**	10

---

[80] As a matter of fact, increasing her repetition rank would allow **O** to play the two alternatives for move 3 within a single play. But increasing the Opponent's rank usually yields redundancies (Clerbout, 2014a; 2014b) making things harder to understand for readers not familiar with the dialogical approach; hence our choice to divide the example into different simple plays.

9	$B(x)$: **prop**		0.2	$x : A$	8

P wins.

Explanation:

The second play starts like the first one until move 2. Then:

- Move 3: this time **O** challenges the thesis by asking for the right-hand part.
- Move 4: **P** responds, stating that $B(x) \supset C(x)$ is a proposition, provided that x : A.
- Move 5: **O** challenges the preceding move by granting the proviso and asking **P** to respond (this kind of move is governed by a Subst-D rule).
- Move 6: **P** responds by stating that $\mathrm{B}(x) \supset \mathrm{C}(x)$ is a proposition.
- Move 7: **O** challenges move 6 by asking the left-hand part, as specified by the formation rule for material implication.
  To defend against this challenge, **P** needs to make an elementary move. But since **O** has not played it yet, **P** cannot defend it at this point. Thus:
- Move 8: **P** launches a counterattack against initial concession 0.2 by granting the proviso $x : A$ (that has already been conceded by **O** in move 5), making use of the same kind of statement-substitution (Subst-D) rule deployed in move 5.
- Move 9: **O** answers to move 8 and states that $B(x)$ is a proposition.
- Move 10: **P** can now defend the challenge initiated with move 7 and win this dialogue.

Once again, there is another possible choice for the Opponent because of her move 7: she could ask the right-hand part. This would yield a dialogue similar to the one above except that the last moves would be about $C(x)$ instead of $B(x)$.

Concluding on the formation-play example:

By displaying these various possibilities for the Opponent, we have entered the strategic level. This is the level at which the question of the good formation of the thesis gets a definitive answer, depending on whether the Proponent can always win—that is, whether he has a winning strategy. The basic notions related to this level of strategies are to be found in our presentation of standard dialogical logic (see section III.5 or chapter V.1); section VII.5 below will deal with local reasons in this strategy level in the framework of dialogues for immanent reasoning and section VII.7 will introduce strategic reasons.

VII.1.2 The rules for local reasons: synthesis and analysis

Now that the dialogical account of formation rules has been clarified, we may further develop our analysis of plays by introducing local reasons. Let us do so by providing the rules that prescribe the synthesis and analysis of local reasons. For more details on each rule, see section VI.3.

Table 19: synthesis rules for local reasons

	Move	**Challenge**	**Defence**

Conjunction	**X** ! $A \wedge B$	**Y** ? $L^{\wedge}$ or **Y** ? $R^{\wedge}$	**X** $p_1: A$ (resp.) **X** $p_2: B$
Existential quantifiation	**X** ! $(\exists x : A)B(x)$	**Y** ? $L^{\exists}$ or **Y** ? $R^{\exists}$	**X** $p_1: A$ (resp.) **X** $p_2: B(p_1)$
Subset separation	**X** ! $\{x : A \mid B(x)\}$	**Y** ? L or **Y** ? R	**X** $p_1: A$ (resp.) **X** $p_2: B(p_1)$
Disjunction	**X** ! $A \vee B$	**Y** ?$^{\vee}$	**X** $p_1: A$ or **X** $p_2: B$
Implication	**X** ! $A \supset B$	**Y** $p_1: A$	**X** $p_2: B$
Universal quantification	**X** ! $(\forall x: A)B(x)$	**Y** $p_1: A$	**X** $p_2: B(p_1)$
Negation	**X** ! $\neg A$ Also expressed as **X** ! $A \supset \bot$	**Y** $p_1: A$	**X** ! $\bot$ (**X** gives up[81])

Table 20: analysis rules for local reasons

	Move	Challenge	Defence
Conjunction	**X** $p: A \wedge B$	**Y** ? $L^{\wedge}$ or **Y** ? $R^{\wedge}$	**X** $L^{\wedge}(p)^{X}: A$ (resp.) **X** $R^{\wedge}(p)^{X}: B$
Existential quantifiation	**X** $p: (\exists x: A)B(x)$	**Y** ? $L^{\exists}$ or **Y** ? $R^{\exists}$	**X** $L^{\exists}(p)^{X}: A$ (resp.) **X** $R^{\exists}(p)^{X}: B(L^{\exists}(p)^{X})$
Subset separation	**X** $p: \{x : A \mid B(x)\}$	**Y** ? L or **Y** ? R	**X** $L^{\{\dots\}}(p)^{X}: A$ (resp.) **X** $R^{\wedge}(p)^{X}: B(L^{\{\dots\}}(p)^{X})$
Disjunction	**X** $p: A \vee B$	**Y** ?$^{\vee}$	**X** $L^{\vee}(p)^{X}: A$ or **X** $R^{\vee}(p)^{X}: B$
Implication	**X** $p: A \supset B$	**Y** $L^{\supset}(p)^{Y}: A$	**X** $R^{\supset}(p)^{X}: B$
Universal quantification	**X** $p: (\forall x: A)B(x)$	**Y** $L^{\forall}(p)^{Y}: A$	**X** $R^{\forall}(p)^{X}: B(L^{\forall}(p)^{Y})$
Negation	**X** $p: \neg A$ Also expressed as **X** $p: A \supset \bot$	**Y** $L^{\neg}(p)^{Y}: A$ **Y** $L^{\supset}(p)^{Y}: A$	**X** $R^{\neg}(p)^{X}: \bot$ **X** $R^{\supset}(p)^{X}: \bot$

---

[81] The reading of stating bottom as giving up stems from (Keiff, 2007).

Anaphoric instructions: dealing with cases of anaphora

One of the most salient features of the CTT framework is that it contains the means to deal with cases of anaphora,[82] and some of the demonstrations for the exercises of chapter II required them, as in exercise 4 formalizing Barbara in CTT (p. 44).

Recall the formalization of exercise 4, where the projection **fst**(z) can be seen as the tail of the anaphora whose head is z:

$$\begin{array}{ll} (\forall z : (\exists x : D)A)B[\mathbf{fst}(z)] \text{ true} & \textit{premise 1} \\ (\forall z : (\exists x : D)B)C[\mathbf{fst}(z)] \text{ true} & \textit{premise 2} \\ \hline (\forall z : (\exists x : D)A)C[\mathbf{fst}(z)] \text{ true} & \textit{conclusion} \end{array}$$

In dialogues for immanent reasoning, when a local reason has been made explicit, this kind of anaphoric expression is formalized through *instructions*, which provides a further reason for introducing them. For example if a is the local reason for the first premise we have

$$\mathbf{P}\ p : (\forall z : (\exists x : D)A(x))B(L^{\exists}(L^{\forall}(p)^{\mathbf{O}}))$$

However, since the thesis of a play does not bear an explicit local reason (we use the exclamation mark to indicate there is an implicit one), it is possible for a statement to be bereft of an explicit local reason. When there is no explicit local reason for a statement using anaphora, we cannot bind the instruction $L^{\forall}(p)^{\mathbf{O}}$ to a local reason p. We thus have something like this, with a blank space instead of the anaphoric local reason:

$$\mathbf{P}\ !\ (\forall z : (\exists x : D)A(x))B(L^{\exists}(L^{\forall}(\ \ )^{\mathbf{O}}))$$

But this blank stage can be circumvented: the challenge on the universal quantifier will yield the required local reason: **O** will provide $a : (\exists x : D)A(x)$, which is the local reason for z. We can therefore bind the instruction on the missing local reason with the corresponding variable—z in this case—and write

$$\mathbf{P}!\ (\forall z : (\exists x : D)A(x))B(L^{\exists}(L^{\forall}(z)^{\mathbf{O}}))$$

We call this kind of instruction, *Anaphoric instructions*. For the substitution of Anaphoric instructions the following two cases are to be distinguished**:**

Substitution of Anaphoric Instructions 1

Given some Anaphoric instruction such as $L^{\forall}(z)^{\mathbf{Y}}$, once the quantifier $(\forall z : A)B(\dots)$ has been challenged by the statement $a : A$, the occurrence of $L^{\forall}(z)^{\mathbf{Y}}$ can be substituted by a. The same applies to other instructions.

In our example we obtain:

$$\begin{array}{c} \mathbf{P}!\ (\forall z : (\exists x : D)A(x))B(L^{\exists}(L^{\forall}(z)^{\mathbf{O}})) \\ \mathbf{O}\ a : (\exists x : D)A(x) \\ \mathbf{P}\ b : B(L^{\exists}(L^{\forall}(z)^{\mathbf{O}})) \\ \mathbf{O}\ ?\ a\ /\ L^{\forall}(z)^{\mathbf{O}} \\ \mathbf{P}\ b : B(L^{\exists}(a)) \\ \dots \end{array}$$

---

[82] See (Sundholm, 1986, pp. 501-503) and (Ranta, 1994, pp. 77-99).

Substitution of Anaphoric Instructions 2

Given some Anaphoric instruction such as $L^{\forall}(z)^{\mathbf{Y}}$, once the instruction $L^{\forall}(c)$—resulting from an attack on the universal $\forall z : \varphi$—has been resolved with $a : \varphi$, then any occurrence of $L^{\forall}(z)^{\mathbf{Y}}$ can be substituted by a. The same applies to other instructions.

VII.2 Global Meaning in dialogues for immanent reasoning

We here provide the structural rules for dialogues for immanent reasoning, which determine the global meaning in such a framework. They are for the most part similar in principle to the precedent logical framework for dialogues; the rules concerning instructions are an addition for dialogues for immanent reasoning.

The structural rules for formal dialogues (as opposed to material dialogues; see chapter X) are of three kinds: starting rules (SR0 and SR2i), the Socratic rules that are player dependent rules for elementary statements (SR5), and global rules that are player independent procedural rules (SR1, 2ii, 3-4 and 6-7).

VII.2.1 Structural Rules

SR0: Starting rule

The start of a *formal dialogue of immanent reasoning* is a move where **P** states the *thesis*. The thesis can be stated under the condition that **O** commits herself to certain other statements called *initial concessions*; in this case the thesis has the form ! $A\ [B_1, \ldots, B_n]$, where A is a statement with implicit local reason and $B_1, \ldots, B_n$ are statements with or without implicit local reasons.

A dialogue with a thesis proposed under some conditions starts if and only if **O** accepts these conditions. **O** accepts the conditions by stating the *initial concessions* in moves numbered 0.1, …, 0.*n* before choosing the repetition ranks.

After having stated the thesis (and the initial concessions, if any), each player chooses in turn a positive integer called the *repetition rank* which determines the upper boundary for the number of attacks and of defences each player can make in reaction to each move during the play.

SR1: Development rule

The Development rule depends on what kind of logic is chosen: if the game uses intuitionistic logic, then it is SR1i that should be used; but if classical logic is used, then SR1c must be used.

SR1i: Intuitionistic Development rule, or Last Duty First

Players play one move alternately. Any move after the choice of repetition ranks is either an attack or a defence according to the rules of formation, of synthesis, and of analysis, and in accordance with the rest of the structural rules.

If the logical constant occurring in the thesis is not recorded by the table for local meaning, then either it must be introduced by a nominal definition, or the table for local meaning needs to be enriched with the new expression.[83]

Players can answer only against the *last non-answered* challenge by the adversary.

---

[83] If the logical constant occurring in the thesis is not recorded by the table for local meaning, then either it must be introduced by a nominal definition based on some logical constant already present in the particle rules, or the table for local meaning needs to be enriched with the new expression.

Note: This structural rule is known as the Last Duty First condition, and makes dialogical games suitable for intuitionistic logic, hence the name of this rule.

SR1c: Classical Development rule

Players play one move alternately. Any move after the choice of repetition ranks is either an attack or a defence according to the rules of formation, of synthesis, and of analysis, and in accordance with the rest of the structural rules.

If the logical constant occurring in the thesis is not recorded by the table for local meaning, then either it must be introduced by a nominal definition, or the table for local meaning needs to be enriched with the new expression.

*Note: The structural rules with SR1c (and not SR1i) produce strategies for classical logic. The point is that since players can answer to a list of challenges in any order (which is not the case with the intuitionistic rule), it might happen that the two options of a **P**-defence occur in the same play—this is closely related to the classical development rule in sequent calculus allowing more than one formula at the right of the sequent.*

SR2: Formation rules for formal dialogues

SR2i: Starting a formation dialogue

A formation-play starts by challenging the thesis with the formation request **O** $?_{\mathbf{prop}}$; **P** must answer by stating that his thesis is a proposition.

SR2ii: Developping a formation dialogue

The game then proceeds by applying the formation rules up to the elementary constituents of **prop**/**set**.

After that **O** is free to use the other particle rules insofar as the other structural rules allow it.

Note: The constituents of the thesis will therefore not be specified before the play but as a result of the structure of the moves (according to the rules recorded by the rules for local meaning).

SR3: Resolution of instructions

1. A player may ask his adversary to carry out the prescribed instruction and thus bring forward a suitable local reason in defence of the proposition at stake. Once the defender has replaced the instruction with the required local reason we say that the instruction has been resolved.
2. The player index of an instruction determines which of the two players has the right to choose the local reason that will resolve the instruction.
    a. If the instruction $\mathcal{I}$ for the logical constant $\mathcal{K}$ has the form $\mathcal{I}^{\mathcal{K}}(p)^{\mathbf{X}}$ and it is **Y** who requests the resolution, then the request has the form **Y** ?…/ $\mathcal{I}^{\mathcal{K}}(p)^{\mathbf{X}}$, and it is **X** who chooses the local reason.
    b. If the instruction $\mathcal{I}$ for the logic constant $\mathcal{K}$ has the form $\mathcal{I}^{\mathcal{K}}(p)^{\mathbf{Y}}$ and it is player **Y** who requests the resolution, then the request has the form **Y** p_{i} / $\mathcal{I}^{\mathcal{K}}(p)^{\mathbf{Y}}$, and it is **Y** who chooses the local reason.
3. In the case of a sequence of instructions of the form $\pi[\mathcal{I}_{\mathrm{i}}(...(\mathcal{I}_{\mathrm{k}}(p))...)]$, the instructions are resolved from the inside ($\mathcal{I}_{\mathrm{k}}(p)$) to the outside ($\mathcal{I}_{\mathrm{i}}$).

This rule also applies to functions.

SR4: Substitution of instructions

Once the local reason b has been used to resolve the instruction $\mathscr{I}^{\mathscr{K}}(p)^{\mathbf{X}}$, and if the same instruction occurs again, players have the right to require that the instruction be resolved with b. The substitution request has the form $?b/\mathscr{I}_{\mathrm{k}}(p)^{\mathbf{X}}$. Players cannot choose a different substitution term (in our example, not even **X**, once the instruction has been resolved).

This rule also applies to functions.

SR5: Socratic rule and definitional equality

The following points are all parts of the Socratic rule, they all apply.

SR5.1: Restriction of P statements

P cannot make an elementary statement if **O** has not stated it before[84], except in the thesis.

An elementary statement is either an elementary proposition with implicit local reason, or an elementary proposition and its local reason (not an instruction).

SR5.2: Challenging elementary statements in formal dialogues

Challenges of elementary statements with implicit local reasons take the form:

$$\boldsymbol{X}\ !\, A$$
$$\boldsymbol{Y}\ ?_{reason}$$
$$\boldsymbol{X}\ a : A$$

Where A is an elementary proposition and a is a local reason.[85]

P cannot challenge **O**'s elementary statements, except if **O** provides an elementary initial concession with implicit local reason, in which case **P** can ask for a local reason, or in the context of transmission of equality.

SR5.3: Definitional equality

O may challenge elementary **P**-statements; **P** then answers by stating a definitional equality expressing the equality between a local reason and an instruction both introduced by **O** (for non-reflexive cases, that is when **O** provided the local reason as a resolution of an instruction), or a reflexive equality of the local reason introduced by **O** (when the local reason was not introduced by the resolution of an instruction, that is either as such in the initial concessions or as the result of a synthesis of a local reason). We thus distinguish two cases of the Socratic rule:

1. non-reflexive cases;[86]
2. reflexive cases.[87]

These rules do not cover cases of transmission of equality. The Socratic rule also applies to the resolution or substitution of functions, even if the formulation mentions only instructions.

---

[84] This yields an asymmetry in the structural rules. For a discussion on the consequences of this feature and how it is closely linked to the symmetry of the particle rules, see section IV.3, p. 67, and section XI.3.

[85] Note that **P** is allowed to make an elementary statement only as a thesis (Socratic rule); he will be able to respond to the challenge on an elementary statement only if **O** has provided the required local reason in her initial concessions.

[86] See below, section VII.3, for an illustration.

[87] See for instance $\mathcal{P}_1$ in VII.6.10 for an illustration.

SR5.3.1: Non-reflexive cases of the Socratic rule

We are in the presence of a *non-reflexive case* of the Socratic rule when **P** responds to the challenge with the indication that **O** gave the same local reason for the same proposition when she had to resolve or substitute instruction $\mathcal{I}$.

Here are the different challenges and defences determining the meaning of the three following moves:

Table 21: Non-reflexive cases of the Socratic rule

	Move	**Challenge**	**Defence**
SR5.3.1a	**P** $a : A$	**O** $? = a$	**P** $\mathcal{I} = a : A$
SR5.3.1b	**P** $a : A(b)$	**O** $? = b^{A(b)}$	**P** $\mathcal{I} = b : D$
SR5.3.1c	**P** $\mathcal{I} = b : D$ (this statement stems from **SR5.3.1b)**	**O** $? = A(b)$	**P** $A(\mathcal{I}) = A(b)$: ***prop***

Presuppositions:

(i) The response prescribed by SR5.3.1a presupposes that **O** has stated A or $a = b : A$ as the result of the resolution or substitution of instruction $\mathcal{I}$ occurring in $\mathcal{I} : A$ or in $\mathcal{I} = b : A$.

(ii) The response prescribed by SR5.3.1b presupposes that **O** has stated A and $b : D$ as the result of the resolution or substitution of instruction $\mathcal{I}$ occurring in $a : A(\mathcal{I})$.

(iii) SR5.3.1c assumes that **P** $\mathcal{I} = b : D$ is the result of the application of SR5.3.1b. The further challenge seeks to verify that the replacement of the instruction produces an equality in **prop**, that is, that the replacement of the instruction with a local reason yields an equal proposition to the one in which the instruction was not yet replaced. The answer prescribed by this rule presupposes that **O** has already stated $A(b)$: **prop** (or more trivially $A(\mathcal{I}) = A(b)$: **prop**).

The **P**-statements obtained after defending elementary **P**-statements cannot be attacked again with the Socratic rule (with the exception of SR5.3.1c), nor with a rule of resolution or substitution of instructions.

SR5.3.2: Reflexive cases of the Socratic rule

We are in the presence of a *reflexive case* of the Socratic rule when **P** responds to the challenge with the indication that **O** adduced the same local reason for the same proposition, though that local reason in the statement of **O** is not the result of any resolution or substitution.

The attacks have the same form as those prescribed by SR5.3.1 (see Table 21). Responses that yield reflexivity presuppose that **O** has previously stated the same statement or even the same equality.

The response obtained cannot be attacked again with the Socratic rule.

SR6: Transmission of definitional equality

Transmission of definitional equality I: Substitution within dependent or independent statements. The expression "type" refers to either **prop** or **set**. For more explanations on this structural rule, see below, section VII.2.2.

Table 22: Transmission of definitional equality I: Substitution within dependent or independent statements

Move	Challenge	Defence
X $b(x) : B(x)\ [x : A]$	**Y** $a = c : A$	**X** $b(a) = b(c) : B(a)$
X $b(x) = d(x) : B(x)\ [x : A]$	**Y** $a : A$	**X** $b(a) = d(a) : B(a)$
X $B(x) : type\ [x : A]$	**Y** $a = c : A$	**X** $B(a)=B(c) : type$
X $B(x)=D(x) : type\ [x : A]$	**Y** $?_{B(x)=D(x)}\ a : A$ or **Y** $?_{B(x)=D(x)}\ a = c : A$	**X** $B(a)=D(a) : type$ or **X** $B(a)=D(c) : type$
X $A = B : type$	**Y** $?_{A=D}\ a : A$ or **Y** $?_{A=D}\ a = c : A$	**X** $a : B$ or **X** $a = c : B$

Table 23: Transmission of definitional equality II

	Move	Challenge	Defence
Type-reflexivity	**X** $A : type$	**Y** $?_{type}$- *refl*	**X** $A = A : type$
Type-symmetry	**X** $A = B : type$	**Y** $?_B$-*symm*	**X** $B = A : type$
Type-transitivity	**X** $A = B : type$ **X** $B = C : type$	**Y** $?_A$-*trans*	**X** $A = C : type$
Reflexivity	**X** $a : A$	**Y** $?a\_refl$	**X** $a = a : A$
Symmetry	**X** $a = b : A$	**Y** $?_b$-*symm*	**X** $b = a : A$
Transitivity	**X** $a = b : A$ **X** $b = c : A$	**Y** $?_a$-*trans*	**X** $a = c : A$

SR7: Winning rule for plays

The player who makes the last move wins**.** If the last **O-**move in the play is $\perp$ then **P** can bring forward the local reason $you_{gave\ up}(n)$ in support for any statement that he has not defended before **O** stated $\perp$ at move n (even if that statement is $\perp$, see for instance VII.6.2 for an illustration).

VII.2.2 Rules for the transmission of definitional equality

As can be expected, definitional equality is transmitted by reflexivity, symmetry[88], and transitivity. Definitional equalities however can also be used in order to carry out a substitution within dependent statements—they can in fact be seen as a special form of application of the substitution rule for dependent statement Subst-D presented in the first section for local meaning, with the formation rules (VII.1.1, p. 106). We use the expression "type" as encompassing **prop** and **set**.

---

[88] Symmetry used here is not the same notion as the symmetry of section IV.3.

Reading adjuvant for the fourth rule (dependent statements) in Table 22:

If **X** stated that $B(x)$ and $D(x)$ are equal propositional functions, provided that x is an element of the set *A*—that is, **X** *B*(*x*)=*D*(*x*) : **prop** [*x* : *A*]—, then **Y** can carry out two kinds of attacks:

1. Stating himself that some local reason, say *a*, can be adduced for *A*—$\boldsymbol{Y}\ a\colon A$—, and request at the same time of **X** that he replaces x with a in *B*(*x*)=*D*(*x*), that is stating *B*(*a*)=*D*(*a*) : **prop**.
2. Stating himself an equality such as *a* = *c*: *A,* and request at the same time **X** to carry out the corresponding substitutions in *B*(*x*)=*D*(*x*), that is to state **X** *B*(*a*)=*D*(*c*) : **prop**.

Reading adjuvant for Table 23:

In order to trigger reflexivity, transitivity, and symmetry from some equality statements the challenger can attack an equality by asking for each of these properties. For example, if **X** stated *A* = *B* : **prop/set***,* **Y** can ask **X** to state the commutated equality *B* = *A* : **prop/set** by calling on *symmetry.* The notation of such an attack is as follows: **Y** $?_{B\text{-}symm}$. Similarly, **Y** $?_{A\text{-}refl}$ and **Y** $?_{A\text{-}trans}$ respectively request reflexivity and transitivity.

VII.3 Example: $(\forall \boldsymbol{x}\colon \boldsymbol{D})\big(\boldsymbol{Q}(\boldsymbol{x}) \supset \boldsymbol{Q}(\boldsymbol{x})\big)$

$\mathcal{P}_1$	O			P			
					$!\,(\forall x\colon D)\big(Q(x) \supset Q(x)\big)$	0	
	1	$m \coloneqq 1$			$n \coloneqq 2$	2	
synthesis of local reason	3	$d_1\colon D$	0		$d_2\colon Q(d_1) \supset Q(d_1)$	4	
Analysis of local reason	5	$L^{\supset}(d_2)\colon Q(d_1)$	4		$R^{\supset}(d_2)\colon Q(d_1)$	8	*Before answering, request the resolution of the instruction.*
	7	$d_{2.1}\colon Q(d_1)$		5	$?\ldots/L^{\supset}(d_2)$	6	
	9	$?\ldots/R^{\supset}(d_2)$	8		$d_{2.1}\colon Q(d_1)$	10	*Statement allowed by move 7*
SR5.3	11	$?= d_{2.1}$	10		$L^{\supset}(d_2) = d_{2.1}\colon Q(d_1)$	12	*SR5.3.1: non-reflexive case between instruction and local reason*

P wins

VII.4 Solved exercises for the play level in dialogues for immanent reasoning

Build a play for the following theses in dialogues for immanent reasoning.

1. $B \vee A\ [c\colon A \vee B]$
   That is, build a play for $B \vee A$ with the initial concession $c\colon A \vee B$.
2. $\big((A \vee (A \supset\perp)) \supset\perp\big) \supset\perp$
   Alternative notation: $\neg\neg(A \vee \neg A)$; for a play of this game in the standard framework, see section IV.5. For all the plays in immanent reasoning, see section VII.6.
3. $(A \wedge (A \supset\perp)) \supset\perp$
   Alternative notation: $\neg(A \wedge \neg A)$; for all the plays, see section VII.6.
4. $\big((A \supset B) \supset A\big) \supset A$
5. $\big(A \wedge (B \supset\perp)\big) \supset \big((A \supset B) \supset\perp\big)$

6. $*(A \wedge B) \wedge C\ [c: A \wedge (B \wedge C)]$
7. $*(B \wedge A) \supset C\ [c: (A \wedge B) \supset C]$
   As for the other cases, section VII.6 provides all the plays through the heuristical procedure for building a winning strategy; but the theses 5 and 6 will also have their strategic reason built in section VII.7.4, with an exposition of the tree-shaped core and the core turned into a natural deduction style demonstration in section IX.5.
8. $(\forall x: D)A(x) \supset \bot\ [(\exists x: D)(A(x) \supset\bot)]$
9. $(\exists x: D)A(x) \wedge (\exists x: D)B(x)\ \big[(\exists x: D)\big(A(x) \wedge B(x)\big)\big]$
10. $(\exists x: D)(\exists y: D)\big(A(x) \supset B(y)\big)\ [A(a) \supset B(b); a: D\ ; b: D]$
11. $(\exists x: D)B(x) \wedge (\exists x: D)P(x)\ [B(a) \wedge P(b); a: D\ ; b: D]$
12. $(\exists x: D)(\exists y: D)A(x, y)[(\exists x: D)A(x, x)]$
13. $(\forall x: D)(\forall y: D)\big(A(x, y) \wedge A(y, x)\big)[(\forall x: D)(\forall y: D)A(x, y)]$
14. $*(\exists x: D)\big(A(x) \supset (\forall x: D)A(x)\big)\ \big[\big((\exists x: D)(A(x) \supset\bot)\big) \vee \big((\forall x: D)A(x)\big); a: D\big]$
    The graphic (tree) presentation of the strategy is also provided for this thesis.

In general, the solutions will be found in section (VII.6) which provides the **P**-winning strategies through the heuristical procedure, thus providing all the relevant plays for each thesis.

VII.5 The strategy level in dialogues for immanent reasoning

The strategy level in dialogues for immanent reasoning is analogous to that of the standard framework: the core of a **P**-strategy can be built by means of the heuristical procedure which allows to find *all* of the *relevant* plays for a given thesis. However, since the local and structural rules have been modified in the framework for dialogues for immanent reasoning in order to incorporate features of CTT, the plays from which the strategy emerges will be modified in consequence. The procedure and the assumptions are thus the same as in chapter V—to which we refer the reader—but the result of their application will differ, especially because of local reasons and instructions, absent of the standard dialogical framework. A few remarks are due before proceeding to the exercises.

Adding to the assumptions on O's move preferences:

1. When **O** has to choose a local reason she will always choose a new one.[89]
2. **O** will challenge instructions (if there are any) before carrying out other moves.[90]
   A direct consequence of this—added to the assumption that **O** will always challenge first, if she can (see p. 77)—is that in the case that **P** challenges a material implication or a universal, **O** will first counterattack the instructions $L^{\supset}$ or $L^{\forall}$ before responding to the challenge.

---

[89] This assumption is analogous to the assumption in section V.2 that **O** chooses a new constant when she can. The reason is the same: it is the best possible choice for **O**. Indeed, here also **P** is restricted by the Socratic rule, so he needs to rely on **O**'s choices in order to copy a move. In such a context, the only way to (try to) block the use of this kind of equality is to always choose, whenever possible, a new local reason.

[90] The reason is similar to the previous one: it is better for **O** to force **P** to makes his choice as soon as possible.

Recalling the O-decisions:

O has a choice and takes a decision when	
O challenges a…	**O defends a…**
Conjunction Existential	Disjunction Implication

Recalling left and right decisions:

We speak of a *left-decision* when:

- **O** defends the left side of a disjunction;
- **O** challenges the left side of a conjunction or existential;
- **O** counterattacks instead of defending an implication.

We speak of a *right decision* when:

- **O** defends the right side of a disjunction;
- **O** challenges the right side of a conjunction or existential;
- **O** defends an implication.

Getting rid of infinite ramifications in a strategy:

When the Opponent has to choose a local reason for a previously unresolved instruction, it will be a member of some set which, unless otherwise specified, may be infinite. The Opponent can then choose among an infinite number of members when asked to replace the instruction with a local reason,[91] though once she has chosen one she must keep it for the rest of the play (recall the statement-substitution rule).

Repetition ranks ensure that plays are finite (they fix an upper boundary to the number of challenges and defences each player can play in reaction to a move). But strategies take all the possible plays for a thesis into account, which easily yields infinite strategies. These are reduced down to a finite length (the core) by disregarding redundant variants.

Local reasons and proof-objects:

Local reasons are not the dialogical counterparts of CTT proof-objects—strategic reasons are, which will be presented in section VII.7—, and thus are not enough to establish the connection between the dialogical and the CTT approaches.

The connection between our dialogical games and CTT is not to be found at the level of plays, but at the level of strategies where the CTT notion of proof and the dialogical notion of **P**-winning strategies come together. More precisely, demonstrations in Natural deduction in general, and in CTT in particular, correspond within the dialogical framework to **P**-winning strategies (see chapter IX for how this correspondence is built).

---

[91] See (Clerbout, 2014a; 2014c): if there is a move by which the Opponent can force her victory, then nothing prevents her from playing it as soon as she has a chance to. Whether this move is a challenge or a defence, the repetition rank 1 is enough to allow her to play it in accordance with *SR*1*i*.

Disregarding formation rules for formal plays

The dialogical rules allow the players to enquire about the type of expressions used in a dialogue; in particular they allow to ask whether an expression is a proposition or not. This leads to plays that use formation rules (see VII.1.1).

The scope of the present study only covers the fragment of CTT involving logically valid propositions and formal plays (see chapter IX); so generally in the formal enquiries which we are dealing with here, we presuppose that the logical structure of the expressions whose validity must be demonstrated have been well typed. We will therefore set aside the formation rules in considering the strategic level for formal plays.

Disregarding irrelevant variations in the order of O-moves.

A **P**-strategy must account for every possible way for **O** to play, and in particular it must deal with any order in which **O** might play her moves. Since we look for a winning **P**-strategy, we can select any particular order of **O**-moves without losing anything in terms of **P**'s victory: the very definition of a winning strategy states that **P** must win in every play proceeding from an **O**-choice, so the order of **O**-moves does not influence the result.

Special care must nevertheless be taken, especially in an intuitionistic framework (because of the restrictions of *SR*1*i* ('last duty first'), **O** can only answer to **P**'s last non-answered challenge): we do not want a play in which **O** loses because she played poorly. Since the extensive form of the strategy will contain all of the possible plays—the ones in which **O** played well and those in which she did not—, if we select some plays and discard the others we should take care not to retain one of those plays in which **O** did not play well. When extracting a particular order of **O**'s moves in all the possible order of moves, we shall select a play such that every **P**-challenge is immediately followed by the **O**-defence, for it tends to be strategically safer for **O** to defend immediately (and be sure not to lose the chance of making that move later on) and delay possible moves involving counterattacks. By doing so we explicitly get rid of the cases in which **O** loses only because she poorly chose the order of her moves.

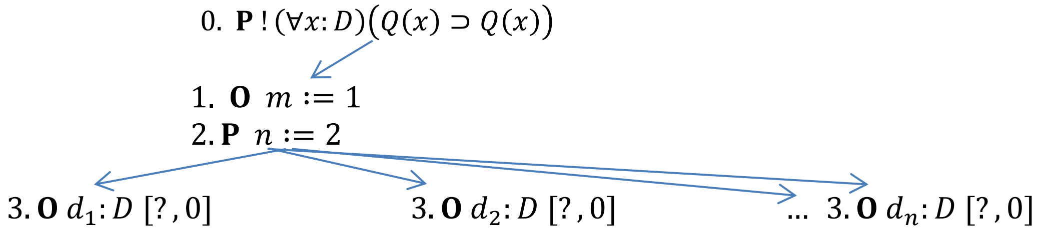
Once we have removed all the redundant information for developing a demonstration, what remains is the *core* $\mathbb{C}$ of the strategy.

Example: obtaining the core for $(\forall x\colon D)Q(x) \supset Q(x)$

From the play in section VII.3, one can build a graphic (tree) presentation of the core, provided infinite ramifications are reduced to a finite number, which can be carried out in two steps: getting rid of ramifications first from repetition ranks, then from choice of names.

Step 1: getting rid of ramifications from repetition ranks

Here we present the core of the strategy where we delete all the branches where **O** chooses a repetition rank bigger than 1



$4.\ \mathbf{P}\ d_2 : Q(p_1) \supset Q(p_1)\ [!, 3]$

$5.\ \mathbf{O}\ L^{\supset}(d_2) : Q(d_1) [?, 4]$

$6.\ \mathbf{P}\ ?\ldots/L^{\supset}(d_2)\ \ [?, 5]$

$4.\ \mathbf{P} d_2 : Q(p_1) \supset Q(p_1)\ [!, 3]$

$5.\ \mathbf{O}\ L^{\supset}(d_2) : Q(d_1)\ [?, 4]$

6. …

4. …

$7.\ \mathbf{O}\ d_{2.1} : Q(d_1)[!, 6]$

$8.\ \mathbf{P}\ R^{\supset}(d_2) : Q(d_1)[!, 5]$

$9.\ \mathbf{O}\ ?\ldots/R^{\supset}(d_2)[?, 8]$

$10.\ \mathbf{P}\ d_{2.1} : Q(d_1)[!, 9]$

$11.\ \mathbf{O}\ ?= d_{2.1}\ [?, 10]$

$12.\ \mathbf{P}\ L^{\supset}(d_2) = d_{2.1} : Q(d_1)[!, 11]$

P wins

$7.\ \mathbf{O}\ d_{2.1} : Q(d_1)[!, 6]$

8. …

…

$7.\ \mathbf{O}\ d_{2.1} : Q(d_1)[!, 6]$

8. …

Step 2: getting rid of choices of names

Delete all but one of the branches of the previous tree triggered by **O**'s choices of a local reason.

$0.\ \mathbf{P}\ !\ (\forall x : D)\big(Q(x) \supset Q(x)\big)$

$1.\ \mathbf{O}\ \ m := 1$

$2.\ \mathbf{P}\ \ n := 2$

$3.\ \mathbf{O}\ d_1 : D\ [?, 0]$

$4.\ \mathbf{P}\ d_2 : Q(p_1) \supset Q(p_1)\ [!, 3]$

$5.\ \mathbf{O}\ L^{\supset}(d_2) : Q(d_1)[?, 4]$

$6.\ \mathbf{P}\ ?\ldots/L^{\supset}(d_2)\ \ [?, 5]$

$7.\ \mathbf{O}\ d_{2.1} : Q(d_1)[!, 6]$

$8.\ \mathbf{P}\ R^{\supset}(d_2) : Q(d_1)[!, 5]$

$9.\ \mathbf{O}\ ?\ldots/R^{\supset}(d_2)[?, 8]$

$10.\ \mathbf{P}\ d_{2.1} : Q(d_1)[!, 9]$

$11.\ \mathbf{O}\ ?= d_{2.1}\ [?, 10]$

$12.\ \mathbf{P}\ L^{\supset}(d_2) = d_{2.1} : Q(d_1)[!, 11]$

P wins

Litterature on the link between CTT and the dialogical framework:

Nicolas Clerbout (2014a; 2014b; 2014c) showed how to extract winning strategies for **P** out of the extensive form of strategies, and he developed the corresponding proof. Clerbout & Rahman (2015) extended the result to the CTT-demonstrations.

For other publications on the development of such algorithms see (Felscher, 1985), (Keiff, 2007), (Rahman & Tulenheimo, 2009), (Rahman, Clerbout, & Keiff, 2009) and (Cardascia, 2016). Laurent Keiff (2007) was the first to have developped this kind of method for standard dialogical logic.

VII.6 Exercises and solutions[92]

Build a demonstration for the theses provided in section VII.4 using the heuristic procedure.

Provide also the graphic (tree) presentation for the theses marked with an asterisk *.

Solutions

The exercises have been developed considering the following elements:

- the assumptions for the procedure of building a winning strategy (see sections V.2.1 and VII.5). We thus assume that **O**'s repetition rank is 1.
- As much as possible, local reasons will follow the notation d_1, $d_{1.2}$, etc., in order to keep track of the complex local reason from which they originate.
  For example d_1 *indicates that the local reason is the first component of the complex local reason* d. *We will use this notation as long as it does not hinder the choices of the players.*
- In order to shorten the length of the plays we will not deploy challenges of the form $? \ldots = A(b): type$.
- In order to implement more clearly the Intuitionistic Development rule SR1i (section VII.2.1) we present the negation in its implication form—instead of writing $p : \neg A$ we write $p: A \supset\bot$.

VII.6.1 $\boldsymbol{B \vee A\ [c: A \vee B]}$

$\mathcal{P}_1$

O			P		
0.1	$c: A \vee B$			$! B \vee A\ [c: A \vee B]$	0
1	$m := 1$			$n := 2$	2
3	$?\vee$	0		$c_1: A$	8
5	$L^{\vee}(c): A\ [\delta_1, \ldots]$		0.1	$?\vee$	4
7	$c_1: A$		5	$? \ldots / L^{\vee}(c)$	6
9	$? = c_1$	8		$L^{\vee}(c) = c_1: A$	10

P wins

$\mathcal{P}_2$

O			P		
0.1	$c: A \vee B$			$! B \vee A\ [c: A \vee B]$	0
1	$m := 1$			$n := 2$	2
3	$?\vee$	0		$c_2: B$	8
5	$R^{\vee}(c): A\ [\delta_1, \delta_2]$		0.1	$?\vee$	4
7	$c_2: B$		5	$? \ldots / R^{\vee}(c)$	6
9	$? = c_2$	8		$R^{\vee}(c) = c_2: B$	

P wins

VII.6.2 $\boldsymbol{\big((A \vee (A \supset\bot)) \supset\bot\big) \supset\bot}$

$\mathcal{P}_1$

O	P

---

[92] This section has been developped by Steephen Eckoubili.

				$!((A \lor (A \supset\bot)) \supset\bot) \supset\bot$	0
1	$m \coloneqq 1$			$n \coloneqq 2$	2
3	$d_1: (A \lor (A \supset\bot)) \supset\bot$	0		$you_{gave\ up}(7\ [\mathcal{P}_{1R}]): \bot$	8 $[\mathcal{P}_{1R}]$
	…			$L^{\supset}(d_1): A \lor (A \supset\bot)$	4
5	$?\ldots/L^{\supset}(d_1)$	4		$d_{1.1}: A \lor (A \supset\bot)$	6

$\mathcal{P}_{1L}$

7	$?^{\lor}$	6		$R^{\lor}(d_{1.1}): A \supset\bot$	8
9	$?\ldots/R^{\lor}(d_{1.1})$	8		$d_{1.1.2}: A \supset\bot$	10
11	$L^{\supset}(d_{1.1.2}): A$	10			
13	$d_{1.1.2.1}: A$		11	$?\ldots/L^{\supset}(d_{1.1.2})$	12
[7]	$[?^{\lor}]$	[6]		$L^{\lor}(d_{1.1}): A$	14
15	$?\ldots/L^{\lor}(d_{1.1})$	14		$d_{1.1.2.1}: A$	16
17	$?= d_{1.1.2.1}$	16		$L^{\supset}(d_{1.1.2}) = d_{1.1.2.1}: A$	18

P wins

$\mathcal{P}_{1R}$

7	$R^{\supset}(d_1): \bot$			…	

P wins

VII.6.3 $(\boldsymbol{A} \land (\boldsymbol{A} \supset\bot)) \supset\bot$

$\mathcal{P}_1$

O			**P**		
				$!(A \land (A \supset\bot)) \supset\bot$	0
1	$m \coloneqq 1$			$n \coloneqq 2$	2
3	$d_1: A \land (A \supset\bot)$	0		$you_{gave\ up}(15\ [\mathcal{P}_{1R}]): \bot$	16 $[\mathcal{P}_{1R}]$
5	$L^{\land}(d_1): A$		3	$?L^{\land}$	4
7	$d_{1.1}: A$		5	$?\ldots/L^{\land}(d_1)$	6
9	$R^{\land}(d_1): A \supset\bot$		3	$?\ R^{\land}$	8
11	$d_{1.2}: A \supset\bot$		7	$?\ldots/R^{\land}(d_1)$	10
	…		11	$L^{\supset}(d_{1.2}): A$	12
13	$?\ldots/L^{\supset}(d_{1.2})$	12		$d_{1.1}: A$	14

$\mathcal{P}_{1L}$

15	$?= d_{1.1}$	14		$L^{\land}(d_1) = d_{1.1}: A$	16

P wins

$\mathcal{P}_{1R}$

15	$R^{\supset}(d_{1.2}): \bot$			…	

P wins

VII.6.4 $((\boldsymbol{A} \supset \boldsymbol{B}) \supset \boldsymbol{A}) \supset \boldsymbol{A}$

$\mathcal{P}_1$

O			**P**		
				$!((A \supset B) \supset A) \supset A$	0
1	$m \coloneqq 1$			$n \coloneqq 2$	2
3	$d_1: (A \supset B) \supset A$	0		$d_{1.1.1}: A$ $d_{1.2}: A$	10 $[\mathcal{P}_{1L}]$ 10 $[\mathcal{P}_{1R}]$

	…			$L^{\supset}(d_1): A \supset B$	4
5	$? \dots/L^{\supset}(d_1)$	4		$d_{1.1}: A \supset B$	6

$\mathcal{P}_{1L}$

7	$L^{\supset}(d_{1.1}): A$	6			
9	$d_{1.1.1}: A$		7	$? \dots/L^{\supset}(d_{1.1})$	8
11	$? = d_{1.1.1}$	10		$L^{\supset}(d_{1.1}) = d_{1.1.1}: A$	12

P wins

$\mathcal{P}_{1R}$

7	$R^{\supset}(d_1): A$			…	
9	$d_{1.2}: A$		7	$? \dots/R^{\supset}(d_1)$	8
11	$? = d_{1.2}$	10		$R^{\supset}(d_1) = d_{1.2}: A$	12

P wins

VII.6.5 $\left(\boldsymbol{A} \wedge (\boldsymbol{B} \supset \bot)\right) \supset \left((\boldsymbol{A} \supset \boldsymbol{B}) \supset \bot\right)$

$\boldsymbol{\mathcal{P}_1}$

O			**P**		
				$!\left(A \wedge (B \supset \bot)\right) \supset \left((A \supset B) \supset \bot\right)$	0
1	$m := 1$			$n := 2$	2
3	$d_1: A \wedge (B \supset \bot)$	0		$d_2: (A \supset B) \supset \bot$	4
5	$L^{\supset}(d_2): A \supset B$	4		$you_{gave\ up}(21\ [\mathcal{P}_{1RR}]): \bot$	22 [$\mathcal{P}_{1RR}$]
7	$d_{2.1}: A \supset B$		5	$? \dots/L^{\supset}(d_2)$	6
9	$L^{\wedge}(d_1): A$		3	$?\ L^{\wedge}$	8
11	$d_{1.1}: A$		9	$? \dots/L^{\wedge}(d_1)$	10
13	$R^{\wedge}(d_1): B \supset \bot$		3	$?\ R^{\wedge}$	12
15	$d_{1.2}: B \supset \bot$		13	$? \dots/R^{\wedge}(d_1)$	14
	…		7	$d_{1.1}: A$	16

$\mathcal{P}_{1L}$

17	$? = d_{1.1}$	16		$L^{\wedge}(d_1) = d_{1.1}: A$	18

P wins

$\mathcal{P}_{1R}$

17	$R^{\supset}(d_{2.1}): B$			…	
19	$d_{2.1.2}: B$		17	$? \dots/R^{\supset}(d_{2.1})$	18
	…		15	$d_{2.1.2}: B$	20

$\mathcal{P}_{1RL}$

21	$? = d_{2.1.2}$	20		$R^{\supset}(d_{2.1}) = d_{2.1.2}: B$	22

P wins

$\mathcal{P}_{1RR}$

21	$R^{\supset}(d_{1.2}): \bot$			…	

P wins

VII.6.6 $*(\boldsymbol{A} \wedge \boldsymbol{B}) \wedge \boldsymbol{C}\ [\boldsymbol{c}: \boldsymbol{A} \wedge (\boldsymbol{B} \wedge \boldsymbol{C})]$

$\boldsymbol{\mathcal{P}_1}$

O			**P**		
0.1	$c: A \wedge (B \wedge C)$			$!\,(A \wedge B) \wedge C\ [c: A \wedge (B \wedge C)]$	0
1	$m := 1$			$n := 2$	2

3	$?L^{\wedge}$	0		$d_1 : A \wedge B$	4
5	$?L^{\wedge}\ [\delta_1, \ldots]$	4		$L^{\wedge}(d_1) : A$	6
7	$?\ldots/L^{\wedge}(d_1)$			$c_1 : A$	12
9	$L^{\wedge}(c) : A$		0.1	$?L^{\wedge}$	8
11	$c_1 : A$		9	$?\ldots/L^{\wedge}(c)$	10
13	$? = c_1$	12		$L^{\wedge}(c) = c_1 : A$	14

P wins

$\mathcal{P}_2$

	O			**P**	
0.1	$c : A \wedge (B \wedge C)$			$!\,(A \wedge B) \wedge C\ [c : A \wedge (B \wedge C)]$	0
1	$m := 1$			$n := 2$	2
3	$?L^{\wedge}\ [\delta_2, \ldots]$	0		$d_1 : A \wedge B$	4
5	$?R^{\wedge}\ [\delta_1, \delta_2]$	4		$R^{\wedge}(d_1) : B$	6
7	$?\ldots/R^{\wedge}(d_1)$			$c_{2.1} : B$	16
9	$R^{\wedge}(c) : B \wedge C$		0.1	$?R^{\wedge}$	8
11	$c_2 : B \wedge C$		9	$?\ldots/R^{\wedge}(c)$	10
13	$L^{\wedge}(c_2) : B$		11	$?L^{\wedge}$	12
15	$c_{2.1} : B$		13	$?\ldots/L^{\wedge}(c_2)$	14
17	$? = c_{2.1}$	16		$L^{\wedge}(c_2) = c_{2.1} : B$	18

P wins

$\mathcal{P}_3$

	O			**P**	
0.1	$c : A \wedge (B \wedge C)$			$!\,(A \wedge B) \wedge C\ [c : A \wedge (B \wedge C)]$	0
1	$m := 1$			$n := 2$	2
3	$?R^{\wedge}\ [\delta_2, \delta_3]$	0		$c_{2.2} : C$	12
5	$R^{\wedge}(c) : B \wedge C$		0.1	$?R^{\wedge}$	4
7	$c_2 : B \wedge C$		5	$?\ldots/R^{\wedge}(c)$	6
9	$R^{\wedge}(c_2) : C$		7	$?R^{\wedge}$	8
11	$c_{2.2} : C$		9	$?\ldots/R^{\wedge}(c_2)$	10
13	$? = c_{2.2}$	12		$R^{\wedge}(c_2) = c_{2.2} : C$	14

P wins

The core of the strategy can be found section VII.7.4.

VII.6.7 $*(\boldsymbol{B} \wedge \boldsymbol{A}) \supset \boldsymbol{C}\ [\boldsymbol{c} : (\boldsymbol{A} \wedge \boldsymbol{B}) \supset \boldsymbol{C}]$

$\mathcal{P}_1$

	O			**P**	
0.1	$c : (A \wedge B) \supset C$			$!\ (B \wedge A) \supset C\ [c : (A \wedge B) \supset C]$	0
1	$m := 1$			$n := 2$	2
3	$d_1 : B \wedge A$	0		$c_2 : C$	18 $[\mathcal{P}_{1R}]$
5	$L^{\wedge}(d_1) : B$		3	$?L^{\wedge}$	4
7	$d_{1.1} : B$		5	$?\ldots/L^{\wedge}(d_1)$	6
9	$R^{\wedge}(d_1) : A$		3	$?\ R^{\wedge}$	8
11	$d_{1.2} : A$		7	$?\ldots/R^{\wedge}(d_1)$	10

	...		0.1	$L^{\supset}(c): A \wedge B$	12
13	$? \ldots / L^{\supset}(c)$	12		$c_1: A \wedge B$	14

$\mathcal{P}_{1L}$

15	$? L^{\wedge}\ [\boldsymbol{\delta_1}, \ldots]$	14		$L^{\wedge}(c_1): A$	16
17	$? \ldots / L^{\wedge}(c_1)$	16		$d_{1.2}: A$	18
19	$? = d_{1.2}$	18		$R^{\wedge}(d_1) = d_{1.2}: A$	20

P wins

$\mathcal{P}_{1R}$

15	$R^{\supset}(c): C$			...	
17	$c_2: C$		15	$? \ldots / R^{\supset}(c)$	16
19	$? = c_2$	18		$R^{\supset}(c) = c_2: C$	20

P wins

$\boldsymbol{\mathcal{P}_2}$

O			**P**		
0.1	$c: (A \wedge B) \supset C$			$!\ (B \wedge A) \supset C\ [c: (A \wedge B) \supset C]$	0
1	$m := 1$			$n := 2$	2
3	$d_1: B \wedge A$	0		$c_2: C$	18 [$\mathcal{P}_{1R}$]
5	$L^{\wedge}(d_1): B$		3	$? L^{\wedge}$	4
7	$d_{1.1}: B$		5	$? \ldots / L^{\wedge}(d_1)$	6
9	$R^{\wedge}(d_1): A$		3	$?\ R^{\wedge}$	8
11	$d_{1.2}: A$		7	$? \ldots / R^{\wedge}(d_1)$	10
	...		0.1	$L^{\supset}(c): A \wedge B$	12
13	$? \ldots / L^{\supset}(c)$	12		$c_1: A \wedge B$	14

$\mathcal{P}_{2L}$

15	$? R^{\wedge}\ [\delta_1, \delta_2]$	14		$R^{\wedge}(c_1): B$	16
17	$? \ldots / R^{\wedge}(c_1)$	16		$d_{1.1}: B$	18
19	$? = d_{1.1}$	18		$L^{\wedge}(d_1) = d_{1.1}: B$	20

P wins

$\mathcal{P}_{2R}$

15	$R^{\supset}(c): C$			...	
17	$c_2: C$		15	$? \ldots / R^{\supset}(c)$	16
19	$? = c_2$	18		$R^{\supset}(c) = c_2: C$	20

P wins

The core of the strategy can be found section VII.7.4.

VII.6.8 $\boldsymbol{((\forall x: D)A(x)) \supset \bot\ [(\exists x: D)(A(x) \supset \bot)]}$

$\boldsymbol{\mathcal{P}_1}$

O			**P**		
0.1	$!(\exists x: D)(A(x) \supset \bot)$			$!((\forall x: D)A(x)) \supset \bot$ $[(\exists x: D)(A(x) \supset \bot)]$	0
1	$m := 1$			$n := 2$	2
3	$d_1: (\forall x: D)A(x)$	0		$you_{gave\ up}(19\ [\mathcal{P}_{1R}]): \bot$	20[$\mathcal{P}_{1R}$]

5	$c_1 : d$		0.1	$?\, L^{\exists}$	4
7	$c_2 : A(c_1) \supset \bot$		0.1	$?\, R^{\exists}$	6
13	$R^{\forall}(d_1) : A(c_1)$		3	$L^{\forall}(d_1) : D$	8
9	$?\, .../L^{\forall}(d_1)$	8		$c_1 : D$	10
11	$? = c_1$	10		$c_1 = c_1 : D$	12
15	$d_{1.2} : A(c_1)$		13	$?\, .../R^{\forall}(d_1)$	14
	...		7	$L^{\supset}(c_2) : A(c_1)$	16
17	$?\, .../L^{\supset}(c_2)$	16		$d_{1.2} : A(c_1)$	18

$\mathcal{P}_{1L}$

19	$? = d_{1.2}$	18		$R^{\forall}(d_1) = d_{1.2} : A(c_1)$	20

P wins

$\mathcal{P}_{1R}$

19	$R^{\supset}(c_2) : \bot$			...	

P wins

VII.6.9 $(\exists \boldsymbol{x} : \boldsymbol{D})\boldsymbol{A}(\boldsymbol{x}) \wedge (\exists \boldsymbol{x} : \boldsymbol{D})\boldsymbol{B}(\boldsymbol{x})\big[(\exists \boldsymbol{x} : \boldsymbol{D})\big(\boldsymbol{A}(\boldsymbol{x}) \wedge \boldsymbol{B}(\boldsymbol{x})\big)\big]$

$\boldsymbol{\mathcal{P}}_1$

O			**P**		
0.1	$!\, (\exists x : D)\big(A(x) \wedge B(x)\big)$			$!\, (\exists x : D)A(x) \wedge (\exists x : D)B(x)$ $\big[(\exists x : D)\big(A(x) \wedge B(x)\big)\big]$	0
1	$m \coloneqq 1$			$n \coloneqq 2$	2
3	$?\, L^{\wedge}$	0		$d_1 : (\exists x : D)A(x)$	4
5	$?\, L^{\exists}\ [\delta_1, ...]$	4		$L^{\exists}(d_1) : D$	6
7	$?\, .../L^{\exists}(d_1)$	6		$c_1 : D$	10
9	$c_1 : D$		0.1	$?\, L^{\exists}$	8
11	$? = c_1$	10		$c_1 = c_1 : D$	12

P wins

$\boldsymbol{\mathcal{P}}_2$

O			**P**		
0.1	$!\, (\exists x : D)\big(A(x) \wedge B(x)\big)$			$!\, (\exists x : D)A(x) \wedge (\exists x : D)B(x)$ $\big[(\exists x : D)\big(A(x) \wedge B(x)\big)\big]$	0
1	$m \coloneqq 1$			$n \coloneqq 2$	2
3	$?\, L^{\wedge}\ [\delta_2, ...]$	0		$d_1 : (\exists x : D)A(x)$	4
5	$?\, R^{\exists}\ [\delta_1, \delta_2]$	4		$R^{\exists}(d_1) : A(c_1)$	14
7	$c_1 : D$		0.1	$?\, L^{\exists}$	6
9	$c_2 : A(c_1) \wedge B(c_1)$		0.1	$?\, R^{\exists}$	8
11	$L^{\wedge}(c_2) : A(c_1)$		9	$?\, L^{\exists}$	10
13	$c_{2.1} : A(c_1)$		11	$?\, .../L^{\wedge}(c_2)$	12
15	$?\, .../R^{\exists}(d_1)$	14		$c_{2.1} : A(c_1)$	16
17	$? = c_{2.1}$	16		$L^{\wedge}(c_2) = c_{2.1} : A(c_1)$	18

P wins

$\boldsymbol{\mathcal{P}}_3$

O	**P**

0.1	$!\,(\exists x: D)(A(x) \wedge B(x))$			$!\,(\exists x: D)A(x) \wedge (\exists x: D)B(x)$ $[(\exists x: D)(A(x) \wedge B(x))]$	0
1	$m \coloneqq 1$			$n \coloneqq 2$	2
3	$?\,R^{\wedge}\ [\delta_2, \delta_3]$	0		$d_2: (\exists x: D)B(x)$	4
5	$?\,L^{\exists}\ [\delta_3, \ldots]$	4		$L^{\exists}(d_2): D$	6
7	$?\,\ldots/L^{\exists}(d_2)$	6		$c_1: D$	10
9	$c_1: D$		0.1	$?\,L^{\exists}$	8
11	$?= c_1$	10		$c_1 = c_1: D$	12

P wins

$\mathcal{P}_4$

O			**P**		
0.1	$!\,(\exists x: D)(A(x) \wedge B(x))$			$!\,(\exists x: D)A(x) \wedge (\exists x: D)B(x)$ $[(\exists x: D)(A(x) \wedge B(x))]$	0
1	$m \coloneqq 1$			$n \coloneqq 2$	2
3	$?\,R^{\wedge}\ [\delta_2, \delta_3]$	0		$d_2: (\exists x: D)B(x)$	4
5	$?\,L^{\exists}\ [\delta_3, \delta_4]$	4		$R^{\exists}(d_2): B(c_1)$	14
8	$c_1: D$		0.1	$?\,L^{\exists}$	6
9	$c_2: A(c_1) \wedge B(c_1)$		0.1	$?\,R^{\exists}$	8
11	$R^{\wedge}(c_2): B(c_1)$		9	$?\,R^{\exists}$	10
13	$c_{2.2}: B(c_1)$		11	$?\,\ldots/R^{\wedge}(c_2)$	12
15	$?\,\ldots/R^{\exists}(d_2)$	14		$c_{2.2}: B(c_1)$	16
17	$?= c_{2.2}$	16		$R^{\wedge}(c_2) = c_{2.2}: B(c_1)$	18

P wins

VII.6.10 $\boldsymbol{(\exists x: D)(\exists y: D)(A(x) \supset B(y))[A(a) \supset B(b); a: D\ ; b: D]}$

$\mathcal{P}_1$

O			**P**		
0.1 0.2 0.3	$!A(a) \supset B(b)$ $a: D$ $b: D$			$!\,(\exists x: D)(\exists y: D)(A(x) \supset B(y))$ $[A(a) \supset B(b); a: D\ ; b: D]$	0
1	$m \coloneqq 1$			$n \coloneqq 2$	2
3	$?\,L^{\exists}\ [\delta_1, \ldots]$	0		$a: D$	4
5	$?= a$	4		$a = a: D$	6

P wins

$\mathcal{P}_2$

O			**P**		
0.1 0.2 0.3	$!A(a) \supset B(b)$ $a: D$ $b: D$			$!\,(\exists x: D)(\exists y: D)(A(x) \supset B(y))$ $[A(a) \supset B(b); a: D\ ; b: D]$	0
1	$m \coloneqq 1$			$n \coloneqq 2$	2
3	$?\,R^{\exists}\ [\delta_1, \delta_2]$	0		$d_2: (\exists y: D)(A(a) \supset B(y))$	4
5	$?\,L^{\exists}\ [\delta_2, \ldots]$	4		$b: D$	6
7	$?= b$	6		$b = b: D$	8

P wins

$\boldsymbol{\mathcal{P}}_3$

O			**P**		
0.1 0.2 0.3	$!A(a) \supset B(b)$ $a: D$ $b: D$			$!\,(\exists x: D)(\exists y: D)(A(x) \supset B(y))$ $[A(a) \supset B(b); a: D\ ; b: D]$	0
1	$m \coloneqq 1$			$n \coloneqq 2$	2
3	$?\, R^{\exists}\ [\delta_1, \delta_2]$	0		$d_2: (\exists y: D)(A(a) \supset B(y))$	4
5	$?\, R^{\exists}\ [\delta_2, \delta_3]$	4		$R^{\exists}(d_2): A(a) \supset B(b)$	6
7	$?\, \ldots/R^{\exists}(d_2)$	6		$d_{2.2}: A(a) \supset B(b)$	8
9	$L^{\supset}(d_{2.2}): A(a)$	8		$R^{\supset}(d_{2.2}): B(b)$	12
11	$d_{2.2.1}: A(a)$		9	$?\, \ldots/L^{\supset}(d_{2.2})$	10
13	$?\, \ldots/R^{\supset}(d_{2.2})$	12		$d_{2.2.2}: B(b)$	16 [$\mathcal{P}_{3R}$]
	…		0.1	$d_{2.2.1}: A(a)$	14

$\mathcal{P}_{3L}$

15	$? = d_{2.2.1}$	14		$L^{\supset}(d_{2.2}) = d_{2.2.1}: A(a)$	16

P wins

$\mathcal{P}_{3R}$

15	$d_{2.2.2}: B(b)$	6		…	
17	$? = d_{2.2.2}: B(b)$	16		$d_{2.2.2} = d_{2.2.2}: B(b)$	18

P wins

VII.6.11 $(\exists \boldsymbol{x}: \boldsymbol{D})\boldsymbol{B}(\boldsymbol{x}) \wedge (\exists \boldsymbol{x}: \boldsymbol{D})\boldsymbol{P}(\boldsymbol{x})[\boldsymbol{B}(\boldsymbol{a}) \wedge \boldsymbol{P}(\boldsymbol{b}); \boldsymbol{a}: \boldsymbol{D}\ ; \boldsymbol{b}: \boldsymbol{D}]$

$\boldsymbol{\mathcal{P}}_1$

O			**P**		
0.1 0.2 0.3	$!B(a) \wedge P(b)$ $a: D$ $b: D$			$!\,(\exists x: D)B(x) \wedge (\exists x: D)P(x)$ $[B(a) \wedge P(b); a: D\ ; b: D]$	0
1	$m \coloneqq 1$			$n \coloneqq 2$	2
3	$?\, L^{\wedge}$	0		$d_1: (\exists x: D)B(x)$	4
5	$?\, L^{\exists}\ [\delta_1, \ldots]$	4		$L^{\exists}(d_1): D$	6
7	$?\, \ldots/L^{\exists}(d_1)$	6		$a: D$	8
9	$? = a$	8		$a = a: D$	10

P wins

$\boldsymbol{\mathcal{P}}_2$

O			**P**		
0.1 0.2 0.3	$!B(a) \wedge P(b)$ $a: D$ $b: D$			$!\,(\exists x: D)B(x) \wedge (\exists x: D)P(x)$ $[B(a) \wedge P(b); a: D\ ; b: D]$	0
1	$m \coloneqq 1$			$n \coloneqq 2$	2
3	$?\, L^{\wedge}\ [\delta_2, \ldots]$	0		$d_1: (\exists x: D)B(x)$	4
5	$?\, R^{\exists}\ [\delta_1, \delta_2]$	4		$R^{\exists}(d_1): B(a)$	6
7	$?\, \ldots/R^{\exists}(d_1)$	6		$c_1: B(a)$	10
9	$c_1: B(a)$		0.1	$?\, L^{\exists}$	8
11	$? = c_1$	10		$c_1 = c_1: B(a)$	12

P wins

$\mathcal{P}_3$

O			P		
0.1 0.2 0.3	$!B(a) \wedge P(b)$ $a: D$ $b: D$			$!\,(\exists x: D)B(x) \wedge (\exists x: D)P(x)$ $[B(a) \wedge P(b); a: D\ ; b: D]$	0
1	$m \coloneqq 1$			$n \coloneqq 2$	2
3	$?\,L^{\wedge}\ [\delta_2, \delta_3]$	0		$d_2: (\exists x: D)P(x)$	4
5	$?\,L^{\exists}\ [\delta_3, \dots]$	4		$L^{\exists}(d_2): D$	6
7	$?\,\dots/L^{\exists}(d_2)$	6		$b: D$	8
9	$? = b$	8		$b\ = b: D$	10

P wins

$\mathcal{P}_4$

O			P		
0.1 0.2 0.3	$!\,B(a) \wedge P(b)$ $a: D$ $b: D$			$!\,(\exists x: D)B(x) \wedge (\exists x: D)P(x)$ $[B(a) \wedge P(b); a: D\ ; b: D]$	0
1	$m \coloneqq 1$			$n \coloneqq 2$	2
3	$?\,L^{\wedge}\ [\delta_2, \delta_3]$	0		$d_2: (\exists x: D)P(x)$	4
5	$?\,R^{\exists}\ [\delta_3, \delta_4]$	4		$R^{\exists}(d_2): P(b)$	6
7	$?\,\dots/R^{\exists}(d_2)$	6		$c_2: P(b)$	10
9	$c_2: P(b)$		0.1	$?\,R^{\wedge}$	8
11	$? = c_2$	10		$c_2\ = c_2: P(b)$	12

P wins

VII.6.12 $(\exists \boldsymbol{x}: \boldsymbol{D})(\exists \boldsymbol{y}: \boldsymbol{D})\boldsymbol{A}(\boldsymbol{x}, \boldsymbol{y})[(\exists \boldsymbol{x}: \boldsymbol{D})\boldsymbol{A}(\boldsymbol{x}, \boldsymbol{x})]$

$\mathcal{P}_1$

O			P		
0.1	$!\,(\exists x: D)A(x, x)$			$!(\exists x: D)(\exists y: D)A(x, y)$ $[(\exists x: D)A(x, x)]$	0
1	$m \coloneqq 1$			$n \coloneqq 2$	2
3	$?\,L^{\exists}\ [\delta_1, \dots]$	0		$d_1: D$	6
5	$d_1: D$		0.1	$?\,L^{\exists}$	4
7	$? = a$	6		$d_1\ = d_1: D$	8

P wins

$\mathcal{P}_2$

O			P		
0.1	$!\,(\exists x: D)A(x, x)$			$!(\exists x: D)(\exists y: D)A(x, y)$ $[(\exists x: D)A(x, x)]$	0
1	$m \coloneqq 1$			$n \coloneqq 2$	2
3	$?\,R^{\exists}[\delta_1, \delta_2]$	0		$d_3: (\exists y: D)A(d_1, y)$	8
5	$d_1: D$		0.1	$?\,L^{\exists}$	4
7	$d_2: A(d_1, d_1)$		0.1	$?\,R^{\exists}$	6
9	$?\,L^{\exists}\ [\delta_2, \dots]$	8		$L^{\exists}(d_3): D$	10
11	$?\,\dots/L^{\exists}(d_3)$	10		$d_1: D$	12
13	$? = d_1$	12		$d_1\ = d_1: D$	14

P wins

$\mathcal{P}_3$

O			**P**		
0.1	$!(\exists x : D)A(x,x)$			$!(\exists x : D)(\exists y : D)A(x,y)$ $[(\exists x : D)A(x,x)]$	0
1	$m := 1$			$n := 2$	2
3	$?R^{\exists}[\delta_1, \delta_2]$	0		$d_3 : (\exists y : D)A(d_1, y)$	8
5	$d_1 : D$		0.1	$?L^{\exists}$	4
7	$d_2 : A(d_1, d_1)$		0.1	$?R^{\exists}$	6
9	$?R^{\exists}\ [\delta_2, \delta_3]$	8		$R^{\exists}(d_3) : A(d_1, d_1)$	10
11	$?\ldots/R^{\exists}(d_3)$	10		$d_2 : A(d_1, d_1)$	12
13	$? = d_2$	12		$d_2 = d_2 : A(d_1, d_1)$	14

P wins

VII.6.13 $(\forall \boldsymbol{x} : \boldsymbol{D})(\forall \boldsymbol{y} : \boldsymbol{D})\big(\boldsymbol{A}(\boldsymbol{x},\boldsymbol{y}) \wedge \boldsymbol{A}(\boldsymbol{y},\boldsymbol{x})\big)[(\forall \boldsymbol{x} : \boldsymbol{D})(\forall \boldsymbol{y} : \boldsymbol{D})\boldsymbol{A}(\boldsymbol{x},\boldsymbol{y})]$

$\mathcal{P}_1$

O			**P**		
0.1	$!(\forall x : D)(\forall y : D)A(x,y)$			$!(\forall x : D)(\forall y : D)(A(x,y) \wedge A(y,x))$ $[(\forall x : D)(\forall y : D)A(x,y)]$	0
1	$m := 1$			$n := 2$	2
3	$d_1 : D$	0		$d_2 : (\forall y : D)\big(A(d_1, y) \wedge A(y, d_1)\big)$	4
5	$L^{\forall}(d_2) : D$	4		$R^{\forall}(d_2) : A(d_1, d_{2.1}) \wedge A(d_{2.1}, d_1)$	8
7	$d_{2.1} : D$		5	$?\ldots/L^{\supset}(d_2)$	6
9	$?\ldots/R^{\forall}(d_2)$	8		$d_{2.2} : A(d_1, d_{2.1}) \wedge A(d_{2.1}, d_1)$	10
11	$?L^{\wedge}\ [\delta_1, \ldots]$	10		$L^{\wedge}(d_{2.2}) : A(d_1, d_{2.1})$	12
13	$?\ldots/L^{\wedge}(d_{2.2})$	12		$c_{2.2} : A(d_1, d_{2.1})$	26
17	$c_2 : (\forall y : D)A(d_1, y)$		0.1	$d_1 : D$	14
15	$? = d_1$	14		$d_1 = d_1 : D$	16
23	$R^{\forall}(c_2) : A(d_1, d_{2.1})$		17	$L^{\forall}(c_2) : D$	18
19	$?\ldots/L^{\forall}(c_2)$	18		$d_{2.1} : D$	20
21	$? = d_{2.1}$	20		$L^{\forall}(d_2) = d_{2.1} : D$	22
25	$c_{2.2} : A(d_1, d_{2.1})$		23	$?\ldots/R^{\forall}(c_2)$	24
27	$? = c_{2.2}$	26		$R^{\forall}(c_2) = c_{2.2} : A(d_1, d_{2.1})$	28

P wins

$\mathcal{P}_2$

O			**P**		
0.1	$!(\forall x : D)(\forall y : D)A(x,y)$			$!(\forall x : D)(\forall y : D)(A(x,y) \wedge A(y,x))$ $[(\forall x : D)(\forall y : D)A(x,y)]$	0
1	$m := 1$			$n := 2$	2
3	$d_1 : D$	0		$d_2 : (\forall y : D)\big(A(d_1, y) \wedge A(y, d_1)\big)$	4
5	$L^{\forall}(d_2) : D$	4		$R^{\forall}(d_2) : A(d_1, d_{2.1}) \wedge A(d_{2.1}, d_1)$	8
7	$d_{2.1} : D$		5	$?\ldots/L^{\supset}(d_2)$	6
9	$?\ldots/R^{\forall}(d_2)$	8		$d_{2.2} : A(d_1, d_{2.1}) \wedge A(d_{2.1}, d_1)$	10
11	$?R^{\wedge}\ [\delta_1, \delta_2]$	10		$R^{\wedge}(d_{2.2}) : A(d_{2.1}, d_1)$	12

13	$? \ldots / R^{\wedge}(d_{2.2})$	12		$c_{2.2}: A(d_{2.1}, d_1)$	26
17	$c_2: (\forall y: D)A(d_{2.1}, y)$		0.1	$d_{2.1}: D$	14
15	$? = d_{2.1}$	14		$L^{\forall}(d_2) = d_{2.1}: D$	16
23	$R^{\forall}(c_2): A(d_{2.1}, d_1)$		17	$L^{\forall}(c_2): D$	18
19	$? \ldots / L^{\forall}(c_2)$	18		$d_1: D$	20
21	$? = d_1$	20		$d_1 = d_1: D$	22
25	$c_{2.2}: A(d_{2.1}, d_1)$		23	$? \ldots / R^{\forall}(c_2)$	24
27	$? = c_{2.2}$	26		$R^{\forall}(c_2) = c_{2.2}: A(d_{2.1}, d_1)$	28

P wins

VII.6.14 $(\exists \boldsymbol{x}: \boldsymbol{D})\big(\boldsymbol{A}(\boldsymbol{x}) \supset (\forall \boldsymbol{x}: \boldsymbol{D})\boldsymbol{A}(\boldsymbol{x})\big)$ $\Big[\big((\exists \boldsymbol{x}: \boldsymbol{D})(\boldsymbol{A}(\boldsymbol{x}) \supset \bot)\big) \vee \big((\forall \boldsymbol{x}: \boldsymbol{D})\boldsymbol{A}(\boldsymbol{x})\big); \boldsymbol{a}: \boldsymbol{D}\Big]$

$\boldsymbol{\mathcal{P}}_1$

O			P		
0.1 0.2	$!\big((\exists x: D)(A(x) \supset \bot)\big) \vee \big((\forall x: D)A(x)\big)$ $a: D$			$!\,(\exists x: D)\big(A(x) \supset (\forall x: D)A(x)\big)$ $\Big[\big((\exists x: D)(A(x) \supset \bot)\big)$ $\vee \big((\forall x: D)A(x)\big); a: D\Big]$	0
1	$m := 1$			$n := 2$	2
3	$?\, L^{\exists}\ [\delta_1, \ldots]$	0		$a: D$	4
5	$? = a$	4		$a = a: D$	6

P wins

$\boldsymbol{\mathcal{P}}_2$

O			P		
0.1 0.2	$!\big((\exists x: D)(A(x) \supset \bot)\big) \vee \big((\forall x: D)A(x)\big)$ $a: D$			$!\,(\exists x: D)\big(A(x) \supset (\forall x: D)A(x)\big)$ $\Big[\big((\exists x: D)(A(x) \supset \bot)\big) \vee \big((\forall x: D)A(x)\big)\Big]$	0
1	$m := 1$			$n := 2$	2
3	$?\, R^{\exists}\ [\delta_1, \delta_2]$	0		$d_2: A(c_{1.1}) \supset (\forall x: D)A(x)$	14
5	$c_1: (\exists x: D)(A(x) \supset \bot)\ [\delta_2, \ldots]$		0.1	$?\vee$	4
7	$L^{\exists}(c_1): D$		5	$?\, L^{\exists}$	6
9	$c_{1.1}: D$		7	$? \ldots / L^{\exists}(c_1)$	8
11	$R^{\exists}(c_1): A(c_{1.1}) \supset \bot$		5	$?\, R^{\exists}$	10
13	$c_{1.2}: A(c_{1.1}) \supset \bot$		11	$? \ldots / R^{\exists}(c_1)$	12
15	$L^{\supset}(d_2): A(c_{1.1})$	14		$you_{gave\ up}(21[\mathcal{P}_{2R}]): (\forall x: D)A(x)$	22 $[\mathcal{P}_{2R}]$
17	$d_{2.1}: A(c_{1.1})$		15	$? \ldots / L^{\supset}(d_2)$	16
	…		13	$L^{\supset}(c_{1.2}): A(c_{1.1})$	18
19	$? \ldots / L^{\supset}(c_{1.2})$	18		$d_{2.1}: A(c_{1.1})$	20

$\mathcal{P}_{2L}$

21	$? = d_{1.2}$	20		$L^{\supset}(d_2) = d_{2.1}: A(c_{1.1})$	22

P wins

$\mathcal{P}_{2R}$

21	$R^{\supset}(c_2): \bot$			…	

P wins

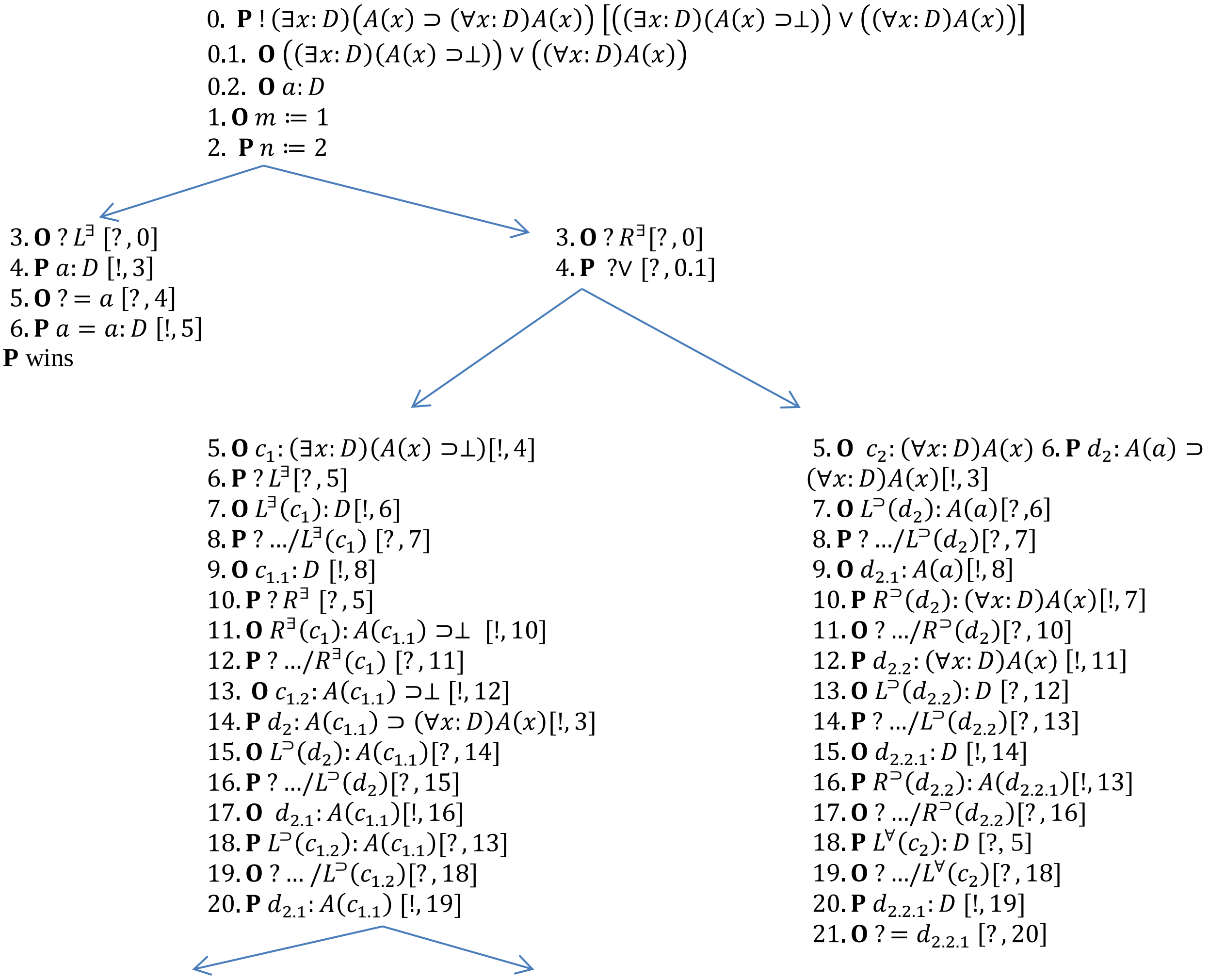
$\boldsymbol{\mathcal{P}}_3$

O	P

0.1 0.2	$!\big((\exists x{:}D)(A(x)\supset\bot)\big)\vee\big((\forall x{:}D)A(x)\big)$ $a{:}D$			$!(\exists x{:}D)\big(A(x)\supset(\forall x{:}D)A(x)\big)$ $\big[\big((\exists x{:}D)(A(x)\supset\bot)\big)$ $\vee\big((\forall x{:}D)A(x)\big)\big]$	0
1	$m \coloneqq 1$			$n \coloneqq 2$	2
3	$?\,R^{\exists}\;[\delta_1,\delta_2]$	0		$d_2{:}\,A(a)\supset(\forall x{:}D)A(x)$	6
5	$c_2{:}\,(\forall x{:}D)A(x)\;[\delta_2,\delta_3]$		0.1	$?\vee$	4
7	$L^{\supset}(d_2){:}\,A(a)$	6		$R^{\supset}(d_2){:}\,(\forall x{:}D)A(x)$	10
9	$d_{2.1}{:}\,A(a)$		7	$?\,.../L^{\supset}(d_2)$	8
11	$?\,.../R^{\supset}(d_2)$	10		$d_{2.2}{:}\,(\forall x{:}D)A(x)$	12
13	$L^{\supset}(d_{2.2}){:}\,D$	12		$R^{\supset}(d_{2.2}){:}\,A(d_{2.2.1})$	16
15	$d_{2.2.1}{:}\,D$		13	$?\,.../L^{\supset}(d_{2.2})$	14
17	$?\,.../R^{\supset}(d_{2.2})$	16		$c_{2.1}{:}\,A(d_{2.2.1})$	26
23	$R^{\forall}(c_2){:}\,A(d_{2.2.1})$		5	$L^{\forall}(c_2){:}\,D$	18
19	$?\,.../L^{\forall}(c_2)$	18		$d_{2.2.1}{:}\,D$	20
21	$?\!=d_{2.2.1}$	20		$L^{\supset}(d_{2.2})=d_{2.2.1}{:}\,D$	22
25	$c_{2.1}{:}\,A(d_{2.2.1})$		23	$?\,.../R^{\forall}(c_2)$	24
27	$?\!=c_{2.1}$	26		$R^{\forall}(c_2)=c_{2.1}{:}\,A(d_{2.2.1})$	28

P wins

The graphic (tree) presentation of the core yields the following:



0. **P** $!(\exists x{:}D)\big(A(x)\supset(\forall x{:}D)A(x)\big)\,\big[\big((\exists x{:}D)(A(x)\supset\bot)\big)\vee\big((\forall x{:}D)A(x)\big)\big]$
0.1. **O** $\big((\exists x{:}D)(A(x)\supset\bot)\big)\vee\big((\forall x{:}D)A(x)\big)$
0.2. **O** $a{:}\,D$
1. **O** $m \coloneqq 1$
2. **P** $n \coloneqq 2$

3. **O** $?\,L^{\exists}\;[?\,,0]$
4. **P** $a{:}\,D\;[!,3]$
5. **O** $?\!=a\;[?\,,4]$
6. **P** $a=a{:}\,D\;[!,5]$
P wins

3. **O** $?\,R^{\exists}[?\,,0]$
4. **P** $?\vee\;[?\,,0.1]$

5. **O** $c_1{:}\,(\exists x{:}D)(A(x)\supset\bot)[!,4]$
6. **P** $?\,L^{\exists}[?\,,5]$
7. **O** $L^{\exists}(c_1){:}\,D[!,6]$
8. **P** $?\,.../L^{\exists}(c_1)\;[?\,,7]$
9. **O** $c_{1.1}{:}\,D\;[!,8]$
10. **P** $?\,R^{\exists}\;[?\,,5]$
11. **O** $R^{\exists}(c_1){:}\,A(c_{1.1})\supset\bot\;\;[!,10]$
12. **P** $?\,.../R^{\exists}(c_1)\;[?\,,11]$
13. **O** $c_{1.2}{:}\,A(c_{1.1})\supset\bot\;[!,12]$
14. **P** $d_2{:}\,A(c_{1.1})\supset(\forall x{:}D)A(x)[!,3]$
15. **O** $L^{\supset}(d_2){:}\,A(c_{1.1})[?\,,14]$
16. **P** $?\,.../L^{\supset}(d_2)[?\,,15]$
17. **O** $d_{2.1}{:}\,A(c_{1.1})[!,16]$
18. **P** $L^{\supset}(c_{1.2}){:}\,A(c_{1.1})[?\,,13]$
19. **O** $?\,...\,/L^{\supset}(c_{1.2})[?\,,18]$
20. **P** $d_{2.1}{:}\,A(c_{1.1})\;[!,19]$

5. **O** $c_2{:}\,(\forall x{:}D)A(x)$ 6. **P** $d_2{:}\,A(a)\supset(\forall x{:}D)A(x)[!,3]$
7. **O** $L^{\supset}(d_2){:}\,A(a)[?\,,6]$
8. **P** $?\,.../L^{\supset}(d_2)[?\,,7]$
9. **O** $d_{2.1}{:}\,A(a)[!,8]$
10. **P** $R^{\supset}(d_2){:}\,(\forall x{:}D)A(x)[!,7]$
11. **O** $?\,.../R^{\supset}(d_2)[?\,,10]$
12. **P** $d_{2.2}{:}\,(\forall x{:}D)A(x)\;[!,11]$
13. **O** $L^{\supset}(d_{2.2}){:}\,D\;[?\,,12]$
14. **P** $?\,.../L^{\supset}(d_{2.2})[?\,,13]$
15. **O** $d_{2.2.1}{:}\,D\;[!,14]$
16. **P** $R^{\supset}(d_{2.2}){:}\,A(d_{2.2.1})[!,13]$
17. **O** $?\,.../R^{\supset}(d_{2.2})[?\,,16]$
18. **P** $L^{\forall}(c_2){:}\,D\;[?,5]$
19. **O** $?\,.../L^{\forall}(c_2)[?\,,18]$
20. **P** $d_{2.2.1}{:}\,D\;[!,19]$
21. **O** $?\!=d_{2.2.1}\;[?\,,20]$

21. **O** $? = d_{1.2}$ $[?, 20]$
22. **P** $L^{\supset}(d_2) = d_{2.1}: A(c_{1.1})$ $[!, 21]$
P wins

21. **O** $R^{\supset}(c_2): \bot$ $[!, 20]$
22. **P** $you_{gave\ up}(21[\mathcal{P}_{2R}]): (\forall x: D)A(x)$ $[!, 15]$
P wins

22. **P** $L^{\supset}(d_{2.2}) = d_{2.2.1}: D[!, 21]$
23. **O** $R^{\forall}(c_2): A(d_{2.2.1})[!, 18]$
24. **P** $? .../R^{\forall}(c_2)[?, 23]$
25. **O** $c_{2.1}: A(d_{2.2.1})$ $[!, 24]$
26. **P** $c_{2.1}: A(d_{2.2.1})[!, 17]$
27. **O** $? = c_{2.1}$ $[?, 26]$
28. **P** $R^{\forall}(c_2) = c_{2.1}: A(d_{2.2.1})[!\, 28]$
P wins

VII.7 Strategic reasons in dialogues for immanent reasoning

The philosophical backbone on which rests the proof of admissibility provided in chapter IX,[93] and which is one of the dialogical framework's greatest strengths, is probably the notion of strategic reason which allows to adopt a global view on all the possible plays that constitute a strategy; but this global view should not be identified with the perspective common in proof theory: strategic reasons are a kind of recapitulation of what can happen for a given thesis and show the entire history of the play by means of the instructions. Strategic reasons thus yield an overview of the possibilities enclosed in a thesis—what plays can be carried out from it—, but without ever being carried out in an actual play: they are only a perspective on all the possible variants of plays for a thesis and not an actual play. In this way the rules of synthesis and analysis of strategic reasons provided below are not of the same nature as the analysis and synthesis of local reasons, they are not produced through challenges and their defence, but are a recapitulation of the plays that can actually be carried out.

The notion of strategic reasons enables us to link dialogical strategies with CTT-demonstrations, since strategic reasons (and not local reasons) are the dialogical counterpart of CTT proof-objects; but it also shows clearly that the strategy level by itself—the only level that proof theory considers—is not enough: a deeper insight is gained when considering, together with the strategy level, the fundamental level of plays; strategic reasons thus bridge these two perspectives, the global view of strategies and the more in-depth and down-to-earth view of actual plays with all the possible variations in logic they allow,[94] without sacrificing the one for the other.

This vindication of the play level is a key aspect of the dialogical framework and one of the purposes of the present study: other logical frameworks lack this dimension, which besides is not an extra dimension appended to the concern for demonstrations, but actually constitutes it, the heuristical procedure for building strategies out of plays showing the gapless link there is between the play level and the strategy level: strategies (and so demonstrations) stem from plays. Thus the dialogical framework can say at least as much as other logical frameworks, and, additionally, reveals limitations of other frameworks through this level of plays.

This final section presenting dialogues for immanent reasoning will be devoted first to a rather informal presentation of strategic reasons, stressing out the philosophical

---

[93] Chapter IX will focus on proving that dialogues for immanent reasoning is an admissible logical framework; it will therefore be rather technical and will address problems that mainly concern logicians, though we will here take care to outline the main philosophical aspects involved.

[94] Among these variations can be counted cooperative games, non-monotony, the possibility of player errors or of limited knowledge or ressources, to cite but a few options the play level offers, making the dialogical framework very well adapted for history and philosophy of logic.

aspects underlying this key notion; then to the synthesis of strategic reasons and showing the link with the introduction rules for CTT; and finally to the analysis of strategic reasons. Once the rules have been provided, two examples will illustrate them and show how to produce a strategic reason for the thesis out of the core of a strategy.

VII.7.1 **Introducing strategic reasons**

Strategic reasons belong to the strategy level, but are elements of the object-language of the play level: they are the reasons brought forward by a player entitling him to his statement. Strategic reasons are a perspective on plays that take into account all the possible variations in the play for a given thesis; they are never actually carried out, since any play is but the actualization of only one of all the possible plays for the thesis: each individual play can be actualized but will be separate from the other individual plays that can be carried out if other choices are made; strategic reasons allow to see together all these possible plays that in fact are always separate. There will never be in any of the plays the complex strategic reason for the thesis as a result of the application of the particle rules, only the local reason for each of the subformulas involved; the strategic reason will put all these separate reasons together as a recapitulation of what can be said from the given thesis.

Consider for instance a conjunction: the Proponent claims to have a strategic reason for this conjunction. This means that he claims that whatever the Opponent might play, be it a challenge of the left or of the right conjunct, the Proponent will be able to win the play. But in a single play with repetition rank 1 for the Opponent, there is no way to check if a conjunction is justified, that is if both of the conjuncts can be defended, since a play is precisely the carrying out of only one of the possible **O**-choices (challenging the left or the right conjunct): to check both sides of a conjunction, two plays are required, one in which the Opponent challenges the left side of the conjunction and another one for the right side. So a strategic reason is never a single play, but refers to the strategy level where all the possible outcomes are taken into account; the winning strategy can then be displayed as a tree showing that both plays (respectively challenging and defending the left conjunct and right conjunct) are won by the Proponent, thus justifying the conjunction.

Let us now study what strategic reasons look like, how they are generated and how they are analyzed.

A strategic *perspective* on a statement

In the standard framework of dialogues, where we do not explicitly have the reasons for the statements in the object-language, the particle rules simply determine the local meaning of the expressions. In dialogues for immanent reasoning, the reasons entitling one to a statement are explicitely introduced; the particle rules (synthesis and analysis of local reasons) govern both the local reasons and the local meaning of expressions. But when building the core of a winning **P**-strategy, local reasons are also linked to the justification of the statements—which is not the case if considering single plays or non-winning strategies, for then only one aspect of the statement may be taken into account during the play, the play providing thus only a partial justification.

Take again the example of a **P**-conjunction, say

$$\mathbf{P}\ w : A \wedge B.$$

In providing a strategic reason w for the conjunction $A \wedge B$, **P** is claiming to have a winning strategy for this conjunction, that is, he is claiming that the conjunction is absolutely justified, that he has a proper reason for asserting it and not simply a *local* reason for stating it. Assuming that **O** has a repetition rank of 1 and has stated both A and B prior to move i, two different plays can be carried out from this point, which we provide without the strategic reason:

Play 12: introducing strategic reasons: stating a conjunction

O			**P**		
	Concessions			Thesis	0
1	$m := 1$			$n := 2$	2
…	…			…	…
…	…			$!A \wedge B$	i

Play 13: introducing strategic reasons: left decision option on conjunction

O			**P**		
	Concessions			Thesis	0
1	$m := 1$			$n := 2$	2
…	…			…	…
…	…			$!A \wedge B$	i
$i+1$	$?\wedge_1$	i		$!A$	$i+2$

Play 14: introducing strategic reasons: right decision option on conjunction

O			**P**		
	Concessions			Thesis	0
1	$m := 1$			$n := 2$	2
…	…			…	…
…	…			$!A \wedge B$	i
$i+1$	$?\wedge_2$	i		$!B$	$i+2$

So if **P** brings forward the strategic reason w to support his conjunction at move i, he is claiming to be able to win *both* Play 13 and Play 14, and yet the actual play will follow into only one of the two plays. Strategic reasons are thus a *strategic perspective* on a statement that is brought forward during actual plays.

An anticipation of the play and a recapitulation of the strategy

Since a strategic reason (w for instance) is brought forward during a play (say at move i), it is clear that the play has not yet been carried out fully when the player claims to be able to defend his statement against whatever challenge his opponent might launch: bringing forward a strategic reason is thus an anticipation on the outcome of the play.

But strategic reasons are not a simple claim to have a winning strategy, they also have a complex internal structure: they can thus be considered as recapitulations of the plays of the winning strategy produced by the heuristic procedure, that is the winning strategy obtained only after running all the relevant plays; this strategy-building process specific to the dialogical framework is a richer process than the one yielding CTT demonstrations—or proof theory in general—, since the strategic reasons will contain traces of *choice dependences*, which constitute their complexity.

Choice dependences link possible moves of a player to the choices made by the other player: a player will play this move if his opponent used this decision-option, that move if the opponent used that decision-option. In the previous example, the Proponent will play move $i + 2$ depending on the Opponent's decision at move $i + 1$, so the strategic object w played at move i will contain these two possible scenarios with the $i + 2$ **P**-move depending of the $i + 1$ **O**-decision. The strategic reason w is thus a *recapitulation* of what would happen if each relevant play was carried out. When the strategic reason makes clearly explicit this choice-dependence of **P**'s moves on those of **O**, we say that it is in a *canonical argumentation form* and is a recapitulation of the statement.

The rules for strategic reasons do not provide the rules on how to play but rather rules that indicate how a winning strategy has been achieved while applying the relevant rules at the play level. Strategic reasons emerge as the result of considering the optimal moves for a winning strategy: this is what a recapitulation is about.

The canonical argumentation form of strategic reasons is closely linked to the synthesis and analysis of local reasons: they provide the recapitulation of all the relevant local reasons that could be generated from a statement. In this respect following the rules for the synthesis and analysis of local reasons (see section VII.1.2), the rules for strategic reasons are divided into synthesis and analysis of strategic reasons, to which we will now turn.

In a nutshell, the synthesis of strategic reasons provides a guide for what **P** needs to be able to defend in order to justify his claim; the analysis of strategic reasons provides a guide for the local reasons **P** needs to make **O** state in order to copy these reasons and thus defend his statement.

Assertions and statements

The difference between local reasons and strategic reasons should now be clear: while local reasons provide a local justification entitling one to his statement, strategic reasons provide an absolute justification of the statement, which thus becomes an *assertion*.

The equalities provided in each of the plays constituting a **P**-winning strategy, and found in the analysis of strategic objects, convey the information required for **P** to play in the best possible way by specifying those **O**-moves necessary for **P**'s victory. This information however is not available at the very beginning of the first play, it is not made explicit at the root of the tree containing all the plays relevant for the **P**-winning strategy: the root of the tree will not explicitly display the information gathered while developing the plays; this information will be available only once the whole strategy has been developped, and each possible play considered. So when a play starts, the thesis is a simple statement; it is only at the end of the construction process of the strategic reason that **P** will be able to have the knowledge required to assert the thesis, and thus provide in any new play a strategic reason for backing his thesis.

The *assertion* of the thesis, making explicit the strategic reason resulting from the plays, is in this respect a *recapitulation* of the result achieved after running the relevant plays, after **P**'s initial simple statement of that thesis. This is what the canonical argumentation form of a strategic object is, and what renders the dialogical formulation of a CTT canonical proof-object.

It is in this fashion that dialogical reasons correspond to CTT proof-objects: introduction rules are usually characterized as the right to assert the conclusion from the premises of the inference, that is, as defining what one needs in order to be entitled to assert the conclusion; and the elimination rules are what can be infered from a given statement. Thus, in the dialogical perspective of **P**-winning strategies, since we are looking at **P**'s entitlements and duties, what corresponds to proof-object introduction rules would define what **P** is required to justify in order to assert his statement, which is the synthesis of a **P**-strategic reason; and what corresponds to proof-object elimination rules would define what **P** is entitled to ask of **O** from her previous statements and thus say it himself by copying her statements, which is the analysis of **P**-strategic reasons. We will thus provide the rules for the synthesis and analysis of strategic reasons (always in the perspective of a **P**-winning strategy), followed by their corresponding CTT rule. We have in this regard a good justification of Sundholm's idea that inferences can be considered as involving an *implicit interlocutor*, but here at the strategy level.

VII.7.2 Rules for the synthesis of P-strategic reasons: Strategic reasons as recapitulations of the local reasons required for a P-winning strategy

P-strategic reasons must be built (*synthesis* of **P**-strategic reasons); they constitute the justification of a statement by providing certain information—choice-dependences—that are essential to the relevant plays issuing from the statement: strategic reasons are a recapitulation of the building of a winning strategy, directly inserted into a play. Thus a strategic reason for a **P**-statement can have the form $p_2^{\mathbf{P}}$ ⟦ $p_1^{\mathbf{O}}$⟧ and indicates that **P**'s choice of p_2 is dependent upon **O**'s choice of p_1. In this respect the synthesis of **P**-strategic reasons differ from the *analysis* of **P**-strategic reasons—dealt with in the following section (VII.7.3)—which provide a guide for all the local reasons **P** needs to ask **O** to provide in order for **P** to justify his statement through the equality rules.

Strategic reasons for **P** are the dialogical formulation of CTT proof-objects, and the canonical argumentation form of strategic reasons correspond to canonical proof-objects. Since in this chapter we are seeking a notion of winning strategy that corresponds to that of a CTT-demonstration, and since these strategies have being identified to be those where **P** wins, we will only provide the synthesis of strategic reasons for **P**.[95]

The rules for the synthesis of **P**-strategic reasons provide the strategic reasons in their *canonical argumentation form*, that is they make explicit the choice-dependences for **P**-moves based on **O**-decisions; the range of possibilities to take into account is dictated by the rules for the synthesis of local reasons, the synthesis of strategic reasons recapitulating what **P** needs to provide in order to have a winning strategy (or demonstration, that is, for his statement to be justified).

Thus for a conjunction, the rules for the synthesis of local reasons stipulates that **P** does not have the choice and must provide the local reason required by **O**. So in order to justify a conjunction, **P** must be able to provide a local reason for both of the conjuncts: the strategic reason for a conjunction will thus have the form $< p_1, p_2 >$, where p_1 and p_2 are the local reasons for each conjunct.

The defence of a disjunction on the other hand gives the choice to **P**, so the strategic reason for a disjunction requires only one local reason.

The defence of an implication can use what **O** had to concede in order to challenge the implication, so the local reason p_2 for the consequent depends on the local reason p_1

---

[95] The table which follows is in fact the dialogical analogue to the *introduction rules* in CTT: dialogically speaking, these rules display the *duties* required by **P**'s assertions—we will come back to this issue in section IX.1.

for the antecedent provided by **O**; the strategic reason for the implication thus has the form $p_2{}^{\mathbf{P}}[\![p_1{}^{\mathbf{O}}]\!]$.

Table 24: Synthesis of strategic reasons for **P**:

	Move	Synthesis of local reasons		Synthesis of strategic reasons
		Challenge	Defence	
Conjunction	$\mathbf{P} ! \ A \wedge B$	$\mathbf{O} ? L^{\wedge}$ or $\mathbf{O} ? R^{\wedge}$	$\mathbf{P}\ p_1 : A$ (resp.) $\mathbf{P}\ p_2 : B$	$\mathbf{P} < p_1, p_2 >: A \wedge B$
Existential quantifiation	$\mathbf{P}! \ (\exists x : A)B(x)$	$\mathbf{O} ? L^{\exists}$ or $\mathbf{O} ? R^{\exists}$	$\mathbf{P}\ p_1 : A$ (resp.) $\mathbf{P}\ p_2 : B(p_1)$	$\mathbf{P} < p_1, p_2 >: (\exists x : A)B(x)$
Subset separation	$\mathbf{P} ! \{x : A \mid B(x)\}$	$\mathbf{O} ? L$ or $\mathbf{O} ? R$	$\mathbf{P}\ p_1 : A$ (resp.) $\mathbf{P}\ p_2 : B(p_1)$	$\mathbf{P} < p_1, p_2 >: \{x : A \mid B(x)\}$
Disjunction	$\mathbf{P} ! A \vee B$	$\mathbf{O} ?^{\vee}$	$\mathbf{P}\ p_1 : A$ or $\mathbf{P}\ p_2 : B$	$\mathbf{P}\ p_1 : A \vee B$ or $\mathbf{P}\ p_2 : A \vee B$
Implication	$\mathbf{P} ! \ A \supset B$	$\mathbf{O}\ p_1 : A$	$\mathbf{P}\ p_2 : B$	$\mathbf{P}\ p_2{}^{\mathbf{P}}[\![p_1{}^{\mathbf{O}}]\!] : A \supset B$
Universal quantification	$\mathbf{P} ! \ (\forall x : A)B(x)$	$\mathbf{O}\ p_1 : A$	$\mathbf{P}\ p_2 : B(p_1)$	$\mathbf{P}\ p_2{}^{\mathbf{P}}[\![p_1{}^{\mathbf{O}}]\!] : (\forall x : A)B(x)$

For negation, we must bear in mind that we are considering **P**-strategies, that is, plays in which **P** wins, and we are not providing particle rules with a proper challenge and defence, but we are adopting a strategic perspective on the reason to provide backing a statement; thus the response to an **O**-challenge on a negation cannot be **P** ! ⊥, which would amount to **P** losing; this statement “$\mathbf{P}\ n^{\mathbf{O}}[\![p_1{}^{\mathbf{O}}]\!] : \neg A$” indicates that **P**’s strategic reason for the negation is based on **O**’s move n (where **O** is forced to state ⊥), move n which is dependent upon **O**’s choice p_1 as local reason for the antecedent of the negation. This yields the following rule for the synthesis of the strategic reason for negation:

Table 25: synthesis of the strategic reason for negation

	Move	Challenge	Defence	Strategic reason (synthesis)
Negation	$\mathbf{P} ! \neg A$ Also expressed as $\mathbf{P} ! \ A \supset \bot$	$\mathbf{O}\ p_1 : A$	$\mathbf{O} ! \bot$ **P**’s successful defence of the negation amounts to a switch such that **O** must now state that she has a local reason for A. However this move leads her to give up by bringing forward $\bot$ (n)	$\mathbf{P}\ n^{\mathbf{O}}[\![p_1{}^{\mathbf{O}}]\!] : \neg A$ The move **O** $p_1 : A$, allows **P** to force her to give up in move n, which leads to **P**'s victory.

Correspondence between the synthesis of strategic reasons and CTT introduction rules

Since we are considering a **P**-winning strategy, we are searching what **P** needs to justify in order to justify his thesis, which is the point of the synthesis rules for strategic reasons. This search corresponds to the CTT introduction rules, since these determine what one needs in order to carry out an inference. The following table displays the correspondence between the procedures of synthesis of a strategic reason and an introduction rule, though in a notation closer to the original one of Martin-Löf for the logical interpretation of the Π- and Σ-operators than to the more modern general notation used in chapter II.

Table 26: correspondence between synthesis of strategic reasons and introduction rules

	Synthesis of P-strategic reasons:	**CTT-introduction rule:**
Existential quantification	$\boldsymbol{P}\ !\ (\exists x : A)B(x)$ $\begin{array}{c:c} \boldsymbol{O}\ ?L^{\exists} & \boldsymbol{O}\ ?R^{\exists} \\ \boldsymbol{P}\ p_1 : A & \boldsymbol{P}\ p_2 : B(p_1) \end{array}$ $\boldsymbol{P}\ \langle p_1, p_2 \rangle : (\exists x : A)B(x)$	$(\exists x : A)B(x)\ \boldsymbol{true}$ $\dfrac{p_1 : A \qquad p_2 : B(p_1)}{\langle p_1, p_2 \rangle : (\exists x : A)B(x)}$
Conjunction	$\boldsymbol{P}\ !\ A \wedge B$ $\begin{array}{c:c} \boldsymbol{O}\ ?\ L^{\wedge} & \boldsymbol{O}\ ?R^{\wedge} \\ \boldsymbol{P}\ p_1 : A & \boldsymbol{P}\ p_2 : B \end{array}$ $\boldsymbol{P}\ \langle p_1, p_2 \rangle : A \wedge B$	$A \wedge B\ \boldsymbol{true}$ $\dfrac{p_1 : A \qquad p_2 : B}{\langle\ p_1, p_2 \rangle : A \wedge B}$
Disjunction	$\boldsymbol{P}\ !A \vee B$ $\boldsymbol{O}\ ?^{\vee}$ $\begin{array}{c:c} \boldsymbol{P}\ !p_1 : A & \boldsymbol{P}\ !p_2 : B \\ \boldsymbol{P}\ p_1 : A \vee B & \boldsymbol{P}\ p_2 : A \vee B \end{array}$	$A \vee B\ \boldsymbol{true}$ $\dfrac{p_1 : A}{\boldsymbol{i}(p_1) : A \vee B} \qquad \dfrac{p_2 : B}{\boldsymbol{j}(p_2) : A \vee B}$
Implication	$\boldsymbol{P}\ !A \supset B$ $\boldsymbol{O}\ p_1 : A$ $\boldsymbol{P}\ p_2 : B$ $\boldsymbol{P}\ !{p_2}^{\boldsymbol{P}}\llbracket {p_1}^{\boldsymbol{O}} \rrbracket : A \supset B$	$A \supset B\ \boldsymbol{true}$ $\dfrac{\begin{array}{c}(x : A) \\ p_2(x) : B\end{array}}{(\lambda x)p_2(x) : A \supset B}$
Universal quantification	$\boldsymbol{P}!\ (\forall x : A)B(x)$ $\boldsymbol{O}\ p_1 : A$ $\boldsymbol{P}\ p_2 : B(p_1)$ $\boldsymbol{P}\ {p_2}^{\boldsymbol{P}}\llbracket {p_1}^{\boldsymbol{O}} \rrbracket : (\forall x : A)B(x)$	$(\forall x : A)B(x)\ \boldsymbol{true}$ $\dfrac{\begin{array}{c}(x : A) \\ p_2(x) : B(x)\end{array}}{(\lambda x)p_2(x) : (\forall x : A)B(x)}$
Negation	$\boldsymbol{P}\ !\neg A$ $\boldsymbol{O}\ \ p_1 : A$ $\vdots$ $\boldsymbol{O}\ !\ \bot\ (n)$	$\neg A\ \boldsymbol{true}$ $(x : A)$ $\vdots$ $\underline{p_2 : \bot}$

	$\boldsymbol{P}\ n^{\boldsymbol{O}}[\![p_1{}^{\boldsymbol{O}}]\!]: \neg A$	$(\lambda x)p_2(x): \neg A$

Dependences

In the case of material implication and universal quantification, a winning **P**-strategy literally displays the procedure by which the Proponent chooses the local reason for the consequent *depending* on the local reason chosen by the Opponent for the antecedent. What the canonical argumentation form of a strategic object does is to make explicit the relevant *choice-dependence* by means of a *recapitulation* of the plays stemming from the thesis.

This corresponds to the general description of proof-objects for material implications and universally quantified formulas in CTT: a method which, given a proof-object for the antecedent, yields a proof-object for the consequent.

VII.7.3 Rules for the analysis of P-strategic reasons: Strategic reasons as recapitulations of procedures of analysis and record of instructions

The analysis of **P**-strategic reasons focuses on this other essential aspect of **P**'s activity while playing: not determining *what* he needs in order to justify his statement—that aspect is dealt with by the synthesis of **P**-strategic reasons—, but determining *how* he will be able to defend his statement through **O**'s statement and through those alone; that is, the analysis of **P**-strategic reasons are a direct consequence of the Socratic rule: since **P** must defend his thesis using only the elements provided by **O**, **P** must be able to analyze **O**'s statements and find the elements he needs for the justification of his own statements, so as to force **O** to bring these elements forward during the play.

In this regard, the analysis of strategic reasons constitute both the analogue of the *elimination rules* in CTT and the *equality rules* of a type, to which we now turn.

Analysis of strategic reasons providing the equalities that will guide P's choices

As explained in chapter II, the *equality rules* for a type A determine the equalities between its elements which are introduced by the evaluation rules for the selector for A (the non-canonical operators that render the proof-objects of elimination rules for that type). Thus in the CTT-setting, the equality rules serve to justify the elimination rules by showing how the evaluation of the selectors (their computation) yield canonical elements of A.

In the dialogical framework, the analysis of strategic reasons renders the instructions (the analogue of selectors) and the equality rules for the resolutions of those instructions in such a way that in a winning strategy the result of the resolution of an instruction for an elementary A (challenged by the Opponent) is a local reason stated by the Opponent.

The strategic reason for the thesis then recapitulates not only the relevant processes of synthesis and analysis but it also includes a record of the resolution of the instructions manifesting how the process of analysis and synthesis intertwine. Accordingly, "recording resolutions" amounts to making explicit the equalities between instructions and the local reasons required by the justification of the thesis.

Since the winning strategies corresponding to CTT-demonstrations are **P**-winning strategies, the rules for the recapitulation of the procedures of analysis are fixed for **O**. From the dialogical point of view, the rules corresponding to CTT elimination rules display what **P** is entitled to state on the basis of **O**'s statements: a **P**-winning strategy includes all the **O**-statements brought forward while challenging the thesis. The procedures of analysis leading to a **P**-winning strategy thus prescribe how to analyze the **O**-statements into reasons that can be copied by **P** and that contribute to the justification of **P**'s thesis.

The case of the strategic reason for disjunction

The most striking example for the analysis of **P**-strategic reasons is probably the case of a **P**-winning strategy based on a disjunction stated by **O**: let us take the case of a **P**-winning strategy for some thesis π such that **O** has to defend at some point the disjunction $d: B \vee A$, and such that **P** needs **O** to defend this disjunction in order to win; in this context **P** has a winning strategy for his thesis because he is entitled to defend his own commitments—undertaken while stating the thesis—by analyzing precisely those statements that **O** is forced to concede while defending either side of $B \vee A$.

The rule for the analysis of local reasons for a disjunction states that the defender may choose which local reason to provide supporting the disjunct he chose. Thus if **O** states the disjunction $d: B \vee A$, and **P** challenges this disjunction, then **O** can choose to defend it by stating either $L^{\vee}(d)^{\boldsymbol{O}}: B$ or $R^{\vee}(d)^{\boldsymbol{O}}: A$. So when **P** analyzes **O**'s statement during the play, he must take **O**'s choice into account and track how **O** will resolve the instruction so that **P** can copy that resolution and not the corresponding instruction. The analysis of the **P**-strategic reason for a disjunction has thus the following form:

$$L^{\vee}(d)^{\boldsymbol{O}} = {d_1}^{\boldsymbol{P},\boldsymbol{O}} | R^{\vee}(d)^{\boldsymbol{O}} = {d_2}^{\boldsymbol{P},\boldsymbol{O}}$$

This strategic reason is constituted of two equalities, the left one being $L^{\vee}(d)^{\boldsymbol{O}} = {d_1}^{\boldsymbol{P},\boldsymbol{O}}$ and the right one being $R^{\vee}(d)^{\boldsymbol{O}} = {d_2}^{\boldsymbol{P},\boldsymbol{O}}$, separated by a bar **|** which indicates that it is **O**, not **P,** who has the choice between the two instructions. The equalities state that the instruction used by **O** to defend her disjunction (left or right side) will be resolved into a local reason chosen by **O** (d_1 and d_2 respectively), local reason that will be copied by **P** in order to support a later claim—C for instance—, the superscripts $\boldsymbol{P}$ and $\boldsymbol{O}$ indicating that the labelled player brings the local reason (${d_1}^{\boldsymbol{P}}$ for instance) or the instruction ($L^{\vee}(d)^{\boldsymbol{O}}$ for instance) forward in the play. The strategic reason for the thesis will in this respect not only *display the instructions* resulting from the analysis of d, but it will also *record the resolution of those instructions* that yield the reasons for the elementary propositions that justify the thesis.

That is, statements of the form **P** $L^{\vee}(d)^{\boldsymbol{O}} = {d_1}^{\boldsymbol{P},\boldsymbol{O}} | R^{\vee}(d)^{\boldsymbol{O}} = {d_2}^{\boldsymbol{P},\boldsymbol{O}}: C$ indicate that the strategic reason for C is the outcome of a disjunction stated by **O** such that whatever be the local reason she decides to defend her disjunction with, this reason will provide the strategic reason for **P** to state C: if the local reason adduced by **O** in defence of the disjunction is the one that is resolved by the left instruction, and if **O** uses d_1 to resolve the left instruction, then **P** will copy that local reason and use d_1 in order to defend C. The same holds for the right side, *mutatis mutandis*.

It is once again clear that a strategic reason is only a perspective on a statement which is not actually carried out in a play: both sides of the disjunction are taken into account without determining which one will be taken, thus allowing for the indetermination of **O**'s choice.

The rules of analysis of strategic reasons

Here is the table providing the rules for the analysis of **P**-strategic reasons, in which we also recall the analysis rules for local reasons, so as to make the link between the two as clear as possible. Since the rules for the analysis of strategic reasons will display equalities, we assume that the instructions are resolved into its elementary constituents.

Table 27: analysis rules for **P**-strategic reasons

	Move	Analysis of local reasons		Analysis of P-strategic reasons
		Challenge	Defence	
Conjunction	$\mathbf{O}\ p: A \wedge B$	$\mathbf{P}\ ?L^{\wedge}$ or $\mathbf{P}\ ?R^{\wedge}$	$\mathbf{O}\ L^{\wedge}(p)^{O}: A$ (resp.) $\mathbf{O}\ R^{\wedge}(p)^{O}: B$	$\mathbf{P}\ L^{\wedge}(p)^{\boldsymbol{O}} = p_1{}^{\boldsymbol{P,O}}: A$ (resp.) $\mathbf{P}\ R^{\wedge}(p)^{\boldsymbol{O}} = p_2{}^{\boldsymbol{P,O}}: B$
Existential quantifiation	$\mathbf{O}\ p: (\exists x: A)B(x)$	$\mathbf{P}\ ?L^{\exists}$ or $\mathbf{P}\ ?R^{\exists}$	$\mathbf{O}\ L^{\exists}(p)^{O}: A$ (resp.) $\mathbf{O}\ R^{\exists}(p)^{O}: B(L^{\exists}(p)^{O})$	$\mathbf{P}\ L^{\exists}(p)^{\boldsymbol{O}} = p_1{}^{\boldsymbol{P,O}}: A$ (resp.) $\mathbf{P}\ R^{\exists}(p)^{\boldsymbol{O}} = p_2{}^{\boldsymbol{P,O}}: B(p_1{}^{\boldsymbol{P,O}})$
Subset separation	$\mathbf{O}\ p: \{x : A \mid B(x)\}$	$\mathbf{P}\ ?L$ or $\mathbf{P}\ ?R$	$\mathbf{O}\ L^{\{\dots\}}(p)^{O}: A$ (resp.) $\mathbf{O}\ R^{\wedge}(p)^{O}: B(L^{\{\dots\}}(p)^{O})$	$\mathbf{P}\ L^{\{\dots\}}(p)^{\boldsymbol{O}} = p_1{}^{\boldsymbol{P,O}}: A$ (resp.) $\mathbf{P}\ R^{\wedge}(p)^{\boldsymbol{O}} = p_2{}^{\boldsymbol{P,O}}: B\left(p_1{}^{\boldsymbol{P,O}}\right)$
Disjunction	$\mathbf{O}\ p: A \vee B$	$\mathbf{P}\ ?^{\vee}$	$\mathbf{O}\ L^{\vee}(p)^{O}: A$ or $\mathbf{O}\ R^{\vee}(p)^{O}: B$	$\mathbf{P}\ L^{\vee}(d)^{\boldsymbol{O}} = d_1{}^{\boldsymbol{P,O}} \mid R^{\vee}(d)^{\boldsymbol{O}} = d_2{}^{\boldsymbol{P,O}}: C$
Implication	$\mathbf{O}\ p: A \supset B$	$\mathbf{P}\ L^{\supset}(p)^{P}: A$	$\mathbf{O}\ R^{\supset}(p)^{O}: B$	$\boldsymbol{P}\ R^{\supset}(p)^{\boldsymbol{O}} = p_2{}^{\boldsymbol{P,O}} \left[\!\left[L^{\supset}(p)^{P} = p_1{}^{\boldsymbol{P,O}} \right]\!\right]: B$
Universal quantification	$\mathbf{O}\ p: (\forall x: A)B(x)$	$\mathbf{P}\ L^{\forall}(p)^{P}: A$	$\mathbf{O}\ R^{\forall}(p)^{O}: B(L^{\forall}(p)^{P})$	$\boldsymbol{P}\ R^{\forall}(p)^{\boldsymbol{O}} = p_2{}^{\boldsymbol{P,O}}[\![L^{\forall}(p)^{\boldsymbol{P}} = p_1{}^{\boldsymbol{P,O}}]\!]: B(p_1{}^{\boldsymbol{P,O}})$
Negation	$\mathbf{O}\ p: \neg A$ Also expressed as $\mathbf{O}\ \ p: A \supset\perp$	$\mathbf{P}\ L^{\neg}(p)^{P}: A$	$\mathbf{O}\ R^{\neg}(p)^{O}: \perp$	$\boldsymbol{P}\ L^{\neg}(p)^{\boldsymbol{P}} = p_1^{\boldsymbol{P,O}}: A$
		$\mathbf{P}\ L^{\supset}(p)^{P}: A$	$\mathbf{O}\ R^{\supset}(p)^{O}: \perp$	$\mathbf{P}\ you_{gave\ up}(n) \left[\!\left[L^{\supset}(p)^{P} = p_1{}^{\boldsymbol{P,O}} \right]\!\right]: C$

Note that the analysis of strategic reasons for negation is divided into two presentations of negation, **O** $p: \neg A$ and **O** $p: A \supset\perp$, which, at the play level, are governed by SR7 (see p. 115). The first presentation yields **O** stating $\perp$, that is giving up, and therefore the play ends with **P** winning without further ado. Thus the strategic reason is constituted by the resolution of the instruction for A with the means provided by **O** $(L^{\neg}(p) \ = p_1{}^{\boldsymbol{O}})$.

The second presentation on the other hand, allows **P** to back any proposition C with the local reason '$you_{gave\ up}(n)$' once **O** has stated $\perp$ at move n. Thus the strategic reason for any proposition C is constituted by '$you_{gave\ up}(n)$', provided that **O** has provided **P** with the means for resolving the instruction $L^{\supset}(p)$.

Correspondence between the analysis of strategic reasons and CTT equality and elimination rules

We will not present here the table of correspondences since they can be reconstructed by the reader emulating the table of correspondence for procedures of synthesis. Let us only indicated that:

$$\mathbf{P}\ you_{gave\ up}(n) \left[\!\left[L^{\supset}(p)^{P} = p_1{}^{\boldsymbol{P,O}} \right]\!\right] : C$$

corresponds to the CTT-elimination-rule for *absurdity*, that is:

$$\underline{\perp \text{ true}}$$

$$C \text{ true}$$

interpreted as the fact that we shall never get an element of $\bot$, defined as the empty $\mathbb{N}_0$. More precisely, if $c : \mathbb{N}_0$, then the proof-object of C is "R_0" understood as an "aborted programme"[96]

$$\frac{c : \mathbb{N}_0}{R_0(c) : C(c).}$$

In this respect the dialogical reading of the abort-operator is that a player gives up, and the reason for the other player to state C is that the antagonist gave up.

VII.7.4 Examples for building a strategic reason

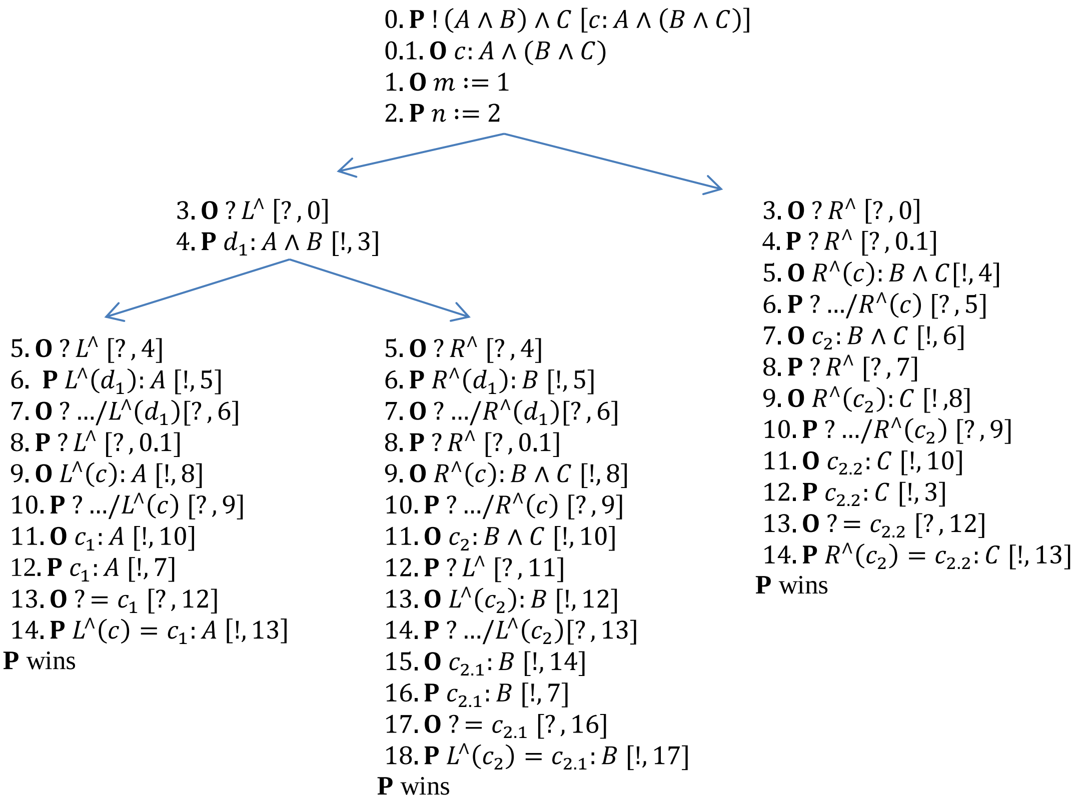
First example: $\boldsymbol{d}: (\boldsymbol{A} \wedge \boldsymbol{B}) \wedge \boldsymbol{C}\ [\boldsymbol{c}: \boldsymbol{A} \wedge (\boldsymbol{B} \wedge \boldsymbol{C})]$

How to build the strategic reason for the thesis $(A \wedge B) \wedge C\ [c: A \wedge (B \wedge C)]$, that is for $(A \wedge B) \wedge C$ provided the initial concessions $c: A \wedge (B \wedge C)$?

Building the strategic reason for the thesis:

The thesis is a conjunction. In order to have a strategic reason for the thesis, **P** must have a winning strategy: in this case he must win whether the left conjunct or the right is requested when **O** challenges the thesis; in other words, **P** must be able to defend both: the form of the strategic reason will be an ordered pair $\langle \dots, \dots \rangle$ (see the synthesis of a strategic reason for a conjunction above).

We know **P** has a winning strategy (see the heuristic procedure section VII.6.6); the core can thus be represented in a tree:



0. **P** $!\,(A \wedge B) \wedge C\ [c: A \wedge (B \wedge C)]$
0.1. **O** $c: A \wedge (B \wedge C)$
1. **O** $m := 1$
2. **P** $n := 2$

3. **O** $?L^{\wedge}\ [?, 0]$
4. **P** $d_1: A \wedge B\ [!, 3]$

5. **O** $?L^{\wedge}\ [?, 4]$
6. **P** $L^{\wedge}(d_1): A\ [!, 5]$
7. **O** $?\ .../L^{\wedge}(d_1)[?, 6]$
8. **P** $?L^{\wedge}\ [?, 0.1]$
9. **O** $L^{\wedge}(c): A\ [!, 8]$
10. **P** $?\ .../L^{\wedge}(c)\ [?, 9]$
11. **O** $c_1: A\ [!, 10]$
12. **P** $c_1: A\ [!, 7]$
13. **O** $?{=}\ c_1\ [?, 12]$
14. **P** $L^{\wedge}(c) = c_1: A\ [!, 13]$
P wins

5. **O** $?R^{\wedge}\ [?, 4]$
6. **P** $R^{\wedge}(d_1): B\ [!, 5]$
7. **O** $?\ .../R^{\wedge}(d_1)[?, 6]$
8. **P** $?R^{\wedge}\ [?, 0.1]$
9. **O** $R^{\wedge}(c): B \wedge C\ [!, 8]$
10. **P** $?\ .../R^{\wedge}(c)\ [?, 9]$
11. **O** $c_2: B \wedge C\ [!, 10]$
12. **P** $?L^{\wedge}\ [?, 11]$
13. **O** $L^{\wedge}(c_2): B\ [!, 12]$
14. **P** $?\ .../L^{\wedge}(c_2)[?, 13]$
15. **O** $c_{2.1}: B\ [!, 14]$
16. **P** $c_{2.1}: B\ [!, 7]$
17. **O** $?{=}\ c_{2.1}\ [?, 16]$
18. **P** $L^{\wedge}(c_2) = c_{2.1}: B\ [!, 17]$
P wins

3. **O** $?R^{\wedge}\ [?, 0]$
4. **P** $?R^{\wedge}\ [?, 0.1]$
5. **O** $R^{\wedge}(c): B \wedge C[!, 4]$
6. **P** $?\ .../R^{\wedge}(c)\ [?, 5]$
7. **O** $c_2: B \wedge C\ [!, 6]$
8. **P** $?R^{\wedge}\ [?, 7]$
9. **O** $R^{\wedge}(c_2): C\ [!, 8]$
10. **P** $?\ .../R^{\wedge}(c_2)\ [?, 9]$
11. **O** $c_{2.2}: C\ [!, 10]$
12. **P** $c_{2.2}: C\ [!, 3]$
13. **O** $?{=}\ c_{2.2}\ [?, 12]$
14. **P** $R^{\wedge}(c_2) = c_{2.2}: C\ [!, 13]$
P wins

---

[96] See (Martin-Löf, 1984, pp. 66-67) .

To build the strategic reason for the thesis, let us start with the right conjunct, C. We must look at the end of a branch in the winning strategy where C is the last **P**-move: this is move 14 at the outmost right branch; but move 14 defends against **O**'s challenge in move 13 on the local reason chosen by **P** for his elementary move 12, which synthesizes the local reason for the right-hand side of the thesis (challenged move 3). **P** chose this local reason because that is the one **O** herself chose (move 11); this choice dependence is what the equality move 14 expresses. The strategic reason for the thesis can have this form (leaving the left-hand side of the conjunction for later):

$$\langle \ldots, R^{\wedge}(c_2) = c_{2.2} \rangle : (A \wedge B) \wedge C[c : A \wedge (B \wedge C)]$$

The explicit rendering of the embeddings encoded by $R^{\wedge}(c_2)$ yields:

$$\langle \ldots, R^{\wedge}\big(R^{\wedge}(c)\big) = c_{2.2} \rangle : (A \wedge B) \wedge C \;\; [c : A \wedge (B \wedge C)]$$

Let us now explicit the left-hand side of the strategic reason for the conjunction: since the left side of the thesis is also a conjunction ($A \wedge B$), the argumentative canonical form of the strategic reason will also be a pair:

$$\langle \langle \ldots, \ldots \rangle, R^{\wedge}\big(R^{\wedge}(c)\big) = c_{2.2} \rangle : (A \wedge B) \wedge C \; [c : A \wedge (B \wedge C)]$$

This pair will be made of the left and the right of d_1, the local reason synthesized by **P** move 3 for the complex statement $A \wedge B$. Thus the strategic reason will be:

$$\langle \langle L^{\wedge}(d_1), R^{\wedge}(d_1) \rangle, R^{\wedge}\big(R^{\wedge}(c)\big) = c_{2.2} \rangle : (A \wedge B) \wedge C \; [c : A \wedge (B \wedge C)]$$

The outmost left branch of the tree tells us that

$$L^{\wedge}(c) = c_1 / L^{\wedge}(d_1) : A,$$

and the middle branch tells us that

$$L^{\wedge}(c_2) = c_{2.1} / \, R^{\wedge}(d_1) : B$$

and the explicit rendering of the embeddings encoded by $L^{\wedge}(c_2)$ yields:

$$L^{\wedge}(R^{\wedge}(c)) = c_{2.1} / \, R^{\wedge}(d_1) : B$$

So putting it now all together and implementing the substitution of $L^{\wedge}(d_1)$ and $R^{\wedge}(d_1)$ we obtain:

$$\langle \langle L^{\wedge}(c) = c_1, L^{\wedge}\big(R^{\wedge}(c)\big) = c_{2.1} \rangle, R^{\wedge}\big(R^{\wedge}(c)\big) = c_{2.2} \rangle : (A \wedge B) \wedge C[c : A \wedge (B \wedge C)]$$

If we do not take into consideration the equalities we obtain the following strategic reason for the thesis:

$$\langle \langle L^{\wedge}(c), L^{\wedge}(R^{\wedge}(c)) \rangle, R^{\wedge}\big(R^{\wedge}(c)\big) \rangle : (A \wedge B) \wedge C \; [c : A \wedge (B \wedge C)]$$

Second example: $(B \wedge A) \supset C \; [c : (A \wedge B) \supset C]$

Develop a demonstration of $(B \wedge A) \supset C$, given the global assumption $c : (A \wedge B) \supset C$, out of the strategic core.

Resolution: first, we start by displaying the core in the tree presentation (see section VII.6.7 for the heuristical procedure):

$0.$ **P** $!\,(B \wedge A) \supset C\ [c : (A \wedge B) \supset C]$
0.1 **O** $c : (A \wedge B) \supset C$
$1.$ **O** $m := 1$
$2.$ **P** $n := 2$
$3.$ **O** $d_1 : B \wedge A\ [?, 0]$
$4.$ **P** $?\,L^{\wedge}\ [?, 3]$
$5.$ **O** $L^{\wedge}(d_1) : B\ [!, 4]$
$6.$ **P** $?\ldots/L^{\wedge}(d_1)[?, 5]$
$7.$ **O** $d_{1.1} : B\ [!, 6]$
$8.$ **P** $?\ R^{\wedge}\ [?, 3]$
$9.$ **O** $R^{\wedge}(d_1) : A\ [!, 8]$
$10.$ **P** $?\ldots/R^{\wedge}(d_1)\ \ [?, 9]$
$11.$ **O** $d_{1.2} : A\ [!, 10]$
$12.$ **P** $L^{\supset}(c) : A \wedge B\ [?, 0.1]$
$13.$ **O** $?\ldots/L^{\supset}(c)\ [?, 12]$
$14.$ **P** $c_1 : A \wedge B\ [!, 13]$

$15.$ **O** $?\,L^{\wedge}\ [?, 14]$	$15.$ **O** $?\,R^{\wedge}\ [?, 14]$	$15.$ **O** $R^{\supset}(c) : C\ [!, 12]$
$16.$ **P** $L^{\wedge}(c_1) : A\ [!, 15]$	$16.$ **P** $R^{\wedge}(c_1) : B\ [!, 15]$	$16.$ **P** $?\ldots/R^{\supset}(c)[?, 15]$
$17.$ **O** $?\ldots/L^{\wedge}(c_1)\ [?, 16]$	$17.$ **O** $?\ldots/R^{\wedge}(c_1)\ [?, 16]$	$17.$ **O** $c_2 : C\ [!, 16]$
$18.$ **P** $d_{1.2} : A\ [!, 17]$	$18.$ **P** $d_{1.1} : B\ [!, 17]$	$18.$ **P** $c_2 : C\ [!, 3]$
$19.$ **O** $?= d_{1.2}\ [?, 18]$	$19.$ **O** $?= d_{1.1}\ [?, 18]$	$19.$ **O** $?= c_2\ [?, 18]$
$20.$ **P** $R^{\wedge}(d_1) = d_{1.2} : A\ [!, 19]$	$20.$ **P** $L^{\wedge}(d_1) = d_{1.1} : B\ [!, 19]$	$20.$ **P** $R^{\supset}(c) = c_2 : C\ [!, 19]$
P wins	**P** wins	**P** wins

Recapitulation: building the strategic reason for the thesis. The thesis is an implication, so the form of the strategic reason will be

$$\ldots^{\mathbf{P}}[\![\ldots^{\mathbf{O}}]\!]$$

The right-hand side of the strategic reason is what **O** brings forward to back her stating the antecedent of the implication: this is the synthesis of the local reason she has to carry out in order to challenge the thesis, move 3:

$$\ldots^{\mathbf{P}}[\![d_1^{\mathbf{O}}]\!]$$

But the antecedent for the implication (the thesis) is a conjunction $(B \wedge A)$, so d_1 is a reason for a conjunction, that is, d_1 is a pair:

$$\langle L^{\wedge}(d_1), R^{\wedge}(d_1)\rangle : B \wedge A$$

With the equalities (moves 5-7 and 9-11) we obtain the analysis of a strategic reason for conjunction:

$$\langle L^{\wedge}(d_1) = d_{1.1}, R^{\wedge}(d_1) = d_{1.2}\rangle : B \wedge A$$

This is therefore what **O** entitles **P** to state by challenging his implication (the thesis).

The left-hand side of the strategic reason for the thesis backs the consequent of the implication: **P** has here to play an elementary statement which therefore needs to be grounded in **O**-moves; this is the purpose of the equality move 20. So the form of the strategic reason will be:

$$\left(R^{\supset}(c) = c_2\right)^{\mathbf{P}}[\![\langle L^{\wedge}(d_1) = d_{1.1}, R^{\wedge}(d_1) = d_{1.2}\rangle]\!]^{\mathbf{O}} : (B \wedge A) \supset C\ [c : (A \wedge B) \supset C]$$

But $R^{\supset}(c)$ depends on **P** stating $L^{\supset}(c) : A \wedge B$ move 12, a conjunction; the strategic reason for it is a pair, and move 14 shows that $L^{\supset}(c) = c_1$ and the two left branches provide the internal structure of c_1:

$$\langle L^{\wedge}(c_1), R^{\wedge}(c_1)\rangle : A \wedge B$$

The resolution of the instructions and the final equalities provide the structure of $L^{\supset}(c) = c_1$:

$$\langle L^{\wedge}\big(L^{\supset}(c)\big)^{\mathbf{P}} = R^{\wedge}(d_1)^{\mathbf{P,O}}, R^{\wedge}\big(L^{\supset}(c)\big)^{\mathbf{P}} = L^{\wedge}(d_1)^{\mathbf{P,O}}\rangle : A \wedge B$$

So the analysis of the initial concession $c\colon (A \wedge B) \supset C$ yields:

$$(R^{\supset}(c) = c_2)^{\mathbf{O}} \ \left[\!\left[\langle L^{\wedge}\big(L^{\supset}(c)\big)^{\mathbf{P}} = R^{\wedge}(d_1)^{\mathbf{P,O}}, R^{\wedge}\big(L^{\supset}(c)\big)^{\mathbf{P}} = L^{\wedge}(d_1)^{\mathbf{P,O}}\rangle\right]\!\right]$$

So the strategic reason for the thesis $(\boldsymbol{B} \wedge \boldsymbol{A}) \supset \boldsymbol{C}\ [\boldsymbol{c}\colon (\boldsymbol{A} \wedge \boldsymbol{B}) \supset \boldsymbol{C}]$ becomes:

$$\left(R^{\supset}(c)^{\mathbf{O}} \left[\!\left[\langle L^{\wedge}\big(L^{\supset}(c)\big)^{\mathbf{P}} = R^{\wedge}(d_1)^{\mathbf{P,O}}, R^{\wedge}\big(L^{\supset}(c)\big)^{\mathbf{P}} = L^{\wedge}(d_1)^{\mathbf{P,O}}\rangle\right]\!\right]\right)^{\mathbf{P}} [\![\langle L^{\wedge}(d_1), R^{\wedge}(d_1)\rangle]\!]^{\mathbf{O}}$$

VIII. THE REMARKABLE CASE OF THE AXIOM OF CHOICE[97]

It is rightly said that the principle of set theory known as the *Axiom of Choice* is "probably the most interesting and in spite of its late appearance, the most discussed axiom of mathematics, second only to Euclid's Axiom of Parallels which was introduced more than two thousand years ago" (Fraenkel, Bar-Hillel, & Levy, 1973).[98]

According to Ernst Zermelo's formulation of 1904, the Axiom of Choice amounts to the claim that, given any family *A* of non-empty sets, it is possible to select a single element from each member of *A*. The selection process is carried out by a function ***f*** with domain in *M*, such that for any non-empty set *M* in *A*, ***f***(*M*) is an element of *M*. The axiom encountered resistance from its very beginnings, and has triggered heated foundational discussions concerning, among others, mathematical existence and the notion of mathematical objects, in particular of mathematical functions. With time however, the foundational and philosophical reluctances faded and were replaced with a kind of praxis-driven view accepting the Axiom of Choice as a kind of postulate—rather than as an axiom whose truth is manifest—necessary for the practice and development of mathematics.

The foundational discussions around the Axiom of Choice (AC) found an unexpected revival around 1980 when Per Martin-Löf showed that in constructive logic—a logic that does not presuppose the Excluded Middle—the AC in its intensional version is logically valid, and that this logical truth follows naturally (almost trivially) from the constructive meaning of the quantifiers involved. More than being a simple postulate, this form of constructive "evidence" is what makes it a proper *axiom*. As for the extensional version, it can also be proved, though either the Excluded Middle or the unicity of the function must be assumed. Martin-Löf's proof, for which he was awarded the prestigious Kolmogorov prize, showed that the old discussions were rooted in an even older conceptual problem: the tension there is between *intension* and *extension*.

Jaako Hintikka has in turn tackled the AC with a game-theoretical interpretation[99]—though he did not take Martin-Löf's proof into account, presumably because he was not in favour of the constructivist approaches. As opposed to Hintikka's game-theoretical approach, the *dialogical* take on the AC does not require an unaxiomatizable language such as the one underlying Independent Friendly logic (IF-logic). Hintikka is certainly right in stressing the aptness of the game-theoretical interpretation of the AC, yet we contend that he is mistaken in what concerns the theory of meaning such an interpretation requires. As pointed out by Jovanovic (2013), Hintikka's claim, that a game-theoretical perspective on Zermelo's AC in a first-order logic was perfectly acceptable for Constructivists, has found sound confirmation in the dialogical approach to Constructive Type Theory, albeit no underlying IF-semantics is required in this framework. Ironically, Hintikka's formulation of the AC, fully spelled out, yields Martin-Löf's own CTT-formulation, making it constructivist-friendly after all.

---

[97] This section is based on (Clerbout & Rahman, 2015) and (Rahman, Clerbout, & Jovanovic, 2015), where we developed a complete demonstation of AC but with a slighlty different dialogical setting.

[98] See also (Bell, 2009).

[99] See for example (Hintikka, 1996b; 2001).

By dealing with the case of the AC in dialogues for immanent reasoning, we intend to exhibit the expressive power of this constructivist game-theoretical framework based on equality in action.

Our point in bringing up this case study is thus to show two perks of our framework, dialogues for immanent reasoning:

- first, that it is possible, through the CTT approach, to distinguish at the object-language level the *intensional formulation* of the AC—demonstrable in CTT—and its *extensional formulation*—demonstrable only if it is assumed there is only one choice function. This will be expounded on in a first section (VIII.1).
- Second, that the dialogical formulation of the intensional version of the AC—also called the *Principle of Dependent Choices*—plainly shows how the evidence of its logical truth amounts to developing a winning strategy in which the consequent of the relevant main implication follows from an interactive take on the meaning of the antecedent. This will be in a following section (VIII.2).

  The construction of this meaning is relevant to immanent reasoning or equality in action, since it is built out of the interaction of the two players generating dependences between moves according to the choices attached with the embedding of an existential quantifier—defender's choice—within a universal quantifier—challenger's choice. In such a perspective, the logical truth of the Principle of Dependent Choices is rooted in the equality between the reasons grounding the antecedent on the one hand and the reasons grounding the consequent on the other.

VIII.1 The intensional and extensional versions of the Axiom of Choice

In order to show the expressive power of CTT, and in particular how it allows to exhibit quantifier interdependence at the object-language level, we will present the CTT proof of the Axiom of Choice.

Zermelo's formulation

The Axiom of Choice (AC) was first introduced by Zermelo (1904) in order to prove Cantor's theorem: every set can be rendered in such a way as to be well ordered. Zermelo gave two formulations of this axiom: one in 1904, and a second one in 1908. The second formulation is relevant for our discussion, since it relates to both Martin-Löf's and the game-theoretical formalization, and is spelled out in the following fashion:

> *A set S that can be decomposed into a set of disjoint parts $A, B, C, \ldots$, each containing at least one element, possesses at least one subset S_1 having exactly one element with each of the parts $A, B, C, \ldots$, considered.*[100] (Zermelo, 1908b)

The AC immediately attracted a lot of attention and both of its formulations were criticized by constructivists such as René-Louis Baire, Émile Borel, Henri-Léon Lebesgue and Luitzen Egbertus Jan Brouwer. The first objections were related to the non-predicative character of the axiom, for a certain choice function was supposed to exist

---

[100] Cited from the English translation (van Heijenoort, 1967, p. 186). The original is in (Zermelo, 1908a, p. 110)

without constructively showing that it does. The axiom however found its way into the ZFC set theory and was finally accepted by the majority of mathematicians because of its usefulness in different branches of mathematics.

Martin-Löf's proof: separating an extensional and an intensional version

Martin-Löf's proof of the AC (provided below, p. 150) is produced in a constructive setting (CTT) and brings together two seemingly incompatible perspectives on this axiom, namely

1. Bishop's surprising observation from (1967): A choice function exists in *constructive mathematics*, because a choice is implied by the very meaning of existence; and
2. the proofs by Diaconescu (1975) and by Goodman and Myhill (1978) that the AC implies the Excluded Middle.

These two perspectives seem incompatible because Constructivists do not presuppose the Excluded Middle: Bishop's observation would thus seem to entail the Excluded Middle in a constructive setting if we also accept the second perspective. The solution resides in distinguishing two versions of the AC: an intensional and an extensional version.

In his (2006) paper, Martin-Löf thus shows that there are versions of the AC that are perfectly acceptable for Constructivists, namely those in which the choice function is defined *intensionally*. But in order to be able to formulate the intensional version of the AC, the axiom must be expressed within a CTT-framework. Such a setting allows the extensional and the intensional formulations of the axiom to be compared: we find that it is in fact the *extensional* version of the AC that implies the Excluded Middle, and that Bishop's remark is compatible with the intensional version of the AC.[101]

Bishop and choice in the meaning of existence

In harmony with Bishop's remark that the meaning of existence entails a choice, the truth of the AC follows rather naturally in CTT from the meaning of the quantifiers: take for instance the proposition $(\forall x : A)\ P(x)$ where $P(x)$ is of the type proposition (**prop**), provided that x is an element of the set A. If the proposition is true in a constructive setting, then there is a proof for it. Such a proof is a function that renders a proof of $P(x)$ for every element x of A. Thus Bishop's remark should be understood as follows:

1. since in a constructive setting the truth of a universal quantification amounts to the existence of a proof, and
2. since this proof is a function,
3. then the truth of a universal quantification amounts in a constructive setting to the existence of such a function.

From this CTT characteristic the proof of the AC can be developed quite straightforwardly. If we recall that in the CTT-setting

- the existence of a function from A to B amounts to the existence of a proof-object for the universal *every A is B*; and that
- the proof of the proposition $B(x)$, existentially quantified over the set A, amounts to a pair such that the first element of the pair is an element of A and the second element of the pair is a proof of $B(x)$;

---

[101] See for instance this observation of (Martin-Löf, 2006, p. 349): "[…] this is not visible within an extensional framework, like Zermelo-Fraenkel set theory, where all functions are by definition extensional."

then a full-fledged formulation of the AC—more precisely, of the *Principle of Dependent Choices* (PDC)—follows, in which we explicit the set over which the existential quantifiers are defined:

Figure 1: the intensional formulation of the Axiom of Choice:

$$(\forall x\colon A)\big(\exists y : B(x)\big)C(x,y) \supset \big(\exists f : (\forall x : A)B(x)\big)(\forall x : A)C\big(x, f(x)\big)$$

Martin-Löf's proof of the intensional version of the Axiom of Choice: [102]

> *The usual argument in intuitionistic mathematics, based on the intuitionistic interpretation of the logical constants, is roughly as follows: to prove* $(\forall x)(\exists y)C(x,y) \supset (\exists f)(\forall x)C(x, f(x))$*, assume that we have a proof of the antecedent. This means we have a method which, applied to an arbitrary* x*, yields a proof of* $(\exists y)C(x,y)$*. Let* f *be the method which, to an arbitrarily given* x*, assigns the first component of this pair. Then* $C(x, f(x))$ *holds for an arbitrary* x*, and hence, so does the consequent. The same idea can be put into symbols getting a formal proof in intuitionistic type theory. Let* $A\colon set$*,* $B(x)\colon set\ (x\colon A)$*,* $C(x,y)\colon set\ (x\colon A, y\colon B(x))$*, and assume* $z\colon (\Pi x\colon A)(\Sigma y\colon B(x))C(x,y)$*. If* x *is an arbitrary element of* A*, i.e.* $x\colon A$*, then by* Π*- elimination we obtain*
>
> $$Ap(z,x)\colon \big(\Sigma y\colon B(x)\big)C(x,y)$$
>
> *We now apply left projection to obtain*
>
> $$p\big(Ap(z,x)\big)\colon B(x)$$
>
> *and right projection to obtain*
>
> $$q\big(Ap(z,x)\big)\colon C\left(x, p\big(Ap(z,x)\big)\right).$$
>
> *By* λ*-abstraction on x (or* Π*-introduction), discharging* $x\colon A$*, we have*
>
> $$(\lambda x)p\big(Ap(z,x)\big)\colon (\Pi x\colon A)B(x)$$
>
> *and by* Π*- equality*
>
> $$Ap\left((\lambda x)p\big(Ap(z,x)\big), x\right) = \ p\big(Ap(z,x)\big)\colon Bx$$
>
> *By substitution*[103] *we get*
>
> $$C\left(x, Ap\left((\lambda x)p\big(Ap(z,x)\big), x\right)\right) = C(x, p(Ap(z,x)))$$[104]
>
> *and hence by equality of sets*
>
> $$q(Ap(z,x))\colon C\left(x, Ap\left((\lambda x)p\big(Ap(z,x)\big), x\right)\right)$$
>
> *where* $(\lambda x)p\big(Ap(z,x)\big)$ *is independent of* x*. By abstraction on* x
>
> $$(\lambda x)p(Ap(z,x))\colon (\Pi x\colon A)C\left(x, Ap\left((\lambda x)p\ \big(Ap(z,x)\big), x\right)\right)$$
>
> *We now use the rule of pairing (that is* Σ*-introduction) to get*
>
> $$\Big((\lambda x)p\big(Ap(z,x)\big), (\lambda x)q\big(Ap(z,x)\big)\Big)\colon \big(\Sigma f\colon (\Pi x\colon A)B(x)\big)(\ \Pi x\colon A)C\big(x, Ap(f,x)\big)$$
>
> *(note that in the last step, the new variable* f *is introduced and substituted for* $(\lambda x)p\big(Ap(z,x)\big)$ *in the right member). Finally by abstraction on* z*, we obtain*

---

[102] For the formal demonstration spelled out as a Natural deduction tree, see (Clerbout & Rahman, 2015, p. 78) and (Rahman, Clerbout, & Jovanovic, 2015, p. 204).

[103] making use of $C(x,y)\colon set\ \big(x\colon A, y\colon B(x)\big)$.

[104] That is, $C(x, Ap((\lambda x)\ p\ (Ap(z,x), x) \ = \ C\left(x, p\big(Ap(z,x)\big)\right)\colon set.$

$$(\lambda z)\left((\lambda x)p\big(Ap(z,x)\big),(\lambda x)q\big(Ap(z,x)\big)\right):(\Pi x{:}\,A)\big(\Sigma y{:}\,B(x)\big)C(x,y)$$
$$\supset\big(\Sigma f{:}\,(\Pi x{:}\,A)B(x)\big)(\Pi x{:}\,A)C\big(x,Ap(f,x)\big)^{105}$$

Martin-Löf (2006) further shows that—from a constructive point of view—what is wrong with the AC is the extensional formulation of it, formulation that Hintikka seems to assume and that can be expressed in the following way:

Figure 2: the extensional version of the Axiom of Choice:

$$(\forall x{:}\,A)\big(\exists y : B(x)\big)C(x,y)\supset\big(\exists f{:}\,(\forall x{:}\,A)B(x)\big)\left(Ext(f)\wedge(\forall x{:}\,A)C\big(x,f(x)\big)\right)$$

Where $Ext(f)$ is defined as $(\forall i,j{:}\,A)\big(\ i =_A j \rightarrow f(i) = f(j)\big)$.

Thus, from the constructive point of view, what is really wrong with the classical formulation of the AC is the assumption that from the truth that all of the As are Bs we can obtain a function that satisfies extensionality. In fact, as shown by Martin-Löf (2006), the classical version holds, even constructively, if we assume that there is only one such choice function in the set at stake:

Figure 3: the constructive extensional formulation of the AC:

$$(\forall x{:}\,A)\big(\exists!\, y{:}\,B(x)\big)C(x,y)\supset\big(\exists f{:}\,(\forall x{:}\,A)B(x)\big)\left(Ext(f)\wedge(\forall x{:}\,A)C\big(x,f(x)\big)\right)$$

Conclusion on the two formulations of the AC

Let us retain that if we take

$$(\forall x :\ A)\left(\exists y : B(x)C(x,y)\supset\big(f{:}\,(\forall x{:}\,A)B(x)\big)(\forall x{:}\,A)C\big(x,f(x)\big)\right)$$

to be the formalization of the AC, or more precisely of the PDC, then that axiom is not only unproblematic for Constructivists but it is also a theorem. In fact, it is the explicit language of CTT that allows a fine-grained distinction between the two formulations of the AC, equivalent only from the outside. This is due to the expressive power of CTT that allows expressing at the object-language level *quantifier interdependence*.

What is more, Hintikka's insight that the validity of the AC results from the fact that a winning strategy for the antecedent amounts to the existence of a suitable function, which seems to sum up the idea behind the demonstration of Martin-Löf[106], is that what makes apparent the truth of the PDC is the game-theoretical approach to the meaning of the quantifiers. However, despite Hintikka's own developments, it is the dialogical approach to CTT that actually does the job. So now that the usefulness of CTT has been illustrated and the two versions of the AC defined, let us proceed to our second point: showing how the PDC in the framework of dialogues for immanent reasoning explicit the interdependency of the quantifiers through equality in action.

VIII.2 The Principle of Dependent Choices within dialogues for immanent reasoning[107]

Since dialogues for immanent reasoning is a dialogical framework that incorporates the features of CTT, it is at least as expressive as CTT, and can thus also differentiate between an extensional version of the AC and an intensional version (the PDC). The

---

[105] (Martin-Löf, 1984, pp. 50-51)

[106] See (Jovanovic, 2013).

[107] The proof stems from (Clerbout & Rahman, 2015).

particle rules provide the meaning of the logical constants; the existential and the universal quantifiers have their meaning determined notably by which player has the choice—the challenger for the universal quantification, the defender for the existential quantification. What is more, the whole structure of dialogues for immanent reasoning rests on this fundamental rule, SR4 or Socratic rule (see section VII.2.1, p. 113), that explicits the dynamics of the choices of the players in interaction: the Proponent will copy the Opponent's choices and reasons in order to provide local reasons for his own statements.

The PDC embeds existential quantification within universal quantification in the antecedent of the implication, and universal quantification within existential quantification in the consequent. In dialogues for immanent reasoning, just like in the standard dialogical framework, demonstrations are obtained by providing a **P**-strategy (see III.5 and in chapter V.1), that is **P** must be able to win whatever be **O**'s choices.

Figure 4: the thesis for the PDC in dialogues for immanent reasoning:

$$\mathbf{P}\ !\ (\forall x{:}A)\big(\exists y : B(x)\big)C(x,y) \supset \big(\exists f : (\forall x : A)B(x)\big)(\forall x : A)C\big(x,f(x)\big)$$

Since the thesis is stated by **P** and is an implication, **O** will challenge it by stating the antecedent, **P** will be defending it by stating the consequent:

$$\mathbf{O}\ !\ (\forall x{:}A)\big(\exists y : B(x)\big)C(x,y)$$

$$\mathbf{P}\ !\ \big(\exists f : (\forall x : A)B(x)\big)(\forall x : A)C\big(x,f(x)\big)$$

In this fashion, **O** will have to defend the embedding of the existential within the universal, and **P** will have to defend the embedding of the universal within the existential.

In other words, **O** will have to defend the embedding of her choice (defence of the existential) within a **P**-choice (defence of a universal):

$$\mathbf{O}\ !\ (\forall x{:}A)^{\mathbf{P}\ choice}\big(\exists y : B(x)\big)^{\mathbf{O}\ choice}C(x,y)$$

and **P** will have to defend the embedding of an **O**-choice (defence of a universal) within her choice (defence of an existential):

$$\mathbf{P}\ !\ \Big(\exists f : (\forall x : A)^{\mathbf{O}\ choice}B(x)\Big)^{\mathbf{P}\ choice}(\forall x : A)C\big(x,f(x)\big)$$

Playing his moves in an optimal fashion will thus allow **P** to ask **O** to choose first and then copy her choices in order to build an equality through interaction and be able to win. The basic idea is that **P** can copy **O**'s choice for y in the antecedent for his defence of $f(x)$ in the consequent since both are equal objects of type $B(x)$, for any $x : A$. Thus a winning strategy for the implication follows simply from the meaning of the antecedent: this meaning is defined by the dependences generated by the interaction of choices involved in the embedding of an existential quantifier within a universal quantifier.

The following two dialogue tables display the relevant plays for building a winning strategy. These two plays are triggered by **O**'s decision-options concerning the existential in the consequent (stated by **P**): she can ask for the left or for the right (move 5), **P** must be able to win in both cases if he is to have a winning strategy.

Here is a reminder of some particle rules of dialogues for immanent reasoning:

Table 28: recalling the synthesis of local reasons

	Move	**Challenge**	**Defence**
Implication	$\mathbf{X}\ !\ A \supset B$	$\mathbf{Y}\ p_1{:}A$	$\mathbf{X}\ p_2{:}B$

Existential quantifiation	$\mathbf{X}\,!\,(\exists x : A)B(x)$	$\mathbf{Y}\,?\,L^{\exists}$ or $\mathbf{Y}\,?\,R^{\exists}$	$\mathbf{X}\,p_1 : A$ (resp.) $\mathbf{X}\,p_2 : B(p_1)$
Universal quantification	$\mathbf{X}\,!\,(\forall x : A)B(x)$	$\mathbf{Y}\,p_1 : A$	$\mathbf{X}\,p_2 : B(p_1)$

We assume by now that the reader is familiar enough with the dialogical framework and with this kind of table presentation; the comments will therefore be more to the point in explaining what is specific to the PDC. Are highlighted the moves where **P** defends his choice of a local reason by stating an equality, the crucial move defining immanent reasoning where **P** says "my reason is the same as yours".

Play 15: the PDC (left decision-option)

O			**P**		
0.1 0.2	$C(x, y)$: **set** $[x : A, y : B(x)]$ $B(x)$: **set** $[x : A]$			$! \ (\forall x : A)\ (\exists y : B(x))\ C(x,y) \supset (\exists f : (\forall x : A)\ B(x))\ (\forall x : A)\ C(x, f(x))$	0
1	m:= 1			n:= 2	2
3	$d_1 : (\forall x : A)\ (\exists y : B(x))\ C(x, y)$	0		$d_2 : (\exists f : (\forall x : A)\ B(x))\ (\forall x : A)\ C(x,f(x))$	4
5	$?_L$	4		$L^{\exists}(d_2) : (\forall x : A)\ B(x)$	6
7	? --- / $L^{\exists}(d_2)$	6		$g_1 : (\forall x : A)\ B(x)$	8
9	$L^{\forall}(g_1) : A$	8		$R^{\forall}(g_1) : B(a)$	24
11	$a : A$		13	? --- / $L^{\forall}(g_1)$	10
17	$R^{\forall}(d_1) : (\exists y : B(a))\ C(a, y)$		3	$L^{\forall}(d_1) : A$	12
13	? --- / $L^{\forall}(d_1)$	12		$a : A$	14
15	$? = a$	14		$L^{\forall}(g_1) = a : A$	16
19	$v : (\exists y : B(a))\ C(a, y)$		17	? --- / $R^{\forall}(d_1)$	18
21	$L^{\exists}(v) : B(a)$		19	$?_L$	20
23	$v_1 : B(a)$		21	? --- / $L^{\exists}(v)$	22
25	? --- / $R^{\forall}(g_1)$	24		$v_1 : B(a)$	26
27	$? = v_1$	26		$L^{\exists}(v) = v_1 : B(a)$	28
29	$? = a^{B(a)}$	28		$L^{\forall}(g_1) = a : A$	30
31	$? = B(a)$	30		$! \ B(L^{\forall}(g_1)) = B(a)$: **set**	34
33	$! \ B(a)$: **set**		0.2 Subst-D	$a : A$	32

Commentary:

- **Move 3:** Once the thesis has been stated and the repetition ranks established, **O** challenges the material implication by stating the left component.
- **Move 4: P** states the right component of the material implication.
- **Moves 5 and 6: O** has here the choice between asking for the left or for the right component of the existential. The present play describes the development of the play triggered by the left choice.
- **Moves 7-13** follow from a straightforward application of the dialogical rules.
- **Move 15: O** asks for the local reason that corresponds to the instruction stated by **P** at move 14.

- **Move 16: P** answers by recalling that **O** used the same reason, namely a, in defence of the claim that A holds (that it is not empty). And indeed this is what **O** did when she resolved $L^{\forall}(g_1)$ with a. This led **P** to implement the Socratic rule and state the equality $L^{\forall}(g_1) = a : A$.
- **Moves 27-28 and 29-30:** deal with the same kind of situation.
- **Moves 30-31:** Once move 30 has established that a—occurring in $B(a)$—constitutes a local reason for holding A, according to the Socratic rule S4.1c, **O** can launch an attack requesting **P** to show that $B(a)$ and $B(L^{\forall}(g_1))$ are equal propositions.
- **Moves 32-34: P** applies the rule for substitution within dependent statements (Subst-D) on the first concession, forcing **O** to state the condition that allows **P** to answer to the attack of move 31 and win the play by applying the Socratic rule to **O**'s move 25.

Notice that when **P** *attacks* **O***'s initial concession* 0.2 *he has to state a : A (move 32); according to the provisos of the Socratic rule however (see SR4.1, p. 114),* **O** *cannot attack this statement again: move 16 establishes* **P***'s right to state this equality.*

Let us now develop the second play, in which **O** went for the right decision-option at move 5.

Play 16: the PDC (right decision-option)

O				**P**	
0.1 0.2	$C(x, y)$: **set** $[x : A, y : B(x)]$ $B(x)$: **set** $[x : A]$			$! (\forall x : A) (\exists y : B(x)) C(x,y) \supset (\exists f : (\forall x : A) B(x)) (\forall x : A) C(x, f(x))$	0
1	m:= 1			n:= 2	2
3	$d_1 : (\forall x : A) (\exists y : B(x)) C(x, y)$	0		$d_2 : (\exists f : (\forall x : A) B(x)) (\forall x : A) C(x,f(x))$	4
5	$?_R$	4		$R^{\exists}(d_2) : (\forall x : A) C(x, L^{\exists}(d_2)(x)$	6
7	$? \text{---} / R^{\exists}(d_2)$	6		$g_2 : (\forall x : A) C(x, g_1(x))$	8
9	$L^{\forall}(g_2) : A$	8		$R^{\forall}(g_2) : C(x, g_1(a))$	30
11	$a : A$		11	$? \text{---} / L^{\forall}(g_2)$	10
17	$R^{\forall}(d_1) : (\exists y : B(a)) C(a, y)$		3	$L^{\forall}(d_1) : A$	12
13	$? \text{---} / L^{\forall}(d_1)$	12		$a : A$	14
15	$? = a$	16		$! L^{\forall}(g_2) = a : A$	16
19	$v : (\exists y : B(a)) C(a ,y)$		17	$? \text{---} / R^{\forall}(d_1)$	18
21	$L^{\exists}(v) : B(a)$		19	$?_L$	20
23	$t_1 : B(a)$		21	$? \text{---} / L^{\exists}(v)$	22
25	$R^{\exists}(v) : C(a, L^{\exists}(v))$		19	$?_R$	24
27	$R^{\exists}(v) : C(a, t_1)$		25	$? t_{1\,/} L^{\exists}(v)$	26
29	$t_2 : C(a, t_1)$		27	$? \text{---} / R^{\exists}(v_2)$	28
31	$? \text{---} / R^{\forall}(g_2)$	30		$t_2 : C(a, g_1(a))$	32
33	$? \text{---} / g_1(a)$	32		$t_2 : C(a, t_1)$	34
35	$? = t_2$	34		$R^{\exists}(v) = t_2 : C(a, t_1)$	36
37	$? = a^{C(a,\, t1)}$	36		$L^{\exists}(v) = t_1 : B(a)$	38
39	$? = C(a, t_1)$	38		$C(a, L^{\exists}(v)) = C(a, t_1)$: **set**	44
41	$C(a, y)$: **set** $[y : B(x)]$		0.1 Subst-D	$a : A$	40

43	$C(a, t_1)$: **set**		41 Subst-D		42

Remark: This procedure differs from the one in the previous play only with moves 40 and 44: since $C(x, y)$ is a dyadic predicate, **P** must apply substitution twice before obtaining of **O** the right to state the winning equality at move 44.

Since **P** has a way of winning regardless of **O**'s choices, that is he can win in any case, **P** has a winning strategy for the PDC.

Conclusion on the Axiom of Choice

From the constructive point of view, functions are rules of correspondence. Rules of correspondence only make sense if we know how the correspondences are to be carried out. From the dialogical (and more generally from the game theoretical) point of view a function is the result of a player choosing an object of the domain and the defender choosing the suitable match, which can be seen as carrying out a rule of correspondence. Intensional functions however cannot be directly produced from these interactions, though extensionality is also brought forward by interaction. Moreover, the dialogical take on constructivism is strongly linked to the view that in order to understand a play, and a winning strategy constituted by the relevant plays, it is not enough to know the rules of the game; it is not even enough to believe that there is a winning strategy behind the moves: what we need is to be able to describe the moves in such a way that it makes their contribution to the winning strategy understandable, that is, how the content involved in each move constitutes that strategy; in other words, a proof which would be beyond our description capacities would not produce knowledge at all. This is what Hintikka's use of Wittgenstein's notion of *human-playable games* amounts to in his (2001) work; it is what the dialogical demonstration of the PDC amounts to.[108]

Let us conclude by quoting some beautiful lines of Poincaré, his response to what he considered a purely "formalistic" approach to mathematics:

> *Si vous assistez à un partie d'échecs, il ne vous suffira pas, pour comprendre la partie, de savoir les règles de la marche des pièces. Cela vous permettrait seulement de reconnaître que chaque coup a été joué conformément à ces règles et cet avantage aurait vraiment bien peu de prix. C'est pourtant ce que ferait le lecteur d'un livre de Mathématiques, s'il n'était que logicien. Comprendre la partie, c'est tout autre chose ; c'est savoir pourquoi le joueur avance telle pièce plutôt que telle autre qu'il aurait pu faire mouvoir sans violer les règles du jeu. C'est apercevoir la raison intime qui fait de cette série de coups successifs une sorte de tout organisé. À plus forte raison, cette faculté est-elle nécessaire au joueur lui-même, c'est-à-dire à l'inventeur. (Poincaré, La valeur de la science, 1905, p. 27).*[109]

---

[108] See section XI.1 for further details on game definiteness and its crucial role in the dialogical framework.

[109] "If you are present at a game af chess, it will not suffice, for the understanding of the game, to know the rules for moving the pieces. That will only enable you to recognize that each move has been made conformably to these rules, and this knowledge will truly have very little value. Yet this is what the reader of a book on mathematics would do if he were a logician only. To understand the game is wholly another matter; it is to know why the player moves this piece rather than that otber which he could have moved without breaking the rules of the game. It is to perceive the inward reason which makes of this series of successive moves a sort of organized whole. This faculty is still more necessary for the player himself, that is, for the inventor." (Poincaré, 1907, p. 22).

IX. FROM DIALOGICAL STRATEGIES TO CTT-DEMONSTRATIONS AND BACK

IX.1 General transformation principles between CTT and immanent reasoning

In a nutshell, we take from (Rahman, Clerbout, & Keiff, 2009) the following two correspondences within a **P**-winning strategy, provided some exceptions to be discussed below:

1. The result of applying a particle rule to a **P**-move corresponds to the application of an introduction rule of a CTT-demonstration rule (provided we read the **P**-moves "bottom-up").

Table 29: correspondence between dialogical rules on P-moves and introduction rules

APPLICATION OF A DIALOGICAL RULE TO A **P**-	CORRESPONDS IN CTT TO THE *INTRODUCTION RULE FOR*
disjunction	disjunction
conjunction	conjunction
existential	existential
subset separation	subset separation
implication	implication
universal	universal

2. The result of applying a particle-rule to an **O**-move corresponds to the application of an elimination rule of a CTT-demonstration.

Table 30: correspondence between dialogical rules on **O**-moves and elimination rules

APPLICATION OF A DIALOGICAL RULE TO AN **O**-	CORRESPONDS IN CTT TO THE *ELIMINATION RULE FOR*
disjunction *(may be related to a case-dependent **P**-statement)*	disjunction
conjunction	conjunction
existential	existential
subset separation	subset separation
implication	implication
universal	universal

Notice that from the perspective of a **P**-winning strategy, challenges and defences of **P**-statements on the one hand are *duties*, which can be understood as what **P** must bring forward in order to develop a dialogical demonstration for a given particle rule; but challenges and defences on **O**-statements, on the other hand, can be understood as what **P** is *entitled* to.[110] If *duties* (or *commitments*) are understood as the normative force involved by the deployment of introduction rules of a CTT-framework, and *entitlements* as the normative force involved by the deployment of elimination rules of a CTT-

---

[110] See the introduction, section I.2.3.

framework, then the correspondence between CTT rules and dialogical rules follows naturally.

Exceptions to the general correspondence

The exceptions to the general principle that **P**-statements correspond to introduction rules and **O**-statements to elimination rules are the following:

P-statements depending upon O ⊥ moves correspond to ⊥-eliminations

When **P** makes a statement (elementary or not) adducing the local reason $you_{gave\ up}(n)$ after **O** has previously stated ⊥—the **P**-statement results of applying the elimination rule for ⊥.

P-elementary statements defended with $I = p_i{:}\,type$ correspond to definitional equalities

If an elementary **P**-statement has been challenged and defended with a statement of the form $I = p_i{:}\,type$—"I" standing for an instruction and "p_i" for a local reason—the algorithm presented in this chapter which transforms dialogical strategies into CTT demonstrations (and reversewise), will first introduce in the demonstration tree this equality as the application of a definitional equality, and will remove the **P**-elementary statement that triggered the defence. So the algorithm first inserts the equality then eliminates it.

Dialogical plays use profusely definitional equalities. In fact, in the context of immanent reasoning every use of a move based on the Socratic rule is based on such form of equality. Yet, the natural deduction demonstrations that are not in normal form do not in general require coming back to the equality backing the coordination of introduction and elimination rules.

Within the standard Natural deduction setting Π- and Σ-equalities are made explicit when eliminations introduce non-canonical proof-objects. In the dialogical setting this corresponds to the cases in which **P** chooses a resolution of an instruction (or function) that mirrors the resolution of an *embedded* instruction (or function) occurring already either in the initial concession or in the main thesis.

Let us call the resulting equalities *anaphoric-based equalities* (for short *A-equalities*). Thus, we will only retain in the tree A-equalities. Strictly speaking, plays within the core are carried out in "normal form".

P-statements which are case-dependent of O-statements set the conditions that allow drawing the conclusion of a disjunction elimination rule

In fact, case-dependent **P**-statements correspond to either one of the disjunction eliminations achieved by introductions that follow from stating each of the components of the disjunction.[111]

Applications of the Statement-Substitution rule

They correspond to substitutions on hypotheticals occurring as global assumptions (initial concessions by **O**).

---

[111] See section VII.7.3 on the argumentation form of a strategic object, in which we explicitly discuss the case of the disjunction (p. 141).

Transmission of Equality

The transmission rules for definitional equality will be introduced unmodified in the demonstration tree with their direct CTT counterpart no matter whether they have been applied to **P**-moves or to **O**-moves.

IX.2 Terminology

Let recall some terminology from previous sections and introduce some new ones.

Concessions

A *concession* is either:

(a) a move that **O** conceded as conditioning the claim of the thesis. We call this also *initial concession*. It corresponds to the notion of *global assumption* of proof-theory including *epistemic assumptions* and premises.

(b) any other **O**-move brought forward during the development of a play, while challenging a **P**-implication or a **P**-universal, or while defending an **O**-disjunction or subset separation. We call this also *local concession*. It corresponds to the notion of *local assumption* of proof-theory.

Nodes

Nodes descending from a statement: for a statement π occurring in the dialogical core $\mathbb{C}$, the nodes *descending from* π are all the nodes which are related to π by a chain of applications of dialogical rules.

Branches

Left and right branches: when the dialogical core (or the demonstration) we are building splits, we speak of the *left* and *right branches of the core* (or demonstration).We may sometimes assign an order on the branches from left to right and speak of the first branch, second branch, etc.

Dependent moves

Dependent moves: a move M *depends on* the move M' if there is a chain of applications of game rules that leads from M' to M.

Case-dependent move

Let π be some statement and p some local reason. We say that in the core $\mathbb{C}$ the move M_j **P**! π is *case-dependent upon* move $M_{i<j}$ **O**! $p : \phi$ if ϕ is a disjunction and move M_j depends upon move $M_{i<j}$.

More precisely the move **P**! π is *case-dependent* upon **O**'s disjunction ϕ iff the local reason(s) that occur in the defence of **P**'s move π is definitionally equal to one of the instructions for ϕ or if **P** is dispensed to defend π by **O**'s move ⊥ which results from the defence of ϕ.

As we will discuss below, the point in distinguishing case-dependent moves is that these moves set—from the strategic point of view—the conditions for the conclusion of a disjunction elimination rule.

A-equalities

A-equalities are anaphoric-based equalities, which are built when **P** chooses a resolution of an instruction (or function) that mirrors the resolution of an *embedded* instruction (or function) occurring already either in the initial concession or in the main thesis. See above, section IX.1.

IX.3 Part 1: from strategies to CTT demonstrations

IX.3.1 Extracting the core

We start by a method for extracting the parts of a winning strategy relevant for making it correspond to a demonstration in natural deduction; we call it *the core of a winning strategy*—or, simply, the *core* (see sections V.2 and VII.5).

In order to extract the core, we develop a method in order to

1. extract the finite part of a winning strategy;
2. disregard the formation rules involving the non-logical constants (since we are dealing with logical inferences);
3. disregard different orders of moves of the Opponent.

Once this has been achieved, we will describe the algorithm allowing us to transform the core into a CTT-demonstration. And finally we will prove the adequacy of the algorithm.

The rationale behind the operation of extracting the core is the following: because $\mathcal{S}^{\star}$ is the extensive form of a **P**-winning strategy, we know that **P** wins in every branch, so that to some extent which local reasons **O** choose**s** for the instructions does not matter.

> Let us take as example the case of a universal quantification stated by **P**. Since we assume that **P** has a winning strategy, he has a method to successfully defend his statement no matter which local reason **O** chooses for $L^{\forall}(p) : A$ (where A is a set). This yields a natural deduction description of the Introduction rule for universal quantification with an implicit interlocutor: whatever **O** brings forward as proof-object for the antecedent **P** has a method to transform it into a proof-object of the consequent. Hence, it is harmless to keep only one *representative* of the possible choices by **O** because the existence of a winning **P**-strategy ensures that there is indeed a successful method for every possible **O-**choice.

IX.3.2 General procedure and EPI

The next step is to apply transformations to this core until we obtain a CTT demonstration.[112]

The transformation algorithm will *re-write* the tree that represents the core $\mathbb{C}$ of the strategy by means of a step-by-step procedure specified below. One important issue is that the re-writing procedure will ignore the following:

- The players' identities.
- The moves where the choices of the repetition ranks are made explicit.
- Questions. Strictly speaking, only statements will be incorporated in the demonstration resulting from the translation algorithm. Thus, questions will not be re-written as separate step; they however have an important role in the transformation-procedure, which will be described below.

---

[112] The section strongly relies on (Clerbout & Rahman, 2015), implementing in the CTT-framework the ideas developed in (Rahman, Clerbout, & Keiff, 2009) and (Clerbout, 2014a; 2014b) in the standard framework for FOL.

Thus, important information will appear to be lost during the transformation procedure, though actually the procedure is only making it implicit : the other direction of the transformation (from CTT demonstrations to strategies) will show how to bring this information back, showing that dialogues for immanent reasoning make explicit what is implicit in a CTT demonstration.

Remarks:

1. We adopt here the natural deduction style of demonstrations used by Martin-Löf rather than the turnstile notation deployed in the introductory chapter on CTT.
2. The dialogical demonstrations will assume that if there are initial concessions, they have already been granted by the Opponent.

EPI: Resolution and Substitution of Instructions (SR3-4)

The operation we describe hereafter consists in replacing the challenged instruction with the local reason (chosen as response to the challenge) while placing move n in the demonstration under way, and ignoring those equalities that are not *A-equalities*.[113]

One of the cases for which it is important that the algorithm does not ignore the question mark '?' is the application of the structural rules *SR*3-4 (see section VII.2.1) allowing the resolution and substitution of instructions by local reasons. The algorithm described below takes them into account through the following operation which we shall refer as *Endowing local reasons to Instructions* (EPI):

EPI operation: Endowing local reasons to Intructions

Assume that some instruction occurs in move number n.

1. Scan the core $\mathbb{C}$: if move n is challenged by a question of the form "? --- /", or "? -π-/" or "?I/p" for some instruction *I* and some local reason p, then scan $\mathbb{C}$ in search for the defence.
2. Write the replacement-process in the following way (only once): the instruction at the bottom and the resolution on the top, *without the request*.
3. Once such a replacement has been carried out it will be systematically implemented in every further stage of the construction of the demonstration.

The EPI operation ensures that equalities which stem from the resolution of non-embedded instructions will be taken into account during the procedure. Recall that the operation is not necessary for the so-called *A*-equalities—which correspond to resolutions of embedded instructions—because the algorithm already accounts for them as definitional equalities (as explained above, section IX.1, and described in details in step B.2.b of the algorithm).

Remark

The distinction between resolutions of non-embedded instructions and *A*-equalities is a technical by-product of the conceptual work on equalities and on immanent reasoning carried out in the present study, and which was not fully grasped in (Clerbout & Rahman, 2015); one consequence of this distinction is that the EPI operation will not, in general,

---

[113] However, as specified below, the equalities that are not deployed in the demonstration tree are nevertheless useful at the strategic level in order to find out the justification of elementary **P**-statements.

need to be applied as much as it was in (Clerbout & Rahman, 2015). Thus the present version of the algorithm is more precise when it comes to account for definitional equalities when building CTT demonstrations out of dialogical winning strategies.

IX.3.3The algorithm

The procedure we describe hereafter is a kind of rearrangement of (some of) the nodes in $\mathbb{C}$ which eventually produces a CTT demonstration. For convenience we assume that we have an unmodified "copy" of $\mathbb{C}$ to which we can refer to while the procedure goes on. The last stage (**D**) of the procedure requires some explanation: we provide it after the procedure.

Step A. Initial stage: placing the conclusion and premises.

Let π be the thesis stated by **P** and $\gamma_1, \dots, \gamma_n$ be the initial concessions by **O**, if any. Place π as the conclusion of the demonstration under construction and$\gamma_1, \dots, \gamma_n$ as its upmost premisses:[114]

$$\begin{array}{c} \gamma_1, \dots, \gamma_n \\ \vdots \\ \pi \end{array}$$

Then go to step **B**.

Step B.

Consider the lowest expression π_i just added in the branch of the demonstration tree under construction (in the first stage it will be the thesis).

Find the move in the core $\mathbb{C}$ that responds to it (in the first stage of the procedure it will be a challenge of the thesis).

Scan $\mathbb{C}$ in order to identify the challenge and the defence resulting from the application of the particle rule relevant to this expression. Then:

B.0. EPI-operation

If applicable, implement the EPI-operation to the expressions present in what we have so far in the branch of the demonstration; that is, replace instructions by local reasons, place within the demonstration the equalities used either for statement-substitutions or substitutions involving instructions.[115] Recall that the EPI does not need to be applied to *A*-equalities (these are dealt with in step **B.2.b**).

B.1. Moving to step C

If the relevant challenge and defence have already been accounted for in the branch being constructed, then go to **C**. Otherwise go to **B.2**.

B.2. Accounting for the relevant challenge and defence

If the defence is a **P**-elementary expression, apply stage **B.2.a**. Otherwise, implement step **B.2.b**.

---

[114] According to the above Remark 2, we ignore the first steps of a dialogue where the initial assumptions (if any) of the thesis are written to the right of the thesis, and start the algorithm assuming straightaway that the initial concessions have already been settled and the thesis displays the local reason that resulted after **O** stated the initial concessions.

[115] Ramifications will be dealt with further on in the algorithm.

B.2.a. P-elementary expressions

The present step concerns occurrences of **P**-elementary expressions π_i in the core that *are not* dependent upon an **O** ⊥-move.

Place π_i in the demonstration, draw an inference line above it and label it *SR* (short for the application of some form of the Socratic rule, as defined in rule SR5).

If the corresponding **O**-move (that allowed **P** to state π_i) has already been accounted for in the branch, then rearrange it so that it is placed as the premiss of the application of the rule. Find the **O**-move by searching for the relevant equality. *All* of the equalities in the core will indicate precisely the **O**-move relevant for the elementary expression at stake.

Go back to **B**.

B.2.b. Other expressions

The present step concerns non **P**-elementary expressions π_i.

Draw an inference line above π_i and label it with the relevant name of the rule.

Place the defence—and the challenge, if relevant[116]—as the premiss (or premisses) for the application of the rule, according to the following conventions:

1) In the cases of the Introduction rule for implication, negation, or universal quantification, the defence is the immediate premiss and the challenge is placed upwards as an assumption such that:
    i. the defence depends on that assumption,
    ii. the assumption is numbered and marked as discharged at the inference step, and
    iii. the assumption is still in the scope of previously placed assumptions such as the premisses of the demonstration placed in stage **A**.

2) Here we apply the correspondences (given above) between player-moves and natural deduction steps in the following way:[117]
    i. If π_i is an **O**-implication move, we are facing an elimination rule: the premiss is constituted by that move and its challenge. The conclusion is the defence. Similarly for negations and universals.
    ii. If π_i is a **P**-conjunction or a **P**-existential move, we are dealing with an introduction rule: each premiss is constituted by one of the defences. The conclusion is the challenged conjunction or existential.
    Dually, if π_i is an **O**-conjunction or an **O**-existential move, we are in presence of an elimination-rule, so that π_i is the premiss of each of the inferences with each of the defences as conclusion.
    iii. If π_i is a case-dependent **P**-statement, then rewrite each of the **O**-defences of the relevant disjunction as a local assumption upon which a copy of π_i will be made dependent. Draw an inference line and copy below a third copy of the case-dependent statement π_i.

---

[116] That is, when the particle rule applied is a challenge to an implication, a negation, or a universal.

[117] Recall that those moves in $\mathbb{C}$ have the form of defences and challenges established by the particle rules or structural rules.

iv. If π_i is an **O**-elementary statement, it is either initial concessions or the result of an elimination rule, in which case draw an inference line above π_1 and place the relevant move as premiss (see Table 29 and Table 30 above).

v. If π_i is a **P**-statement (elementary or not) that does not need to be defended because **O** stated ⊥ (allowing **P** to adduce the local reason $you_{gave\ up}(n)$ for π_i), then it corresponds to the application of an elimination rule for ⊥. In such a case, scan for the move ⊥, place it in the demonstration as a premiss of a ⊥-elimination, draw an inference line and write π_i below it. If π_i is either an implication or a universal delete the **O**-challenge to it.

vi. If π_i is a **P**-statement that displays an *A*-equality, place it in the demonstration as the conclusion of a Σ-(Π-)equality-rule with the moves that lead to that equality as a premiss.[118]

vii. If π_i is a substitution-move based on an *A*-equality, place that equality and the expression in which the substitution has been carried out as premiss of the application of a substitution rule. Similarly for statement-substitution moves.

viii. Apply the EPI operation to the newly added expressions (if applicable). Move to the first (starting from the left) newly opened branch if relevant and go back to **B**.

At this stage multiple premises can occur. Those premises are not dependent upon one another (with the exception of the premises of the elimination of a disjunction) and are placed on the same level: each one opening a new branch in the demonstration. In such cases all the premisses that were placed at some previous step in the translation must be copied and pasted for each newly opened branch.

Step C.

If the situation is the one of **B.1** and no new expression has been added to the branch under construction, then:

C.1. Rearrangements

Perform any rearrangement required to match the notational convention of natural deduction trees and go to **C.2.**

C.2. Dealing with *SR*

If the branch does not feature applications of *SR* then go to **C.3**. Otherwise for each application of *SR* in the branch, remove its conclusion and the associated inference line.[119] Go to **C.3**.

---

[118] Recall the remarks above (section IX.1, 'Exceptions') concerning **P**-elementary statements defended with $I = p_i : type$.

[119] *SR*-rules display two copies of the same elementary expression, one as premise and one as conclusion. In the standard presentation of natural deduction (used in the present text) this is not necessary, unless we make use of some other recent presentations of natural deductions that introduce explicitly axioms of the form *A* ├ *A*.

C.3. Moving in the algorithm

Move to the next branch to which stages **C.1** and **C.2** has not yet been applied and go back to **B**.

If there are no such branches left, go to **D**.

Step D. Inserting proof-objects and stopping the procedure.

Going from top to bottom, replace in the demonstration at hand the dialogical local reasons with CTT proof-objects in accordance with the CTT rules. The point is that once the demonstration has been built we do not have local reasons any more but strategic objects—the latter but not the former correspond to proof-objects.

Then stop the procedure.

Checking method

The table of correspondences between strategic objects and proof-objects (section IX.1) can be used as checking method using the following steps:

Extract the strategic object of the thesis from the core. Use the correspondences of the table and provide the proof-object for it. Compare with the result of the procedure.

Remarks:

We have designed the algorithm so that the branches in the demonstration under construction are dealt with sequentially. However, it is possible to process them all at the same time in parallel.

The concluding stage **D** is necessary because, as discussed all over our study (see for instance section VII.5), dialogical local reasons differ from CTT proof-objects, which correspond to strategic reasons.

IX.3.4Adequacy of the translation algorithm

We must ensure that the algorithm is adequate: given the core of a winning **P**-strategy it must always yield a CTT demonstration. Let us first describe the general idea behind the demonstration, that in fact is an almost literal reproduction of the one developed in (Clerbout & Rahman, 2015, pp. 49-52), with small changes due to the present take on equalities.

The translation procedure ultimately consists in rearranging the nodes of the original dialogical core $\mathbb{C}$. We must ensure that the reordering results in a derivation which complies with the CTT rules. We noticed that during this reordering, the procedure introduces what we may call "gaps" which we have marked with vertical dots. Take for example the first step of such a transformation procedure. In this step the thesis of the core provides the conclusion of the demonstration and the concessions provide the assumptions, though we still do not know at this point of the process what corresponds to the steps between the assumptions and the conclusion. Accordingly, we start by simply linking the assumptions and the conclusion with vertical dots. The idea behind the adequacy of the algorithm is that all these gaps will eventually be filled and that it will be done in a way which observes the CTT rules.

The last part of this statement is easily checked. Let us assume that all the gaps are indeed removed. Then we can easily see that the resulting derivation is such that every rule applied in it is a CTT rule. We have indeed associated every application of a dialogical rule to a CTT rule, with the following exceptions: the rules involving elementary statements by **P** (the *SR*-rules) and the rules for Resolution and Substitution for Instructions that do not involve *A*-equalities. But applications of these three rules will also eventually be removed. Indeed the last stage of the algorithm replaces local reasons with proof-objects, so applications of the *SR*-rules regarding instructions will eventually be removed. Recall also that **P**-statements (elementary or not) that are dependent upon $\bot$-eliminations correspond to those eliminations.

So far so good—though the critical task of checking that the CTT rules are *properly* applied still remains. This process must show the important fact that following the algorithm will eventually remove the gaps, as it was assumed above. In order to ground this assumption let us temporarily consider an extension of the CTT calculus which includes the rules *SR*-rules as well as a new rule called *Gap.* In relation to the the *SR*-rules, recall that in the so-called *full-presentation* of CTT, every leaf of a demonstration starts with an axiom of the form $A \vdash A$; thus, the introduction of *SR* is not at all foreign to the framework of a CTT-demonstration. In relation to *Gap*, it either allows to link (with the help of vertical dots) two nodes of the demonstration without a dialogical rule explaining such a link, or to introduce an expression as the last step of a sequence of vertical dots. We will show that when following the algorithm, each of the applications of the rule *Gap* will be replaced by applications of a suitable CTT rule or by applications of a *SR*-rule. We will then simply need to show that when no dialogical rule is applied to the corresponding node from $\mathbb{C}$, the expression will not introduce additional gaps: the rearranging in the stage **C.1** of the algorithm is harmless. Once we have reached this point, and after all the applications of a *SR*-rule have been removed, we are assured to have a proper CTT demonstration.

Accordingly, let us show first that the gaps introduced during the process of building the CTT demonstration are temporary and will be progressively removed bottom-up:

Algorithm-Lemma (AL)

For any stage of the translation procedure, there is a corresponding node in the original dialogical core $\mathbb{C}$ *for every expression resulting from a gap.*

Proof

This proof is a straightforward induction which also establishes that newly introduced gaps at a given stage of the translation have the “right shape”, so that they will be filled by a proper application of a rule later on.

The *base case* is trivial: the initial stage *A* of the algorithm stipulates that the first expression resulting from an application of the rule *Gap* is the thesis, which is obviously a node in $\mathbb{C}$ to which a dialogical rule is applied.

Inductive Hypothesis. Assume that AL holds for every application of the rule *Gap* up to this step in the translation procedure, say after n steps. We show that the Proposition holds for the gaps introduced at step $n + 1$ and that they have the correct “shape” in relation to the development of a CTT demonstration. This is done by cases, depending on the form of the last expression introduced at this point. For simplicity and brevity we only spell out two cases:

Case 1: The associated node in $\mathbb{C}$ is a **P**-disjunction $p : A \vee B$ which is not case-dependent, and the fragment of the derivation at stake at step n is:

$$\begin{gathered}\vdots\\ p\colon A \vee B\end{gathered}$$

Then, according to the algorithm, the result at step $n+1$ is:

$$\dfrac{\begin{matrix}\vdots\\ L^{\vee}(p)\colon A\end{matrix}}{p\colon A\vee B}\ I\vee$$

We next recall that we must have **O** challenging the disjunction at some place in the core: if there is a **P**-move in $\mathbb{C}$ which **O** does not challenge—though she could—then the core contains branches which do not represent terminal plays. However this is not possible since we have assumed $\mathbb{C}$ to be the core of a **P**-winning strategy. For the same reason, the core must feature the successful defence by **P** of one of the disjuncts, say A. Thus, the newly added expression filling up the dots introduced by *Gap* does indeed correspond to a node in $\mathbb{C}$.

Case 2: The associated node in $\mathbb{C}$ is a **P**-conjunction $p\colon A\wedge B$ which is not case-dependent. After step n we then have:

$$\begin{matrix}\vdots\\ p\colon A\wedge B\end{matrix}$$

so that according to the algorithm the result at step $n+1$ is:

$$\dfrac{\begin{matrix}\vdots\\ L(p)\colon A\end{matrix}\qquad\qquad \begin{matrix}\vdots\\ R(p)\colon B\end{matrix}}{p\colon A\wedge B}\ I\wedge$$

Just like in the previous case, we must have **O** challenging the **P** conjunction at some place in the core—otherwise $\mathbb{C}$ would contain non-terminal plays and we would have a contradiction—resulting in a ramification in which each branch contains the statements by **P** of one of the conjuncts. The demonstration underway thus follows the CTT rule and the new expressions filling up the dots introduced by *Gap* correspond to these nodes in $\mathbb{C}$.

The construction of the demonstration thus proceeds by progressively filling up the temporary gaps until it reaches a stage at which no further gap is introduced. Except for the initial assumptions of the demonstration, the cases in which no gaps are introduced are reduced to cases of atomic expressions. But these come either from an Elimination rule for $\perp$ or from the application of some *SR*-rule, that is, precisely the cases for which the premisses must already have been processed.

Summing up, the demonstration by induction of AL shows that the algorithm builds a derivation by introducing temporary gaps and then progressively filling them up until no further gap occurs. Moreover, this construction has been developed in such a way that the derivation complies with the proceedings of what we have called the extended CTT calculus (which includes the *SR*-rules).

Finally, as we have pointed out at the beginning of this section, the applications of the rules that do not strictly pertain to CTT are removed to guarantee that only CTT rules are applied in the resulting derivation. From all this together we have the following corollary:

Algorithm-Corollary

*Let $\mathbb{C}$ be the core of a winning **P**-strategy in the game for $p : \phi$ under initial concessions $\gamma_1, \dots, \gamma_n$. The result of applying the translation algorithm to $\mathbb{C}$ is a CTT demonstration of ϕ under the hypotheses $\gamma_1, \dots, \gamma_n$.*

This concludes the study of the process by the means of which dialogical strategies lead to CTT-demonstrations. For the demonstration of the equivalence between dialogical games and CTT, we need to consider the converse direction, namely from a CTT demonstration to a **P**-winning strategy. We tackle this issue in the next sections.

IX.4 Part 2: from CTT demonstrations to strategies

In this chapter we will consider the other direction of the equivalence between the valid fragments of the CTT framework and the dialogical framework. That is, we will show that if there is a CTT demonstration for ϕ, then there is a winning **P**-strategy in the dialogical game for ϕ.

The demonstration, quite unsurprisingly, rests on developing a translation procedure which is the converse of the previous one. That is, we will present a procedure transforming a given CTT demonstration and we will show that the result is the core of a winning **P**-strategy—which is then expanded to a full-fledged winning strategy.

A core is expanded to a full strategy by adding branches accounting for variations in the order of the moves of the other player and in the local reasons he chooses. We will not give the specifics of that particular operation because it does not present any difficulty, and they have already been given in details in (Clerbout, 2014a; 2014c). We would rather focus on the way the initial CTT demonstration is transformed and on the proof that the result is the core of a winning **P**-strategy.

For the latter, we need to prove that the transformation results in a tree in which each branch represents a play won by **P**. In other words, we need to show that in the resulting tree each branch represents a legal sequence of moves ending with a **P**-move, or with **O** stating ⊥. We also need to check that the tree has all the necessary information to be a core which can be expanded to a full strategy. That is to say, we must make sure that no possible play for **O** is ignored, excepting those varying in the order of the moves or the names of the local reasons.

The development of the next sections follows the proof by (Clerbout & Rahman, 2015) with the sole exception of the last step in which the equalities are introduced in the core for every **P**-move that is not a result of a SR4-rule (that is those elementary statements of **P** that do not involve resolution of instructions).

IX.4.1 Transformation procedure

As in the first part, we first need here to design a transformation procedure. We will start with an informal description of the task and of the ideas underlying the procedure. Then we will provide the detailed algorithm.

Guidelines

In general there are two main obstacles such a procedure must overcome:

1. CTT is not an interactive-based framework. In particular the notions of players, challenges and defences are not present in CTT.
2. The progression of a CTT demonstration differs quite greatly from the progression of a dialogical core. Most notably, the production of ramifications on the one hand and the order of expressions on the other hand do not match in the two approaches.

These are just descriptions of the fundamental differences between a CTT demonstration and a dialogical core. There obviously are many other aspects which our translation method must take into account. Let us give further explanations on the topics on which the desired transformation procedure must operate.

From CTT judgements to dialogical statements

To begin with we need to enrich the CTT demonstration with the players' identities. We need for that a way to figure out which expressions are stated by which player. In fact, there is a subtlety in this process because some steps in a CTT demonstration may be associated with either players, **P** or **O**; see below, "Identical statements by the two players", for more details on this issue. But the general idea underlying the process is otherwise quite simple. The starting point is that the conclusion of the CTT demonstration is to be stated by **P** because it is the expression at stake: in a dialogue, that is the thesis. Moreover, the hypotheses of the demonstration, that is, the undischarged assumptions that may occur at the leaves of the CTT demonstration, correspond to initial concessions made by **O**.

From there, it is quite straightforward to associate the other steps in the CTT demonstration with players by using the correspondences between the CTT and the dialogical rules used in the precedent sections. By means of illustration, suppose some step in the CTT demonstration has been associated to player **X** and suppose that the expression results from an application of the $\supset$-Introduction rule. Then the assumption discharged by applying the Introduction rule is to be associated to player **Y** (it will occur in the core as the challenge by **Y**), and the expression immediately preceding the inference line is to be associated to player **X** (it will occur in the core as the defence by **X**).

Identical statements by the two players

Because the CTT framework is not based on interaction, it does not distinguish between the two players. The point is that a CTT demonstration may very well feature expressions occurring only once, while two instances (or more) would be needed for a dialogical demonstration, that is, for the construction of a dialogical core. Elementary expressions associated to **P**, and which do not result from the application of the $\perp$–Elimination rule, are one example. More generally, an expression may be used in a demonstration when applying the two kinds of rules: for example it can be used first when applying an Elimination rule and later on when applying an Introduction rule. In such cases, this expression is likely to occur as stated by the two players in a dialogical core (intuitively, this is because of the correspondence between Elimination rules and **O**-applications of rules on the one hand, and Introduction rules and **P**-applications of rules on the other hand). These consideration show the need of adding occurrences of expressions, but as stated by a different player.

Dialogical instructions and local reasons

Next we need to account for the difference between CTT proof-objects on the one hand, and dialogical local reasons and instructions on the other hand. More precisely, we need to go from the CTT perspective on applications of rules to the dialogical perspective. In the CTT framework, applications of rules manifest themselves by specific operations defining the way proof-objects are obtained from other proof-objects. In the dialogical approach, meaning explanations are given in terms of local reasons and instructions at the other (prior) level of plays in which interaction prevails over the set-theoretic operations.

To perform this change of perspective, we start by substituting an arbitrary local reason p for the proof-object in the conclusion of the demonstration (and for that one only); in other words, we choose an arbitrary local reason for the thesis of the dialogical core we are building. Also, if relevant, we substitute local reasons for proof-objects in the expressions corresponding to initial concessions by **O**.

From there, it is a trivial matter to replace the other proof-objects occurring in the demonstration with the appropriate dialogical *instructions*. We simply look which analysis rule (section VII.1.2, Table 20) is applied to know which subscript must be associated to the letters L and R which will result in a proper dialogical instruction. For example, an instruction of the form $L^{\wedge}(\ldots)$—or $R^{\wedge}(\ldots)$—is substitued for the proof-object of the conclusion resulting from an application of the ∧-Elimination rule in the initial CTT demonstration.

Once we have a way to replace proof-objects with dialogical instructions, we are able to introduce local reasons as well. To do so we introduce the moves involving resolutions of instructions. We do so as we replace CTT proof-objects with dialogical instructions: every time we determine the dialogical instruction replacing the CTT proof-object, we also choose a local reason resolving the instruction. As a result, an expression "$\alpha : \phi$", where α is a proof-object, will be replaced by an instruction of the form "$I : \phi$ " where I is an instruction. Immediately after that, another version of the same move is added in the structure, but with a local reason instead of the instruction I. By doing this immediately we can progressively replace proof-objects with simple instructions relative to local reasons, instead of having embedded instructions getting more and more (unnecessarily) complex.[120]

Adding questions

At this point we have obtained a tree-like structure featuring a substantial number of expressions which differ only by the player identity, or maybe by the instruction and local reason.

Still, some aspects are missing to read the structure at hand in terms of interaction. To put it simply, the structure lacks challenges consisting in questions. For example, that two expressions **X** ! $I : \phi$ (for some instruction I) and **X** ! $p : \phi$ (for some local reason p) following each other in the structure does not make dialogical sense until the question **Y** $I/?$ is placed between them; only then can we speak of an interaction in which **Y** asks **X** for the resolution of the instruction I and **X** chooses p for the resolution. Similarly with other questions such as $?_{\vee}$, $?L$, $?R$, etc., depending on the rule at stake.

The next step in the translation procedure is therefore to include questions in the relevant way so that one can accurately speak of interaction through the application of dialogical rules. However, the result still cannot be called a dialogical core. For that we need to overcome the difference in the production of ramifications between the CTT framework and dialogical strategies.

Rearranging the branches and order of the moves

Recall that we are dealing with a tree-like structure written "upside-down", that is, where the root of the tree (the conclusion of the demonstration we started with) is at the bottom and the leaves are at the top.

The most important transformation that remains is reorganising the tree at hand so that we obtain a good candidate for a core of a **P**-winning strategy. This means we aim

---

[120] Suppose for example that we have introduced an instruction $L^{\vee}(p)$. If we do not immediately decide for a local reason, say q, resolving this instruction, then the next instruction will be of the form $I(L^{\vee}(p))$ instead of the simpler $I(q)$—for some I.

for a tree in which branches are linear representations of plays in such a way that ramifications represent choices of **O** between different moves (since we are interested in **P**-strategies). The CTT framework distinguishes between rules applied to one or more expressions. In the latter case, a ramification is produced but not in the former case. But since there is no explicit notion of interaction and strategy (in the game-theoretical sense) in CTT, it is obvious that ramifications may not correspond to differences due to possible choices by a player, that are taken into account in a strategy for his adversary.

A typical example are the differences between the CTT Elimination rules for material implication and universal quantification on the one hand, and their dialogical counterpart on the other hand. In CTT these rules have (at least) two premisses: first, the complex expression, and, second, a judgement of the form $a : A$ when A is the antecedent or the set which is quantified over. Each of these two premisses opens a branch in the demonstration. But in a dialogical game, one is the statement, the other is the challenge against it; consequently they occur in the same play and hence, in the same branch of a strategy. Notice that this will also happen with other rules including those equality assertions that in the CTT-demonstration result from the application of equality rules.

The goal in this step of the transformation is thus to reorganise the tree in order to overcome this discrepancy. We must also take some additional precautions (such as adding the choices of repetition ranks) so that the branches in the new tree do indeed represent plays.

Once this has been accomplished we reintroduce applications of the Socratic Rule (*SR*) to the **P-**elementary statements resulting from the resolution of some instruction. In other words, once all the previous steps have been carried out we reintroduce those equalities arising from **O**'s-choices while resolving instructions that have not been already implemented in the CTT demonstration. Recall that the standard CTT demonstrations deploy Σ- and Π-equalities only when the Elimination rules might produce a non-canoncial proof-object.

We shall stop the general explanations here. All the details are given in the full description of the translation algorithm given below. Let us simply mention here that the procedure is meant to obtain the core of a **P**-winning strategy after all these modifications. This is something that must be proved, which we do afterwards.

IX.4.2 The algorithm

Let us precise now the details of the procedure: we start with a CTT demonstration *D* of an expression *E* under a set *H* of global hypotheses or epistemic assumptions (that is, assertions that may include "given" proof-objects).

Step A. From judgements to moves

First we enrich the initial demonstration with player identities and the sign !

A1. Adding the players' identities

Rewrite the conclusion *E* as **P** ! *E* . Then, for every $h \in H$ occurring as a leaf of *D*, rewrite *h* as **O** ! *h*. Go to **A2**.

A2. Using-up expressions

Scan D bottom-up. When there is no unused expression left, go to **A3.** Otherwise, let $E1$ be the (left-most[121]) unused expression **X** ! $E1$. Then,

(1) If **X** is **O** and $E1$ results in D from applying an Introduction rule, then insert **P** ! $E1$ as the conclusion of the rule preceding $\mathbf{O}! E1$. Consider the latter as used and go back to **A2**.

(2) If **X** is **P** and $E1$ results in D from applying an Elimination rule other than for $\vee$ or Σ, then insert $\mathbf{O}$! $E1$ as the conclusion of the rule preceding **P** ! $E1$. Consider the latter as used and go back to **A2**.

(3) Otherwise use the correspondences between CTT and dialogical rules given in section IX.1 to rewrite the expressions allowing the application of the rule with the adequate player.[122]
In doing so, observe the following constraints:
   1. an expression can be labelled as a **P**-move *and* an **O**-move,
   - each player can be assigned at most once to an expression.

After this, consider the expression as used and go back to **A2**.

A3. Corresponding O-moves

Scan the demonstration at hand. For each elementary statement by **P** which has no counterpart by **O** apply one of the following,

1. if it is the result of an application of the $\bot$-Elimination rule, then leave it as it is.
2. If there is no corresponding **O**-move, then insert one immediately below the **P**-move, and insert the expression **P** *Socratic Rule* at the leaf of the current branch.

Then go to **A4**.

A4. Separating double labels

If there are leaves with the double label **O** ! / **P** !, separate them into two expressions such that the **P-**move is placed as the leaf. Go to step **B**.

Step B. Instructions and local reasons

Next we introduce local reasons in place of proof-objects. This is done in the following way.

B1. Introducing arbitrary local reasons

In the conclusion $\mathbf{P}! E$, replace the proof-object with an arbitrary local reason p. Then, for each initial concession **O** ! h occurring at a leaf of the demonstration, substitute, if relevant, an arbitrary local reason for the proof-object. Consider these expressions as processed and go to **B2**.

---

[121] This accounts for the fact that D may have several branches.

[122] For example, given the situation

$$\frac{\alpha : (\Pi x : A) \qquad a : A}{\mathbf{O}! \ldots} \; E\Pi$$

The deployment of the procedure described by **A3** yields:

$$\frac{\mathbf{O}! \, \alpha : (\Pi x : A) \quad \mathbf{P}! \, a : A}{\mathbf{O}! \ldots} \; E\Pi$$

B2. Processing all the expressions

Scan the demonstration bottom-up. If there is no expression left unprocessed, go to **C**. Otherwise take the leftmost expression **X** ! $E2$ with a local reason which has not been processed so far, and

- Use the correspondences between the CTT and the dialogical rules given in the previous sections to substitute the adequate instructions for the proof-object (or proof-objects) in the premiss (premisses) of the rule whose application results in **X** ! $E2$.
- For each instruction introduced in that way, copy the expression at stake, replacing the instruction by an arbitrary local reason. Place the version with the local reason immediately above the expression with the instruction.
- Consider **X** ! $E2$ as processed and go back to **B2**.

Step C. Adding questions

Scan the demonstration and identify the applications of rules for which the dialogical counterpart features a question. For each expression understood as a defence according to such a rule, add the corresponding challenge performed by the adversary immediately below the expression.

Go to **D**.

Step D. Moving the Opponent's initial concessions

Consider each leaf of the demonstration at hand which is an initial concession by **O**—that is, an undischarged assumption of the initial demonstration D which has been identified as an **O**-move. Remove it and place it below the conclusion **P** ! E. In case of multiple occurrences, keep only one occurrence.

Go to **E**.

Step E. Removing non-dialogical splits

Scan the demonstration top-down. Going from the left to the right, check each point where two different branches join. Depending on the case, apply one of the following,

1. If the ramification is such that the two branches are opened by two **O**-moves relevant for a rule dealing with a logical constant, then ignore them and proceed downwards.
2. Otherwise, "cut" one and "paste" it above the other one, according to the following convention:
   - If both branches have a **P**-move as the leaf, or if both have an **O**-move as the leaf, then pick any one of the branch to be cut and pasted;
   - Otherwise pick the one with a **P**-move at the leaf to be cut and pasted.

Go to **F**.

Step F. Reordering the nodes.

Scan the tree structure at hand bottom-up. Starting from the thesis **P** ! E, change the order of the expressions according to the following conditions,

- each **O**-move is a reaction—as specified by the dialogical rules—to the **P**-move placed immediately below;

- a question or a statement which is a challenge always occurs before (i.e. closer to the root) a defence reacting to it.
- Ramifications are preserved so that each branch is opened with an **O**-move as a reaction to a **P**-move which is immediately below.

Go to **G**.

Step G. Introducing equalities by means of the Socratic rule

Search for nodes labelled *SR* (*Socratic Rule*). Depending on the case, apply one of the following:

- If the **O**-expression copied by **P** is the result of applying a CTT equality rule, apply the relevant case defined in SR5.3.1: Non-reflexive cases of the Socratic rule (see section VII.2.1).
- If the **O-**expression copied by **P** is *not* the result of applying a CTT equality rule, apply the relevant case defined in SR5.3.2: Reflexive cases of the Socratic rule (section VII.2.1).

Explanation

The first case occurs when the **O-**expression is the result of a two step process in the CTT demonstration: the first step being an application of an elimination rule, and the second an application of an equality rule computing the proof-object to a given value. In the context of a dialogical play, this corresponds to **O** resolving or substituting an instruction. As explained in the description of the Socratic rule, this corresponds to a non reflexive case.

The second case occurs when the **O**-expression is an assumption, when a value for the proof-object is directly introduced without being computed by means of a CTT equality rule. In the context of a dialogical play, this corresponds to **O** conceding an elementary expression directly, without going through the resolution of substitution of an instruction. In this context, the Socratic rule specifies that **P** will apply the reflexive case of the rule.

Go to **H.**

Step H. Adding ranks

Insert an expression $\mathbf{O}\ n := 1$ immediately above the thesis $\mathbf{P}\ !E$. Then insert an expression $\mathbf{P}\ m := k$ above the one just inserted. Choose k to be the highest number of times a given rule is applied by **P** to the same expression in the tree.

The procedure stops.

IX.4.3 Adequacy of the algorithm

We have developed the algorithm transforming a CTT demonstration into a winning strategy. It remains to show that the algorithm indeed does so, in other words that applying the algorithm to a given a CTT demonstration results in the core of a **P**-winning strategy.

To be more specific, the point is to show that the result of applying the algorithm to a CTT demonstration is a tree in which,

1. each branch represents a play: the sequence of moves in each branch complies with the game rules,
2. each play in the tree is won by **P**,

3. the tree describes all the relevant alternatives for a core. In other words: there is no significantly different course of action for **O** that would be disregarded in the resulting tree.[123]

Proposition 1

Each branch in the resulting tree represents a play.

We need to show that in each branch the sequence of moves complies with the rules of dialogical games.

Proof

Because the translation observes a correspondence between CTT rules and dialogical particle rules (recall that step C of the algorithm is used to insert questions), we simply need to check that the dialogical structural rules are observed. We leave the Winning Rule aside for now since it is the topic we address in the next Proposition.

So for the Starting Rule *SR*0, steps D and H of the algorithm ensure that every sequence of moves in the tree starts with the initial concessions of **O**, which are followed by the thesis stated by **P**, and then by the choices of repetition ranks.

As for the Intuitionistic Development Rule, step F of the algorithm guarantees that each move following the repetition ranks in a sequence is played in reaction to a previous move. The condition in step F according to which **O**-moves immediately follow the **P**-move to which they react ensures that the intuitionistic restriction of the *Last Duty First* is observed.[124] Moreover the choice of the repetition ranks prescribed by step H ensures that the players do not perform unauthorised repetitions.

As for the Socratic rule, no challenge against an elementary **O**-statement is added when applying the algorithm. Moreover, in the case of elementary statements made by **P**, step A3 and G of the procedure ensure that, if needed, a corresponding move by **O** and the adequate challenges and defences are added.

As for the rules related to the Resolution of Instructions, step B of the algorithm (in combination with step C introducing questions) guarantees that instructions are resolved according to the structural rules SR3 and SR4—recall that we ignore the formation dialogues since we are focusing on that fragment of CTT in which verifying the well formation is assumed successful.

Now that we have established that the branches represent plays because they comply with the dialogical rules, we must assess the situation in relation to victory and show that:

Proposition 2

Each branch of the resulting tree represents a play won by **P**.

---

[123] By significantly different we mean other than relative to the order of **O**'s moves, or the choice of local reasons to replace instructions.

[124] This is known since (Felscher, 1985). See also more details in (Clerbout, 2014c).

Proof

We must check that the leaf of each branch is either:

1. an elementary statement by **P** preceded by a **O**-statement of ⊥,
2. an elementary statement by **P** (different to the previous case),
3. a **P**-equality that results from applying the Socratic Rule.

But all this is guaranteed by steps A3, E, F and G of the algorithm.

Finally, it remains to show that the tree describes all the relevant courses of actions for the Opponent underlying the core of a **P**-strategy:

Proposition 3

*There is no **P**-move in the tree remaining unanswered by **O** and there is no rule that would allow leaving such a **P**-move without a response.*

Proof

We know from the initial demonstration *D* and steps A1-A4 of the algorithm that every statement made by **P** in the resulting tree occurs as the result of an Introduction rule, of the Elimination rule for the Σ operator or of the Elimination rule for disjunction. In the case of complex statements, the correspondence with dialogical particle rules, together with the addition of questions via step C of the algorithm, ensure that they are challenged and that when they are themselves played as challenges they are answered. In the case of elementary statements, we know from the proof of the preceeding Proposition (2) that they are challenged if **O** may challenge them.

Moreover, all the possible challenges allowed by the particle rules are covered by the CTT rules they correspond to. For this reason, the only remaining possible variations left to **O** are the order of her moves and the choice of local reasons for the Resolution of Instructions. But these variations are the ones which are not relevant to build the *core* of a **P**-strategy. In other words the correspondence between CTT rules and the particle rules ensure that the starting demonstration *D* already contains the variations which are relevant to the core of a **P**-strategy.

The adequacy of our translation procedure, which amounts to the second direction of the equivalence between CTT demonstrations and dialogical strategies, is then a direct consequence of these three Propositions.

Corollary

*The result of applying the algorithm that transcribes a CTT demonstration into a tree of **P**-terminal plays constitutes the core of a winning **P**-strategy.*

IX.5 Exercices

Transform the core of the winning strategy for the following theses into a natural-deduction demonstrations by deploying the procedure described in the preceding sections.

1. $(A \wedge B) \wedge C \ [c\colon A \wedge (B \wedge C)]$
2. $(B \wedge A) \supset C \ [c\colon (A \wedge B) \supset C]$

Solution to exercise 1: demonstration of $(A \wedge B) \wedge C \ [c\colon A \wedge (B \wedge C)]$

Recall from section VII.7.4 the core of the winning strategy for the thesis.

Figure 5: Exercise 1 - the core

$0.\ \mathbf{P}\ !\ (A \wedge B) \wedge C\ [c: A \wedge (B \wedge C)]$
$0.1.\ \mathbf{O}\ c: A \wedge (B \wedge C)$
$1.\ \mathbf{O}\ m := 1$
$2.\ \mathbf{P}\ n := 2$

$3.\ \mathbf{O}\ ?\ L^{\wedge}\ [?\,,0]$
$4.\ \mathbf{P}\ d_1: A \wedge B\ [!,3]$

$5.\ \mathbf{O}\ ?\ L^{\wedge}\ [?\,,4]$
$6.\ \ \mathbf{P}\ L^{\wedge}(d_1): A\ [!,5]$
$7.\ \mathbf{O}\ ?\ .../L^{\wedge}(d_1)[?\,,6]$
$8.\ \mathbf{P}\ ?\ L^{\wedge}\ [?\,,0.1]$
$9.\ \mathbf{O}\ L^{\wedge}(c): A\ [!,8]$
$10.\ \mathbf{P}\ ?\ .../L^{\wedge}(c)\ [?\,,9]$
$11.\ \mathbf{O}\ c_1: A\ [!,10]$
$12.\ \mathbf{P}\ c_1: A\ [!,7]$
$13.\ \mathbf{O}\ ?{=}\ c_1\ [?\,,12]$
$14.\ \mathbf{P}\ L^{\wedge}(c) = c_1: A\ [!,13]$
P wins

$5.\ \mathbf{O}\ ?\ R^{\wedge}\ [?\,,4]$
$6.\ \mathbf{P}\ R^{\wedge}(d_1): B\ [!,5]$
$7.\ \mathbf{O}\ ?\ .../R^{\wedge}(d_1)[?\,,6]$
$8.\ \mathbf{P}\ ?\ R^{\wedge}\ [?\,,0.1]$
$9.\ \mathbf{O}\ R^{\wedge}(c): B \wedge C\ [!,8]$
$10.\ \mathbf{P}\ ?\ .../R^{\wedge}(c)\ [?\,,9]$
$11.\ \mathbf{O}\ c_2: B \wedge C\ [!,10]$
$12.\ \mathbf{P}\ ?\ L^{\wedge}\ [?\,,11]$
$13.\ \mathbf{O}\ L^{\wedge}(c_2): B\ [!,12]$
$14.\ \mathbf{P}\ ?\ .../L^{\wedge}(c_2)[?\,,13]$
$15.\ \mathbf{O}\ c_{2.1}: B\ [!,14]$
$16.\ \mathbf{P}\ c_{2.1}: B\ [!,7]$
$17.\ \mathbf{O}\ ?{=}\ c_{2.1}\ [?\,,16]$
$18.\ \mathbf{P}\ L^{\wedge}(c_2) = c_{2.1}: B\ [!,17]$
P wins

$3.\ \mathbf{O}\ ?\ R^{\wedge}\ [?\,,0]$
$4.\ \mathbf{P}\ ?\ R^{\wedge}\ [?\,,0.1]$
$5.\ \mathbf{O}\ R^{\wedge}(c): B \wedge C[!,4]$
$6.\ \mathbf{P}\ ?\ .../R^{\wedge}(c)\ [?\,,5]$
$7.\ \mathbf{O}\ c_2: B \wedge C\ [!,6]$
$8.\ \mathbf{P}\ ?\ R^{\wedge}\ [?\,,7]$
$9.\ \mathbf{O}\ R^{\wedge}(c_2): C\ [!\,,8]$
$10.\ \mathbf{P}\ ?\ .../R^{\wedge}(c_2)\ [?\,,9]$
$11.\ \mathbf{O}\ c_{2.2}: C\ [!,10]$
$12.\ \mathbf{P}\ c_{2.2}: C\ [!,3]$
$13.\ \mathbf{O}\ ?{=}\ c_{2.2}\ [?\,,12]$
$14.\ \mathbf{P}\ R^{\wedge}(c_2) = c_{2.2}: C\ [!,13]$
P wins

Let us now launch the procedure for transforming the tree for the winning strategy into a natural deduction tree.

Steps A-B on the thesis

We place the thesis as conclusion and the initial concession as global assumption. Since in the tree the initial concessions occur before a branching, we need to introduce two branches in the demonstration starting both by the same initial concession:

Figure 6: Exercise 1 - Step A

$$\frac{\begin{matrix} c: A \wedge (B \wedge C) \\ \vdots \end{matrix} \qquad\qquad \begin{matrix} c: A \wedge (B \wedge C) \\ \vdots \end{matrix}}{A \wedge (B \wedge C)}$$

We now scan for the lowest expression in the demonstration tree—at this time it is the thesis—and find in the core the responses to it—the responses here are the challenges on the conjunctions; in order to help the reading of the process, we highlight in bold the last challenge-defence pair:

Figure 7: Exercise 1 - Step B on the thesis[125]

$$c: A \wedge (B \wedge C) \qquad\qquad c: A \wedge (B \wedge C)$$

---

[125] To help the reader we add this step with the challenges in request form on the conjunction, though the algorithm does not include the questions in the demonstration tree. We will not put them further on.

$$\dfrac{\begin{array}{c}\vdots\\ ?_L\,[?\,;\mathbf{0}]\\ \vdots\\ \boldsymbol{d_1\colon A\wedge B}\end{array}\qquad\qquad \begin{array}{c}\vdots\\ ?_R\,[?\,;\mathbf{0}]\\ \vdots\\ \boldsymbol{c_{2.2}\colon C}\end{array}}{A\wedge(B\wedge C)}\ I\wedge$$

Whereas the left of the two expressions added in this step is a conjunction, the right is an elementary expression: steps **B.2.b** and **B.2.a** apply respectively. Let us start with the latter, and therefore begin with the right branch.

Building the right branch: steps B-C

Since $c_{2.2}\colon C$ is a **P-**elementary expression we know that it must be the result of Socratic rule (here noted *SR*).

Figure 8: Exercise 1 - Step B.2.a

$$\dfrac{\begin{array}{c}c\colon A\wedge(B\wedge C)\\ \vdots\\ \\ d_1\colon A\wedge B\end{array}\qquad\qquad \begin{array}{c}c\colon A\wedge(B\wedge C)\\ \vdots\\ \dfrac{}{\boldsymbol{c_{2.2}\colon C}}\ \boldsymbol{SR}\end{array}}{A\wedge(B\wedge C)}\ I\wedge$$

We now look in the core for the **O-**move that allowed **P** to state $c_{2.2}\colon C$**.** In order to do so, we search for the equality move that defended the challenge upon $c_{2.2}\colon C$—it is the equality move 14, and it provides the information that this statements uses move 11.

Figure 9: Exercise 1 - Step B.2.a

$$\dfrac{\begin{array}{c}c\colon A\wedge(B\wedge C)\\ \\ \vdots\\ \\ d_1\colon A\wedge B\end{array}\qquad\qquad \begin{array}{c}c\colon A\wedge(B\wedge C)\\ \vdots\\ \dfrac{\boldsymbol{c_{2.2}\colon C}}{\boldsymbol{c_{2.2}\colon C}}\ SR\end{array}}{A\wedge(B\wedge C)}\ I\wedge$$

We know that **O**’s elementary moves are either initial concessions or the result of an elimination rule: here move 11 is an answer to **P**’s challenge on the conjunction $B\wedge C$ stated move 7; so **O**’s move $c_{2.2}\colon C$ is the result of an elimination rule on the conjunction.

Figure 10: Exercise 1 – Step B.2.b (iv)

$$\dfrac{\begin{array}{c}c\colon A\wedge(B\wedge C)\\ \\ \vdots\\ \\ \\ d_1\colon A\wedge B\end{array}\qquad\qquad \begin{array}{c}c\colon A\wedge(B\wedge C)\\ \vdots\\ \dfrac{\dfrac{\boldsymbol{c_2\colon B\wedge C}}{\boldsymbol{c_{2.2}\colon C}}\ E\wedge}{c_{2.2}\colon C}\ SR\end{array}}{A\wedge(B\wedge C)}\ I\wedge$$

The move $c_2: B \wedge C$ is the lowest of the expressions just added, so we apply once more **step B** and search in the core for the moves that triggered it: a defence to **P**'s challenge on the initial concession $c: A \wedge (B \wedge C)$—a conjunction again: the rule allowing for the statement $c_2: B \wedge C$ is therefore a conjunction elimination rule (we skip here the instructions and replace them straight).

Figure 11: Exercise 1 - Step B.2.b

$$\dfrac{\begin{array}{c} c: A \wedge (B \wedge C) \\ \vdots \\ d_1: A \wedge B \end{array} \qquad \dfrac{\dfrac{\dfrac{\boldsymbol{c}: \boldsymbol{A} \wedge (\boldsymbol{B} \wedge \boldsymbol{C})}{\boldsymbol{c_2}: \boldsymbol{B} \wedge \boldsymbol{C}} E\wedge}{c_{2.2}: C} E\wedge}{c_{2.2}: C} SR}{A \wedge (B \wedge C)} I\wedge$$

Step C to the right branch. The premiss for infering $c_2: B \wedge C$ being already in the tree, we move to step C for this branch: we now delete the *SR* and leave only one copy of the elementary expression.

Figure 12: Exercise 1 - Step C to the right

$$\dfrac{\begin{array}{c} c: A \wedge (B \wedge C) \\ \vdots \\ d_1: A \wedge B \end{array} \qquad \dfrac{\dfrac{c: A \wedge (B \wedge C)}{c_2: B \wedge C} E\wedge}{c_{2.2}: C} E\wedge}{A \wedge (B \wedge C)} I\wedge$$

Building the left branches: steps B-C

We now turn our attention to the left branch and take the lowest expression last added in the branch of the demonstration: it is a conjunction stated by **P**, so it is the result of a conjunction introduction rule. Since it is an introduction rule, the premisses must be constituted by the members of the conjunction stated by **P**, which opens two different branches, one for each side of the conjunction:

Figure 13: Exercise 1 - step B to the left

$$\dfrac{\dfrac{\begin{array}{c} c: A \wedge (B \wedge C) \\ \vdots \\ \boldsymbol{L^{\wedge}(d_1)}: \boldsymbol{A} \end{array} \qquad \begin{array}{c} c: A \wedge (B \wedge C) \\ \vdots \\ \boldsymbol{R^{\wedge}(d_1)}: \boldsymbol{B} \end{array}}{\boldsymbol{d_1}: \boldsymbol{A} \wedge \boldsymbol{B}} I\wedge \qquad \dfrac{\dfrac{c: A \wedge (B \wedge C)}{c_2: B \wedge C} E\wedge}{c_{2.2}: C} E\wedge}{A \wedge (B \wedge C)} I\wedge$$

With the resolution of the instructions we obtain:

Figure 14: Exercise 1 - step B to the left with instructions

$$\begin{array}{c} c: A \wedge (B \wedge C) \\ \vdots \\ \dfrac{\boldsymbol{c_1}: \boldsymbol{A}}{\boldsymbol{L^{\wedge}(d_1)}: \boldsymbol{A}} EPI \end{array} \qquad \begin{array}{c} c: A \wedge (B \wedge C) \\ \vdots \\ \dfrac{\boldsymbol{c_{2.1}}: \boldsymbol{B}}{\boldsymbol{R^{\wedge}(d_1)}: \boldsymbol{B}} EPI \end{array} \; I\wedge \qquad \dfrac{\dfrac{c: A \wedge (B \wedge C)}{c_2: B \wedge C} E\wedge}{} E\wedge$$

$$\dfrac{\dfrac{}{d_1: A \land B} \qquad\qquad \dfrac{}{c_{2.2}: C}}{A \land (B \land C)}\, I\land$$

Since they are elementary expressions, they result from a Socratic rule (we also apply the EPI operation):

Figure 15: Exercise 1 - step B2a on the left

$$\dfrac{\dfrac{\dfrac{\begin{matrix} c: A \land (B \land C) \\ \vdots \\ \boldsymbol{c_1: A}\end{matrix}}{\boldsymbol{c_1: A}}\, SR \quad \dfrac{\begin{matrix} c: A \land (B \land C) \\ \vdots \\ \boldsymbol{c_{2.1}: B}\end{matrix}}{\boldsymbol{c_{2.1}: B}}\, SR}{d_1: A \land B}\, I\land \qquad \dfrac{\dfrac{c: A \land (B \land C)}{c_2: B \land C}\, E\land}{c_{2.2}: C}\, E\land}{A \land (B \land C)}\, I\land$$

O’s move $c_1: A$ is a response to **P**'s challenge $?_L$ on the initial concession $c: A \land (B \land C)$, and **O**'s move $c_{2.1}: B$ is a response to **P**’s challenge on the left side of **O**'s conjunction $c_2: (B \land C)$—so in the demonstration tree they respectively correspond to the left and right eliminations of the conjunctions $c: A \land (B \land C)$ and $c_2: B \land C$.

Figure 16: Exercise 1 - step B2b (iv) on the left

$$\dfrac{\dfrac{\dfrac{\dfrac{\boldsymbol{c: A \land (B \land C)}}{\boldsymbol{c_1: A}}\, \boldsymbol{E\land}}{c_1: A}\, SR \quad \dfrac{\dfrac{\begin{matrix} c: A \land (B \land C) \\ \vdots \\ \boldsymbol{c_2: B \land C}\end{matrix}}{\boldsymbol{c_{2.1}: B}}\, E\land}{c_{2.1}: B}\, SR}{d_1: A \land B}\, I\land \qquad \dfrac{\dfrac{c: A \land (B \land C)}{c_2: B \land C}\, E\land}{c_{2.2}: C}\, E\land}{A \land (B \land C)}\, I\land$$

Implementing step B on the last unaccounted for expression inserted in the tree, $c_2: B \land C$, yields the information that c_2 is a response to **P**’s challenge $?_R$ on the initial concession $c: A \land (B \land C)$.

Figure 17: Exercise 1 - Finishing step B to the left

$$\dfrac{\dfrac{\dfrac{\dfrac{c: A \land (B \land C)}{c_1: A}\, E\land}{c_1: A}\, SR \quad \dfrac{\dfrac{\dfrac{\boldsymbol{c: A \land (B \land C)}}{\boldsymbol{c_2: B \land C}}\, E\land}{c_{2.1}: B}\, E\land}{c_{2.1}: B}\, SR}{d_1: A \land B}\, I\land \qquad \dfrac{\dfrac{c: A \land (B \land C)}{c_2: B \land C}\, E\land}{c_{2.2}: C}\, E\land}{A \land (B \land C)}\, I\land$$

Step C to the left branch. Every expression in the branch has been accounted for, so we now move on to step C: we delete the *SR* inferences and leave only one copy of the elementary expression.

Figure 18: Exercise 1 - Step C to the left

$$\dfrac{\dfrac{\dfrac{c: A \wedge (B \wedge C)}{c_1: A}E\wedge \qquad \dfrac{\dfrac{c: A \wedge (B \wedge C)}{c_2: B \wedge C}E\wedge}{c_{2.1}: B}E\wedge}{d_1: A \wedge B}I\wedge \qquad \dfrac{\dfrac{c: A \wedge (B \wedge C)}{c_2: B \wedge C}E\wedge}{c_{2.2}: C}E\wedge}{A \wedge (B \wedge C)}I\wedge$$

Step D: inserting proof-objects

We've reached the final step, and now replace the local reasons with proof-objects and obtain the CTT-demonstration in the style of natural deduction tree.

Figure 19: Exercise 1 - step D

$$\dfrac{\dfrac{\dfrac{c: A \wedge (B \wedge C)}{\mathbf{fst}(c): A}E\wedge \qquad \dfrac{\dfrac{c: A \wedge (B \wedge C)}{\mathbf{snd}(c): B \wedge C}E\wedge}{\mathbf{fst}\big(\mathbf{snd}(c)\big): B}E\wedge}{\langle \mathbf{fst}(c), \mathbf{fst}\big(\mathbf{snd}(c)\big)\rangle : A \wedge B}I\wedge \qquad \dfrac{\dfrac{c: A \wedge (B \wedge C)}{\mathbf{snd}(c): B \wedge C}E\wedge}{\mathbf{snd}(\mathbf{snd}\big((c)\big): C}E\wedge}{\langle\langle \mathbf{fst}(c), \mathbf{fst}\big(\mathbf{snd}(c)\big)\rangle, \mathbf{snd}\big(\mathbf{snd}(c)\big)\rangle : A \wedge (B \wedge C)}I\wedge$$

Recall from section VII.7.4 that the *strategic reason* for the thesis $d: (A \wedge B) \wedge C$ $[c: A \wedge (B \wedge C)]$ is $\langle\langle L^{\wedge}(R^{\wedge}(c), L^{\wedge}(c)\rangle, R^{\wedge}\big(R^{\wedge}(c)\big)\rangle : (A \wedge B) \wedge C$, which is the dialogical counterpart to the proof-object in the demonstration tree.

Solution to exercise 2: $(B \wedge A) \supset C$ $[c: (A \wedge B) \supset C]$

We start by displaying the core. As in section VII.7.4 we repeated move 16:

Figure 20: Exercise 2 - the core

0. **P** $! (B \wedge A) \supset C \; [c : (A \wedge B) \supset C]$
0.1 **O** $c: (A \wedge B) \supset C$
1. **O** $m := 1$
2. **P** $n := 2$
3. **O** $d_1: B \wedge A$ [?,0]
4. **P** $?L^{\wedge}$ [?,3]
5. **O** $L^{\wedge}(d_1): B$ [!,4]
6. **P** $?\ldots/L^{\wedge}(d_1)$[?,5]
7. **O** $d_{1.1}: B$ [!,6]
8. **P** $?\ R^{\wedge}$ [?,3]
9. **O** $R^{\wedge}(d_1): A$ [!,8]
10. **P** $?\ldots/R^{\wedge}(d_1)$ [?,9]
11. **O** $d_{1.2}: A$ [!,10]
12. **P** $L^{\supset}(c): A \wedge B$ [?,0.1]
13. **O** $?\ldots/L^{\supset}(c)$ [?,12]
14. **P** $c_1: A \wedge B$ [!,13]

15. **O** $?L^{\wedge}$ [?,14]	15. **O** $?R^{\wedge}$ [?,14]	15. **O** $R^{\supset}(c): C$ [!,12]
16. **P** $L^{\wedge}(c_1): A$ [!,15]	16. **P** $R^{\wedge}(c_1): B$ [!,15]	16. **P** $?\ldots/R^{\supset}(c)$[?,15]
17. **O** $?\ldots/L^{\wedge}(c_1)$ [?,16]	17. **O** $?\ldots/R^{\wedge}(c_1)$ [?,16]	17. **O** $c_2: C$ [!,16]
18. **P** $d_{1.2}: A$ [!,17]	18. **P** $d_{1.1}: B$ [!,17]	18. **P** $c_2: C$ [!,3]

19. **O** $? = d_{1.2}$ $[?, 18]$	19. **O** $? = d_{1.1}$ $[?, 18]$	19. **O** $? = c_2$ $[?, 18]$
20. **P** $R^{\wedge}(d_1) = d_{1.2}: A$ $[!, 19]$	20. **P** $L^{\wedge}(d_1) = d_{1.1}: B$ $[!, 19]$	20. **P** $R^{\supset}(c) = c_2: C$ $[!, 19]$
P wins	**P** wins	**P** wins

Step A-B on the thesis

Step A. We place the thesis $(B \wedge A) \supset C$ as conclusion and the initial concession $c: (A \wedge B) \supset C$ as global assumption:

Figure 21: Exercise 2 - Step A

$$\frac{\begin{array}{c} c: (A \wedge B) \supset C \\ \vdots \end{array}}{(B \wedge A) \supset C}$$

Step B. We scan now for the lowest expression in demonstration tree—the thesis at this stage—and find in the core the response to it—a challenge on the implication. The challenge on the **P**-implication is the local assumption $d_1: B \wedge A$ (after resolution of instructions), which is one of the premisses of the implication introduction rule; the second premiss is the **P**-move $c_2: C$ occurring move 18 on the outmost right branch of the core.

Figure 22: Exercise 2 - step B on the thesis

$$\frac{\begin{array}{c} c: (A \wedge B) \supset C \\ \vdots \end{array} \qquad \dfrac{\begin{array}{c} \boldsymbol{d_1: B \wedge A} \\ \vdots \end{array}}{\boldsymbol{c_2: C}}}{(B \wedge A) \supset C}\ I\supset$$

Step B: building the branches

Since the defence $c_2: C$ of the implication is a **P**-elementary expression, it must be a copy of a move made by **O**; move 18 is a copy of move 17, as recorded by the equality of move 20. This yields the following *SR* indication:

Figure 23: Exercise 2 - step B.2.a

$$\frac{\begin{array}{c} c: (A \wedge B) \supset C \\ \vdots \end{array} \qquad \dfrac{\begin{array}{c} d_1: B \wedge A \\ \vdots \\ \boldsymbol{c_2: C} \end{array}}{\boldsymbol{c_2: C}}\ SR}{(B \wedge A) \supset C}\ I\supset$$

Now, the core informs us move 17 results from the resolution of the right part of the initial concession $c: (A \wedge B) \supset C$, an implication: it thus results from the implication elimination applied to the concession; since the concession is an implication, **P** must state the antecedent in order to challenge it, which yields the **P**-move 14 $c_1: A \wedge B$.

Figure 24: Exercise 2 - Step B.2.b

$$\begin{array}{cc} & d_1: B \wedge A \\ & \vdots \\ \hline \boldsymbol{c: (A \wedge B) \supset C} & \boldsymbol{c_1: A \wedge B} \end{array}\ E\supset$$

$$\dfrac{\dfrac{\dfrac{}{\boldsymbol{c_2: C}}\;SR}{c_2: C}\;I\supset}{(B \wedge A) \supset C}$$

We know that **O** has two options to respond to a challenge on an implication, namely, stating the consequent, or launching a counterattack on the challenge. The defensive move has been implemented in the previous step, so we must now deal with the counterattack. Since the **P-**challenge is a conjunction, there are two possible challenges, namely, left, or right, each of them opening a new branch. The branches of the core trigger two branches in the demonstration tree by applying twice the conjunction introduction rule, corresponding to move 18 in the outmost left branch and move 18 in the middle branch respectively.

Figure 25: Exercise 2 - step B.2.b

$$\dfrac{\dfrac{\dfrac{c: (A \wedge B) \supset C \qquad \dfrac{\begin{matrix} d_1: B \wedge A \\ \vdots \\ \boldsymbol{d_{1.2}: A} \end{matrix} \qquad \begin{matrix} d_1: B \wedge A \\ \vdots \\ \boldsymbol{d_{1.1}: B} \end{matrix}}{\boldsymbol{c_1: A \wedge B}}\;I\wedge}{c_2: C}\;E\supset}{c_2: C}\;SR}{(B \wedge A) \supset C}\;I\supset$$

Since **P**-moves $d_{1.2}: A$ and $d_{1.1}: B$ are elementary, they are the result of the application of the Socratic Rule.

Figure 26: Exercise 2 - step B.2.a

$$\dfrac{\dfrac{\dfrac{c: (A \wedge B) \supset C \qquad \dfrac{\dfrac{\begin{matrix} d_1: B \wedge A \\ \vdots \\ \boldsymbol{d_{1.2}: A} \end{matrix}}{\boldsymbol{d_{1.2}: A}}\;SR \qquad \dfrac{\begin{matrix} d_1: B \wedge A \\ \vdots \\ \boldsymbol{d_{1.1}: B} \end{matrix}}{\boldsymbol{d_{1.1}: B}}\;SR}{c_1: A \wedge B}\;I\wedge}{c_2: C}\;E\supset}{c_2: C}\;SR}{(B \wedge A) \supset C}\;I\supset$$

In the core, move 5 **O** ! $L^{\wedge}(d_1): B$ and move 9 **O** ! $R^{\wedge}(d_1): A$ convey the information that the **O-**moves result from applying the conjunction elimination rule to the local assumption $d_1: B \wedge A$.

Figure 27: Exercise 2 - final step B

$$\dfrac{\dfrac{\dfrac{c: (A \wedge B) \supset C \qquad \dfrac{\dfrac{\dfrac{\boldsymbol{d_1: B \wedge A}}{\boldsymbol{d_{1.2}: A}}\;E\wedge}{d_{1.2}: A}\;SR \qquad \dfrac{\dfrac{\boldsymbol{d_1: B \wedge A}}{\boldsymbol{d_{1.1}: B}}\;E\wedge}{d_{1.1}: B}\;SR}{c_1: A \wedge B}\;I\wedge}{c_2: C}\;E\supset}{c_2: C}\;SR}{}\;I\supset$$

$$\frac{}{(B \wedge A) \supset C}$$

Step C: eliminating *SR*

We follow now the step C and delete the *SR* to leave only one copy of the elementary expression.

Figure 28: Exercise 2 - Step C

$$\dfrac{\dfrac{c:(A \wedge B) \supset C \qquad \dfrac{\dfrac{d_1: B \wedge A}{d_{1.2}: A}E\wedge \qquad \dfrac{d_1: B \wedge A}{d_{1.1}: B}E\wedge}{c_1: A \wedge B}I\wedge}{c_2: C}E\supset}{(B \wedge A) \supset C}I\supset$$

Step D: inserting proof-objects

We have reached the final step: we replace the local reasons with proof-objects and obtain the CTT-demonstration in the style of natural deduction tree.

Figure 29: Exercise 2 - Step D

$$\dfrac{\dfrac{c:(A \wedge B) \supset C \qquad \dfrac{\dfrac{d_1: B \wedge A}{\mathbf{snd}(d_1): A}E\wedge \qquad \dfrac{d_1: B \wedge A}{\mathbf{fst}(d_1): B}E\wedge}{\langle \mathbf{snd}(d_1), \mathbf{fst}(d_1)\rangle: A \wedge B}I\wedge}{\mathbf{ap}(c, \langle \mathbf{snd}(d_1), \mathbf{fst}(d_1)\rangle): C}E\supset}{\lambda d_1.[\mathbf{ap}(c, \langle \mathbf{snd}(d_1), \mathbf{fst}(d_1)\rangle)]: (B \wedge A) \supset C}I\supset$$

Although inserting proof-objects can be done without using the strategic reason by refering to the CTT rules presented in chapter II, one might want to delve into the relationship between strategic reasons for a dialogical thesis and proof-objects for a CTT conclusion. We shall thus illustrate this relationship in the context of this exercise.

Recall from section VII.7.4 the strategic reason for the thesis:

$$\left(R^{\supset}(c)^{\mathbf{O}} \left[\!\!\left[\langle L^{\wedge}\left(L^{\supset}(c)\right)^{\mathbf{P}} = R^{\wedge}(d_1)^{\mathbf{P,O}}, R^{\wedge}\left(L^{\supset}(c)\right)^{\mathbf{P}} = L^{\wedge}(d_1)^{\mathbf{P,O}}\rangle\right]\!\!\right]\right)^{\mathbf{P}} [\![\langle L^{\wedge}(d_1), R^{\wedge}(d_1)\rangle]\!]^{\mathbf{O}}$$

First, it is important to notice that dialogical strategic reasons and CTT proof-objects, despite having a close relationship warranted by the translation procedure, are at the same time two quite different kinds of objects. The main difference is that the strategic reason keeps track of the interactive process involved in the context of a dialogical game. In particular, it keeps track of which player is responsible for which strategic reason: in this case, **P** is responsible for the strategic reason of the consequent as it stems from the strategic reason for the antecedent endorsed by **O**. This is the basic idea underlying the general form $...^{\mathbf{P}} [\![...^{\mathbf{O}}]\!]$ of the strategic reason. Such aspects are not to be found in proof-objects, since CTT is not a framework based on (strategic) interaction between two players.

Now as we have noticed, this is not to say that one cannot find any kind of relationship between the two objects. As we shall see hereafter, one finds common elements in a strategic reason and the corresponding proof-object through a dialogical reading of CTT proof-objects (or equivalently through a CTT reading of strategic reasons). In a nutshell, strategic reasons give a dialogical, interaction-based reading of the corresponding proof-objects, or, analogously, proof-objects are a kind of linear, interaction-free reading of the corresponding strategic reasons.

In the case in point, the general structure of the strategic reason shows the (strategic) dependence of **P**'s reason for the consequent upon **O**'s reason for the antecendent in a way similar to how the proof-object of the conclusion shows how the proof-object for the consequent results from applying a certain function to the proof-object for the antecedent. Notice how the interactive aspects in the strategic reason are accounted for by means of the players' identities and the equalities.

More precisely on the concrete example at hand, the most obvious connection between the strategic reason for the thesis and the proof-object for the conclusion lies between the dialogical instructions $L^{\wedge}(d_1)$ and $R^{\wedge}(d_1)$ in the strategic reason, and $\mathbf{fst}(d_1)$ and $\mathbf{snd}(d_1)$ in the proof-object.

Once we have noticed this, it is straightforward to read the *left-hand part* of the strategic reason

$$\left(R^{\supset}(c)^{\mathbf{O}}\left[\!\left[\langle L^{\wedge}\left(L^{\supset}(c)\right)^{\mathbf{P}} = R^{\wedge}(d_1)^{\mathbf{P},\mathbf{O}}, R^{\wedge}\left(L^{\supset}(c)\right)^{\mathbf{P}} = L^{\wedge}(d_1)^{\mathbf{P},\mathbf{O}}\rangle\right]\!\right]\right)^{\mathbf{P}}$$

as saying that the reason $R^{\supset}(c)$ for C is obtained from the pair $\langle R^{\wedge}(d_1), L^{\wedge}(d_1)\rangle$ (read as $\langle \mathbf{snd}(d_1), \mathbf{fst}(d_1)\rangle$) given **O**'s reason $\langle L^{\wedge}(d_1), R^{\wedge}(d_1)\rangle$ in the right-hand part, read as $\langle \mathbf{fst}(d_1), \mathbf{snd}(d_1)\rangle$.

The strategic dependence of **P**'s reason upon **O**'s reason is then to be read from the CTT point of view as the existence of a function between the two parts, that is, as a lambda abstraction, which gives us the proof-object $\lambda d_1.\,[\mathbf{ap}(c, \langle \mathbf{snd}(d_1), \mathbf{fst}(d_1)\rangle)]$ for the CTT conclusion.

X. MATERIAL DIALOGUES

As pointed out by Krabbe (1985, p. 297), material dialogues—that is, dialogues in which propositions have content—receive in the writings of Paul Lorenzen and Kuno Lorenz priority over formal dialogues: material dialogues constitute the *locus* where the logical constants are introduced. However in the standard dialogical framework, since both material and formal dialogues marshal a purely syntactic notion of the formal rule—through which logical validity is defined—, this contentual feature is bypassed,[126] with this consequence that Krabbe and others after him considered that, after all, *formal* dialogues had priority over material ones.

As can be gathered from the above discussion, we believe that this conclusion stems from shortcomings of the standard framework, in which local reasons are not expressed at the object-language level. We thus explicitly introduced these local reasons in order to undercut this apparent precedence of a formalistic approach that makes away with the contentual origins of the dialogical project.

And yet the Socratic Rule, as defined in the main chapters of our study (VI-IX), entirely leaves the introduction of local reasons to the Opponent (the Proponent only being allowed to copy what the Opponent introduced). This rule applying to *any* proposition (or set), it can be considered as a formal rule; so if we are to specify the rules for *material* reasoning—to use Peregrin's (2014, p. 228) apt terminology—, the rules specifying the elementary propositions involved in a dialogue must also be defined: whereas in the structural rules for formal dialogues of immanent reasoning (see section VII.2), only the Socratic rules dealt with elementary statements, and without providing any specification on that statement beside the simple fact that it must be the Opponent who introduces them in the dialogue, the structural rules for material dialogues of immanent reasoning will have Socratic rules that are player dependent rules for elementary statements specific to that very statement, but the global rules (player independent) also include rules for elementary statements, specific to those statements (thus providing the material level). These Global rules establish how to synthesize and analyze local reasons during the development of play, once the role of the players have been fixed; they are thus at the structural level and not at the local level, though we still speak of *local* reasons (and *not* of global reasons).

Thus, as mentioned in section VI.2, a local reason prefigures a material dialogue displaying the content of the proposition stated. This aspect makes up the ground level of the normative approach to meaning of the dialogical framework, in which *use*—or dialogical interaction—is to be understood as *use prescibed by a rule*; such a use is what Peregrin (2014, pp. 2-3) calls the *role of a linguistic expression*. Dialogical interaction is this *use*, entirely determined by rules that give it meaning: the linguistic expression of every statement determines this statement by the role it plays, that is by the way it is used, and this use is governed by rules of interaction. The meaning of elementary propositions in dialogical interaction thus amounts to their *role* in the kind of interaction that is governed by the Socratic and Global rules for material dialogues, that is by the specific formulations of the Socratic and Global rules for precisely those very propositions.

It follows that material dialogues are important not only for the general issue on the normativity of logic but also for rendering a language with *content*.

---

[126] (Krabbe, 1985, p. 297)

A thorough study on material dialogues is still work in progress and it seems to be related to recent researches by Piecha & Schröder-Heister (2011) and Piecha (2012) on dialogues with definitions. A study devoted to material dialogues as applied to natural language will require a text of a similar length as the one of the present book; however we can already sketch the main features of material dialogues that include sets of natural numbers and the set $\mathbf{Bool}$. The latter allows for expressing classical truth-functions within the dialogical framework, and it has an important role in the CTT-approach to empirical propositions.[127] Accordingly, this chapter is structured as follows:

1. A first section (X.1) will introduce material dialogues through one of its cases, $\mathbf{Bool}$, which is central for material dialogues as it provides the means for dealing with empirical quantities (the canonical elements of $\mathbf{Bool}$ being the answers to yes or no questions); this will also enable us to provide the truth-functional operators within a dialogical framework.
2. We will then (section X.2) provide further rules on material dialogues, dealing more generally with identity and equality, and provide an application of these rules by demonstrating *within* the system—that is, without presupposing a metalanguage—that *every element of the set* $\mathbf{Bool}$ *is equal to* $\boldsymbol{yes}$ *or to* $\boldsymbol{no}$, which has many consequences in the foundations of mathematics, discussed in the following section.
3. A further section (X.3) will deal more precisely with issues concerning mathematics and logic, where we will introduce natural numbers and broaden the perspective on $\mathbf{Bool}$. We will then test the approach by discussing some case-studies in the domains of the foundations of mathematics and non-classical truth-functional operators.
    a. Developping the consequences of the demonstration of section X.2.3 on $\mathbf{Bool}$, we shall first demonstrate that the two canonical elements $\boldsymbol{yes}, \boldsymbol{no}$ of the set $\mathbf{Bool}$ are different; we shall then approach the fourth axiom of Peano's arithmetic ("*0* is identical to no successor of a natural number": $(\forall x : \mathbb{N})\neg \boldsymbol{Id}(\mathbb{N}, 0, \boldsymbol{s}(x))$). Moreover, such a demonstration gives us the chance to delve into the notion of a universe $\boldsymbol{\mathcal{U}}$ constituted by sets dependent upon the Boolean set $\{\boldsymbol{yes}, \boldsymbol{no}\}$. In other words, while $\boldsymbol{\mathcal{U}}$ is constituted by codes of sets, there is no code for $\boldsymbol{\mathcal{U}}$ itself. Universes constitute the constructivist formulation of the mathematical notion of *sets of sets.*
    b. In relation to the introduction of non-classical truth-functional operators, we will generalize Boolean operators to operators within finite sets and we will put this dialogical framework into work by integrating logics tolerant to some contradictions.
4. Empirical quantities will then (section X.4) be inserted within material dialogues. Empirical content can thus be inserted within the framework of immanent reasoning: it comes from using "empirical quantities" as the outcome of procedures triggered by questions pertaining specifically to that quantity.
5. The final section (X.5) will make explicit the epistemological background underlying the dialogical framework, by stressing the notion of

---

[127] See (Martin-Löf, 2014)

internalization.[128] In this respect, the dialogical framework can be considered as a formal approach to reasoning rooted in the dialogical constitution and "internalization" of content—including empirical content—rather than in the syntactic manipulation of un-interpreted signs. This discussion on material dialogues will provide a new perspective on Willfried Sellars' (1991, pp. 129-194) notion of *Space of Reasons*: the dialogical framework of immanent reasoning enriched with the material level should show how to integrate world-directed thoughts (displaying empirical content) into an inferentialist approach, thereby suggesting that immanent reasoning can integrate within the same epistemological framework the two conflicting readings of the Space of Reasons brought forward by John McDowell (2009, pp. 221-238) on the one hand, who insists in distinguishing world-direct thought and knowledge gathered by inference, and Robert Brandom (1997) on the other hand, who interprets Sellars' work in a more radical anti-empiricist manner. The point is not only that we can deploy the CTT-distinction between *reason* as a premise and *reason* as a piece of evidence justifying a proposition, but also that the dialogical framework allows for distinguishing between the objective justification level targeted by Brandom (1997, p. 129) and the subjective justification level stressed by McDowell. According to our approach the sujective feature corresponds to the play level, where a concrete player brings forward the statement *It looks red to me*, rather than *It is red*. The general epistemological upshot from these initial reflections is that, on our view, many of the worries on the interpretation of the Space of Reasons and on the shortcomings of the standard dialogical approach to meaning (beyond the one of logical constants) have their origin in the neglect of the play level.[129]

X.1 Material dialogues for Bool

Most of the literature differentiating the philosophical perspective underlying the work of Boole and the one of Frege focused on discussing either the different ways both authors understood the relation between logic and psychology or the links between mathematics and logic, or both. According to these studies, Boole's framework has been mainly conceived as a kind of psychologism and a programme for the mathematization of logic, contrasting as well with Frege's radical anti-psychologism as with his logicist project for the foundations of mathematics. These comparative studies have also been combined with the contrast between model-theoretical approaches to meaning, with their associated notion of *varying domains of discourse*, versus inferentialist approaches to meaning, with a *fixed universe of discourse*. It might be argued that while the first approach could be more naturally understood as an offspring of the algebraic tradition of Boole-Schröder, the second approach could be seen as influenced by Frege's *Begriffsschrift.*[130]

---

[128] By "internalization" we mean that the relevant content is made part of the setting of the game of giving and asking for reasons: any relevant content is the content displayed during the interaction. For a discussion on this conception of internalization see (Peregrin, 2014, pp. 36-42).

[129] For some recent literature on those kind of objections to the approach to meaning of the dialogical conception of logic of (Lorenzen & Lorenz, 1978) see (Duthil Novaes, 2015) and (Trafford, 2017), chapter 4, section 2. Our concluding chapter (XI) will deal with this neglect of the play level at length.

[130] Recall the distinction of language *as the universal medium* and *as a calculus* (van Heijenoort, 1967).

However, from the point of view of contemporary classical logic, and after the meta-mathematical perspective of Gödel, Bernays, and Tarski, both Boole's and Frege's view on semantics are subsumed under the same formalization, according to which classical semantics amount to a function of interpretation between the sentences $\boldsymbol{S}$ of a given language $\boldsymbol{L}$ and the set of truth values $\{\mathbf{0},\ \mathbf{1}\}$—let us call such a set the set $\mathbf{Bool}$. This function assumes that the well-formed formulae of $\boldsymbol{S}$ are made dependent upon a domain—either a *local domain of discourse* (in the case of Tarski's-style approach to Boolean-algebra), or a *universal domain* (in the case of Frege). More precisely, this functional approach is based on a separation of cases that simply assumes that the quantifiers and connectives take propositional functions into classical propositions—for a lucid insight on this perspective's limitations see (Sundholm, 2006). In fact, the integration of both views within the same formal semantic closes a gap in Boole's framework, which was already pointed out by Frege: the links between the semantics of propositional and first-order logic.[131]

*Constructive Type Theor*y includes a third (epistemic) paradigm in the framework allowing for a new way of dividing the waters between Boolean operators and logical connectives, and, at the same time, integrating them in a common inferential system in which each of them has a specific role to play. The overall paradigm at stake is the Brouwer–Heyting–Kolmogorov conception of propositions as sets of proofs embedded in the framework in which, thanks to the insight brought forward by the Curry–Howard isomorphism (Howard, 1980), propositions are read as sets and as types.

In a nutshell: the CTT framework takes judgements, not propositions, to yield the minimal unity of knowledge and meaning—as the old philosophical tradition did before the spreading of the metalogical view of Gödel, Tarski and Bernays; see (Sundholm, 1997; 1998; 2001; 2009) (2012; 2013b; 2013a). Within CTT the simplest form of a judgement (the categorical) is an expression of the form

$$a : B$$

where "B" is a proposition and "a" a proof-object on the grounds of which the proposition B is asserted to be true, standing as shorthand for

"*a provides evidence for B*".

In other words, the expression "$a : B$", is the formal notation of the categorical judgment

"*The proposition B is true*",

which is shorthand for

"*There is evidence for B*".

According to this view, a proposition is a set of elements, called proof-objects, that make the proposition B true. Furthermore, we distinguish between *canonical proof-objects* on the one hand, those entities providing a *direct evidence* for the truth the proposition B, and on the other hand *non-canonical proof-objects*, the entities providing indirect pieces of evidence for B.

---

[131] Frege points out that within Boole's approach there is no organic link between propositional and first-order logic : " *In Boole the two parts run alongside one another ; so that one is like the mirror image of the other, but for that very reason stands in no organic relation to it*" (Frege G. , Boole's Logical Calculus and the Concept Script [1880/81], 1979).

This generalization also allows another third reading: a proposition is a *type* and its elements are instances of this type. If we follow this reading proof-objects are conceived as instantiations of the type. This type-reading naturally leads to Brouwer–Heyting–Kolmogorov's constructivism mentioned above: if a proposition is understood as the set of its proofs, it might be the case that we do not have any proof for that proposition at our disposal, but that we neither have a proof for its negation (thus, in such a framework, *third excluded* fails). Notice that the constructivist interpretation requires the intensional rather than the extensional constitution of sets—recall the Aristotelian view that no "form" ("type") can be conceived independently of its instances and reversewise.

Moreover CTT provides a novel way to render the meaning of the set $\{\mathbf{0}, \mathbf{1}\}$ as the type **Bool**. More precisely, the type **Bool** is characterized as the set of the canonical elements **0** and **1**. Thus, each non-canonical element is equal to one of them. But what kind of entities are those (non-canonical) elements that might be equal to $\mathbf{1}$ or $\mathbf{0}$? Since in such a setting **1**, **0** and those equal to them are elements, they are not considered to be of the type proposition; they are rather providers of truth or falsity of a proposition (or a set, according to the Curry–Howard isomorphism between propositions, sets, and types): they are proof-objects that provide evidence for the assertion **Bool true**.

In order to illustrate our point here and to explicit the link with material dialogues, let us take an example outside of mathematics, for instance this *sentence*:

Bachir Diagne is from Senegal.

This sentence differs from the *proposition*, that is, what Frege called the *sense* or *thought* expressed by that sentence, which would be

that Bachir Diagne is from Senegal

So if we take the sentence as expressing the proposition, then we might be able to bring forward some proof-object—some piece of evidence a, his passport or his birth certificate for instance—that renders the proposition *that Bachir Diagne is from Senegal* true. In such a case we have the assertion that the proposition is true on the grounds of the piece of evidence a (the passport), which we can write:

passport : Bachir Diagne is from Senegal

or, in a more general assertion:

That Bachir Diagne is from Senegal **true**
(*there is some piece of evidence that Bachir Diagne is from Senegal*)

In this fashion, if we take the sentence *Bachir Diagne is from Senegal* as related to a *Boolean object*, then this sentence triggers a procedure yielding a non-canonical element, say $\mathbf{X}$ (the proposition), of the set **Bool**. In such a case the sentence would not *express* a proposition, but it could be understood as an *answer* to the question:

Is Bachir Diagne from Senegal ?

the answer being:

yes, *Bachir Diagne is from Senegal*

which would thus yield the outcome **1**. In other words, determining to which of the canonical elements, **1** or **0**, the non-canonical element **X** is equal, would require answering to the question *Is Bachir Diagne from Senegal* ? Thus, in our case, we take it to be equal to **1**.[132] The procedure would amount to the following steps:

Is Bachir Diagne from Senegal ?

↓

yes, *Bachir Diagne is from Senegal*

↓

$$\mathbf{X} = \mathbf{1} : \mathbf{Bool}$$

The arrows indicate that determining to which of the elements **X** is equal actually is the result of an enquiry (in this case an empirical one).

This is not only different from:

passport : Bachir Diagne is from Senegal

but it is also different from:

Bool true

Indeed, while $\mathbf{X} = \mathbf{1} : \mathbf{Bool}$ expresses one of the possible outcomes the non-canonical element **X** can take in **Bool**, **Bool true** expresses the fact that at least one element of the set **Bool** can be brought forward.

Thus, a distinction is drawn between the Boolean object **1** (one of the canonical elements of **Bool**) and the predicate **true** that applies to **Bool**.

Moreover, operations between elements of **Bool** would then not be the logical connectives introduced by natural deduction rules at the right hand side of the colon, but they would be operations between objects occurring at the left hand side of the colon. For example the disjunction "**+** " at left of the colon in

$$A + B = \mathbf{1} : \mathbf{Bool}\ (given\ A = \mathbf{1} : \mathbf{Bool})$$

stands for an operation between the non-canonical **Boolean** objects A and B; whereas the disjunction occurring at right of the colon in the assertion

$$b : A \vee B\ (given\ b : A)$$

expresses the familiar logical connective of disjunction, that is here true since a piece of evidence for one of the disjuncts is provided: the piece of evidence b for A.

---

[132] For the interpretation of empirical propositions see (Martin-Löf, 2014).

Since **Bool** is a type, and since according to the Curry–Howard isomorphism, it is itself a proposition, we can certainly have both, propositional connectives as sets of proof-objects, and have them combined with Boolean operations. This allows us, for example, to demonstrate that each canonical element in **Bool** is identical either to **1** or **0**:

$$(\forall x\colon \mathbf{Bool})\boldsymbol{Id}(\mathbf{Bool}, x, \mathbf{1}) \vee \ \boldsymbol{Id}(\mathbf{Bool}, x, \mathbf{0})\ \boldsymbol{true}$$

This will however require first a presentation of the dialogical rules for **Bool** and the Boolean operators, which will enable us to introduce dialogical rules for material meaning (Specific Socratic rules) through the case of **Bool,** before providing the general notions of material dialogues with the rules for identity and equality (section X.2). We will then have all the elements for dealing with the demonstration of the above proposition, which we shall carry out (see section X.2.3) in the dialogical framework of immanent reasoning.

Dialogical Rules for Boolean Operators

In the dialogical framework, the elements of **Bool** are responses to yes-no questions, so that each element of **Bool** is equal to $\boldsymbol{yes}$ or $\boldsymbol{no}$. Responses such as $b = \boldsymbol{yes}$ or $b = \boldsymbol{no}$ make explicit one of the possible origins of the answer $\boldsymbol{yes}$ (or $\boldsymbol{no}$), namely whether b is or not the case. Here are the Global (player independent) rules for synthesis, analysis, and equalities of the Boolean operators.

Table 31: Global rules for **Bool** and Boolean operators in immanent reasoning

	Move	**Challenge**	**Defence**
Synthesis	**X** ! **Bool**	**Y** $?_{\boldsymbol{Bool}}$	**X** $\boldsymbol{yes}$: **Bool** **X** $\boldsymbol{no}$: **Bool**
Analysis	**X** $p : C(c)[c : \mathbf{Bool}]$	**Y** $? = c^{\mathbf{Bool}}$	**X** $\mathrm{c} = \boldsymbol{yes} : \mathbf{Bool}$ **X** $\mathrm{c} = \boldsymbol{no} : \mathbf{Bool}$
Equalities $\boldsymbol{yes}$	**X** $c = \ \boldsymbol{yes} : \mathbf{Bool}$ … **X** $p : C(c)[c : \mathbf{Bool}]$	**Y** $? =_{reason}\ \boldsymbol{yes}$	**X** $p_1\colon C(\boldsymbol{yes})$
$\boldsymbol{no}$	**X** $c = \ \boldsymbol{no} : \mathbf{Bool}$ … **X** $p : C(c)[c : \mathbf{Bool}]$	**Y** $? =_{reason}\ \boldsymbol{no}$	**X** $p_1\colon C(\boldsymbol{no})$

Note:

In a play, given the statements **P** $p_1\colon C(\boldsymbol{yes})$ or **P**$p_2 : C(\boldsymbol{no})$ (**P**-defences in the equality rules), the play would continue with **O** challenging the elementary statement according to the attack prescribed by the general Socratic rule (see SR5: Socratic rule and definitional equality in section VII.2.1).

Specific Socratic rule for Bool and Boolean operators

When **O** states $a\colon \mathbf{Bool}$, she is stating that a is an element of **Bool**, that is, she is committing to a being either $\boldsymbol{yes}$ or $\boldsymbol{no}$; **P** may challenge this **O**-statement by requesting that she makes her commitment explicit and provides the equality $a = \boldsymbol{yes} : \mathbf{Bool}$ or $a = \boldsymbol{no} : \mathbf{Bool}$. The following table provides the dialogical rule for this interaction; this rule is part of the Socratic rules because it is player dependent,[133] but it is a rule specific to **Bool** and the Boolean operators thus providing their specific meaning: this specific Socratic rule for **Bool** and the Boolean operators provides their material meaning.

Table 32: Specific Socratic rule for **Bool**

	Move	**Challenge**	**Defence**
Specific Socratic rule for Bool	$\mathbf{O}\ a\colon \mathbf{Bool}$	$\mathbf{P}\ ?= a^{\mathbf{Bool}}$	$\mathbf{O}\ a = \boldsymbol{yes}\colon \mathbf{Bool}$
			$\mathbf{O}\ a = \boldsymbol{no}\colon \mathbf{Bool}$

We can now introduce quite smoothly the rules for the classical truth-functional connectives as operations between *elements* of **Bool** (left-hand side of the colon), which are distinct from the usual propositional connectives (right-hand side of the colon). We leave the description for quantifiers to the diligence of the reader, whereby the universal quantifier is understood as a finite sequence of products, and, dually, the existential as a finite sequence of additions.

The dialogical interpretation of the rules below is very close to the usual one: it amounts to the commitments and entitlements specified by the rules of the dialogue: if for instance the response is $\boldsymbol{yes}$ to a (left-hand side) product, then the speaker is also committed to answer $\boldsymbol{yes}$ to further questions on both of the components of that product. Here again, the meaning of the connectives is provided by interaction, where *choice* is a fundamental feature.

Table 33: Global rules for classical truth-functional operators

	Move	**Synthesis of local reasons**		**Synthesis of strategic reasons**
		Challenge	**Defence**	
Product	$\mathbf{X}\ a \times b\colon \mathbf{Bool}$	$\mathbf{Y}\ ?= a \times b$	$\mathbf{X}\ (a \times b) = \boldsymbol{yes}\colon \mathbf{Bool}$	$\mathbf{P}\ \boldsymbol{yes}\ [\![\langle a = \boldsymbol{yes}, b = \boldsymbol{yes}\rangle]\!]^{\mathbf{O}} : \mathbf{Bool}$
			$\mathbf{X}\ (a \times b) = \boldsymbol{no}\colon \mathbf{Bool}$	$\mathbf{P}\ \boldsymbol{no}\ [\![a = \boldsymbol{no} \mid b = \boldsymbol{no}]\!]^{\mathbf{O}} : \mathbf{Bool}$
***Yes*-equality (product)**	$\mathbf{X}\ (a \times b) = \boldsymbol{yes} : \mathbf{Bool}$	$\mathbf{Y}\ ?L^{\times}\boldsymbol{yes}$	$\mathbf{X}\ a = \boldsymbol{yes}\colon \mathbf{Bool}$	
		$\mathbf{Y}\ ?R^{\times}\boldsymbol{yes}$	$\mathbf{X}\ b = \boldsymbol{yes}\colon \mathbf{Bool}$	
***No*-equality (product)**	$\mathbf{X}\ (a \times b) = \boldsymbol{no} : \mathbf{Bool}$	$\mathbf{Y}\ ?^{\times}\ \boldsymbol{no}$	$\mathbf{X}\ a = \boldsymbol{no}\colon \mathbf{Bool}$	
			$\mathbf{X}\ b = \boldsymbol{no}\colon \mathbf{Bool}$	
Addition	$\mathbf{X}\ a + b\colon \mathbf{Bool}$	$\mathbf{Y}\ ?= a + b$	$\mathbf{X}\ a = \boldsymbol{yes}\colon \mathbf{Bool}$	$\mathbf{P}\ \boldsymbol{yes}\ [\![a = \boldsymbol{yes} \mid b = \boldsymbol{yes}]\!]^{\mathbf{O}} : \mathbf{Bool}$
			$\mathbf{X}\ b = \boldsymbol{yes}\colon \mathbf{Bool}$	$\mathbf{P}\ \boldsymbol{no}\ [\![\langle a = \boldsymbol{no}, b = \boldsymbol{no}\rangle]\!]^{\mathbf{O}} : \mathbf{Bool}$

---

[133] For a discussion on player dependence and the way this feature divides the Structural rules and the Particle rules, see section XI.3.

***Yes*-equality (addition)**	$\mathbf{X}\ (a + b) = \boldsymbol{yes} :\ \mathbf{Bool}$	$\mathbf{Y}\ ?^{+}\ \boldsymbol{yes}$	$\mathbf{X}\ \ a = \boldsymbol{yes}: \mathbf{Bool}$	
			$\mathbf{X}\ \ b = \boldsymbol{yes}: \mathbf{Bool}$	
***No*-equality (addition)**	$\mathbf{X}\ (a + b) = \boldsymbol{no} :\ \mathbf{Bool}$	$\mathbf{Y}\ ?\, L^{+}\boldsymbol{no}$	$\mathbf{X}\ \ a = \boldsymbol{no}: \mathbf{Bool}$	
		$\mathbf{Y}\ ?\, R^{+}\boldsymbol{no}$	$\mathbf{X}\ \ b = \boldsymbol{no}: \mathbf{Bool}$	
Implication	$\mathbf{X}\ a \to b: \mathbf{Bool}$	$\mathbf{Y}\ ? =\ a \to b$	$\mathbf{X}(a \to b) = \boldsymbol{yes}: \mathbf{Bool}$	$\mathbf{P}\ \boldsymbol{yes}\ \llbracket\langle a = \boldsymbol{yes}, b = \boldsymbol{yes}\rangle \mid \langle b = \boldsymbol{no}, a = \boldsymbol{no}\rangle\rrbracket^{\mathbf{O}}: \mathbf{Bool}$ $\mathbf{P}\ \boldsymbol{no}\ \llbracket\langle a = \boldsymbol{yes}, b = \boldsymbol{no}\rangle\rrbracket^{\mathbf{O}}: \mathbf{Bool}$
			$\mathbf{X}(a \to b) = \boldsymbol{no}: \mathbf{Bool}$	
***Yes*-equality (implication)**	$\mathbf{X}\ \ (a \to b) = \boldsymbol{yes}: \mathbf{Bool}$	$\mathbf{Y}\ a = \boldsymbol{yes}: \mathbf{Bool}$	$\mathbf{X}\ b = \boldsymbol{yes}: \mathbf{Bool}$	
		$\mathbf{Y}\ b = \boldsymbol{no}: \mathbf{Bool}$	$\mathbf{X}\ a = \boldsymbol{no}: \mathbf{Bool}$	
***No*-equality (implication)**	$\mathbf{X}\ \ (a \to b) = \boldsymbol{no}: \mathbf{Bool}$	$\mathbf{Y}\ ?\, L^{\to}\boldsymbol{no}$	$\mathbf{X}\ a = \boldsymbol{yes}: \mathbf{Bool}$	
		$\mathbf{Y}\ ?\, R^{\to}\boldsymbol{no}$	$\mathbf{X}\ b = \boldsymbol{no}: \mathbf{Bool}$	
Negation	$\mathbf{X}\ {\sim}a: \mathbf{Bool}$	$\mathbf{Y}\ ?\ {\sim}a$	$\mathbf{X}\ {\sim}a = \boldsymbol{yes}: \mathbf{Bool}$	$\mathbf{P}\ \boldsymbol{yes}\ \llbracket a = \boldsymbol{no}\rrbracket^{\mathbf{O}} : \mathbf{Bool}$ $\mathbf{P}\ \boldsymbol{no}\ \llbracket a = \boldsymbol{yes}\rrbracket^{\mathbf{O}} : \mathbf{Bool}$
			$\mathbf{X}\ {\sim}a = \boldsymbol{no}: \mathbf{Bool}$	
***Yes*-equality (negation)**	$\mathbf{X}\ {\sim}a = \boldsymbol{yes}: \mathbf{Bool}$	$\mathbf{Y}\ ?^{\sim}\ \boldsymbol{yes}$	$\mathbf{X}\ a = \boldsymbol{no}: \mathbf{Bool}$	
***No*-equality (negation)**	$\mathbf{X}\ {\sim}a = \boldsymbol{no}: \mathbf{Bool}$	$\mathbf{Y}\ ?^{\sim}\ \boldsymbol{no}$	$\mathbf{X}\ a = \boldsymbol{yes}: \mathbf{Bool}$	

Reading the strategic reason for the product:

If both of the components of the product are affirmative ($\boldsymbol{yes}$ answer), that is if **O** has at some point stated both, then the recapitulation answer is $\boldsymbol{yes}: \mathbf{Bool}$, provided these **O** statements—$\llbracket\langle a = \boldsymbol{yes},\ b = \boldsymbol{yes}\rangle\rrbracket^{\mathbf{O}}$. If at least one of the two components is a denial, that is if **O** has at some point stated one or the other, then the recapitulation answer is $\boldsymbol{no}: \mathbf{Bool}$, provided one of these **O** statements—$\llbracket a = \boldsymbol{no} \mid b = \boldsymbol{no}\rrbracket^{\mathbf{O}}$.

X.2 Material dialogues through identity and equality

From the CTT perspective, the identity predicate $\boldsymbol{Id}$ should be differentiated from

1. definitional equality, which does not express a proposition but introduces real definitions and establishes equivalence relations between pieces of evidence (proof-objects),
2. nominal definitions, which produces linguistic abbreviations, and
3. equality (or Identity) as a relation building up a proposition, that is, the relation we know from first-order logic, and which constitutes a proposition.

We therefore distinguish between

1. the real definition or *judgemental equality* $a = b : A$;
2. the *nominal definition*, for example *"1" stands for successor of "$s(0)$"*;
3. the propositional *identity* $c : a =_A b$, or better $c : \boldsymbol{Id}(A, a, b)$.

In a dialogical setting,

- real definitions express at the object-language level **P**'s right to state b, **O** having already stated both, a and that a defines b. So **P**'s move $a = b : A$, responding to **O**'s request of justification for $b : A$, can be read as "you just conceded $a : A$, and furthermore you conceded that a defines b".
- Nominal definitions allows **P** to deploy the abbreviations established in such kinds of definition.
- **P** is allowed to state the Identity $\boldsymbol{Id}(A, a, b)$ only if he can state that c is equal to the local (reflexivity) reason $\boldsymbol{refl}(A, a)$ —that is if he can state $\boldsymbol{refl}(B, a) = c : \boldsymbol{Id}(A, a, b)$—, and if he can show that the equality $a = b : B$ presupposed by the formation of $\boldsymbol{Id}(A, a, b)$ has been fulfilled.

X.2.1 The generation of Id

The identity predicate $\boldsymbol{Id}(x, y, z)$, where x is a **set**—or a **prop**—and y and z are local reasons for x, can be read as "y is identical to z within x." The dialogical meaning explanation of this identity predicate is provided below through the different rules governing its use (formation rule and Specifi Socratic rule for $\boldsymbol{Id}$), but can be grasped through this fact that **X**'s entitlement to state $\boldsymbol{Id}(A, a, b)$—"a is identical to b within A"—for instance, presupposes that $a : A$ and that $b : A$, which is precisely its formation rule:

Table 34: Formation rule for $\boldsymbol{Id}$

	Move	Challenge	Defence
Formation of the identity predicate	**X** $\boldsymbol{Id}(A, a_i, a_j) : prop$	**Y** $?_{F1}\,\boldsymbol{Id}$	**X** $A :$ **set**
		Y $?_{F2}\,\boldsymbol{Id}$	**X** $a_i : A$
		Y $?_{F3}\,\boldsymbol{Id}$	**X** $a_j : A$

The three available challenges for the statement that $\mathbf{Id}(A, a_i, a_j) : prop$ correspond to the three presuppositions for this statement, namely that the first argument is a **set**, and that the second and third arguments are members of the first argument.

Since the basic cases of $\boldsymbol{Id}$-statements involve elementary propositions, the rules prescribing the dialogical interaction for this kind of statements must be handled by a Socratic rule *specific to that predicate* (thus providing material meaning). In the case of $\boldsymbol{Id}$, **O**'s identity statements can only be challenged by two means: either the rule of Global analysis, or the Leibniz-substitution rule. We will now provide these Specific Socratic rules for $\boldsymbol{Id}$.

X.2.1.1 SR-Id.1 Specific Socratic rules for $\boldsymbol{Id(A, a, a)}$

If **P** states $\boldsymbol{Id}(A, a, a)$, then he must bring forward the definitional equality that conditions statements of propositional intensional identity. Furthermore, the statement

$\mathbf{P}\,!\boldsymbol{Id}(A,a,a)$ commits the Proponent to make explicit the local reason behind his statement, namely, the local reason $\boldsymbol{refl}(A,a)$ specific to $\boldsymbol{Id}$-statements, whose internal structure only depends on a.

Thus, the dialogical meaning of the instruction $\boldsymbol{refl}(A,a)$ amounts to prescribing the definitional equality $a = \boldsymbol{refl}(A,a) : A$ as defence to the challenge $\mathbf{O}\,? = \boldsymbol{refl}(A,a)$. The following table displays the global rules that implement those prescriptions.

Table 35: Specific Socratic rule for the global synthesis of the local reason for $\mathbf{P}!\boldsymbol{Id}(A,a,a)$ and the rules of equality

	Move	**Challenge**	**Defence**
Identity predicate (for reflexivity)	$\mathbf{P}\ !\ \boldsymbol{Id}(A,a,a)$	$\mathbf{O}\ ?_{reason}\,\boldsymbol{Id}$	$\mathbf{P}\,\boldsymbol{refl}(A,a) : \boldsymbol{Id}(A,a,a)$
Rules of equality	$\mathbf{P}\,\boldsymbol{refl}(A,a) : \boldsymbol{Id}(A,a,a)$	$\mathbf{O}\ ? = \boldsymbol{refl}(A,a)$	$\mathbf{P}\,a = \boldsymbol{refl}(A,a) : A$
	$\mathbf{P}\,a = \boldsymbol{refl}(A,a) : A$	$\mathbf{O}\ ? = a$	$\mathbf{P}\,a = a : A$

Note:

This rule presupposes that the well-formation of $\boldsymbol{Id}(A,a,a)$ has been established. What is more, notice that the two lines for the rules of equality follow each other: the defence that **P** must bring forward to answer the first challenge is the object of the second challenge. For an application of these rules, see below, section X.2.3.

The last line is just applying the general Socratic rule for local reasons to the specific case of $\boldsymbol{refl}(A,a)$, and shows that the local reason $\boldsymbol{refl}(A,a)$ is in fact equal to a (see section VII.2.1).

Challenging O-elementary statements

Since in dialogues for immanent reasoning it is **O** who is given the authority to set the local reasons for the relevant sets, **P** can always trigger from **O** the identity statement $\mathbf{O}\,p : \boldsymbol{Id}(A,a,a)$ for any statement $\mathbf{O}\,a : A$ brought forward during a play. This leads to the next table that constitutes *one of the exceptions* to the interdiction on challenges of **O**'s elementary statements.

Table 36: Specific Socratic rule for triggering the reflexivity move $\mathbf{O}!\boldsymbol{Id}(A,a,a)$

	Move	**Challenge**	**Defence**
O-elementary statements I	$\mathbf{O}\,a{:}A$	$\mathbf{P}\,?\boldsymbol{Id}$	$\mathbf{O}\,\boldsymbol{refl}(A,a){:}\boldsymbol{Id}(A,a,a)$

Remark

Notice that it looks as if **P** would not need to use this rule since according to the rule for the synthesis of the local reason for a **P**-identity statement, he can always state $\boldsymbol{Id}(A,a,a)$, provided **O** stated $a : A$. However, in some cases such as when carrying out a substitution based on identity, **P** might need **O** to make an explicit statement of identity suitable for applying a substitution rule.

The next rule (global analysis of **O** $p: \boldsymbol{Id}(A, a, a)$), also allows **P** to challenge an **O**-elementary statement by prescribing how to analyse a local reason p brought forward by **O** in order to support the statement $\boldsymbol{Id}(A, a, a)$.

Table 37: Analysis I: The Global Analysis of **O** $p: \boldsymbol{Id}(A, a, a)$

	Move	Challenge	Defence
Analysis of O-elementary statements	**O** $p: \boldsymbol{Id}(A, a, a)$	**P** $? = p$	**O** $p = \boldsymbol{refl}(A, a): \boldsymbol{Id}(A, a, a)$

The second rule for analysis involves statements of the form $\boldsymbol{Id}(A, a, b)$, so we need general rules for statements that are not restricted to reflexivity. In fact the rules for $\boldsymbol{Id}(A, a, b)$ can be obtained by re-writing the previous rules—with the exception of the rule that triggers statements of reflexivity by **O**.

We will not spell out the rules for $\boldsymbol{Id}(A, a, b)$, but let us stress two important points

(1) the unicity of the local reason $\boldsymbol{refl}(A, a)$;
(2) the non-inversibily of the intensional predicate of identity in relation to judgmental equality.

In relation to the first remark, the unicity of the local reason $\boldsymbol{refl}(A, a)$, the point is that the local reason produced by a process of synthesis for any identity statement is always $\boldsymbol{refl}(A, a)$. In other words, the local reason prescribed by the procedures of synthesis involving the statement ! $\boldsymbol{Id}(A, a, a)$ and the statement $\boldsymbol{Id}(A, a, a)$, is the *same one*, namely $\boldsymbol{refl}(A, a)$.

In relation to the second point, the non-inversibily of the intensional predicate of identity in relation to judgmental equality, it is important to remember that the *global synthesis* rule refers to the commitments undertaken by **P** when he affirms the identity between a and b. Such commitment amounts to i) providing a local reason for such an identity, and ii) stating $a = b : A$.

On the contrary, the rule of *global analysis of an* **O**-*identity statement* prescribes what **P** may require from **O**'s statement. In that case, **P** cannot force **O** to state $a = b: A$ on the sole basis of her having stated $\boldsymbol{Id}(A, a, b)$: this would only be possible with the so-called extensional version of propositional identity—see section X.2.4, or (Nordström, Petersson, & Smith, 1990, pp. 57-61) for a thorough discussion.

The dialogical view of non-reversibility here is that the *rule of synthesis sets the conditions* **P** *must fulfil* when he states and identity, not *what follows from his statement of identity*.

X.2.2 Global Analysis II: Leibniz's substitution rule for Id

Assume that **O** has stated the propositional identity of a and b, that is, **O**! $\boldsymbol{Id}(A, a, b)$, as well as $d : B(a)$; **P** can then force her to further state $B(b)$, by requesting a substitution of indistinguishables. The following table displays the dialogical take on Leibniz-substitution-rule in the context of **Id**-statements.

Table 38: Leibniz-substitution for $\boldsymbol{Id}$

	Move	Challenge	Defence

Leibniz-substitution for $\boldsymbol{Id}$	**O** $c: \boldsymbol{Id}(A, a, b)$ … **O** $d: B(a)$ *The rule assumes* $a: A$, $b: A$, *and* $B(x): prop\ [x: A]$	**P** $?_{Lbz-\boldsymbol{Id}}\, b/a$ *Apply substitution!*	**O** $Lbz\text{–}\boldsymbol{Id}\text{–}subst(c, d): B(d)$ *Application of substitution*

The substitution rule allows the development of a play where **P** brings forward $Lbz\text{-}\boldsymbol{Id}\text{-}subst(c, d)$ in order to defend his statement $B(a)$. Such kind of plays do not stop (like in the case of formal dialogues) with the affirmation of a definitional reflexivity of the form $Lbz\text{-}\boldsymbol{Id}\text{-}subst(c, d) = Lbz\text{-}\boldsymbol{Id}\text{-}subst(c, d): B(a)$; the aim of the play is rather to make it explicit, by means of a series of equalities, that the local reason $Lbz\text{-}\boldsymbol{Id}\text{-}subst(c, d)$ for $B(b)$, is ultimately equal to the one for $B(a)$, since **O** stated that a and b are indistinguishable local reasons in A.

More precisely, the structure of the play should make it patent that the local reason for $B(b)$, if based on an application of the Leibniz substitution rule for **Id,** is $Lbz\text{-}\boldsymbol{Id}\text{-}subst(c, d) = d(a): B(b)$, and that the latter is constituted by the definitional equalities $\boldsymbol{refl}(A, a) = c$ [**O** $c : \boldsymbol{Id}(A, a, b): A$] and $d = d(a/x): B(a/x)\ [a/x : A]$.

The table below displays the full-version of the sequence of moves that leads to such a definitional equality. The table starts with a statement of the form **P** ! $B(b)$, and the challenge $\boldsymbol{O}\ ?_{reason}\, B(b)$, but is *mutatis mutandis* similarly applicable if the starting statement is **P** $\boldsymbol{I} : B(b)$ and the challenge is $O\ ? = \boldsymbol{I}$–where $\boldsymbol{I}$ stands for some instruction.

In practice, we might display a shortened version so that when **P** brings forward $Lbz\text{-}\boldsymbol{Id}\text{-}subst(c, d)$ in order to defend his statement $B(a)$, the response to the challenge **O** ? = $Lbz\text{-}\boldsymbol{Id}\text{-}subst(c, d) : B(b)$ is the non-reflexive definitional equality $Lbz\text{-}\boldsymbol{Id}\text{-}subst(c, d) = d(a): B(b)$ and then we stop.

Table 39: $Lbz\text{-}\boldsymbol{Id}\text{-}subst$ and Definitional Equality

Move	Challenge	Defence
P! $B(b)$	**O** $?_{reason}\, B(b)$	**P** $\boldsymbol{Lbz\text{–}Id\text{–}subst}(c, d): B(b)$
P $\boldsymbol{Lbz\text{–}Id\text{–}subst}(c, d): B(b)$	**O** ? $= \boldsymbol{Lbz\text{–}Id\text{–}subst}(c, d): B(b)$	**P** $\boldsymbol{Lbz\text{–}Id\text{–}subst}(c, d) = \boldsymbol{Lbz\text{–}Id\text{–}subst}(c, d): B(b)$ Provided that **O** $\boldsymbol{Lbz\text{–}Id\text{–}subst}(c, d): B(b)$
P $\boldsymbol{Lbz\text{–}Id\text{–}subst}(c, d) = \boldsymbol{Lbz\text{–}Id\text{–}subst}(c, d): B(b)$	**O** $? = c$	**P** $\boldsymbol{refl}(A, a) = c : A$ *Provided that* $\boldsymbol{O}\ c : \boldsymbol{Id}(A, a, b)$
P $\boldsymbol{Lbz\text{–}Id\text{–}subst}(c, d): B(b)$ … **P** $\boldsymbol{refl}(A, a) = c : A$ $[\boldsymbol{O}\ c : \boldsymbol{Id}(A, a, b)]$	**O** ? $\boldsymbol{refl}(A, a)\ /\ c$	**P** $\boldsymbol{Lbz\text{–}Id\text{–}subst}(\boldsymbol{refl}(A, a), d) : B(b)$
P $\boldsymbol{Lbz\text{–}Id\text{–}subst}(\boldsymbol{refl}(A, a), d) : B(b$	**O** $? = d$	**P** $d = d(a/x): B(a/x)$ $[a/x : A]$

P $\boldsymbol{Lbz\text{-}Id\text{-}subst}(\boldsymbol{refl}(A,a),d)$ $: B(b)$ … **P** $d = d(a/x): B(a/x)$ $[a/x : A]$	**O** ? $\boldsymbol{Lbz\text{-}Id\text{-}subst}$ $(\boldsymbol{refl}(A,a),d) : B(b)$	**P** $d(a) = \boldsymbol{Lbz\text{-}Id\text{-}subst}$ $(\boldsymbol{refl}(A,a),d): B(b)$ *Provided that* **O**! $\boldsymbol{Id}(A,a,b)$

Note

The statements on lines two and four share a common statement (lines five and six as well). It must however be not only noted that the challenges differ, but also that the object of each of the challenges are different. It is important to realize that these rules do not concern *local meaning* but *global meaning*, which is determined by the evolution of the play. Thus, those challenges targetting pairs of statements such that one member of the pair is different to the other, *do not involve using twice the repetition ranks* for challenging the common statement: *they are different attacks to different statements*.

The Leibniz-elimination rule can be inferred from a more general rule provided by (Nordström, Petersson, & Smith, 1990, p. 58). This general rule includes cases of applying substitution to judgements of the form $B(c)$ that expresses that some property holds of some non-canonical proof-object c for an identity judgement. Thus, the importance of this form of judgments is that it makes it possible to justify that some property applies to proof-objects of identities. We will however not develop it here.

X.2.3 Application to Bool: the third (canonical element) excluded

A natural way to combine Boolean operations and elements with propositional connectives is to make use of the identity predicate $\boldsymbol{Id}$.

Let us see how real definitions and identity interact when establishing the validity of the proposition *every element of the set* **Bool** *is equal to* ***yes*** *or to* ***no***. That is, let us build the winning strategy for

$$(\forall x\colon \mathbf{Bool})\boldsymbol{Id}(\mathbf{Bool}, x, \boldsymbol{yes}) \vee \boldsymbol{Id}(\mathbf{Bool}, x, \boldsymbol{no})\ \boldsymbol{true}$$

Notice that this proposition can be interpreted as the classical third-excluded, this proposition indeed saying that there is no third canonical element in **Bool**beside $\boldsymbol{yes}$ and $\boldsymbol{no}$: any element of **Bool** can thus be reduced to either one of them.

We shall proceed as usual by running the plays constituting a **P**-winning strategy (see section VII.5)

While on the one hand winning strategies concern the process of bringing forward the piece of evidence that justifies the proposition involved in the judgement, here *that every element of the set* **Bool** *is equal to* $\boldsymbol{yes}$ *or to* $\boldsymbol{no}$, on the other hand the commitments engaged in asserting that something is one of the pieces of evidence for **Bool**, say, $a + {\sim}a : \mathbf{Bool}$, amounts to answering the question, Which of the canonical elements of **Bool** is this piece of evidence equal to? In our case, $a + {\sim}a = \boldsymbol{yes} : \mathbf{Bool}.$

Notice that since the set **Bool** contains only two elements, universal quantification over **Bool** can be tested by considering each of the elements of the set, each of them triggering a new play.

Play 17: first play for **Bool** third-excluded

$\boldsymbol{\mathcal{P}_1}$	**O**			**P**			
					$!\,(\forall x : \mathbf{Bool})\big(\boldsymbol{Id}(\mathbf{Bool}, x, \boldsymbol{yes}) \vee \boldsymbol{Id}(\mathbf{Bool}, x, \boldsymbol{no})\big)$	0	*Thesis*
	1	$m \coloneqq 1$			$n \coloneqq 2$	2	
Synthesis of local reason for universal quantification (Table 19)	3	$\boldsymbol{yes}: \mathbf{Bool}\ [\delta_{\mathbf{1}}, \ldots]$	0		$\boldsymbol{yes}: \big(\boldsymbol{Id}(\mathbf{Bool}, \boldsymbol{yes}, \boldsymbol{yes}) \vee \boldsymbol{Id}(\mathbf{Bool}, \boldsymbol{yes}, \boldsymbol{no})\big)$	4	
Analysis of local reason for disjunction (Table 20)	5	$?^{\vee}$	4		$L^{\vee}(\boldsymbol{yes}): \boldsymbol{Id}(\mathbf{Bool}, \boldsymbol{yes}, \boldsymbol{yes})$	6	
Resolution of the instruction	7	$?\ldots/L^{\vee}(\boldsymbol{yes})$	6		$\boldsymbol{refl}(\mathbf{Bool}, \boldsymbol{yes}): \boldsymbol{Id}(\mathbf{Bool}, \boldsymbol{yes}, \boldsymbol{yes})$	8	*Table 35* *local reason* $\boldsymbol{Id}(A, a, a)$
Rule of equality (Table 35)	9	$?= \ \boldsymbol{refl}(\mathbf{Bool}, \boldsymbol{yes})$	8		$\boldsymbol{yes} \ = \ \boldsymbol{refl}(\mathbf{Bool}, \boldsymbol{yes}) \ : \ \mathbf{Bool}$	10	
Rule of equality (Table 35)	11	$?= \boldsymbol{yes}$	8		$\boldsymbol{yes} = \boldsymbol{yes}: \boldsymbol{Bool}$	12	

P wins.

Play 18: second play for **Bool** third-excluded

$\boldsymbol{\mathcal{P}_2}$	**O**			**P**		
					$!\,(\forall x : \mathbf{Bool})\big(\boldsymbol{Id}(\mathbf{Bool}, x, \boldsymbol{yes}) \vee \boldsymbol{Id}(\mathbf{Bool}, x, \boldsymbol{no})\big)$	0
	1	$m \coloneqq 1$			$n \coloneqq 2$	2
	3	$\boldsymbol{no}: \mathbf{Bool}\ [\delta_1, \delta_2]$	0		$\boldsymbol{no}: \big(\boldsymbol{Id}(\mathbf{Bool}, \boldsymbol{no}, \boldsymbol{yes}) \vee \boldsymbol{Id}(\mathbf{Bool}, \boldsymbol{no}, \boldsymbol{no})\big)$	4
	5	$?^{\vee}$	4		$R^{\vee}(\boldsymbol{no}): \boldsymbol{Id}(\mathbf{Bool}, \boldsymbol{no}, , \boldsymbol{no})$	6
	7	$?\ldots/L^{\vee}(\boldsymbol{no})$	6		$\boldsymbol{refl}(\mathbf{Bool}, \boldsymbol{no}): \boldsymbol{Id}(\mathbf{Bool}, \boldsymbol{no}, \boldsymbol{no})$	8
	9	$?= \ \boldsymbol{refl}(\mathbf{Bool}, \boldsymbol{no})$	8		$\boldsymbol{no} = \boldsymbol{refl}(\mathbf{Bool}, \boldsymbol{no}): \mathbf{Bool}$	10
	11	$?= \boldsymbol{no}$	8		$\boldsymbol{no} = \boldsymbol{no}: \boldsymbol{Bool}$	12

P wins

Since every **O**-decision option has been considered with plays $\boldsymbol{\mathcal{P}_1}$and $\boldsymbol{\mathcal{P}_2}$, which are both **P**-terminal, **P** has a winning strategy for $(\forall x : \mathbf{Bool})\big(\boldsymbol{Id}(\mathbf{Bool}, x, \boldsymbol{yes}) \vee \boldsymbol{Id}(\mathbf{Bool}, x, \boldsymbol{no})\big)$, which is thus demonstrated (see chapter IX).

Notice that, in this framework, though it is trivial to show

$$\mathbf{P}\ a + {\sim}a \ = \ \boldsymbol{yes} : \mathbf{Bool}$$

we cannot build a winning strategy for:

$$!\ (\forall x : \mathbf{Bool})\ (\boldsymbol{Id}(\mathbf{Bool}, x, \boldsymbol{yes}) \vee \neg \boldsymbol{Id}(\mathbf{Bool}, x, \boldsymbol{yes}))$$

unless we presuppose $\neg\boldsymbol{Id}(\mathbf{Bool}, \boldsymbol{no}, \boldsymbol{yes})$. This will be discussed in the next section.

X.2.4 **The extensional propositional identity $\boldsymbol{Eq}$**

The dialogical rules that prescribe the extensional propositional identity $\boldsymbol{Eq}$ are simpler than the other forms of equality. Once **O** stated an extensional propositional identity, **P** is allowed to ask him to state a definitional equality involving the terms of the relation $\boldsymbol{Eq}$**.** Thus, the rules that prescribe statements of the form $\boldsymbol{Eq}(A, a, b)$ are the same for **P** and **O.** However, the local reason for the resulting proposition, namely the local reason $\boldsymbol{eq}$ for $\boldsymbol{Eq}(A, a, b)$; yet the analysis of $\boldsymbol{eq}$ does not render the local reasons involved in the definitional equality $a = b : A$ on the basis of which the predicate $\boldsymbol{Eq}$ has been stated: $\boldsymbol{eq}$ is thus a kind of analogue of ontological equality. Moreover, every local reason c for $\boldsymbol{Eq}(A, a, b)$ is definitionally equal to $\boldsymbol{eq}$.

Table 40: Formation rules for Eq

	Move	Challenge	Defence
Eq-formation	**X** **Eq**$(A, a_{\mathrm{i}}, a_{\mathrm{j}})$: **prop**	**Y** $?_{F1}$ $\boldsymbol{Eq}$	**X** A: **set**
		Y$?_{F2}$ $\boldsymbol{Eq}$	**X** $a_i : A$
		Y$?_{F3}$ $\boldsymbol{Eq}$	**X** $a_j : A$

Since we are dealing with formation plays, these challenges can all be carried out without requiring the use of repetition ranks.

Table 41: Specific Socratic rule for the Global Synthesis of the local reason for **X**! $\boldsymbol{Eq}(A, a, a)$

	Move	Challenge	Defence
Socratic rule	**X** ! **Eq**$(A, a_{\mathrm{i}}, a_{\mathrm{j}})$	**Y** $?_{reason}$ $\boldsymbol{Eq}$	**X** $\boldsymbol{eq} : \boldsymbol{Eq}(A, a_i, a_j)$
	P $\boldsymbol{eq} : \boldsymbol{Id}(A, a, a)$	**O** ?= $\boldsymbol{eq}$	**P** $\boldsymbol{eq} = \boldsymbol{eq} : A$

Notice that the crucial difference between the $\boldsymbol{Id}$ and $\boldsymbol{Eq}$ is that the rules of synthesis are formulated as player-independent. In the context of an $\boldsymbol{Id}$-statement, **P** cannot force **O** to bring forward $a = b : A$ after **O** $\boldsymbol{eq}$: $\boldsymbol{Eq}(A, a, b)$, whereas in the context of $\boldsymbol{Eq}$, **P** can. Moreover, the defintional equality of $\boldsymbol{eq}$ is $\boldsymbol{eq}$, which is responsable for the "extensionality" of $\boldsymbol{Eq}$ and leads to the undecidability of assertions of the form $a = b : A$—see chapter II.

The rest of the rules follow from adapting the tables for **Id** to **Eq** and **eq**

X.3 Mathematics and logic

X.3.1 Material dialogues for ℕ

Let us display the main specifications yielding material dialogues for the natural numbers. Since we are in the presence of material dialogues, we need to "harmonize" the rules of synthesis and analysis with equality rules *specific* to ℕ. Moreover, the development of a play that includes statements of the form **P** n : ℕ [0 : ℕ] might presuppose some particular nominal definitions that must be described by the structural rules. In the following sections we display the corresponding tables for synthesis, analysis, and equality, and the set of structural rules for the development of a play based on nominal definitions for 1, 2, ... : ℕ.

Synthesis of $\mathbb{N}$

If **X** states that n is a natural number, he is committed to the further statement that its successor is also a natural number.

Table 42: Synthesis of $\mathbb{N}$

	Move	Challenge	Defence
Synthesis of $\mathbb{N}$	$\mathbf{X}\, n{:}\, \mathbb{N}$	$\mathbf{Y}\, ?\, s(n)$	$\mathbf{X}\, s(n){:}\, \mathbb{N}$

Analysis of $\mathbb{N}$

If **X** states that the property C applies to any natural number n, then **X** must also state

1. that it applies to 0 and
2. that it also applies to $C(\boldsymbol{s}(m))$, for any m chosen by **Y**.

Thus, the local reason p is constituted by a *left component* locally grounding the statement $C(0)$, and a *right component*. Thus, two analysis rules are due, the first prescribing the moves yielding the left component of p, and the second those for the right component. The analysis for the right component requires analyzing this right component of p into two further components, namely,

1. the left component of the right component of p, which provides the local reason for **Y**'s statement that the property C applied to m; and
2. the right component of the right component of p, which provides a local reason for **X**'s statement that C also applies to the successor of m.

Table 43: Analysis rules of $\mathbb{N}$

		Move	Challenge	Defence
Analysis of $\mathbb{N}$	**Left**	$\mathbf{X}\, p{:}\, C^{\mathbb{N}}(n)$ *For arbitrary n : $\mathbb{N}$, given $C(z){:}\, \boldsymbol{set}\ [z{:}\, \mathbb{N}]$*	$\mathbf{Y}\, ?\ L^{\mathbb{N}}_{df}C$	$\mathbf{X}\, L^{\mathbb{N}}(p)^{\mathbf{X}} : C(0)$
	Right	$\mathbf{X}\, p : C^{\mathbb{N}}(n)$ … $\mathbf{X}\, L^{\mathbb{N}}(p)^{\mathbf{X}}{:}\, C(0)$ *For arbitrary $n{:}\, \mathbb{N}$, and given $C(z){:}\, \boldsymbol{set}\ [z{:}\, \mathbb{N}]$*	$\mathbf{Y}\, L\left(R^{\mathbb{N}}(p)\right)^{\mathbf{Y}} : C(m^{\mathbf{Y}})$ *For arbitrary m : $\mathbb{N}$, chosen by **Y***	$\mathbf{X}\, R\left(R^{\mathbb{N}}(p)\right)^{\mathbf{X}} : C\left(\boldsymbol{s}(m^{\mathbf{Y}})\right)$

Specific Socratic rules for $\mathbb{N}$

The rules for definitional equality assume that **O** brought forward p_1 and p_2 as local reasons for $C(0)$ and $C\big(\boldsymbol{s}(m)\big)$ respectively.

Table 44: Specific Socratic rules for $\mathbb{N}$

		Move	Challenge	Defence
Socratic rules	**Definitional equality I**	$\mathbf{P}\, p_1 : C(0)$	$\mathbf{O}\, ? = p_1$	$\mathbf{P}\, p_1^{\mathbf{O}} = L^{\mathbb{N}}(p){:}\, C(0)$

for $\mathbb{N}$	**Definitional equality II**	$\mathbf{P}\ p_2 : C(\boldsymbol{s}(m))$	$\mathbf{O}\ ?=p_2$	$\mathbf{P}\ p_2^{\mathbf{O}} = R\left(R^{\mathbb{N}}(p)\right) : C(\boldsymbol{s}(m))$

Rules specific to C

$\mathbf{Y}$'s challenge upon $C(n)$ is *specifically* defined for C. Thus, if C is the predicate "x is an odd number", the rule establishes that the challenge upon, say, $C(\boldsymbol{s}(0))$ is

choose an x such that $\boldsymbol{Id}(\mathbb{N}, (\boldsymbol{s}(0), 2.x^{\mathbf{X}} + 1)$[134]

That is, the precise form of the challenge in our case a challenge would be the challenge of the existential:

$$\mathbf{X}!\ (\exists x : \mathbb{N})\ \boldsymbol{Id}(\mathbb{N}, (\boldsymbol{s}(0), 2.x^{\mathbf{X}} + 1)$$

The response would assume that n stands for $\boldsymbol{s}(0)$, and that $C(n)$ stands for "$\boldsymbol{s}(0)$ is an odd number". This response would thus be the answer to the challenge on the right side of the above existential, the response to the left side being:

$$\mathbf{X}\ !\ 0 : \mathbb{N}$$

Hence, in this example

$$\mathbf{X}\, p : C(n)$$

would stand for

$$\mathbf{X}\, p : (\exists x : \mathbb{N})\ \boldsymbol{Id}(\mathbb{N}, (\boldsymbol{s}(0), 2.x^{\mathbf{X}} + 1)$$

which presupposes

$$Odd(\boldsymbol{s}(0)) = (\exists x : \mathbb{N})\boldsymbol{Id}(\mathbb{N}, \boldsymbol{s}(0), 2.x + 1) : prop\ [\boldsymbol{s}(0) : \mathbb{N}]$$

which presupposes the definition:

$$Odd(y) = (\exists x : \mathbb{N})\boldsymbol{Id}(\mathbb{N}, y, 2.x + 1) : prop\ [y : \mathbb{N}]$$

Structural rules for statements of the form P 1 : ℕ [0 : ℕ], P 2 : ℕ [0 : ℕ], ...

Given a statement of the form $\mathbf{P}\ n : \mathbb{N}\ [0 : \mathbb{N}]$, where n stands for 1, or 2, or ...,[135] **O** can challenge it by means of the attack $?n$. If **P**'s initial statement is $1 : \mathbb{N}\ [0 : \mathbb{N}]$, then **P** can respond to the challenge $?1$ with $\boldsymbol{s}(0) \equiv_{df} 1 : \mathbb{N}$ only if **O** stated $\boldsymbol{s}(0) : \mathbb{N}$; similarly for 2, and so on. In other words:

$$\mathbf{P}\ 1 : \mathbb{N}$$

$$\frac{\mathbf{O}\ ?1 \qquad \mathbf{O}\ \boldsymbol{s}(0) : \mathbb{N}}{\mathbf{P}\ 1 \equiv_{df} \boldsymbol{s}(0) : \mathbb{N}}$$

(The answer cannot be challenged.)

$$\mathbf{P}\ n : \mathbb{N}$$

$$\frac{\mathbf{O}\ ?n \qquad \mathbf{O}\ \boldsymbol{s}\left(\ldots\left(\boldsymbol{s}(\boldsymbol{s}(0))\right)\right) : \mathbb{N}}{\mathbf{P}\ n \equiv_{df} \boldsymbol{s}\left(\ldots\left(\boldsymbol{s}(\boldsymbol{s}(0))\right)\right) : \mathbb{N}}$$

Example

Let us sketch briefly as an example the play relevant for constituting a winning strategy for the thesis $3 : \mathbb{N}\ [0 : \mathbb{N}]$:

O			**P**		
				$3 : \mathbb{N}\ [0 : \mathbb{N}]$	0
1	$m := 1$			$n := 2$	2

---

[134] The rule assumes that multiplication and addition have been defined already.

[135] This kind of statement can for instance result from defending the left side of an existential such as as $(\exists\, x : \mathbb{N})\ \boldsymbol{Id}(\mathbb{N}, (\boldsymbol{s}(0), 2.x^{\mathbf{X}} + 1)$.

3	$0 : \mathbb{N}$			$3 : \mathbb{N}$	4
5	$?\ 3$	4		$s\left(s\left(s(0)\right)\right) \equiv_{df} 3 : \mathbb{N}$	12
7	$s(0) : \mathbb{N}$		3	$?\ s(0)$	6
9	$s\left(s(0)\right) : \mathbb{N}$		7	$?\, s\left(s(0)\right)$	8
11	$s\left(s\left(s(0)\right)\right) : \mathbb{N}$		9	$?\, s\left(s\left(s(0)\right)\right)$	10

P wins.

Description:

- **Moves 0-5:** establish the thesis $3 : \mathbb{N}$, the concession $0 : \mathbb{N}$, and the challenge $?\,3$ on the thesis.
- **Moves 6-7:** **P** applies the rule of synthesis to the concession $0 : \mathbb{N}$. **O** responds to it.
- **Moves 8-11**: **P** applies the rule of synthesis to moves 7 and 9. **O** responds to them.
- **Move 12**: **P** applies the nominal definition of "3" established by the Socratic rule and wins responding to the challenge on 5.

X.3.2 **Beyond Bool: Finite Sets and Large Sets of Answers.**

Let us now come back to the case of **Bool**. A natural extension of the framework is to have a larger set of answers than just the **yes-no** responses of **Bool**. The interpretation scope offered by the generalization we propose here is quite broad, as it can be interpreted as the different degrees of certainty an answer to a question can take; it can also be understood as encoding different possible answers to a question, so that **0** is the answer a, **1** is the answer b, and so on (we will discuss some examples in the next section).

Since the formation rule for a finite set $\mathbb{N}^n$ of n canonical elements (such that n stands for some natural number) has no premisses in the CTT setting, the dialogical formation rule for it amounts to the following:

Table 45: Formation rule of $\mathbb{N}^n$

	Move	**Challenge**	**Defence**
Formation of $\mathbb{N}^n$	**X** $!\ \mathbb{N}^n$	**Y** $?_F\ \mathbb{N}^n$	**X** $\mathbb{N}^n : set$

The rules of synthesis and analysis are a straightforward generalization of the set **Bool** (that is the set $\mathbb{N}^2$).

Table 46: rules for $\mathbb{N}^n$

	Move	**Challenge**	**Defence**
Synthesis for $\mathbb{N}^n$	**X** $!\ \mathbb{N}^n$	**Y** $?\ \mathbb{N}^n$	**X** $m_1 : \mathbb{N}^n$ ... **X** $m_n : \mathbb{N}^n$
Analysis for $\mathbb{N}^n$ and equalities	**X** $p : C(c)\ [c : \mathbb{N}^n]$	**Y** $? = c^{\mathbb{N}^n}$	**X** $c = m_1 : \mathbb{N}^n$ ... **X** $c = m_n : \mathbb{N}^n$

	$\mathbf{X}\, c = m_1 : \mathbb{N}^n$ … $\mathbf{X}\, c = m_n : \mathbb{N}^n$ … $\mathbf{X}\, p : C(c)[c : \mathbb{N}^n]$	$\mathbf{Y}\, c : \mathbb{N}^n$ … $\mathbf{Y}\ ?_{reason}\, C(c)$	$\mathbf{X}\, p_1 : C(m_1)$ … $\mathbf{X}\, p_n : C(m_n)$
	*Then the play continues with **O** challenging the elementary statement according to the attack prescribed by the general Socratic Rule. This procedure yields the remaining equalities.*		

The case of $\mathbb{N}^{\mathbf{0}}$ and $\mathbb{N}^{\mathbf{1}}$

$\mathbb{N}^{\mathbf{0}}$: following our main interpretation, a statement such as $\mathbf{X}!\, \mathbb{N}^0$ should be understood as stating that there is no local reason that can be adduced for the empty set. From a more dialogical point of view, we can coonsider $\mathbb{N}^0$ as the empty set of possible answers to an enquiry: the player thus states that there is no possible answer or solution to the enquiry at stake. Analoguously to the Kolmogorov interpretation of a proposition as a problem associated with all that can count as a solution to it, the natural reading in a dialogical setting would be to understand a proposition as a solution to a problem or enquiry.

Accordingly, the dialogical rule for $\mathbb{N}^0$ is the same as the one for $\bot$, that is the rule for giving up:

- The player who states $\mathbb{N}^0$ (or $p : \mathbb{N}^0$)at move n loses the current play. If it is **O** who states it, **P** can adduce the local reason **O**-*gives up*(n) in support for any statement that he has not defended before **O** stated $\mathbb{N}^0$ at move n.

$\mathbb{N}^{\mathbf{1}}$: if $\mathbb{N}^0$ is in fact the empty set $\bot$, then the unary set is $\top$, inhabited by only one local reason, namely $\boldsymbol{yesyes} : \top$:

- The player who states $\mathbb{N}^1$, can always adduce $\boldsymbol{yesyes}$ as its local reason.

X.3.3 The set Bool and some applications to mathematics and logic

Universes and codes of sets

The main motivation of introducing *universes* is to have a device for dealing with contexts in which the use of sets of sets is required, though we cannot have the set of all sets, since we cannot describe all the possible ways of constituting a set. However, since sets of sets are particularly useful in the foundations of mathematics, Martin-Löf (1984, pp. 47-49) introduces the notion of *universe of small sets*. A universe $\boldsymbol{\mathcal{U}}$ is a set of *codes* of sets: $\boldsymbol{n^n}$ is the code of the set $\mathbb{N}^{\boldsymbol{n}}$. A *small set* is a set with a code. The universe $\boldsymbol{\mathcal{U}}$ has no code in $\boldsymbol{\mathcal{U}}$ (otherwise a paradox would follow). The formation of a universe requires a decoding function $\mathcal{T}$ that yields sets from codes, i.e. the evaluation of $\mathcal{T}(\boldsymbol{n^k})$ yields the set $\mathbb{N}^{\boldsymbol{k}}$ whose code is $\boldsymbol{n^k}$. In the dialogical setting the formation rule can be formulated in the following way:

Table 47: Formation rules for $\boldsymbol{\mathcal{U}}$

	Move	**Challenge**	**Defence**
Formation of $\boldsymbol{\mathcal{U}}$	$\mathbf{X}\,!\, \boldsymbol{\mathcal{U}}$	$\mathbf{Y}\ ?_F\, \boldsymbol{\mathcal{U}}$	$\mathbf{X}\, \boldsymbol{n^0} : \boldsymbol{\mathcal{U}}$ … $\mathbf{X}\, \boldsymbol{n^n} : \boldsymbol{\mathcal{U}}$
	$\mathbf{X}\, \boldsymbol{n^k} : \boldsymbol{\mathcal{U}}$	$\mathbf{Y}\ ?\, \mathcal{T}(\boldsymbol{n^k})$	$\mathbf{X}\ \mathbb{N}^{\mathbf{k}} : set$

The notion of universe allows one to examine from another angle the difference between the canonical elements of $\mathbf{Bool}$, $\boldsymbol{yes}$ and $\boldsymbol{no}$, and the expressions $\boldsymbol{true}$ and $\boldsymbol{false}$ as applied to a proposition. As mentioned above in the case of the empty set, the dialogical setting allows reading the statement

$$\mathbf{X}\,!\;A$$

as expressing that player **X** states that there is a least one possible solution or answer to the enquiry A.

In the case of

$$\mathbf{X}\,!\;\mathbf{Bool}$$

the statement expresses that **X** is committed that at least one of the two possible answers to the enquiry associated with the set **Bool**holds. For example

$$\mathbf{X}\;\mathbf{0}:\;\mathbf{Bool}$$

which is certainly different from establishing that there is no possible answer to the enquiry

$$\mathbf{X}\,!\;\neg\mathbf{Bool}$$

Now, one consequence of this distinction is that in general we cannot demonstrate—that is, we cannot develop a winning strategy—in such a system that the elements of **Bool** are different, or not identical: $\neg \boldsymbol{Id}(\mathbf{Bool}, \boldsymbol{yes}, \boldsymbol{no})$, unless we assume that $\boldsymbol{yes}$ and $\boldsymbol{no}$ are associated to the codes of two disjoint sets, which are elements of a *universe*. In fact it was shown by Jan Smith (1988, pp. 842-843) by means of a metamathematical demonstration, that for any type A, the demonstration of an inequality of the form $\neg \boldsymbol{Id}(A, a, b)$ requires *universes* constituted by codes of sets.

Demonstration that $\boldsymbol{yes}$ and $\boldsymbol{no}$ are not identical in Bool

In order to develop a winnings strategy for $\neg \boldsymbol{Id}(\mathbf{Bool}, \boldsymbol{yes}, \boldsymbol{no})$ —i.e. $\boldsymbol{Id}(\mathbf{Bool}, \boldsymbol{no}, \boldsymbol{yes}) \supset \bot$—we follow the basic ideas of Martin-Löf's (1984, p. 51) and Nordström, Petersson & Smith's (1990, p. 86) demonstration of Peano's fourth axiom.

The main idea is introduce a predicate defined over **Bool**, more precisely the function $\boldsymbol{G}(x)$ that evaluates in the universe $\boldsymbol{\mathcal{U}}$.[136] Since it evaluates in $\boldsymbol{\mathcal{U}}$, the function yield codes, namely, if x is $\boldsymbol{no}$, then it yields $\boldsymbol{n^0}$, and it yields $\boldsymbol{n^1}$, if x is $\boldsymbol{yes}$. The codes $\boldsymbol{n^0}$ and $\boldsymbol{n^1}$ are codes for the empty set $\mathbb{N}^{\mathbf{0}}$ and the unary set $\mathbb{N}^{\mathbf{1}}$ respectively. So, $\boldsymbol{yes}$ and $\boldsymbol{no}$ are associated to two disjoint sets in $\boldsymbol{\mathcal{U}}$: thus, since the predicate $\boldsymbol{G}(x)$ applies to $\boldsymbol{yes}$, but yields the empty set when applied to $\boldsymbol{no}$, then $\boldsymbol{yes}$ and $\boldsymbol{no}$ cannot be identical. Moreover, the assumption that both of the canonical elements of **Bool** are identical would lead to conclude that the empty set is inhabited, and this proves its negation.[137]

---

[136] Since it is a predicate over **Bool**, it follows the rules for the analysis of these kind of statements (in the CTT its definition stems from the elimination rules for **Bool**).

[137] The CTT-demonstration in a nutshell is the following, with the arrow standing for functions, and the original notation of Martin-Löf where **n_0** is the notation for code and **t, f** are the canonical elements of **Bool** (see chapter II):

Define a family of sets $\boldsymbol{G} : \mathbf{Bool} \rightarrow \mathcal{U}$. $\boldsymbol{G}(x) =:_{\mathbf{df}}$ if x then **n_1**, else **n_0** $: \mathcal{U}$ $[x : \mathbf{Bool}]$.

$\boldsymbol{F} : \mathbf{Bool} \rightarrow$ *set*, by $\boldsymbol{F}(x) =:_{\mathbf{df}} \mathfrak{T}(\boldsymbol{G}(x)) :$ *set* $[x : \mathbf{Bool}]$.

$\mathbf{tt} : \mathfrak{T}(\boldsymbol{G}(\mathbf{t})$ (given $\mathbf{tt} : \mathbb{N}^1$, $\boldsymbol{G}(\mathbf{t}) =$ **n_1** $: \mathcal{U}$, and $\mathfrak{T}(\boldsymbol{G}(\mathbf{t})) = \mathbb{N}^1$: *set*, thus $\mathbf{tt} : \boldsymbol{F}(\mathbf{t})$.

In the dialogical setting we formulate a Socratic rule specific to $\boldsymbol{G}(x)$. We also provide the rule of synthesis specific to the unary set $\mathbb{N}^{\mathbf{1}}$.

Table 48: Specific Socratic rules for $\boldsymbol{G}(x)$ and synthesis of $\mathbb{N}^{\mathbf{1}}$

	Move	**Challenge**	**Defence**
Specific Socratic rule to $\boldsymbol{G}(x)$	**X** $\boldsymbol{G}(x){:}\,\boldsymbol{\mathcal{U}}(x{:}\,\mathbf{Bool})$	**Y** $\boldsymbol{no}{:}\,\mathbf{Bool}$	**X** $(\boldsymbol{G}(\boldsymbol{no}) = \boldsymbol{n}^{\mathbf{0}}{:}\,\boldsymbol{\mathcal{U}}$
		Y $\boldsymbol{yes}{:}\,\mathbf{Bool}$	**X** $(\boldsymbol{G}(\boldsymbol{yes}) = \boldsymbol{n}^{\mathbf{1}}{:}\,\boldsymbol{\mathcal{U}}$
	X $\boldsymbol{G}(\boldsymbol{no}) = \boldsymbol{n}^{\mathbf{0}}{:}\,\boldsymbol{\mathcal{U}}$	**Y** $?\ \mathcal{T}(\boldsymbol{G}(\boldsymbol{no}))$	**X** $\mathcal{T}(\boldsymbol{G}(\boldsymbol{no})) = \mathbb{N}^{\mathbf{0}}{:}\,set$
	X $\boldsymbol{G}(\boldsymbol{yes}) = \boldsymbol{n}^{\mathbf{1}}{:}\,\boldsymbol{\mathcal{U}}$	**Y** $?\ \mathcal{T}(\boldsymbol{G}(\boldsymbol{yes}))$	**X** $\mathcal{T}(\boldsymbol{G}(\boldsymbol{yes})) = \mathbb{N}^{\mathbf{1}}{:}\,set$
Synthesis of $\mathbb{N}^{\mathbf{1}}$	**X** $!\ \mathbb{N}^{\mathbf{1}}$	**Y**$?\,\mathbb{N}^{\mathbf{1}}$	**X** $\boldsymbol{yesyes}{:}\,\mathbb{N}^{\mathbf{1}}$

Note
In the case of **Y**'s chalenge of **X** $\boldsymbol{G}(x){:}\,\boldsymbol{\mathcal{U}}(x{:}\,\mathbf{Bool})$, if **Y** is **P**, then the challenge assumes that **O** already conceded $\boldsymbol{yes}, \boldsymbol{no} : \mathbf{Bool}$.

We will only display here the relevant play for the determination of the winning strategy for $\neg\boldsymbol{Id}(\mathbf{Bool}, \boldsymbol{yes}, \boldsymbol{no})$. The thesis is stated under the condition that **O** concedes the codes $\boldsymbol{n}^{\mathbf{0}}$ and $\boldsymbol{n}^{\mathbf{1}}$ are elements of $\boldsymbol{\mathcal{U}}$, the canonical answers (elements) of **Bool** and the special predicate (function) $\boldsymbol{G}(x)$ $[x : \mathbf{Bool}]$ defined by the specific Socratic rule given above.

Play 19: $\boldsymbol{Id}(\mathbf{Bool}, \boldsymbol{yes}, \boldsymbol{no}) \supset \perp \left[\boldsymbol{n}^{\mathbf{0}}, \boldsymbol{n}^{\mathbf{1}}{:}\,\boldsymbol{\mathcal{U}};\ \mathbb{N}^{\mathbf{1}};\ \boldsymbol{yes}, \boldsymbol{no} : \mathbf{Bool};\ \boldsymbol{G}(x){:}\,\boldsymbol{\mathcal{U}}\ [x : \mathbf{Bool}]\right]$

O			**P**		
0.1 0.2 0.3 0.4	$\boldsymbol{n}^{\mathbf{0}}, \boldsymbol{n}^{\mathbf{1}}{:}\,\boldsymbol{\mathcal{U}}$ $!\ \mathbb{N}^{\mathbf{1}}$ $\boldsymbol{yes}, \boldsymbol{no} : \mathbf{Bool}$ $\boldsymbol{G}(x) : \boldsymbol{\mathcal{U}}\ [x : \mathbf{Bool}]$			$!\,\boldsymbol{Id}(\mathbf{Bool}, \boldsymbol{yes}, \boldsymbol{no}) \supset \perp \left[\begin{array}{c}\boldsymbol{n}^{\mathbf{0}}, \boldsymbol{n}^{\mathbf{1}}{:}\,\boldsymbol{\mathcal{U}};\ \mathbb{N}^{\mathbf{1}};\\ \boldsymbol{yes}, \boldsymbol{no} : \mathbf{Bool};\\ \boldsymbol{G}(x){:}\,\boldsymbol{\mathcal{U}}\ [x : \mathbf{Bool}]\end{array}\right]$	0
1	$m := 1$			$n := 2$	2
3	$p_1 : \boldsymbol{Id}(\mathbf{Bool}, \boldsymbol{yes}, \boldsymbol{no})$	0		$\boldsymbol{you}_{\boldsymbol{gave\ up}}(19){:}\,\perp$	20
5	$\boldsymbol{G}(\boldsymbol{yes}) = \boldsymbol{n}^{\mathbf{1}}{:}\,\boldsymbol{\mathcal{U}}$		0.4	$\boldsymbol{yes} : \mathbf{Bool}$	4
7	$\mathcal{T}(\boldsymbol{G}(\boldsymbol{yes})) = \mathbb{N}^{\mathbf{1}}{:}\,set$		5	$?\,\mathcal{T}(\boldsymbol{G}(\boldsymbol{yes}))$	6
9	$\boldsymbol{yesyes} : \mathbb{N}^{\mathbf{1}}$	4	0.4	$?\ \mathbb{N}^{\mathbf{1}}$	8
11	$\boldsymbol{yesyes} : \mathcal{T}(\boldsymbol{G}(\boldsymbol{yes}))$		9, 7	$?\,\mathcal{T}(\boldsymbol{G}(\boldsymbol{yes}))\ /\mathbb{N}\mathbf{1}$	10
13	$\boldsymbol{G}(\boldsymbol{no}) = \boldsymbol{n}^{\mathbf{0}}{:}\,\boldsymbol{\mathcal{U}}$		0.4	$\boldsymbol{no} : \boldsymbol{Bool}$	12
15	$\mathcal{T}(\boldsymbol{G}(\boldsymbol{no})) = \mathbb{N}^{\mathbf{0}}{:}\,set$		13	$?\,\mathcal{T}(\boldsymbol{G}(\boldsymbol{no}))$	14
17	$\boldsymbol{Lbz\text{–}Id\text{–}subst}(p_1, \boldsymbol{yesyes}){:}\,\mathcal{T}(\boldsymbol{G}(\boldsymbol{no}))$		11, 3	$?_{\boldsymbol{Lbz\text{–}Id\text{–}subst}}\ \boldsymbol{no}/\boldsymbol{yes}$	16
19	$\boldsymbol{Lbz\text{–}Id\text{–}subst}(p_1, \boldsymbol{yesyes}) : \mathbb{N}^{\mathbf{0}}$		17, 15	$\mathcal{T}(\boldsymbol{G}(\boldsymbol{no}))/\ \mathbb{N}^{\mathbf{0}}$	18

P wins.

Assume *z* : **Id**(**Bool**, **t**, **f**), then *subst*(*z*, **tt**): ***F***(**f**)).
Hence, (λ.*z*)*subst*(*z*, **tt**) : ¬**Id**(**Bool**, **t**, **f**).

Description of the play

- **Moves 0-5:** After **O**'s challenge (move 3) on the thesis, **P** counter-attacks (4) the concession 0.4, following the prescription of the Socratic rules specific to $\boldsymbol{G}(x)$. **P** can carry out this challenge because of concession 0.3. In fact it is justified in the Copy-cat rule—we skip here the further challenge of **O** asking for a justification and **P**'s answer with the reflexivity $\boldsymbol{yes} = \boldsymbol{yes}: \mathbf{Bool}$.
- **Moves 6-7** follow from implementing the decoding prescription for $\boldsymbol{G}(x)$.
- **Moves 8-11:** After **O** provides the local reason $\boldsymbol{yesyes}$ for the unary set $\mathbb{N}^{\mathbf{1}}$, $\mathbf{P}$ asks $\mathbf{O}$ to substitute $\mathbb{N}^{\mathbf{1}}$ by $\mathcal{T}(\boldsymbol{G}(\boldsymbol{yes}))$, given the equality conceded move 7.
- **Moves 12-15:** **P** repeats moves 4,6, but chooses this time $\boldsymbol{no} : \mathbf{Bool}$—we also skip here the moves leading to the reflexivity $\boldsymbol{no} = \boldsymbol{no} : \mathbf{Bool}$.
- **Moves 16-20:** Move 16 is the crucial move and leads to the victory of **P**; **P** requests of $\mathbf{O}$ to replace $\boldsymbol{yes}$ with $\boldsymbol{no}$ within move 11, given the identity conceded in move 3 and given the *Leibniz-substitution rule* for $\boldsymbol{Id}$. **O**'s response (17) and her concession (15) that $\mathcal{T}(\boldsymbol{G}(\boldsymbol{no}))$ and $\mathbb{N}^{\mathbf{0}}$ are equal sets leads her to state the giving up move 19. Indeed, in move 19 **O** is forced to admit that following her own moves the empty set (of answers) is not empty. In move 20, **P** can answer **O**'s first challenge which was left unanswered by bringing forward the local reason $\boldsymbol{you}_{\boldsymbol{gave\ up}}(19)$ for defending his statement, which happens to be ⊥ (see section VII.2.1, SR7: Winning rule for plays). He thus wins the play.

After a *recapitulation* of the possible moves, **P** can bring forward $\boldsymbol{you}_{\boldsymbol{gave\ up}}(19)$ as a strategic reason for grounding his thesis and thus assert:

$$\boldsymbol{you}_{\boldsymbol{gave\ up}}(19):\ \boldsymbol{Id}(\boldsymbol{Bool}, \boldsymbol{no}, \boldsymbol{yes}) \supset \bot$$

We did not include this in the play, since we did not develop the whole strategy.

The fourth axiom of Peano's arithmetic

The demonstration of the *fourth axiom of Peano's arithmetic*, "0 is identical to no successor of a natural number":

$$(\forall x : \mathbb{N})\neg \boldsymbol{Id}(\mathbb{N}, 0, \boldsymbol{s}(x))$$

is very close to the precedent one: Peano's fourth axiom was demonstrated first by Martin-Löf (1984, p. 51) using strong elimination rules for $\boldsymbol{Id}$. Nordström, Petersson & Smith (1990, p. 86) provide a demonstration without those rules. Instead of a function defined over $\mathbf{Bool}$, what is required is a function $\boldsymbol{H}(x)$, defined over the natural numbers such that the value is the code for the unary set if the x is 0, and it is the code for the empty set if x is the successor of any natural number—thus there will be a predicate that applies to 0 but not to any other natural number, which contradicts that 0 and the successor of a natural number are identical. We leave to the diligent reader the development of both the dialogical rules for $\boldsymbol{H}(x)$ and of the relevant play for building the winning strategy—notice that $\boldsymbol{H}(x)$, will be defined following the rules of analysis for predicates defined over $\mathbb{N}$.

The conceptual background underlying these demonstrations is that in order to demonstrate that the canonical elements of $\mathbf{Bool}$ and $\mathbb{N}$ are different, we need to have a look from the outside of the respective sets and assume that there is a universe such that the Boolean $\mathbf{1}$ amounts to a code for the **truth**, the unary set; and $\mathbf{0}$ amounts to a code for the **false**, namely the empty set. This elucidates George Boole's own use of **1** and **0**, both as selective functions and as the universal domain ⊤ and the empty set ⊥.[138]

Let us now extend the set **Bool** and study some applications for truth-functional non-classical logics

Integrating many-valued logics: Operations within larger sets

Given some finite set $\mathbb{N}^n$ as defined above, we can define operations over it. For example the three-element set $\mathbb{N}^3$ can yield operations that correspond to a three-valued logic, based on the answers, $\boldsymbol{yes}$, **?**, $\boldsymbol{no}$. So $(a+^{\mathbf{3}}b)$ is equal to **?**, if one of the elements is equal to $\boldsymbol{no}$ and the other to **?**, or both a and b are equal to **?**.

More generally, for any set $\mathbb{N}^{\boldsymbol{n}}$ with elements $\mathbf{0}, \mathbf{1}, \ldots, \boldsymbol{n}$, with minimum $\mathbf{0}$ and maximum $\boldsymbol{n}$,[139] and with the help of the following definition of $x \leq y$ and its inverse $x \geq y$:

$$x \leq y = (\exists z{:}\, \mathbb{N})\boldsymbol{Id}(\mathbb{N}, x+z, y) : \boldsymbol{prop}\ [x{:}\,\mathbb{N}, y{:}\,\mathbb{N}]$$
$$x \geq y = (\exists z{:}\, \mathbb{N})\boldsymbol{Id}(\mathbb{N}, x-z, y) : \boldsymbol{prop}\ [x{:}\,\mathbb{N}, y{:}\,\mathbb{N}]$$

we obtain the following :

- $(a \times^{\boldsymbol{n}} b)$ is equal to $a = \boldsymbol{m}$ if $\boldsymbol{m} \leq \boldsymbol{m}' = b$, otherwise it is equal to $\boldsymbol{m}' = b$;
- $(a+^{\boldsymbol{n}}b)$ is equal to $a = \boldsymbol{m}$ if $\boldsymbol{m} \geq \boldsymbol{m}' = b$, otherwise it is equal to $\boldsymbol{m}' = b$;
- $\sim^{\boldsymbol{n}}a$ is equal to $\boldsymbol{n} - \boldsymbol{m}$, where $\boldsymbol{n}$ is the maximum and $\boldsymbol{m} = a$.

What is more, $(a \rightarrow^{\boldsymbol{n}} b)$ can be defined as $\sim^{n}a+^{\boldsymbol{n}}b$.

The dialogical formulation of this generalization is straightforward:

- the defender states that some operation is an element of $\mathbb{N}^{\boldsymbol{n}}$;
- the challenger requests that the defender shows the operation is equal to some element of $\mathbb{N}^{\boldsymbol{n}}$;
- once the defender chooses one of the elements, the challenger can request that he shows that this choice satisfies the $\leq$ (or $\geq$) condition defining that operation.

For the sake of simplicity we will not display this last request. The following example should be enough. Assume that the defender stated

$$\mathbf{X}\ (a \times^{\boldsymbol{n}} b) = \boldsymbol{m} = a{:}\, \mathbb{N}^{\boldsymbol{n}}$$

The challenger can then ask to check if $\boldsymbol{m}$ satifies the $\boldsymbol{m} \leq \boldsymbol{m}'$ condition required by the operator $\times^{\boldsymbol{n}}$.[140] Challenge and defence have the following form

$$\mathbf{Y}\ ?\ \boldsymbol{m} \leq \boldsymbol{m}'$$
(Does m satisfy the condition $\boldsymbol{m} \leq \boldsymbol{m}'$?)
$$\mathbf{X}\ \boldsymbol{m} \leq \boldsymbol{m}'{:}\, \mathbb{N}^{\boldsymbol{n}}$$

---

[138] For a discussion on this ambiguity see (Prior, 1949).

[139] Where $\mathbf{0}$ can be interpreted as corresponding to the lowest truth-value and $\boldsymbol{n}$ to the highest truth-value of some $\boldsymbol{n}$-valued logic.

[140] Again we assume the defintion of "≤" and "≥". The dialogical formulation of it deploys a Socratic rule specific to that relation, namely, if player **X** states $n \leq m$, given $x : \mathbb{N}, y : \mathbb{N}$, then **Y** can ask **X** to choose a $z : \mathbb{N}$ such that $n + z = m$. Similarly for "≥".

Table 49: dialogical rules for operations in many-valued logics

	Move	**Synthesis of local reasons**		**Synthesis of strategic reasons**
		Challenge	**Defence**	
Product	$\mathbf{X}\ (a \times^{\boldsymbol{n}} b): \mathbb{N}^{\boldsymbol{n}}$	$\mathbf{Y}\ ? = (a \times^{\boldsymbol{n}} b)$	$\mathbf{X}\ (a \times^{\boldsymbol{n}} b) = \boldsymbol{m} = a: \mathbb{N}^{\boldsymbol{n}}$ If $\boldsymbol{m} \leq \boldsymbol{m}'$, where $\boldsymbol{m}' = \boldsymbol{b}: \mathbb{N}^{n}$ or $\mathbf{X}\ (a \times^{\boldsymbol{n}} b) = \boldsymbol{m}' = b: \mathbb{N}^{\boldsymbol{n}}$ if $\boldsymbol{m} > \boldsymbol{m}'$	$\mathbf{P}\ (a \times^{\boldsymbol{n}} b) = \boldsymbol{m}\ [\![\langle\ \boldsymbol{m}, \boldsymbol{m}'\rangle]\!]^{\mathbf{O}} : \mathbb{N}^{\boldsymbol{n}}$ if $\boldsymbol{m}\ \leq\ \boldsymbol{m}'$ $\mathbf{P}\ (a \times^{\boldsymbol{n}} b) = \boldsymbol{m}'[\![\langle\ \boldsymbol{m}, \boldsymbol{m}'\rangle]\!]^{\mathbf{O}}: \mathbb{N}^{\boldsymbol{n}}$ if $\boldsymbol{m}\ >\ \boldsymbol{m}'$ given $\mathbf{O}\ \boldsymbol{m} = a: \mathbb{N}^{\boldsymbol{n}},\ \boldsymbol{m}' = b: \mathbb{N}^{\boldsymbol{n}}$
Addition	$\mathbf{X}\ (a +^{\boldsymbol{n}} b): \mathbb{N}^{\boldsymbol{n}}$	$\mathbf{Y}\ ? = (a +^{\boldsymbol{n}} b)$	$\mathbf{X}\ (a +^{\boldsymbol{n}} b) = \boldsymbol{m} = a: \mathbb{N}^{\boldsymbol{n}}$ If $\boldsymbol{m} \geq \boldsymbol{m}'$, where $\boldsymbol{m}' = b: \mathbb{N}^{n}$ $\mathbf{X}\ (a +^{\boldsymbol{n}} b) = \boldsymbol{m}' = a: \mathbb{N}^{\boldsymbol{n}}$ if $\boldsymbol{m} < \boldsymbol{m}'$	$\mathbf{P}\ (a +^{\boldsymbol{n}} b) = \boldsymbol{m}\ [\![\langle\ \boldsymbol{m}, \boldsymbol{m}'\rangle]\!]^{\mathbf{O}} : \mathbb{N}^{\boldsymbol{n}}$ if $\boldsymbol{m}\ \geq\ \boldsymbol{m}'$ $\mathbf{P}\ (a +^{\boldsymbol{n}} b) = \boldsymbol{m}'[\![\langle\ \boldsymbol{m}, \boldsymbol{m}'\rangle]\!]^{\mathbf{O}}: \mathbb{N}^{\boldsymbol{n}}$ if $\boldsymbol{m}\ <\ \boldsymbol{m}'$ given $\mathbf{O}\ \boldsymbol{m} = a: \mathbb{N}^{\boldsymbol{n}}, \boldsymbol{m}' = b: \mathbb{N}^{\boldsymbol{n}}$
Implication	$\mathbf{X}\ (a \rightarrow^{n} b): \mathbb{N}^{n}$	*Equivalent to the rules for addition.*		
Negation	$\mathbf{X}\ {\sim}^{\boldsymbol{n}} a: \mathbb{N}^{\boldsymbol{n}}$	$\mathbf{Y}\ ? = {\sim}^{\boldsymbol{n}} a$	$\mathbf{X}{\sim}^{\boldsymbol{n}} a =\ \boldsymbol{n} - \boldsymbol{m},$ where $\boldsymbol{m}\ =\ a: \mathbb{N}^{\boldsymbol{n}}$	$\mathbf{P}{\sim}^{\boldsymbol{n}} a\ =\ \boldsymbol{n} - \boldsymbol{m}[\![\langle\ \boldsymbol{m}, \boldsymbol{n}\rangle]\!]^{\boldsymbol{O}}: \mathbb{N}^{\boldsymbol{n}}$ given $\mathbf{O}\ \boldsymbol{m}\ =\ a\ :\ \mathbb{N}^{\boldsymbol{n}}$

Integrating many-valued logics: The Logics of Formal Inconsistency and the *White Bullet* Operator

In a recent paper, E. A. Barrio, N. Clerbout & S. Rahman (2017) developed a dialogical reconstruction of the so-called *Logics of Formal Inconsistency* (LFI)—see (Carnielli, Coniglio, & Marcos, 2007). The LFIs are logics tolerant to some amount of inconsistency but in which some versions of explosion (*ex falso*) still hold. Thus, the LFIs are a form of paraconsistent logics, that is, logics where *ex falso sequitur quodlibet* does not generally hold, and so inconsistencies are tolerated. However, the LFIs do not tolerate all forms of inconsistencies but only those considered to be relevant in a context.[141] In fact LFI constitute a whole family of logics distinguished by the kind of inconsistency they allow.

The main result of (Barrio, Clerbout, & Rahman, 2017) is to provide a formal framework applicable to situations in which inconsistent information may appear during

---

[141] Closely related are the important *adaptive logics* of (Batens, 1980) which are contextually sensitive to different inconsistent situations. They however seemed to have a more inferentialist background than the family of paraconsistent logics that came out of the work of Newton da Costa by 1970–see (d'Ottaviano & da Costa, 1970); for an overview of these logics and their origin see (Bonbenrieth Miserda, 1996); for a recent presentation of new developments see (Carnielli & Coniglio, Paraconsistent Logic: Consistency, Contradiction and Negation, 2016). (Beirlaen & Fontaine, 2016) develop a dialogical reconstruction of some adaptive logics.

certain argumentative interactions, but always within some limits, in particular always keeping some "safe" propositions for which inconsistency is not tolerated. This result has been obtained from the dialogical inferentialist point of view: Barrio, Clerbout & Rahman (2017) reconstructed in their section 5 the many-valued semantics of two of the LFIs into structural rules.

This is therefore a nice example of how one can unify a family of logics in one of the frameworks. As already suggested with the case of Boolean operators, we shall embed the truth-functional semantics of one of the logics studied, namely the *Logic of Pragmatic Truth* or *Quasi-Truth* (MPT) of (Coniglio & Silvestrini, 2014), within our general framework. However, a generalization for all of the LFIs seem to be straightforward.

The truth-functional semantics for MPT includes the operators of product, addition, and negation we described above for $\mathbb{N}^{\mathbf{3}}$ (let us here use the standard three values, $\mathbf{0}$, **½, 1**, where $\mathbf{0}$ is the minimum and $\mathbf{1}$ the maximum), and adds a different negation and a new implication, that we indicate with the superscript MPT, as well as a consistency operator. The dialogical formulation of these operators in the lines proposed for $\mathbb{N}^{\boldsymbol{n}}$ is straightforward.

- $(a \times^{\mathbf{3}} b)\colon \mathbb{N}^{\mathbf{MPT}}$;
  $(a \times^{\mathbf{3}} b)$ is equal to $\mathbf{0}$ if $a = \mathbf{0}$, otherwise it is equal to b;
- $(a +^{\mathbf{3}} b)\colon \mathbb{N}^{\mathbf{MPT}}$;
  $(a +^{\mathbf{3}} b)$ is equal to $\mathbf{1}$ if $a = \mathbf{1}$, otherwise it is equal to b;
- $\sim^{\mathbf{3}} a\colon \mathbb{N}^{\mathbf{MPT}}$;
  $\sim^{\mathbf{3}} a$ is equal to $\mathbf{1} - \boldsymbol{m}$, where (where $\boldsymbol{m} = a$).

- $\sim^{\mathbf{MPT}} a\colon \mathbb{N}^{\mathbf{MPT}}$;
  $\sim^{\mathbf{MPT}} a$ is equal to $\mathbf{1}$ if $a = \mathbf{0}$, otherwise it is equal to $\mathbf{0}$.
- $(a \rightarrow^{\mathbf{MPT}} b)\colon \mathbb{N}^{\mathbf{MPT}}$;
  $(a \rightarrow^{\mathbf{MPT}} b)$ is equal to $\mathbf{0}$ if $\begin{cases} b = \mathbf{0} \\ a = \mathbf{1} \end{cases}$ or if $\begin{cases} b = \mathbf{0} \\ a = ½ \end{cases}$ otherwise it is equal to $\mathbf{1}$.
- $a^{\circ}\colon \mathbb{N}^{\mathbf{MPT}}$
  a° is equal to $\mathbf{0}$ if $a = ½$, otherwise it is equal to $\mathbf{1}$.

The idea of the *white-bullet* operator "°", called *consistency operator* is to create a fragment in which some of the truth-functional objects behave like in classical logic, that is like in our framework for $\mathbf{Bool}$. The dialogical reconstruction of this operator deployed the operator "$\mathbf{V}^{\circ}$" which triggers the opening of a subplay in which the rules of the game are classical (Barrio, Clerbout, & Rahman, 2017).

In the present framework we will study the white-bullet operator as another truth-functional operator, that is as a non-canonical element of the three-elements set $\mathbb{N}^{\mathbf{MPT}}$, but also as a function that evaluates the elements of some fixed subset C of non-canonical elements of $\mathbb{N}^{\mathbf{MPT}}$ as the codes of the universe $\boldsymbol{\mathcal{U}}$ (described above) and those codes, whose decoding yields the empty set falsum ($\mathbb{N}^{\mathbf{0}}$, or$\bot$) and the unary set verum ($\mathbb{N}^{\mathbf{1}}$, or $\top$). This provides the insight that "°" triggers a transfer from $\mathbb{N}^{\mathbf{3}}$ to $\mathbf{Bool}$.

Thus the insight we gain here is that $\mathbf{V}^{\circ}$ should be understood as the following function, with $x\colon C^{\mathbf{MPT}}$as an abbreviation for $\{x : \mathbb{N}^{\mathbf{MPT}} \mid \mathrm{C}(x)\}$:

Table 50: dialogical rules from $\mathbb{N}^{\mathbf{3}}$ to **Bool**

	Move	Challenge	Defence
White bullet	**X** $\mathbf{V}^{\circ}(x) : \boldsymbol{\mathcal{U}}\ [x : \boldsymbol{C}^{\mathbf{MPT}}]$	**Y** $a = \mathbf{0} : \boldsymbol{C}^{\mathbf{MPT}}$	**X** $\mathbf{V}^{\circ}(a) = \boldsymbol{n}^{\mathbf{1}} : \boldsymbol{\mathcal{U}}$
		Y $a = \mathbf{1} : \boldsymbol{C}^{\mathbf{MPT}}$	**X** $\mathbf{V}^{\circ}(a) = \boldsymbol{n}^{\mathbf{1}} : \boldsymbol{\mathcal{U}}$
		Y $a = \mathbf{1/2} : \boldsymbol{C}^{\mathbf{MPT}}$	**X** $\mathbf{V}^{\circ}(a) = \boldsymbol{n}^{\mathbf{0}} : \boldsymbol{\mathcal{U}}$
	X $\mathbf{V}^{\circ}(a) : \boldsymbol{\mathcal{U}}$	**Y** $?\mathcal{T}(\mathbf{V}^{\circ}(a))$	**X** $\mathcal{T}(\mathbf{V}^{\circ}(a)) = \top : set$ if $a = \mathbf{0} : \boldsymbol{C}^{\mathbf{MPT}}$
			X $\mathcal{T}(\mathbf{V}^{\circ}(a)) = \top : set$ if $a = \mathbf{1} : \boldsymbol{C}^{\mathbf{MPT}}$
			X $\mathcal{T}(\mathbf{V}^{\circ}(a)) = \bot : set$ if $a = \mathbf{1/2} : \boldsymbol{C}^{\mathbf{MPT}}$

We can also deploy $\boldsymbol{Id}$ within **Bool** for rendering empirical propositions. Let us discuss this issue now.

X.4 Empirical Quantities and Material Dialogues

X.4.1 **Empirical Quantities as Finite Sets of Answers**

As already mentioned in the opening of the chapter, non-canonical elements of the set **Bool** can be used to study the meaning of empirical propositions, though what we need in particular is the notion of *empirical quantity*. This notion has been introduced by Martin-Löf in applying CTT to the empirical realm (Martin-Löf, 2014): whereas quantities of mathematics and logic are determined by computation, empirical quantities are determined by experiments and observation. An example of a mathematical quantity is $2 + 2$; it is determined by a computation yielding the number 4. An example of an empirical quantity is the colour of some object. This is not determined by computation; rather, one must look at the object under normal conditions.

In the dialogical framework, we can consider empirical quantities as answers to a question. For example, give the question

Are Cheryl's eyes blue?

The yes or no answer, obtained through some kind of empirical procedure received in a given context, can be defined over the set **Bool**, namely as being equal to $\boldsymbol{yes}$ or $\boldsymbol{no}$. The following question however might involve many different answers:

What is the colour of Cheryl's eyes?

If $\boldsymbol{X}$ stands for the empirical quantity *Colour of Cheryl's eyes*, we might define the possible answers over some finite set $\mathbb{N}^{\boldsymbol{n}}$ of natural numbers:

$\boldsymbol{X} = 1 : \mathbb{N}^{\boldsymbol{n}}$ if Cheryl's eyes are brown
$\boldsymbol{X} = 2 : \mathbb{N}^{\boldsymbol{n}}$ if Cheryl's eyes are green
$\boldsymbol{X} = 3 : \mathbb{N}^{\boldsymbol{n}}$if Cheryl's eyes are blue

...
$\boldsymbol{X} = n{:}\,\mathbb{N}^{\boldsymbol{n}}$ if Cheryl’s eyes are…

Certainly the question ***Are Cheryl's eyes blue?*** can also be defined over a larger set, if several degrees of colour are to be included as an answer, or alternatively degrees of certainty (definitely blue, quite blue, slightly blue…). Let us assume then another set $\mathbb{N}^{\boldsymbol{j}}$ for the degree of colour:

$\boldsymbol{Y} = 0\_1{:}\,\mathbb{N}^j$ if Cheryl's eyes are dark blue
$\boldsymbol{Y} = 0\_2{:}\,\mathbb{N}^j$ if Cheryl's eyes are light blue.
$\boldsymbol{Y} = 0\_3{:}\,\mathbb{N}^j$ if Cheryl's eyes are green-blue.
...
$\boldsymbol{Y} = 0\_j{:}\,\mathbb{N}^j$ if Cheryl’s eyes are…

The general dialogical rule for an empirical quantity can thus be rendered:

Table 51: general dialogical rule for an empirical quantity

	Move	Challenge	Defence
Empirical quantity	$\mathbf{X}\ \boldsymbol{X} : \mathbb{N}^n$	$\mathbf{Y}\ ?= \boldsymbol{X}$	$\mathbf{X}\, m_1 = \boldsymbol{X} : \mathbb{N}^n$ ... $\mathbf{X}\, m_n = \boldsymbol{X} : \mathbb{N}^n$ (the defender chooses)

Notice that determining the value of the empirical quantity is an empirical procedure, specific to that quantity; the result of carrying out such a procedure is determined by the rules for that quantity. Moreover, the value of two different empirical quantities might be the same: the quantities only indicate that *the way of determining* the answer to the question might be the same. Take for example these two enquiries

(1) *Did Jorge Luis Borges compose the poem “Ajedrez”?*
(2) *Is Ibn al-Haytham the author of* Al-Shukūk ʿalā Batlamyūs *(Doubts Concerning Ptolemy)?*

These two enquiries involve determining the value of the empirical quantity $\boldsymbol{X}$ for (1) and $\boldsymbol{Y}$ for (2), which can be the same: they can both be $\boldsymbol{yes}$ for instance if the underlying set is **Bool.**

This leads to a Socratic rule specific to statements of the form $\boldsymbol{X}, \boldsymbol{Y}, \mathbf{Z} : \mathbb{N}^n$**.** For example, given the set $\mathbb{N}^n$, **P** can defend the challenges

$\mathbf{O}\ ?= \boldsymbol{X}$ with the statement $\mathbf{P}\ m_1 = \boldsymbol{X}{:}\,\mathbb{N}^n$
$\mathbf{O}\ ?= \boldsymbol{Y}$ with the statement $\mathbf{P}\ m_2 = \boldsymbol{Y}{:}\,\mathbb{N}^n$
$\mathbf{O}\ ?= \mathbf{Z}$ with the statement $\mathbf{P}\ m_3 = \mathbf{Z}{:}\,\mathbb{N}^n$

Incompatibility

A system of rules that targets the development of a more complex meaning network might include incompatibility rules formulated as challenges. Thus, instead of establishing the simple use of Copy-cat, the game might include more sophisticated rules specific to a particular empirical quantity. For example, if a player responded $\boldsymbol{yes}$ to the enquiry associated with $\boldsymbol{X}$

(3) *Did the Greek won in 480 BC the sea-battle take of Salamis*?

that is, if he stated $\boldsymbol{yes} = \boldsymbol{X} : \textbf{Bool}$; this player might not be allowed to respond $\boldsymbol{yes}$ to the enquiry associated with $\mathbf{Z}$

(4) *Did Xerxes won in 480 BC the sea-battle of Salamis?*

that is, he might not be entitled to further state $\boldsymbol{yes} = \mathbf{Z} : \textbf{Bool}$. That is, the other player may challenge the right to answer both (3) and (4) with $\boldsymbol{yes}$:

(5) Both answers cannot be $\boldsymbol{yes}$.

that is, she can challenge his two statements by stating that $\neg\big(\boldsymbol{Id}(\textbf{Bool}, \boldsymbol{yes}, \boldsymbol{X}) \wedge \boldsymbol{Id}(\textbf{Bool}, \boldsymbol{yes}, \mathbf{Z})\big)$. The first player would then have to give up. This challenge would be calling upon some *formal incompatibility* between two statements.

Table 52: Formal incompatibility

	Moves	Challenge	Defence
Formal incompatibility	**P** $\boldsymbol{yes} = \boldsymbol{X} : \textbf{Bool}$ *and* **P** $\boldsymbol{yes} = \mathbf{Z} : \textbf{Bool}$	**O** $\neg\big(\boldsymbol{Id}(\textbf{Bool}, \boldsymbol{yes}, \boldsymbol{X}) \wedge \boldsymbol{Id}(\textbf{Bool}, \boldsymbol{yes}, \mathbf{Z})\big)$	**P** $gives\ up$

But there is another kind of incompatibility challenge, calling upon *contentual incompatibility*. Consider for instance (4): if a player answers $\boldsymbol{yes}$, *Xerxes won in 480 BC the sea-battle of Salamis*, then the other player can challenge this through contentual incompatibility: the challenger simply states the formally incompatible answer to the challenged statement: $\boldsymbol{Id}(\textbf{Bool}, \boldsymbol{yes}, \boldsymbol{X})$, *The Greek won in 480 BC the sea-battle of Salamis.* The challenged player must then give up.

Table 53: Contentual incompatibility

	Moves	Challenge	Defence
Contentual incompatibility	**P** $\boldsymbol{yes} = \mathbf{Z} : \textbf{Bool}$	**O** $\boldsymbol{Id}(\textbf{Bool}, \boldsymbol{yes}, \boldsymbol{X})$	**P** $gives\ up$

X.4.2 Dependent Empirical Quantities

Another more sophisticated form of dealing with empirical quantities is to implement a structure where one empirical quantity might depend on another one. For example let us define the empirical quantity $\boldsymbol{Y}$ as the function $b(\boldsymbol{X}) : \mathbb{N}_j^{\boldsymbol{n}}\ [\boldsymbol{X} : \mathbb{N}^{\boldsymbol{n}}]$ such that

$$\boldsymbol{Y} :=_{df} b(\boldsymbol{X}) : \mathbb{N}_j^{\boldsymbol{n}}\ [\boldsymbol{X} : \mathbb{N}^{\boldsymbol{n}}]$$

$$b(\boldsymbol{X}) = j_i : \mathbb{N}^{\boldsymbol{j}}\ \text{, given } \boldsymbol{X} = n_m : \mathbb{N}^{\boldsymbol{n}}$$

$$\ldots$$

$$b(\boldsymbol{X}) = j_k : \mathbb{N}^{\boldsymbol{j}}\ \text{, given } \boldsymbol{X} = n_n : \mathbb{N}^{\boldsymbol{n}}, \text{ if} \ldots$$

Suppose we are interested in determining the meaning of some empirical propositions; this can involve for instance establishing that stating that something has a determinate colour (say, *red*) would presuppose that the player already answered the question whether the object at stake *is coloured or not*.

In this case also the rules of the game might include rules for challenging empirical quantities on the basis of a certain evaluation of another empirical quantity on which the first is dependent; this would be like challenging that something is red by denying that the empirical quantity that yields the evaluation $\boldsymbol{X}$ has a positive response to the question if the object at stake has a colour.

Dependent Empirical Quantities and Futures Contingents

Among empirical quantities are the quantities of future events, the indicator then being whether or not the event occurs, following an analogous practice in mathematics, defining such a quantity $\boldsymbol{X}$ by making it equal to $\boldsymbol{yes}$ if the event occurs and to $\boldsymbol{no}$ if the event does not occur. We are dealing here with empirical quantities because an empirical method is required for determining their value.

Martin-Löf used such empirical quantities of future events for dealing with Aristotle's sea-battle puzzle: the question

Will a sea-battle take place tomorrow?

can have two answers, $\boldsymbol{X} = \boldsymbol{yes}$ tomorrow a sea-battle will take place (the event will occur), or $\boldsymbol{X} = \boldsymbol{no}$ tomorrow a sea-battle will not take place (the event will not occur). In a dialogical setting, if we replace the empirical quantity $\boldsymbol{X}$ by a variable $\boldsymbol{x}$, we would obtain the following thesis (in the form of a hypothetical):

$$\mathbf{P}\ !\ \big(\boldsymbol{Id}(\mathbf{Bool}, \boldsymbol{x}, \boldsymbol{yes}) \vee \boldsymbol{Id}(\mathbf{Bool}, \boldsymbol{x}, \boldsymbol{no})\big)\ [\boldsymbol{x} : \mathbf{Bool}]$$

Since we do have a winning strategy for $\big(\forall x : \mathbf{Bool}\big)\big(\boldsymbol{Id}(\mathbf{Bool}, \boldsymbol{x}, \boldsymbol{yes}) \vee \boldsymbol{Id}(\mathbf{Bool}, \boldsymbol{x}, \boldsymbol{no})\big)$ (see section X.2.3), it is possible to assert this thesis even though for practical reasons we cannot yet determine the value of $\boldsymbol{x}$.

Application in law

This piece of logical analysis finds a nice application[142] in what Leibniz has called *suspensive conditions* or also *moral conditions*,[143] determining some *conditional right* such as:

Primus must pay 100 dinar to Secundus if a ship arrives from Asia
(within a set time frame)

According to Leibniz, this problem should be coupled with both a logical and an epistemic analysis: the contracting parties must not have any information yet whether the antecedent (that a ship arrives from Asia) is true or false (if they know, then the contract is not conditional). The right established by the contract should be considered to be legally binding, despite the fact that the condition has not yet been satisfied.

Klev (2014b) uses in this context Martin-Löf's notion of empirical quantity: let $\boldsymbol{X}$ be an empirical quantity equal to $\boldsymbol{yes}$ if a ship arrives and equal to $\boldsymbol{no}$ if no ship has arrived within a certain time span. The empirical method here that can be used to determine whether $\boldsymbol{X} = \boldsymbol{yes}$ or $\boldsymbol{X} = \boldsymbol{no}$ can for instance consist in standing on the dock for the specified time span and recording the incoming ships. We can now define a function b on the set $\mathbf{Bool} \coloneqq \{\boldsymbol{yes}, \boldsymbol{no}\}$ by setting

---

[142] Traditional legal approaches to conditional right considered suspensive conditions through the notions of existence or legal fiction (Magnier, 2015, p. 72). New recent logical reconstructions of conditional right have been triggered by the work of (Armgardt, 2001; 2008; 2010), such as the studies of (Thiercelin, 2009; 2010), (Magnier, 2013; 2015), (Rahman, 2015).

[143] *Doctrina conditionum* in Leibniz (1964), see also (Armgardt, 2001).

$b(\boldsymbol{no}) = 0$ and
$b(\boldsymbol{yes}) = 100$
where 0 and 100 are to be understood as the amount of money to be paid.

Since $\boldsymbol{X}$, being a (non-canonical) element of $\mathbf{Bool}$, is equal to either $\boldsymbol{yes}$ or $\boldsymbol{no}$, $b(\boldsymbol{X})$ is well defined, it's evaluation being either 0 or 100. So $b(\boldsymbol{X})$ is understood as the amount to be payed by Primus to Secundus $[\boldsymbol{X} : \mathbf{Bool}]$.

Thus, in the dialogical framework using empirical quantities, the thesis stated by **P** would be the following:

P ! *Primus must pay* $b(\boldsymbol{X})$ *dinar to Secundus.*

Notice that this thesis is not hypothetical but has a categorical form: the condition *If a ship arrives* is not given in a hypothesis, it is built straight into the empirical quantity $\boldsymbol{X}$; one should therefore probably speak of *dependent obligation* rather than *conditional right*. The ruling dependents on the value of $\boldsymbol{X}$, though leaving the possibility open of not yet being in a position to determine $\boldsymbol{X}$, but as soon as it is determined, so is $b(\boldsymbol{X})$, and thereby Primus's debt to Secundus:

- if we can determine that $\boldsymbol{X}$ is $\boldsymbol{no}$, then we can assert the debt to be $b(\boldsymbol{no}) = 0$;
- if we can determine that $\boldsymbol{X}$ is $\boldsymbol{yes}$, then we can assert the debt to be $b(\boldsymbol{yes}) = 100$.

What one is obliged to do depends on the value of such a determined empirical quantity.

What is more, Islamic jurists also have intensive discussions on the issue, and have been precursors of Leibniz's rejection of the roman notion of *retroactivity*. As pointed out by Yvon Linant de Bellefonds (1965, pp. 425-430), the Islamic jurists considered that only a restricted set of suspensive (muʿallaq) conditions (taʿliq) yield legally binding contracts. It might be argued that, from the logical point of view, their rejection was based on a hypothetical analysis of conditional right. An indication of this is that transfers of goods are excluded from contracts with suspensive conditions. A suspensive condition—unless a clear time frame was defined—might introduce a too hazardous parameter for the establishment of the juridical act. In fact, if the time frame is clearly defined and the condition not absolutely contingent, then it was considered not falling under suspensive conditions. Thus, contracts stipulating *too vague* conditions such as *If next year I will have a profitable harvest, then B*, where not considered to be legally binding. However, if the condition is set in a clear time frame, then it is not considered as falling under what they understood to be suspensive. In fact, only a reduced set of cases were allowed, including those juridic acts that can in principle be revoked, such as a will: since it can be revoked, the fulfilment of the will might be formulated as including an *explicit* suspensive condition—*tacit* conditions have another structure, see (Linant de Bellefonds, 1965, pp. 429-430).

This might perhaps lead to distinguish between *dependent obligations* (rather than susupensive conditions) and *conditional right* (dependent upon suspensive conditions). In relation to the latter, a possible reconstruction that stresses the hypothetical character of the conditional right and deploys empirical quantities is the following:

$$\mathbf{P}!\,\big(\boldsymbol{Id}(\mathbf{Bool},\boldsymbol{x},\boldsymbol{yes}) \supset \boldsymbol{Id}(\mathbb{N},\boldsymbol{y},\boldsymbol{yes})\big) \\ \wedge \big(\boldsymbol{Id}(\boldsymbol{Bool},\boldsymbol{x},\boldsymbol{no}) \supset \boldsymbol{Id}(\mathbb{N},\boldsymbol{y},\boldsymbol{no})\big)\ [\boldsymbol{x},\boldsymbol{y}\text{:}\,\mathbf{Bool}]$$

Where $\boldsymbol{x}$ is stands for a variable for the empirical quantity $\boldsymbol{X}$, *Ashraf fulfils condition C* [explicitly established as a condition in Zayd's will], and where $\boldsymbol{y}$ stands for a variable for the empirical quantity $\boldsymbol{Y}$, *Ashraf receives 100 dinar after Zayd's death* (according to Zayd's will).

The procedure determining the value of $\boldsymbol{y}$ is eminently empirical: it amounts to decide if the contract is or not legally binding (this amounts to verifying if it the condition meets the requirements settled for *muʿallaq taʿliq* (suspensive condition)). Similar applies to the determination of $\boldsymbol{x}$**.**[144]

X.4.3 **Conclusion on Empirical Quantities**

Local reasons in general, and empirical quantities in particular, take care of some old and new criticisms raised against the standard dialogical approach to meaning formulated by (Lorenzen & Lorenz, 1978)—see sections I.1, p. 14 and XI.4 for further details on these criticisms and our take on the issue.

It is fair to say that the notion of *material dialogues* seem to be underdeveloped in relation to the formal dialogues that gathered much more attention. However, let us stress that the fathers of dialogical logic where aware of the need of a contentual (*material* was the chose term) basis right from the beginning; they tackled the issue with different devices. Lorenz (1970) in particular dedicated to this issue very thorough and deep studies, most of them collected in (Lorenz, 2010a; 2010b). Moreover, the rules for integrating empirical quantities within the dialogical framework described above are directly inspired by the *predicator rules* already discussed in (Lorenz & Mittelstrass, 1967). Predicator rules, the dialogical counterparts of semantic definitions, are part of the play level; the formalistic interpretation of the dialogical framework pointed out in the opening of this chapter comes for a large part from neglecting this level of meaning. Chapter XI, our concluding chapter, will deal more at large on this neglect of the play level.

Predicator rules are part of the *Orthosprache* project of the Erlangen Constructivism proposed by 1970,[145] which challenged at that time also the approach of mainstream analytic theory of meaning. The underlying idea is the explicit and constructive development, *by example* (*exemplarisch*), of a contentual language in order to build a specific scientific terminology (Kamlah & Lorenzen, 1972, pp. 70-111).

The qualification "by example" refers to one of the major tenets of the overall philosophy of language of the Erlangen School, namely the idea that we grasp an individual as exemplifying something—type theoreticians will say, as exemplifying a type (see below):

> *Yet even science cannot avoid the fact that things do not proffer themselves everywhere as different of their own accord, more often in important areas (e.g. in the social or historical sciences) science must decide for itself what it wants to regard* as of the same kind *and what is* of different kind, *and address them accordingly.* […]
>
> *As we have said already, the world does not "consist of objects" (of "things in themselves") which are subsequently named by men* […].
>
> *In the world being disclosed to us all along through language we tend to grasp the*

---

[144] See (Rahman & Iqbal, 2018) for a general dialogical approach to legal reasoning in the context of Islamic jurisprudence.

[145] The term "*Orthosprache*" was introduced by Lorenzen in 1972, quoted in a footnote in the second edition of the *Logische Propädeutik* (Kamlah & Lorenzen, 1972, p. 73 footnote 1) and discussed in the bible of the Erlangen School: *Konstruktive Logik, Ethik und Wissenschaftstheorie* (Lorenzen & Schwemmer, 1975).

> *individual object as individual at the same time that we grasp it* as specimen of… *Further, when we say "This is a bassoon" we mean thereby "This instrument is a bassoon"* […] *or when we say "This is a blackbird", we presuppose that our discussion partner already knows "what kind of an object is meant", that we are talking about birds. (Kamlah & Lorenzen, 1984, p. 37)*

Accordingly, the predicators of the *Orthosprache* are introduced via the study of exemplification instances. Now, a scientific terminology does not only consist in a set of predicators or even of sentences expressing propositions: an adequate scientific language constitutes a system of conceptual interrelations. The main logical device of the *Orthosprache* project is to establish the corresponding transitions by *predicator rules* that normalize the passage from one predicator to the other. Moreover, these transition rules are formulated within a dialogical frame, so that given the predicator rule

$$x \, \varepsilon \, A \quad \Rightarrow \quad x \, \varepsilon \, B$$

(where x is a free variable and "A" and "B" are predicators)

we have: if a player brings forward an object of which predicator A is said to apply, then he is also committed to ascribe the predicator B to the same object. The idea is that if, for example, someone claims *k is a bassoon*, then he is committed to the further claim *k is a musical instrument* (where k is an individual constant: in the *Logische Propädeutik* the application of these norms proceeds by substituting individual constants for free variables). The Constructivists of Erlangen called *material-analytic norms* such transition rules which structure a (fully interpreted) scientific language by setting the *boundaries of a predicator*. *Material-analytical propositions* (or, more literally, *material-analytical truths*) are defined as the universally quantified propositions based on such material-analytic norms (Lorenzen & Schwemmer, 1975, p. 215).

X.5 Some General Epistemological Consequences

X.5.1 **Play Level and Material Dialogues**

The philosophical background to our dialogical approach of Martin-Löf's notion of empirical quantity is set in describing how to *internalize* empirical data into the rules of the play (Peregrin, 2014, pp. 34-36, 100-104); or, to put it in Sellars' words, describing how to place empirical data *within the space of reasons*. As it is well known, Sellars introduces the notion of space of reasons in the context of observational reports such as "This is green". According to Sellars (1991, pp. 129-194), such a report expresses a state of knowledge, if the one who brings forward the reports is able to justify his assertion by calling on some further, more general, knowledge about the reliability of such reports:

> *The essential point is that in characterizing an episode or a state as that of knowing, we are not giving an empirical description of that episode or state; we are placing it in the logical space of reasons, of justifying and being able to justify what one says. (Sellars, 1991, p. 169)*

Brandom's (1994; 2000; 2008) interpretation of the *space of reasons* intends to provide an inferentialist reading to both the internalization of empirical data, and the broader knowledge that is required for the reliability of such reports: the relations in the space of reasons are constituted by possibilities of reaching positions of entitlement or commitment *by inference* from prior positions of entitlement or commitment; thus what language needs to become the games of giving and asking for reasons' vehicle are inferential rules. To be able to *give* reasons we must be able to make claims that can serve as reasons for other claims; hence our language must provide for sentences that *entail* other sentences. To be able to *ask for* reasons, we must be able to make claims that count as a *challenge* to other claims; hence our language must provide for sentences that are *incompatible* with other sentences. Our language must therefore be structured by these entailment and incompatibility relations.

One must also add on the one hand the relation from which commitments and entitlements are inherited: by committing myself to *This is a dog* I also and thereby commit myself to *This is an animal*, and being entitled to *It is raining* I am also and thereby entitled also to *The streets are wet*; and on the other hand the relation of co-inheritance of incompatibilities (*A* is in this relation to *B* iff whatever is incompatible with *B* is incompatible with *A*).

The relation between dialogical logic and the games of asking and giving reasons has already been pointed out by (Keiff, 2007) and (Marion, 2006; 2009; 2010):

> *My suggestion is simply that dialogical logic is perfectly suited for a precisification of these 'assertion games'. This opens the way to a 'game-semantical' treatment of the 'game of giving and asking for reasons': 'asking for reasons' corresponds to 'attacks' in dialogical logic, while 'giving reasons' corresponds to 'defences'. In the Erlangen School, attacks were indeed described as 'rights' and defences as 'duties', so we have the following equivalences:*
>
> *Right to attack ↔ asking for reasons*
>
> *Duty to defend ↔ giving reasons*
>
> *The point of winning 'assertion games', i.e., successfully defending one's assertion against an opponent, is that one has thus provided a justification or reason for one's assertion. Referring to the title of the book* [Making it Explicit], *one could say that playing games of 'giving and asking for reasons' implicitly presupposes abilities that are made explicit through the introduction of logical vocabulary. (Marion, 2010, p. 490)*

An important component for linking Brandom's intepretation of Sellar's space of reasons with the dialogical framework, namely the strategic level, is stressed in section 1.2 of (Keiff, 2007):

> *Traditionnellement, la logique est présentée comme la science des arguments (ou du raisonnement) préservant la vérité, et les objets de cette théorie sont déterminés par rapport à cette propriété : les constantes logiques sont les unités syntaxiques dans les énoncés qui constituent un argument que l'on ne peut altérer tout en garantissant la préservation de la vérité. Ce que l'on peut reformuler en termes brandomiens : les constantes logiques sont définies comme les unités syntaxiques qu'on ne peut altérer tout en préservant l'identité des conditions d'assertabilité. Mais l'approche dialogique détermine son objet de façon plus précise : elle définit les conditions d'assertabilité en termes de stratégies de justification.*[146]

Yet, despite the dialogical framework's close links with Brandom's inferentialism, there is also—as argued by (Clerbout & Rahman, 2015, pp. ix-xi)—an important difference: the play-level. Indeed from dialogical point of view, strategies are constituted by plays: if we are prepared to determine meaning from the point of view of dialogical games, the constitution of the strategy is a process that cannot be left by side. To put it other words, *not every sequence of moves in games of asking for reasons and providing them is necessarily inferential*, only those plays leading to winning strategies are.

---

[146] Logic is traditionnally presented as the science of arguments (or of reasoning) preserving truth, and the objects of this theory are determined according to this property: the logical constants are syntactical units in the expressions constituting an argument that cannot be changed while still warranting the truth preservation. To put it in Brandomian terms: logical constants are defined as syntactical units that cannot be changed while preserving the identity of the assertability conditions. But the dialogical approach determines its object in a more precise fashion: this approach defines the assertability conditions in terms of strategies of justification.

To put in the nice of words of Peregrin (2014, pp. 228-229), the prescription for the interaction of questions and answers at the play level provides *the material by the means the which we reason*, not the material that prescribes *how to reason*:[147]

> *This is a crucial point, because it is often taken for granted that the rules of logic tell us how to reason precisely in the tactical sense of the word. But what I maintain is that this is wrong, the rules do not tell us how to reason, they provide us with things with which, or in terms of which, to reason. (Peregrin, 2014, pp. 228-229)*

This idea that *not every move in the space of reasons is inferential* might be related to McDowell's (2004; 2009) worry concerning Brandom's interpretation of Sellars:

> *Someone can know what colour something is by looking at it only if she knows enough about the effects of different sorts of illumination on colour appearances. The essential thing for our purposes is that the relation of this presupposed knowledge to the knowledge that presupposes it—support in Sellars's second dimension—is not that the presupposing knowledge is inferentially grounded on the presupposed knowledge. (McDowell, 2004)*

> *We only need to register that it is experience that yields the knowledge expressed in observation reports. Recognizing the second dimension puts us in a position to understand observation reports properly. The knowledge they express is not inferentially grounded on other knowledge of matters of fact, but—in the crucial departure from traditional empiricism—it presupposes other knowledge of fact. (McDowell, 2009, p. 223)*

Our reconstruction of the controversy between Brandom and McDowell is based on a double articulation:

- the difference between the play and the strategy level, and
- the difference between dependences upon empirical quantities and dependences as structured by premises-conclusion.

For short, while the dialogical framework allows for interaction through questions and answers that cannot be reduced to the strategy level—though it may well have the general intent of constituting them, see (Keiff, 2004, pp. 41-42)—, the richer language of immanent reasoning can analyze empirical reports as constituted by empirical quantities and the propositions that bear them, that is, as statements involving local reasons adduced in favour of certain propositions. In this regard, we can analyze this kind of report

(1) This apple looks green to me

as the play level statement of some concrete player, say, *Eloise*:

$$Eloise\ \boldsymbol{X} = 3 : \mathbb{N}^{\mathbf{5}}$$

---

[147] In fact Peregrin (2014) uses the dialogical framework to develop a new approach of the issue on the normativity of logic: he understands the normativity of logic not in the sense of prescriptions on how to reason, but rather as providing the material by the means of which we reason. If we link this proposal with the distinction between the play level and strategy level, we can distinguish prescriptions that aim the development of a play and provide the material for reasoning, from those proper to the tactics, considering the optimal means on how to win. These last prescriptions dictate the design of feasible strategies; Peregrin's suggestion leads to dividing the strategy level with tactics singling out the subset of feasible strategies from the whole set of strategies. While tactical considerations try to find the optimal way to achieve victory, normativity in a more general and fundamental level involves the play level, that is, the level where instruments of reasoning and meaning are forged. Moreover, Peregrin links the normativity of logic with another main conceptual tenet of the dialogical framework, namely, the public feature of the speech-acts underlying an argumentative approach to reasoning. See in particular (Peregrin, 2014, pp. 228-229).

(where $\boldsymbol{X}$ is the empirical quantity that encodes the response to the enquiry on *the apple being green*).

Moreover, determining the response to such an empirical quantity $\boldsymbol{X}$ may well be dependent upon another empirical quantity $\boldsymbol{Y}$:

$$Eloise\ \boldsymbol{X}\ :=_{df}\ b(\boldsymbol{Y}) = 3 : \mathbb{N}^{\mathbf{5}}\ [\boldsymbol{Y} : \mathbf{Bool}]$$

(where $\boldsymbol{Y}$ is the empirical quantity that encodes the response to the enquiry on *the apple being coloured*).

Notice that we are here like McDowell making an empirical quantity dependent upon another one by means of a function between those quantities rather that expressing the dependence by means of inferences.[148]

The rules of the play level *internalize* the empirical features by prescribing the rules specific to the empirical quantity at stake. However this does not mean that we cannot move from the statement *it looks green to me* to the assertion *it is green*: a winning strategy is required for this, strategy that can be totally rendered by inferential moves. In a winning strategy, it is sufficient for Eloise to show that she can defend her statement, given the material rules set by the game, against any challenge of her antagonist Abelard playing according to these rules.

X.5.2 The Dialogical Internalization and the Myth of the Given

Let us stress the point that, if our reconstruction of Sellars' observational reports by means of empirical quantities is correct, acknowledging the legitimacy of such reports *does not* fall into the so-called *Myth of the Given* (Sellars, 1956): in our approach, empirical quantities are non-canonical elements of some set in the context of CTT; in such a context there is no way to approach some object without apprehending it as determining what it is. Indeed two main tenets of CTT are

1. No entity without type
2. No type without semantical equality

If we recall the Curry–Howard isomorphism between types and propositions we have

Every entity is bearer of a proposition.

This is what the *internalization* of empirical content within a dialogical stance is about: bringing forward local reasons for a proposition.[149] Moreover, the dialogical approach understands “no type without semantical equality” as making semantical equality result from the interaction of giving and asking for reasons, which would take care of Brandom's (1994; 1997; 2000; The Centrality of Sellars's Two-Ply Account of Observation to the Arguments of 'Empiricism and the Philosophy of Mind', 2002) worries in interpreting observational reports the way McDowell suggested.

The discussion between McDowell and Brandom has interesting parallels with the opposition between Hintikka's (1973) notion of *outdoor-games* and Lorenzen & Lorenz’s

---

[148] In the early stages of the development of the dialogical framework, meaning dependences where normed by means of *transition rules* between *predicators,* at the play level. See section X.4.3.

[149] This is the sense of *internalizatio*n discussed by (Peregrin, 2014, pp. 34-36, 100-104). However, since he does not use the CTT language, he does not have the means for distinguishing the empirical quantity from the set (proposition) it instantiates.

(1978) *indoor-games*. Indeed Hintikka (1973, pp. 77-82), who acknowledges the close links between dialogical logic and game-theoretical semantics, launched an attack against the philosophical foundations of dialogical logic because of its *indoor*—or purely formal— approach to meaning as use. He argues that *formal proof* games are not much help in accomplishing the task of linking the linguistic rules of meaning with the real world:

> *In contrast to our games of seeking and finding, the games of Lorenzen and Stegmüller are 'dialogical games' which are played 'indoors' by means of verbal 'challenges' and 'responses'.* […]
>
> *If one is merely interested in suitable technical problems in logic, there may not be much to choose between the two types of games. However, from a philosophical point of view, the difference seems to be absolutely crucial. Only considerations which pertain to 'games of exploring the world' can be hoped to throw any light on the role of our logical concepts in the meaningful use of language. (Hintikka, 1973, p. 81)*

This kind of worries have been dealt away by integrating Socratic rules specific to a given predicate and by incorporating empirical quantities.

XI. CONCLUDING REMARKS: A PLAIDOYER FOR THE PLAY LEVEL

To some extent, the criticisms the dialogical approach to logic has been subject to have provided an opportunity for clarifying its basic tenets. Moreover, our responses to the objections have highlighted crucial distinctions constituting the originality and flexibility of this logical framework. We will therefore in this concluding chapter consider some recent objections raised against the dialogical framework in order to pinpoint some of its fundamental features, whose importance may not have appeared clearly enough through the main body of the book; namely, dialogue-definiteness, player-independence, and the dialogical conception of propositions. Showing how and why these features have been developped, and specifying their point and the level they operate on, will enable us to vindicate the play level and thus disarm the objections that have been raised against the dialogical framework for having neglected this crucial level.

We shall first come back on the central notion of dialogue-definiteness and on the dialogical conception of propositions, which are essential for properly understanding the specific role and importance of the play level. We shall then be able to address three objections to the dialogical framework, due to a misunderstanding of the notion of *Built-in Opponent*, of the principles of dialogue-definiteness and of player-independence, and of the reflexion on normativity that constitutes the philosophical foundation of the framework; all of these misunderstandings can be reduced to a misappraisal of the play level. We shall then go somewhat deeper in the normative aspects of the dialogical framework, according to the principle that logic has its roots in ethics. A last section before our final words will come back to the origins of immanent reasoning, sketching some important aspects of Dialogical Constructivism which rest on the learning process constituting the core of intersubjectivity.

XI.1 Dialogue-Definiteness and Propositions

The dialogical theory of meaning is structured in three levels, that of the local meaning (determined by the particle rules for the logical constants), of the global meaning (determined by the structural rules), and the strategic level of meaning (determined by what is required for having a winning strategy). The material level of consideration is part of the global meaning, but with particular rules so precise that they determine only one specific expression (through a modified Socratic rule). A characteristic of the local meaning is that the rules are player independent: the meaning is thus defined in the same fashion for each player; they are bound by the same sets of duties and rights when they start a dialogue. This normative aspect is thus constitutive of the play level (which encompasses both the local meaning and the global meaning): it is even what allows one to judge that a dialogue is taking place. In this regard, meaning is immanent to the dialogue: what constitutes the meaning of the statements in a particular dialogue solely rests on rules determining interaction (the local and the global levels of meaning). The strategy level on the other hand is built on the play level, and the notion of demonstration operates on the strategy level (it amounts to having a winning strategy).

Two main tenets of the dialogical theory of meaning can be traced back to Wittgenstein, and ground in particular the pivotal notion of dialogue-definiteness:

1. the internal feature of meaning (the *Unhintergehbarkeit der Sprache*[150]), and
2. the meaning as mediated by language-games.

As for the first Wittgensteinian tenet, the internal feature of meaning, we already mentioned in the introduction that if we relate the notion of internalization of meaning with both language-games and fully-interpreted languages of CTT, then a salient feature of the dialogical approach to meaning can come to fore: the expressive power of CTT allows all these actions involved in the dialogical constitution of meaning to be incorporated as an explicit part of the object-language of the dialogical framework.[151]

In relation to the second tenet, the inceptors of the dialogical framework observed that if language-games are to be conceived as mediators of meaning carried out by social interaction, these language-games must be games actually playable by human beings: it must be the case that *we can actually perform them*,[152] which is captured in the notion of dialogue-definiteness.[153] Dialogue-definiteness is essential for dialogues to be mediators of meaning, but it is also constitutive of what propositions are, as Lorenz clearly puts it:

> […] *for an entity to be a proposition there must exist an individual play, such that this entity occupies the initial position, and the play reaches a final position with either win or loss after a finite number of moves according to definite rules. (Lorenz, 2001, p. 258)*

A proposition is thus defined in the standard presentation of dialogical logic (see chapters III-V) as a dialogue-definite expression, that is, an expression A such that there is an individual play about A, that can be said to be lost or won after a finite number of steps, following given rules of dialogical interaction.[154]

The notion of *dialogue-definiteness* is in this sense the backbone of the dialogical theory of meaning: it provides the basis for implementing the human-playability requirement and the notion of proposition.

Dialogue-definiteness sets apart rather decisively the level of strategies from the level of plays, as Lorenz's notion of dialogue-definite proposition does not amount to a set of winning strategies, but rather to an individual play. Indeed, a winning strategy for a player **X** is a sequences of moves such that **X** wins *independently of the moves of the antagonist* (see section III.5 and chapter V). It is crucial to understand that the

---

[150] See *Tractatus Logico-Philosophicus,* 5.6.

[151] Moreover as discussed in section X.5, the dialogical conception of CTT internalization enables subjective reports such "this looks green *to me*" to be rendered.

[152] As observed by (Marion, 2006, p. 245), a lucid formulation of this point is the following remark of (Hintikka, 1996a, p. 158) who shared this tenet (among others) with the dialogical framework:

> [Finitism] *was for Wittgenstein merely one way of defending the need of language-games as the sense that* [sic] *they had to be actually playable by human beings.* […] *Wittgenstein shunned infinity because it presupposed constructions that we human beings cannot actually carry out and which therefore cannot be incorporated in any realistic language-game.* […] *What was important for Wittgenstein was not just the finitude of the operations we perform in our calculi and other language-games, but the fact that* we can actually perform them. *Otherwise the entire idea of language-games as meaning mediators will lose its meaning. The language-games have to be humanly playable. And that is not possible if they involve infinitary elements. Thus it is the possibility of actually playing the meaning-conferring language-games that is the crucial issue for Wittgenstein, not finitism as such.*

[153] The fact that these language-games must be finite does not rule out the possibility of a (potentially) infinite number of them.

[154] While establishing particle rules the development rules have not been fixed yet, so we might call those expressions *propositional schemata*.

qualification *independently of the moves of the antagonist* amounts to the fact that the one claiming A has to play under the restriction of the Copy-cat rule: if possessing a winning strategy for player **X** involves being in possession of a method (leading to the win of **X**) allowing to choose a move for any move the antagonist might play, then we must assume that the propositions brought forward by the antagonist are justified. There is a winning strategy if **X** can base his moves leading to a win by endorsing himself those propositions whose justification is rooted on **Y**'s authority (see Martin-Löf's discussion of this point in our Preface). For short, the act of endorsing is what lies behind the so-called Copy-cat rule and structures dialogues for immanent reasoning: it ensures that **X** can win whatever the contender might bring forward in order to contest A (within the limits set by the game).

Furthermore, *refuting*, that is bringing up a strategy *against* A, amounts to the dual requirement: that the antagonist **Y** possess a method that leads to the loss of **X** ! A, whatever **X** is can bring forward, and that she can do it under the Copy-cat restriction:

> ***X*** ! *A is refuted, if the antagonist **Y** can bring up a sequence of moves such that she (**Y**) can win playing under the Copy-cat restriction*.

Refuting is thus different and stronger than contesting: while *contesting* only requires that the antagonist **Y** brings forward at least one counterexample in a kind of play where **Y** does not need to justify her own propositions, *refuting* means that **Y** must be able to lead to the loss of **X** ! A, *whatever **X**'s justification of his propositions might be*.

In this sense, the assumption that every play is a finitary open two-person zero-sum game does not mean that either there is a winning strategy for A or a winning strategy *against* A: the play level cannot be reduced to the strategy level.

For instance, if we play with the Last-duty first development rule (see section IV.4.2), **P** will lose the individual plays relevant for the constitution of a strategy for $\vee \neg A$. So $A \vee \neg A$ is *dialogue-definite*, though there is no winning strategy *against* A.

The distinction between the play level and the strategy level thus emerges from the combination of dialogue-definiteness and the Copy-cat rule.

The classical reduction of strategies against A to the falsity of A (by means of the saddle-point theorem) assumes that the win and the loss of a *play* reduce to the truth or the falsity of the thesis. But we claim that the existence of the play level and a loss in one of the plays introduces a qualification that is not usually present in the purely proof-theoretic approach; to use the previous example, we know that **P** does not have a winning strategy for ! $A \vee \neg A$ (playing under the intuitionisitic development rule SR1i), but neither will **O** have one against it if she has to play under the Copy-cat rule herself (notice the switch in the burden of the restriction of the Copy-cat rule when *refuting* a thesis). Let us identify the player who has to play under the Copy-cat restriction by highlighting her moves:

Play 20 *against* **P** ! $A \vee \neg A$

O			**P**		
				! $A \vee \neg A$	0
1	$n \coloneqq 1$			$m \coloneqq 2$	2
3	$?_{\vee}$	0		! A	4
				P wins	

The same obviously happens with the negation of the thesis ($! \neg(A \vee \neg A)$), albeit there is a winning strategy against a contradiction ($! A \wedge \neg A$):

Play 21 *against* **P** $!\ A \wedge\ \neg\ A$

O			**P**		
				$! A \wedge \neg A$	0
1	$n := 2$			$m := 3$	2
3	$?\ L^{\wedge}$	0		$!\ A$	4
5	$?\ R^{\wedge}$	0		$! \neg\ A$	6
7	$! A$			—	
	O wins				

The distinction between the play and the strategy level can be understood as a consequence of introducing the notion of dialogue-definiteness which amounts to a win or a loss at the play level, though strategically seen, the proposition at stake may be (proof-theoretically) undecidable. Hence, some criticisms to the purported lack of dynamics to dialogical logic are off the mark if they are based on the point that "games" of dialogical logic are deterministic:[155] plays are deterministic in the sense that they are dialogue-definite, but strategies are not deterministic in the sense that for every proposition there would either be a winning strategy for it or a winning strategy *against* it.

Before ending this section let us quote quite extensively (Lorenz, 2001), who provides a synopsis of the historical background that lead to the introduction of the notion of dialogue-definiteness and the distinction of the deterministic conception of plays—which obviously operates at the level of plays—from the proof-theoretical undecidable propositions—which operate at the level of strategies:

> […] *It was Alfred Tarski who, in discussions with Lorenzen in 1957/58, when Lorenzen had been invited to the Institute for Advanced Study at Princeton, convinced him of the impossibility to characterize arbitrary (logically compound) propositions by some decidable generalization of having a decidable proof-predicate or a decidable refutation-predicate.*
>
> […] *It became necessary to search for some decidable predicate which may be used to qualify a linguistic entity as a proposition about any domain of objects, be it elementary or logically compound. Decidability is essential here, because the classical characterization of a proposition as an entity which may be true or false, has the awkward consequence that of an undecided proposition it is impossible to know that it is in fact a proposition. This observation gains further weight by L. E. J. Brouwer's discovery that even on the basis of a set of "value-definite", i.e., decidably true or false, elementary propositions, logical composition does not in general preserve value-definiteness. And since neither the property of being proof-definite nor the one of being refutation-definite nor properties which may be defined using these two, are general enough to cover the case of an arbitrary proposition, some other procedure had to be invented which is both characteristic of a proposition and satisfies a decidable concept. The concept looked for and at first erroneously held to be synonymous with argumentation*[[156]] *turned out to be the concept of dialogue about a*

---

[155] For such criticisms see (Trafford, 2017, pp. 86-88).

[156] Lorenz identifies argumentation rules with rules at the strategy level and he would like to isolate the interaction displayed by the moves constituting the play level—see (Lorenz, 2010a, p. 79). We deploy the term *argumentation-rule* for request-answer interaction as defined by the local and structural rules. It is true that nowadays argumentation-rules has even a broader scope including several kinds of communicative interaction and this might produce some confusion on the main goal of the dialogical framework which is in principle, to provide an argumentative understanding of logic rather than the logic of argumentation.

> *proposition A (which had to replace the concept of truth of a proposition A as well as the concepts of proof or of refutation of a proposition A, because neither of them can be made decidable). Fully spelled out it means that for an entity to be a proposition there must exist a dialogue game associated with this entity, i.e., the proposition A, such that an individual play of the game where A occupies the initial position, i.e., a dialogue D(A) about A, reaches a final position with either win or loss after a finite number of moves according to definite rules: the dialogue game is defined as a finitary open two-person zero-sum game. Thus, propositions will in general be dialogue-definite, and only in special cases be either proof-definite or refutation-definite or even both which implies their being value-definite. Within this game-theoretic framework where win or loss of a dialogue D(A) about A is in general not a function of A alone, but is dependent on the moves of the particular play D(A), truth of A is defined as existence of a winning strategy for A in a dialogue game about A; falsehood of A respectively as existence of a winning strategy against A. Winning strategies for A count as proofs of A, and winning strategies against A as refutations of A. The meta-truth of "either 'A is true' or 'A is false' " which is provable only classically by means of the saddlepoint theorem for games of this kind may constructively be reduced to the decidability of win or loss for individual plays about A. The concept of truth of dialogue-definite propositions remains finitary, and it will, as it is to be expected of any adequate definition of truth, in general not be recursively enumerable. The same holds for the concept of falsehood which is conspicuously defined independently of negation. (Lorenz, 2001, pp. 257-258)*

XI.2 The Built-in Opponent and the Neglect of the Play Level

In recent literature Catarina Duthil Novaes (2015) and James Trafford (2017, pp. 102-105) deploy the term *internalization* for the proposal that natural deduction can be seen as having an internalized Opponent, thereby motivating the inferential steps. This form of internalization is called the *built-in Opponent*. The origin of this concept is linked to Göran Sundholm who, by 2000, in order to characterize the fundamental links between natural deduction and dialogical logic, introduced in his lectures and talks the term *implicit interlocutor*. Yet, since the notion of *implicit interlocutor* was meant to link the strategy level with natural deduction, the concept of *built-in Opponent*—being the *implicit interlocutor*'s offspring—inherited the same strategic perspective on *logical truth*. Thus, logical truth can be seen as the encoding of a process through which the Proponent succeeds in defending his assertion against a stubborn *ideal* interlocutor.[157]

From the dialogical point of view however, the ideal interlocutor of the strategy level is the result of a process of selecting the relevant moves from the play level. Rahman, Clerbout & Keiff (2009), in a paper dedicated to the Festschrift for Sundholm, designate the process as *incarnation*, using Jean-Yves Girard's term. Their thorough description of the incarnation process already displays those aspects of the *cooperative endeavour*, which was formulated by Duthil Novaes (2015) and quoted by Trafford (2017, p. 102) as a criticism of the dialogical framework. Their criticism seems to rest on the idea that the dialogues of the dialogical framework are not truly cooperative, since they are reduced to constituting logical truth. If this is really the point of their criticism, it is simply wrong, for the play level would then be completely neglected: the intersubjective in-built and implicit cooperation of the *strategy* level (which takes care of inferences) grows out of the *explicit* interaction of players at the *play* level in relation to the formation-rules; accepting or contesting a local reason is a process by the means of

---

However, once this distinction has been drawn nothing prevents to develop the interface dialogical-understanding of logic/logical structure of a dialogue. In fact, it is our claim that in order to study the logical structure of a dialogue, the dialogical conception of logic provides the right venue.

[157] With "ideal" we mean an interlocutor that always make the optimal choices in order to collaborate in the task of testing the thesis.

which players cooperate in order to determine the meaning associated to the action-schema at stake.[158]

It is fair to say that the standard dialogical framework, not enriched with the language of CTT, did not have the means to fully develop the so-called material dialogues, that is dialogues that deal with content. Duthil Novaes (2015, p. 602)—but not Trafford (2017, p. 102)—seems to be aware that dialogues are a complex interplay of adversarial and cooperative moves, [159] even in Lorenzen and Lorenz' standard formulation. However, since she understands this interplay as triggered by the built-in Opponent at the strategy level, her suggestions or corrections motivated by reflections on the Opponent's role cannot be made explicit in the framework, and the way this role contributes in finally constituting a winning strategy cannot be traced back.[160]

Duthil Novaes' (2015, pp. 602-604) approach leads her to suggest that monotonicity is a consequence of the role of the Opponent as a stubborn adversary, which takes care of the non-defeasibility of the demonstration at stake; from this perspective, she contends that the standard presentations of dialogical logic, being mostly adversarial or competitive, are blind to defeasible forms of reasons and are thus "[…] rather contrived forms of dialogical interaction, and essentially restricted to specific circles of specialists" (Duthil Novaes, 2015, p. 602). But this argument is not compelling when considering the strategy level as being built from the play level: setting aside the point on content mentioned above, if we conceive the constitution of a strategy as the end-result of the complementary role of competition and cooperation taking place at the play level, we do not seem to need—at least in many cases—to endow the notion of inference with non-monotonic features. The play level is the level were cooperative interaction, either constructive or destructive, can take place until the definitive answer—given the

---

[158] In fact, when Trafford (2017) criticizes dialogical logic in his chapter 4, he surprisingly claims that this form of dialogical interaction does not include the case in which the plays would be open-ended in relation to the logical rules at stake, though it has already been suggested—see for instance in (Rahman & Keiff, 2005, pp. 394-403)—how to develop what we called *Structure Seeking Dialogues* (SSD). Moreover, Keiff's (2007) PhD-dissertation is mainly about SSD. The idea behind SSD is roughly the following; let us take some inferential practice we would like to formulate as an action-schema, mainly in a teaching-learning situation; we then search for the rules allowing us to make these inferential practices to be put into a schema. For example: we take the third excluded to be in a given context a sound inferential practice; we then might ask what kind of moves **P** should be allowed to make if he states the third excluded as thesis. It is nonetheless true to say that SSD were studied only in the case of modal logic.

[159] To put it in her own words: "*the majority of dialogical interactions involving humans appear to be essentially cooperative, i.e., the different speakers share common goals, including mutual understanding and possibly a given practical outcome to be achieved*." (Duthil Novaes, 2015, p. 602)

[160] See for instance her discussion of countermoves (Duthil Novaes, 2015, p. 602) : indefeasability means that the Opponent has no available countermove: "A countermove in this case is the presentation of one single situation, no matter how far-fetched it is, where the premises are the case and the conclusion is not—a counterexample." The question then would be to know how to show that the Opponent has no countermove available. The whole point of building winning strategies from plays is to *actually construct* the evidence that there is no possible move for the Opponent that will lead her to win: that is a winning strategy. But when the play level is neglected, the question remains: how does one know the Opponent has no countermove available? It can actually be argued that the mere notion of countermove tends to blurr the distinction between the level of plays and of strategies: a *counter*move makes sense if it is 'counter' to a winning strategy, as if the players were playing at the strategy level, but that is something we explicitly reject, see for instance section III.5; see also our discussion of strategic reasons in section VII.7 which inserts strategy level considerations within the *play* level, not the other way round. At the play level, on the other hand, there are only simple *moves*: these can be challenges, defences, counterattacks, but *countermoves* do not make any sense.

structural and material conditions of the rules of the game—has been reached.[161] The strategy level is a recapitulation that retains the end result.

These considerations should also provide an end to Trafford's (2017, pp. 86-88) search for *open-ended* dialogical settings: open-ended dialogical interaction, to put it bluntly, is a property of the play level. Certainly the point of the objection may be to point out either that this level is underdeveloped in the literature—a fact that we acknowledge with the provisos formulated above—, or that the dialogical approach to meaning does not manage to draw a clean distinction between local and strategic meaning—the section on *tonk* below intends to make this distinction as clear as possible.

At this point of the discussion we can say that the role of the (built-in) Opponent in Lorenzen and Lorenz' dialogical logic has been fully misunderstood. Indeed, the role of *both interlocutors* (implicit or not) is not about assuring logical truth by checking the non-defeasibility of the demonstration at stake, but their role is about implementing both the dialogical definiteness of the expressions involved and the internalization of meaning.[162]

XI.3 Pathological cases and the Neglect of the Play Level

The notorious case of *tonk* has been several times addressed as a counterargument to inferentialism and also to the “indoor-perspective” of the dialogical framework. This also seems to constitute the background of how Trafford (2017, p. 86) for instance reproduces the circularity objection against the dialogical approach to logical constants. At this point of the discussion, Trafford (2017, pp. 86-88) is clearly aware of the distinction between the rules for local meaning and the rules of the strategy level, though he points out that the local meaning is vitiated by the strategic notion of justification. This is rather surprising as (Rahman & Keiff, 2005), (Rahman, 2012), (Rahman, Clerbout, & Keiff, 2009), and (Rahman & Redmond, 2016) have shown it is precisely the case of *tonk* that provides a definitive answer to the issue.

In this respect, three well distinguished levels of meaning are respectively determined by specific rules:

- the local meaning of an expression establishes how a statement involving such an expression is to be attacked and defended (through the particle rules);
- the global meaning of an expression results from structural rules prescribing how to develop a play having this expression for thesis;
- the strategy rules (for **P**) determine what options **P** must consider in order to show that he does have a method for winning whatever **O** may do—in accordance with the local and structural rules.

It can in a quite straightforward fashion be shown (see below) that an inferential formulation of rules for *tonk* correspond to *strategic* rules that *cannot be constituted* by the formulation of *particle* rules. The player-independence of the particle rules—responsible for the branches at the strategy level—do not yield the strategic rules that the inferential rules for *tonk* are purported to prescribe.

For short, the dialogical take on *tonk* shows precisely how distinguishing rules of local meaning from strategic rules makes the dialogical framework immune to *tonk*. As this distinction is central to the dialogical framework and illustrates the key feature of

---

[161] See (Rahman, 2015) and (Rahman & Iqbal, 2018).

[162] Notice that if the role of the Opponent in adversial dialogues is reduced to checking the achievement of logical truth, one would wonder what the role of the Opponent might be in more coperation-featured dialogues: A *soft* interlocutor ready to accept weak arguments?

player-independence of particle rules, we will now develop the argument; we will then be able to contrast this pathological *tonk* case to another case, that of the black-bullet operator.

XI.3.1 The *tonk* challenge and player-independence of local meaning

To show how the dialogical framework is immune to *tonk* through the importance and priority it gives to the play level, winning strategies are linked to semantic tableaux. According to the dialogical perspective, if tableaux rules (or any other inference system for that matter) are conceived as describing the core of strategic rules for **P** (see section V.2 for the core of a strategy), then the tableaux rules should be justified by the play level, and not the other way round: the *tonk* case clearly shows that contravening this order yields pathological situations. We will here only need conjunction and disjunction for dealing with *tonk*.[163]

A systematic description of the winning strategies available for **P** in the context of the possible choices of **O** can be obtained from the following considerations: if **P** is to win against any choice of **O**, we will have to consider two main different dialogical situations, namely those

(a) in which **O** has uttered a complex formula, and those
(b) in which **P** has uttered a complex formula.

We call these main situations the **O**-cases and the **P**-cases, respectively. In both of these situations another distinction has to be examined:

(i) **P** wins by *choosing*
  i.1. between two possible challenges *in the **O**-cases* (a), or
  i.2. between two possible defences *in the **P**-cases* (b),

iff he can win with *at least one* of his choices.

(ii) When **O** can *choose*
  ii.1. between two possible defences *in the **O**-cases* (a), or
  ii.2. between two possible challenges *in the **P**-cases* (b),

P wins iff he can win *irrespective* of **O**'s choices.

The description of the available strategies will yield a version of the semantic tableaux of Beth that became popular after the landmark work on semantic-trees by Raymond Smullyan (1968), where **O** stands for **T** (left-side) and **P** for **F** (right-side), and where situations of type ii (and not of type i) will lead to a branching-rule.

Table 54: tableaux and **P**-winning strategies for conjunction and disjunction

(**P**)-Chooses	(**O**)-Chooses
$\dfrac{(\mathbf{P})\ A \vee B}{\begin{array}{c}\langle \mathbf{O}?\ \rangle\ (\mathbf{P})A \\ \langle \mathbf{O}?\ \rangle\ (\mathbf{P})B\end{array}}$	$\dfrac{(\mathbf{P})A \wedge B}{\langle \mathbf{O}?\wedge_1\rangle(\mathbf{P})A \mid \langle \mathbf{P}?\wedge_2\rangle(\mathbf{P})B}$
The expressions of the form ⟨**X** ... ⟩ constitute interrogative utterances.	The expressions of the form ⟨**X** ... ⟩ constitute interrogative utterances.
$(\mathbf{O})A \wedge B$	$(\mathbf{O})A \vee B$

---

[163] (Clerbout, 2014a; 2014b) worked out the most thorough method for linking winning strategies and tableaux.

$\langle \mathbf{P}\ ?\wedge_1 \rangle$ $(\mathbf{O})A$ $\langle \mathbf{P} ?\wedge_2 \rangle$ $(\mathbf{O})B$	$\langle \mathbf{P}? \rangle$ $(\mathbf{O})A \mid (\mathbf{O})B$

However, as mentioned above, tableaux are not dialogues. The main point is that dialogues are built bottom up, from local to global meaning, and from global meaning to validity. This establishes the priority of the play level over the winning strategy level. From the dialogical point of view, Prior's original *tonk* contravenes this priority.

Let us indeed temporarily assume that we can start not by laying down the local meaning of *tonk*, but by specifying how a winning strategy for *tonk* would look like with the help of **T**(left)-side and **F**(right)-side tableaux-rules (or sequent-calculus) for logical constants; in other words, let us assume that the tableaux-rules are necessary and sufficient to set the meaning of *tonk*.

Prior's *tonk* rules are built for half on the disjunction rules (taking up only its introduction rule), and for half on the conjunction rules (taking up only its elimination rule). This renders the following tableaux version for the undesirable tonk:[164]

$$\frac{(\mathbf{O})\ [\text{or } (\mathbf{T})]\ AtonkB}{(\mathbf{O})\ [(\mathbf{T})]\ B} \qquad \frac{(\mathbf{P})\ [\text{or } (\mathbf{F})]\ AtonkB}{(\mathbf{P})[(\mathbf{F})]\ A}$$

Tonk is certainly a nuisance: if we apply the cut-rule, it is possible to obtain a closed tableau for $\mathbf{T}A$, $\mathbf{F}B$, for any A and B. Moreover, there are closed tableaux for both $\{\mathbf{T}A,\ Atonk\neg A\}$ and $\{\mathbf{T}A, \neg(Atonk\neg A)\}$.

From the dialogical point of view, the rejection of *tonk* is linked to the fact there is no way to formulate rules for its local meaning that meet the condition of being player-independent: if we try to formulate rules for local meaning matching the ones of the tableaux, the defence yields a different response, namely the tail of *tonk* if the defender is **O**, and the head of *tonk* if the defender is **P**:

Table 55: **O**-tonk rule for challenge and defence

O-move	**Challenge**	**Defence**
O ! $AtonkB$	**P** $?_{tonk}$	**O** ! B

Table 56: **P**-tonk rule for challenge and defence

P-move	**Challenge**	**Defence**
P ! $AtonkB$	**O** $?_{tonk}$	**P** ! A

The fact that we need two sets of rules for the challenge and the defence of a *tonk* move means that the rule that should provide the local meaning of *tonk* is *player-dependent*, which should not be the case.

Summing up, within the dialogical framework *tonk*-like operators are rejected because there is no way to formulate player-independent rules for its local meaning that justify the tableaux rules designed for these operators. The mere possibility of writing

---

[164] Cf. (Rahman, 2012, pp. 222-224).

tableaux rules that cannot be linked to the play level rules shows that the play level rules are not vitiated by strategic rules.

This brief reflection on *tonk* should state our case for both, the importance of distinguishing the rules of the play level from those of the strategy level, and the importance of including in the rules for the local meaning the feature of *player-independence*: it is the player-independence that provides the meaning explanation of the strategic rules, not the other way round.

XI.3.2The black-bullet challenge and dialogue-definiteness

Now, Trafford (2017, pp. 37-41) contests the standard inferentialist approach to the meaning of logical constants by recalling the counterexample of Stephen Read, the *black-bullet* operator. Indeed, Read (2008; 2010) introduces a different kind of pathological operator, the black-bullet •, a zero-adic operator that says of itself that it is false. Trafford (2017, p. 39 footnote 35) suggests that the objection also extends to CTT; this claim however is patently wrong, since those counterexamples would not meet the conditions for the constitution of a type.[165] Within the dialogical framework, though player-independent rules for black-bullet can be formulated (as opposed to *tonk*), they do not satisfy dialogue-definiteness.

Let us have the following tableaux rules for the black-bullet, showing that it certainly is pathological: they deliver closed tableaux for both • and ¬ •:

$$\frac{(\mathbf{P})\bullet}{\begin{array}{c}\langle \mathbf{O}?\rangle\\ (\mathbf{P})\bullet\supset\bot\end{array}} \qquad\qquad \frac{(\mathbf{O})\bullet}{\begin{array}{c}\langle \mathbf{P}?\rangle\\ (\mathbf{O})\bullet\supset\bot\end{array}}$$

We can in this case formulate the following player-independent rules:

Table 57: black-bullet player-independent particle rules

Move	Challenge	Defence
X ! •	**Y** $?_{\bullet}$	**X**! •⊃⊥

The black-bullet operator seems therefore to meet the dialogical requirement of player-independent rules, and would thus have local meaning. But if it does indeed have player-independent rules, the further play on the defence (which is a negation) would require that the challenger concedes the antecedent, that is black-bullet itself:

Table 58: deploying the black-bullet challenges

Y			**X**		
	…			…	
				! •	i
$i+1$	$?_{\bullet}$	i		! •⊃⊥	$i+2$

---

[165] Klev (2017, p. 12 footnote 7) points out that the introduction rule of such kind of operator fails to be meaning-giving because the postulated canonical **set** $\Lambda(A)$ occurs negatively in its premiss, and that the restriction avoiding such kind of operators have been already formulated by (Martin-Löf, 1971, pp. 182-183), and by (Dybjer, 1994).

$i+3$	! •	$i+2$			
			$i+3$	?•	$i+4$

Obviously, this play sequence can be carried out indefinitely, regardless of which player initially states black-bullet. So the apparently acceptable player-independent rules for playing black-bullet would contravene dialogue-definiteness; and the only way of keeping dialogue-definiteness would be to give up player-independence![166]

Conclusion: the meaning of expressions comes from the play level

The two pathological cases we have discussed, the *tonk* and the black-bullet operators, stress the difference between the play level and the strategy level and how the meaning provided by rules at the strategy level does not carry to the local meaning. Thus, from the dialogical point of view, the rules determining the meaning of any expression are to be rooted at the play level, and at this level *what is to be admitted and rejected as a meaningful expression amounts to the formulation of a player-independent rule, that prescribe the constitution of a dialogue-definite proposition (where that expression occurs as a main operator).*

Notice that if we include material dialogues (see chapter X), the distinction between logical operators and non-logical operators is not important any more. If we enrich the dialogical framework with the CTT-language, this feature comes more prominently to the fore. What the dialogical framework adds to the CTT framework is, as pointed out by Martin-Löf (2017a; 2017b), to set a pragmatic layer where normativity finds its natural place.

XI.4 Other than logical constants

In his recent book, Jaroslav Peregrin (2014) marshals the distinction between the play level and the strategy level (that he calls *tactics*) in order to offer another insight, more general, into the issue of normativity mentioned at that start of our volume (see section I.2). Indeed, Peregrin understands the normativity of logic not in the sense of a prescription on *how to reason*, but rather as *providing the material by the means of which* we reason.

> *It follows from the conclusion of the previous section that the rules of logic cannot be seen as tactical rules dictating feasible strategies of a game; they are the rules constitutive of the game as such. (MP does not tell us how to handle implication efficiently, but rather what implication is.) This is a crucial point, because it is often taken for granted that the rules of logic tell us how to reason precisely in the tactical sense of the word. But what I maintain is that this is wrong, the rules do not tell us how to reason, they provide us with things with which, or in terms of which, to reason. (Peregrin, 2014, pp. 228-229)*

Peregrin endorses at this point the dialogical distinction between rules for plays and rules for strategies. In this regard, the prescriptions for developping a *play* provide the *material* for reasoning, that is, the material allowing a play to be developped, and without which there would not even be a play; whereas the prescriptions of the *tactical* level (to use his terminology) prescribe how to win, or how to develop a winning-strategy:

---

[166] We could provide at the local level of meaning a set of player-independent rules, and add some special structural rule in order to force dialogue-definiteness—see (Rahman, 2012, p. 225); however, such kinds of rules would produce a mismatch in the formation of black-bullet: the formulation of the particle rule would have to assume that black-bullet is an operator, but the structural rule would have to assume it is an elementary proposition.

> *This brings us back to our frequently invoked analogy between language and chess. There are two kinds of rules of chess: first, there are rules of the kind that a bishop can move only diagonally and that the king and a rook can castle only when neither of the pieces have previously been moved. These are the rules constitutive of chess; were we not to follow them, we have seen (Section 5.5) we would not be playing chess. In contrast to these, there are tactical rules telling us what to do to increase our chance of winning, rules advising us, e.g., not to exchange a rook for a bishop or to embattle the king by castling. Were we not to follow them, we would still be playing chess, but with little likelihood of winning. (Peregrin, 2014, pp. 228-229)*

This observation of Peregrin plus his criticism on the standard approach to the dialogical framework, according to which this framework would only focus on *logical constants* (Peregrin, 2014, pp. 100, 106)—a criticism shared by many others since (Hintikka, 1973, pp. 77-82)[167]—naturally leads to the main subject of our book, namely immanent reasoning, or linking CTT with the dialogical framework.

The criticism according to which the focus would be on logical constants and not on the meaning of other expressions does indeed fall to some extent on the standard dialogical framework, as little studies have been carried out on material dialogues in this basic framework;[168] but the enriched CTT language in material dialogues deals with this shortcoming.

Yet this criticism seems to dovetail this other criticism, summoned by Martin-Löf as starting point in his Oslo lecture:

> *I shall take up criticism of logic from another direction, namely the criticism that you may phrase by saying that traditional logic doesn't pay sufficient attention to the social character of language. (Martin-Löf, 2017a, p. 1)*

The focus on the social character of language not only takes logical constants into account, of course, but it also considers other expressions such as elementary propositions or questions, as well as the acts bringing these expressions forward in a dialogical interaction, like statements, requests, challenges, or defences—to take examples from the dialogical framework—and how these acts made by persons intertwine and call for—or put out of order—other specific responses by that person or by others. In this regard, the social character of language is put at the core of immanent reasoning through the normativity present in dialogues: normativity involves, within immanent reasoning, rules of interaction which allow us to consider assertions as the result of having intertwined rights and duties (or permissions and obligations). This central normative dimension of the dialogical framework at large, which stems from questionning what is actually being done when implementing the rules of this very framework, entails that objections according to which the focus would be only on logical constants will always be, from the dialogical perspective, slightly off the mark.

XI.5 Normativity and the Dialogical Framework: A New Venue for the Interface Pragmatics-Semantics

In his Oslo and Stockholm lectures, Martin-Löf's (2017a; 2017b) delves in the structure of the deontic and epistemic layers of statements within his view on dialogical

---

[167] For a response to this see our chapter X on material dialogues.

[168] This kind of criticism does not seem to have been aware of (Lorenz, 1970; 2009; 2010a; 2010b), carrying out a thorough discussion on predication from a dialogical perspective, which discusses the interaction between perceptual and conceptual knowledge. However, perhaps it is fair to say that this philosophical work has not been integrated into the dialogical logic—we will come back to this subject below.

logic. In order to approach this normative aspect which pervades logic up to its technical parts, let us discuss the following extracts of “Assertion and Request”:[169]

> […] *we have this distinction, which I just mentioned, between, on the one hand, the social character of language, and on the other side, the non-social* […] *view of language. But there is a pair of words that fits very well here, namely to speak of the monological conception of logic, or language in general, versus a dialogical one. And here I am showing some special respect for Lorenzen, who is the one who introduced the very term dialogical logic.*
>
> *The first time I was confronted with something of this sort was when reading Aarne Ranta's book* Type-Theoretical Grammar *in (1994). Ranta there gave two examples, which I will show immediately. The first example is in propositional logic, and moreover, we take it to be constructive propositional logic, because that does matter here, since the rule that I am going to show is valid constructively, but not valid classically. Suppose that someone claims a disjunction to be true, asserts, or judges, a disjunction to be true. Then someone else has the right to come and ask him, Is it the left disjunct or is it the right disjunct that is true? There comes an opponent here, who questions the original assertion, and I could write that in this way:*
>
> $$?\ \vdash A \vee\ B\ true$$
>
> *And by doing that, he obliges the original assertor to answer either that A is true that is, to assert either that A is true or that B is true, so he has a choice, and we need to have some symbol for the choice here.*
>
> $$\textit{(Dis)} \qquad \frac{\vdash A \vee B\ true \qquad\quad ?\vdash\ A \vee B\ true}{\vdash A\ true \mid\ \vdash\ B\ true}$$
>
> *Ranta's second example is from predicate logic, but it is of the same kind. Someone asserts an existence statement,*
>
> $$\vdash\ (\exists x\ :\ A)B(x)\ true$$
>
> *and then someone else comes and questions that*
>
> $$?\vdash\ (\exists x\ :\ A)B(x)\ true$$
>
> *And in that case the original assertor is forced, which is to say, he must come up with an individual from the individual domain and also assert that the predicate B is true of that instance.*
>
> […] *So, what are the new things that we are faced with here? Well, first of all, we have a new kind of speech act, which is performed by the| oh, I haven't said that, of course I will use the standard terminology here, either speaker and hearer, or else respondent and opponent, or proponent and opponent, as Lorenzen usually says, so that's terminology but the novelty is that we have a new kind of speech act in addition to assertion.*
>
> […] *So, let's call them rules of interaction, in addition to inference rules in the usual sense, which of course remain in place as we are used to them.*
>
> […] *Now let's turn to the request mood. And then it's simplest to begin directly with the rules, because the explanation is visible directly from the rules. So, the rules that involve request are these, that if someone has made an assertion, then you may question his assertion, the opponent may question his assertion.*
>
> $$\textit{(Req1)} \qquad \frac{\vdash\ C}{?\ \vdash_{may}\ C}$$
>
> *Now we have an example of a rule where we have a may. The other rule says that if we have the assertion* $\vdash\ C$*, and it has been challenged, then the assertor must execute his knowledge how to do* C*. And we saw what that amounted too in the two Ranta examples, so I will write this schematically that he will continue by asserting zero, one, or more we have two in the existential case so I will call that schematically by* $C0$.
>
> $$\textit{(Req2)} \qquad \frac{\vdash\ C \qquad\quad ?\vdash\ C}{}$$

---

[169] Transcription of (Martin-Löf, 2017a, pp. 1-3 ; 7).

$$\vdash_{must}\ C'$$

The Oslo and the Stockholm lectures of Martin-Löf (2017a; 2017b) contain challenging and deep insights in dialogical logic, and the understanding of *defences as duties* and *challenges as rights* is indeed at the core of the deontics underlying the dialogical framework.[170] More precisely, the rules Req1 and Req2 do both, they condense the local rules of meaning, and they bring to the fore the normative feature of those rules, which additionally provides a new understanding for Sunholm's notion of *implicit interlocutor*: once we make explicit the role of the interlocutor, the deontic nature of logic comes out.[171] Moreover, as Martin-Löf points out, and rightly so, they should not be called *rules of inference* but *rules of interaction.*

Accordingly, a dialogician might wish to add players **X** and **Y** to Req2, in order to stress both that the dialogical rules do not involve inference but *interaction*, and that they constitute a new approach to the action-based background underlying Lorenzen's (1955) *Operative Logik*. This would yield the following, where we substitute the horizontal bar for an arrow:[172]

$$\text{(Req2)}\qquad \begin{array}{ccc} \vdash^{\mathbf{X}} C & & ?\ \vdash^{\mathbf{Y}}_{may}\ C \\ & \Downarrow & \\ & ?\ \vdash^{\mathbf{X}}_{must}\ C' & \end{array}$$

Such a rule does indeed condense the rules of local meaning, but it still does not express the choices while defending or challenging; yet it is the distribution of these choices that determines for example that the meaning of a disjunction is different from that of a conjunction: while in the former case (disjunction) the defender *must choose* a component, the latter (conjunction) requires of the challenger that, *her right to challenge is bounded to her duty to choose* the side to be requested (though she might further on request the other side). Hence, the rules for disjunction and conjunction (if we adapt them to Martin-Löf's rules) would be the following:

$$\text{(Dis)}\qquad \begin{array}{ccc} \vdash^{\mathbf{X}} D & & ?\ \vdash^{\mathbf{Y}}_{may}\ \mathrm{D} \\ & \Downarrow & \\ \begin{array}{c} \vdash^{\mathbf{X}}_{must}\ D' \\ \mathit{choose} \\ \mathit{one\ of\ the\ components} \\ \mathit{of\ the\ disjunction}\ D \end{array} & & \end{array}$$

$$\text{(Conj)}\qquad \begin{array}{ccc} \vdash^{\mathbf{X}} C & & ?_{left}\ \vdash^{\mathbf{Y}}_{may}\ \mathrm{C} \\ & \Downarrow & \\ & \begin{array}{c} \vdash^{\mathbf{X}}_{must}\ C' \\ \mathit{assert} \\ \mathit{the\ left\ of}\ C \end{array} & \end{array} \qquad \text{(Conj)}\qquad \begin{array}{ccc} \vdash^{\mathbf{X}} C & & ?_{right}\ \vdash^{\mathbf{Y}}_{may}\ \mathrm{C} \\ & \Downarrow & \\ & \begin{array}{c} ?\ \vdash^{\mathbf{X}}_{must}\ C'' \\ \mathit{assert} \\ \mathit{the\ right\ of}\ C \end{array} & \end{array}$$

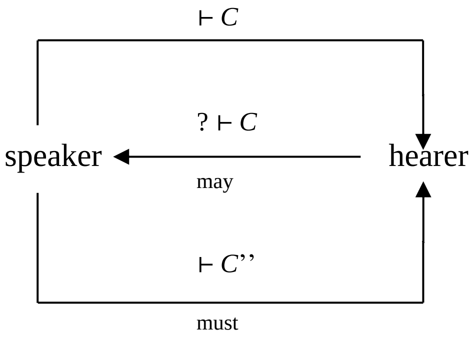
---

[170] See (Lorenz, 1981, p. 120), who uses the expressions *right to attack* and *duty to defend.*

[171] This crucial insight of Martin-Löf on dialogical logic and on the deontic nature of logic seems to underly recent studies on the dialogical framework which are based on Sundholm's notion of the *implicit interlocutor*, such as (Duthil Novaes, 2015) and (Trafford, 2017).

[172] In the context of Operative Logik operations are expressed by means of arrows of the form " ⇒".

These rules can be considered as inserting in the rules the back and forth movement described by Martin-Löf (2017a, p. 8) with the following diagram:



Notice however that these rules only determine the local meaning of disjunction and conjunction, not their global meaning. For example, while classical and constructive disjunction share the same rules of local meaning, they differ at the global level of meaning: in a classical disjunction the defender may come back on the choice he made for defending his disjunction, though in a constructive disjunction this is not allowed, once a player has made a choice he must live with it.

What is more, these rules are not rules of inferences (for example rules of introduction and elimination): they become rules of inference only when we focus on the choices **P** must take into consideration in order to claim that he has a winning strategy for the thesis. Indeed, as mentioned at the start of the present chapter (XI.3.1) strategy rules (for **P**) determine what options **P** must consider in order to show that he has a method for winning whatever **O** does, in accordance with the rules of local and global meaning.

The introduction rules on the one hand establish what **P** has to bring forward in order to assert it, when **O** challenges it. Thus in the case of a disjunction, **P** must *choose* and *assert* one of the two components. So, **P**'s obligation lies in the fact that he must choose, and so **P**'s *duty to choose* yields the introduction rule. Compare this with the conjunction where it is the *challenger* who has the *right to choose* (and who does not assert but request his choice). But in both cases, defending a disjunction and defending a conjunction, only one conclusion will be produced, not two: in the case of a conjunction, the challenger will ask one after the other (recall that it is an interaction taking place within a dialogue where each step alternates between moves of each of the players).

The elimination rules on the other hand prescribe what moves **O** must consider when she asserted the proposition at stake. So if **O** asserted a disjunction, **P** must be able to win whatever the choices of **O** be.

The case of the universal quantifier adds the *interdependence of choices* triggered by the *may*-moves and the *must*-moves: if the thesis is a universal quantifier of the form $(\forall x : A)\, B(x)$, **P** must assert $B(a)$, for *whatever* a **O** may chose from the domain A: this is what correspond to the introduction rule. If it is **O** who asserted the universal quantifier, and if she also conceded that, $a : A$, then **P** may challenge the quantifier by choosing $a : A$, *and request* of **O** that she asserts $B(a)$; this is how the elimination rule for the universal quantifier are introduced in the dialogical framework (for details see chapter VII).

These distinctions can be made explicit if we enrich the first-order language of standard dialogical logic with expressions inspired by CTT. The first task is to introduce statements of the form "$p : A$". On the right-hand side of the colon is the proposition A, on the left-hand side is the *local reason* p brought forward to back the proposition *during a play*. The local reason is therefore *local* if the force of the assertion is limited to the level of plays. But when the assertion "$p : A$" is backed by a *winning strategy,* the judgement asserted draws its justification precisely from that strategy, thus endowing p with the status of a strategic reason that, in the most general cases, encodes an arbitrary choice of **O**.

The rock bottom of the dialogical approach is still the play level notion of dialogue-definiteness of the proposition, namely

> *For an expression to count as a proposition A there must exist an individual play about the statement **X** ! A, in the course of which **X** is committed to bring forward a local reason to back that proposition, and the play reaches a final position with either win or loss after a finite number of moves according to definite local and structural rules.*[173]

The deontic feature of logic is here built directly within the dialogical concept of statements about a proposition. More generally, the point is that, as observed by Martin-Löf (2017a, p. 9), according to the dialogical conception, logic belongs to the area of ethics.

One way of explaining how this important aspect has been overseen or misunderstood might be that the usual approaches to the layers underlying logic got the order of priority between the deontic notions and the epistemic notions the wrong way round.[174]

The Oslo and Stockholm lectures of Martin-Löf propose a fine analysis of the inner and outer structure of the statements of logic from the point of view of speech-act theory, that put the order of priority mentioned above right; in doing so it pushes forward one of the most cherished tenets of the dialogical framework, namely that *logic has its roots in ethics.*

In fact, Martin-Löf's insights on dialogical logic as re-establishing the historical links of ethics and logic provides a clear answer to Wilfried Hodge's (2008)[175] sceptical view in his section 2 as to what the dialogical framework's contribution is. Hodge's criticism seems to target the *mathematical* interest of a dialogical conception of logic, rather than a philosophical interest which does not seem to attract much of his interest.

In lieu of a general plaidoyer for the dialogical framework's philosophical contribution to the foundations of logic and mathematics, which would bring us too far, let us highlight these three points which result from the above discussions:[176]

1) the dialogical interpretation of epistemic assumptions offers a sound venue for the development of inference-based foundations of logic;
2) the dialogical take on the interaction of epistemic and deontic notions in logic, as well as the specification of the play level's role, display new ways of implementing the interface pragmatics-semantics within logic.
3) the introduction of *knowing how* into the realm of logic is of great import (Martin-Löf, 2017a; 2017b).

Obviously, formal semantics in the Tarski-style is blind to the first point, misunderstands the nature of the interface involved in the second, and ignores the third.

---

[173] See above, section XI.1, for a discussion on dialogue-definiteness.

[174] See (Martin-Löf, 2017b, p. 9).

[175] See also (Hodges, 2001) and (Trafford, 2017, pp. 87-88).

[176] See also our Preface and Introduction (chapter I).

XI.6 A brief historical study on material dialogues

Allow us now a brief historic interlude on the distinction between statements of the form $a{:}A$ and of the form $A(a)\ true$ (i.e. $b(a) : A(a)$, where $a{:}B$), $B{:}set$ and $A(x){:}prop\ [x : B]$. As discussed in (Rahman & Clerbout, 2015, pp. 145-146), this distinction is close to the (Lorenz & Mittelstrass, 1967) reconstruction of Plato's notion of *correct naming* in the *Cratylus* (Plato, 1997),[177] and has some links with the dialogical notion of *predicator rule,* which lies at the very basis of material dialogues.

Lorenz & Mittelstrass point out two fundamental speech-acts: *naming* (ὀνομάζειν) and *stating* (λέγειν). The first speech-act, *naming*, amounts to the act of subsuming an entity under a concept, while the second, *stating*, establishes a proposition about a previously named entity. If the naming has been correctly carried out, the (named) entity reveals the concept it instantiates (*names reveal objects for what they are*). *Stating truly* concerns the truth of a proposition that has been constituted by instantiating a propositional function with a suitable element of a genus (a correctly named instantiation of a genus). Thus both acts, naming and stating, involve judgements: while on the one hand *naming* corresponds to the assertion that an entity instantiates a given genus, it says *what the entity is,* and has therefore the following form

$$a{:}A\ [A{:}genus]$$

on the other hand, *stating* corresponds to the act of building a proposition, such as $A(a)$, out of the propositional function $A(x)$ and the *genus* B.

In other words, the correct form of the result of an act of stating, or *saying how something is,* amounts to the judgment

$$A(a){:}prop\ [a{:}B]$$

that presupposes $A(x){:}prop\ [x{:}B],\ B{:}genus$. In this regard,

- the act of naming $a : A$ is said to be true iff a instantiates A; and
- the proposition $A(a)$ is $true$ iff a is one of the entities to which the propositional function $A(x)$ applies (i.e, if a is of the genus B), and it is the case that $A(x)$ can be said of a.

So the resulting proposition (in the context of our example) $A(a)$ is true if $A(x)$ can be said of the entity a. In such a case $A(a)$ would be *stated truly*.

The specialized literature harshly criticized Plato's claim, not that the results of acts of predication could be qualified as true or false, but that the results of acts of naming could be also.[178] According to this criticism, while truth applies to propositions, it does

---

[177] For an endorsement of this interpretation see (Luce, 1969).

[178] Viktor Ilievski (2013, pp. 12-13) provides a condensed formulation of this kind of criticisms:

Socrates next proceeds briefly to discuss true and false speech, with an intention to point out to Hermogenes that there is a possibility of false, incorrect speech. It is a matter of very basic knowledge of logic that truth-value is to be attributed to propositions, or more precisely utterances, specific uses of sentences. Plato's Socrates acknowledges that, but he, somewhat surprisingly, ascribes truth-value to the constituents, or parts of the statements as well, on the assumption that whatever is true of the unit, has to be true of its parts as well. This seems to be an example of flagrant error in reasoning, known as the fallacy of division. Why would Plato's Socrates commit such a fallacy in the course of what seems to be a valid and not apply to entities. (Lorenz & Mittelstrass, 1967, pp. 6-12) defended the old master by suggesting to read these passages as presupposing that in both cases we have the *same kind* of acts of predication,[179] that is, both acts of predication can be qualified as true even if they do not involve the same form of predicator rule.

Resorting to the CTT-setting in order to develop the (Lorenz & Mittelstrass, 1967) interpretation, it follows that claiming both acts of predication can be qualified as true does not necessarily entail that *both* involve propositional functions: Plato's claim can be defended by carefully distinguishinig both constituents of a judgement involving a specific propositional function, namely

1. the act of asserting that a given entity exemplifies the genus presupposed by the formation of that propositional function, and
2. the act of asserting a proposition that results from substituting the variable of the relevant propositional function by a suitable instantiation.

According to this analysis, it is possible to endorse at the same time the following claims of Plato:

1. acts of *naming* and *stating* involve different acts of judgement;
2. both naming and stating can be qualified as *true.*
3. *Neither 1 nor 2* assume[180] that the truth of the result of an act of predication always involves a prescription on how to constitute a propositional function out of another one; rather, *stating*, that is saying *how something is*, presupposes *what it is*, that is *naming*.[181]

Thus, on or view, whereas the act of predication $t\varepsilon S$ (*naming*) can be reconstructed as $t{:}\,S;$ the act of predication $t\varepsilon P$ (*stating*), can be reconstructed as $P(t)\ [t : S]\ true$.

This reconstruction makes explicit the (Lorenz & Mittelstrass, 1967, p. 6) point that *stating* presupposes *naming*. Indeed, let us take the expression man, and use it ambiguously again to express both the assertion $man\ true$ (where $man : genus$), and the assertion $Man(a)\ true$ (where $Man(a){:}\ prop\ [a : living\ being]$). From what we presented earlier on CTT, both make perfect sense:

- $man\ true$ iff man can be instantiated, and thus asserting that a exemplifies man amounts to the truth of man—provided a is indeed such an element;[182]
- $Man(a)\ true$ if a is an instantiation of the genus $living\ being$, presupposed by the formation of the propositional function $Man(x)$; there is a method that takes

---

stable argument? One obvious answer would be that the very theory he is about to expound presupposes the notion of names as independent bearers of meaning and truth, linguistic microcosms encapsulating within themselves both truth-value and reference. In other words, the theory of true and false names has to presuppose that names do not only refer or designate, or even do not only refer and sometimes suggest descriptions, but that they always necessarily represent descriptions of some kind.

[179] *It follows that a true sentence SP really does consist of the ' true parts ' S and P, i.e. t ε S and t ε P. In case of a false sentence SP, however, the second part t ε P is false, while the first part t ε S should ex definitione be considered as true, because any sentence is necessarily a sentence about something (Soph. 262e), namely the subject of it. The subject has to be effectively determined, i.e. it must be a thing correctly named, before one is going to state something about it.* (Lorenz & Mittelstrass, 1967, p. 6)

[180] As (Lorenz & Mittelstrass, 1967, p. 13) seem to:

> *Names, i.e. predicates, are tools with which we distinguish objects from each other. To name objects or to let an individual fall under some concept is on the other hand the means to state something about objects, i.e. to teach and to learn about objects, as Plato prefers to say.*

[181] See section II.1.1.

[182] In fact (Lorenz & Mittelstrass, 1967, p. 6) pointed out, and rightly so, that both acts presuppose a contextually given entity.

> us from $a : living\ being$ to $Man(a)$. Moreover, the falsity of $Man(a)$ also presupposes that a is of the suitable genus presupposed by the propositional function $Man(x)$.[183]

If we follow this interpretation the fact that the judgement $Man(a)\ true$ presupposes $Man(x) : prop\ [x : living\ being]$ makes explicit the relation between *naming* and *stating*.[184]

In the dialogical framework we might say that the formation presupposed is the formation leading to specifying the Socratic rule for material dialogues. These considerations call for further generalization: the difference between the two *Cratylus* speech-acts dovetails the difference between categorical assertions, that involve independent types, and hypotheticals, that involve dependent ones. Let us summarize our suggestions in the following table:[185]

Table 59: Comparing Categorical and Hypothetical judgements in CTT and Plato

Categorical Judgements		**Hypothetical Judgements**	
$t\varepsilon S$		$t\varepsilon P$	
CTT	*Cratylus*	CTT	*Cratylus*
$\boldsymbol{c : B}$	***naming*** **(ὀνομάζειν)**	$\boldsymbol{c(a): B(a)\ [a: A]}$	***stating*** **(λέγειν)**
c is of type B	B names c c is B c exemplifies (the genus) B	*Presupposes the formation rule*: The propositional function $B(x)$ yields a *proposition provided* x *is an element of the set* A	B is predicated of a under the condition that a exemplifies A *Presupposes the predicator rule*: $B(x)$ yields a proposition provided x exemplifies A
***c*:*B* true iff:**	**$\boldsymbol{c}$ is $\boldsymbol{B\ true}$ iff:**	**$\boldsymbol{B(a)\ true}$ iff:**	**$\boldsymbol{B(a)\ true}$ iff:**
c is a canonical element of B, or c is generated from a canonical one	c correctly exemplifies B	B applies to a	$B(x)$ is correctly said of a

---

[183] Cf. (Lorenz & Mittelstrass, 1967, p. 6).

[184] (Lorenz & Mittelstrass, 1967, pp. 6-7) claim that *being correct* and *being true* are to be considered as synonymous.

[185] The table is based on preliminary results of an ongoing research project by S. Rahman and Fachrur Rozie. Let us point out that we do not claim herewith that the CTT-notion of *type* is the same as Plato's notion of *genus*, but rather that they play the same role in judgements involving *type*/*genus*. The claim is that we can establish a kind of parallelism between the CTT use of judgements involving independent and dependent types on the one hand, and Plato's distinction on the other hand between acts of naming and acts of asserting a proposition. For a detailed comparison between the CTT notion of *type* and Plato's notion of *genus*, a detailed study is due.

presupposes the formation rule of the *independent type* B	*presupposes* the formation rule of the *genus* B (and not of $B(x)$)	*presupposes* the formation rule of the set A and the propositional function $B(x)$ over the set A: $B(x) : prop\ [x : A]$ where $B(x)$ is a *dependent type* upon A.	*presupposes* the formation rule of (the genus) A and the *predicator rule*[186] for $B(x)$: $x\ exemplifies\ A \Leftarrow B(x)$ where B(x) is defined over the *genus* A

Identifying versus Individuating

As pointed out in in section II.1, the categorical predication of the form $a : B$ does not have not the function-argument form that modern logic inherited from Frege, and responds to the question on *what something is*. The modern view conflating both forms of predication into a functional form can perhaps be put in the following terms: the act of predicating amounts to a process by the means of which some objects are selected from a given domain. If the process is set to select exactly one, then we have an *individuating predication*. According to this view, Frege-Russell's claim that individual constants can be substituted *salva veritate* by definite descriptions can be seen as pushing forward the idea that at the very end there is only one fundamental form of predication that amounts to a process of selecting by description, either exactly one or a whole class of objects of the given domain. This certainly is very different from Plato and Aristotle: according to Plato's approach the process associated to the act of naming cannot be reduced to relating some kind of sign with an object but establishes what that object is; in other words, the focus in Plato's naming is on *identifying*, not *individuating*. Moreover, it is not only the case that in the tradition of Plato the acts are different but, according to the wise insight of our forefathers, the source of knowledge is not a given indiscriminate domain.

This might provide some further insight on the division of waters between rigid designation and definite descriptions brought forward by Saul Kripke and Hilary Putnam (with some variation). Indeed, the theory of rigid designation assumes an indiscriminate domain of objects within which by some kind of individuation act of baptism exactly one individual is selected. According to this approach the selection happens in such a way that despite the fact that the initial act of selection might have been carried out by an individuating description (but not necessarily so), the further identification occurs by some kind of causally transmitted indexation.

If we push our parallelism forward, we could say that in Plato there is also *a chain of propagation* of some (perhaps initial) act of *identification*. However, the chain might propagate a mistake, and the role of the philosopher is to stop the further dissemination of the error and investigate about its *correctness* by the means of dialectical interaction.

---

[186] For the notion of *predicator rul*e see (Lorenz & Mittelstrass, 1967). We use here a slightly modified version: the original idea is that those kinds of rules establish how to constitute a predicate from a different one. We propose a more basic predicator rule, namely, a rule that establishes how a predicate is ascribed to a certain kind of objects (the genus underlying the predicate).

Thus the rigid-designation theory is a remarkable change from Plato's distinction, as it considers after all that both definite descriptions and rigid designators have the same role: they are individuating operations. Plato's theory of naming and stating establishes two different acts, one identifying while the other establishing how an object is.

As pointed out by (Lorenz & Mittelstrass, 1967, p. 6), the act of establishing presupposes a domain already set by the act of identification.[187] Thus, acts of individuation, that did not attract much neither Plato's nor Aristotle's attention, might find their place as special cases of acts of establishing how things are (if our aim is to insert them in their framework). If that is so, then acts of individuation presuppose a given domain.

This certainly does not meet the perspective of those that strictly distinguish between individuation by flexible designation (or description) and individuation by rigid designation (or indexation). According to this view, both individuation acts are to be understood as two different irreducible ways of selecting objects within a domain, one by selecting an individual and the other as a unary set. The first form of individuation cannot be imported as such in the framework of the *Cratylus*,[188] since it would assume that we could *individuate an object without any knowledge of what it is.*[189]

XI.7 Intersubjectivity, Dialogues, and Learning

The book *Logic, Language and Method. On Polarities in Human Experience,* published in 2010, includes papers written by Kuno Lorenz in a period extending over more than thirty years. These papers have planted the seeds for his further penetrating work (Lorenz, 2009; 2010a; 2010b), which can be considered as philosophical variations on *Das Dialogische Prinzip* (2010b, pp. 509-520) underlying what is often known as *Dialogical Constructivism*.

In the framework of Dialogical Constructivism, the analysis of the notion of intersubjectivity starts by the study of a situation where two persons are engaged in the process of acquiring a common action-competence in a situation of teaching and learning;[190] what is at stake then is not simply mirroring an individual competence in another individual, but rather it is a *procedure* which incorporates from the very beginning this dialogical situation.[191] Immanent reasoning, being an offspring of Dialogical Constructivism, inherits its philosophical background and sensitivity. In this regard, the rules of the play level are not actualizations by themselves, but are rather

---

[187] It can be seen as an early formulation of what is nowadays known as the *principle of comprehension* or *set-specification*.

[188] The proviso "as such" leaves the door open to the possibility that acts of individuation could be carried out over an already separated subset. However, it is not clear if this will work out in Plato's framework, and it seems that it will not work at all in CTT.

[189] In contemporary modal logic we have also a more sophisticated theory of dealing with individuation, namely the theory of individuals as *World-Lines* of the late Jaakko Hintikka (1969), but the discussion of it will go far beyond the aims of this short excursus. (Tulenheimo, 2017) provides both a thorough critical discussion of Hintikka's formulations and a new approach inspired by the notion of world-lines. Tulenheimo's distinction between *world-bounded local objects,* and individuals, might perhaps be asocciated to the distinction between identification (in a world) and Individuating (establishing world links).

[190] The bibliographic background of this section is based mainly on (Lorenz, 2010a, pp. 2017-2018) chapter *Procedural Principles of the Erlangen School. On the Interrelation between the principles of method, of dialogue, and of reason*.

[191] The act of executing must be distinguished from taking the action as an object: while executing an action, actor and execution are said to be indistinguishable.

procedures for actualizing some action consisting in dealing with an object or appropriating that object, be it in a situation of teaching and learning, or any dialogical situation.

A consequence of Lorenz's (2017b, pp. 509-520) general dialogical principle is that the interface semantics-pragmatics should be understood

1. neither as the result of the *semantization of pragmatics*—where deontic, epistemic, ontological, and temporal modalities become truth-functional operators;
2. nor as the result of the *pragmatization of semantics*—where a propositional kernel, when put into use, is complemented by moods yielding assertions, questions, commands and so on.

Lorenz's view (2010a, pp. 71-79) is that the differentiation of semantic and pragmatic layers is the result of the articulation within one and the same utterance: each utterance displays in principle both features: it *signifies* (semantic layer) and it *communicates* (pragmatic layer).

Take for example one-word sentences such as:

Rabbit!
Water!

With these utterances the speaker is conveying at the same time *what* the object is and *how* the object is. But while the first aspect (*what*) is related to *object-constitution*, the second (*how*) is related to *object-description;* or, if we use the terminology of Wittgenstein's *Tractatus*, the first aspect relates to the *act of showing* and the second to the *act of saying*. Object-description is carried out by the use of predicates on an already constituted domain of objects. Lorenz recalls here Plato's *Cratylus* (388b), in which these two acts and their interdependence are distinguished as *naming* (which has the role of *indicating*) and *establishing* (with the role of *communicating*).[192] Lorenz's view is that each utterance of a sentence has this double nature, not only one-word sentences. Thus:

(a) *Sam is smoking.*

has both roles, indicating as well as communicating; though according to this analysis, uttering such a sentence does not yield any ambiguity: uttering it simply displays within one movement object-constitution (or construction) and object-description (or attribution).

While the first, object-constitution, involves differentiating parts of a whole (including the processes of partitioning a whole by synthesis and analysis), the second, object-description, involves stating that a certain relation holds. In this regard, *attribution* is not a relation, but a means for stating that relations hold of objects. The usual procedure for representing attribution by using extensional class-membership relations thus blurs this distinction.

According to the language of immanent reasoning (borrowed from CTT),

(a) *Sam is smoking*

can be read as expressing either

(b) ! *Sam* : *Smoking*

or

(c) ! *d*(*Sam*) : *Smoking*(*Sam*) (*Sam* : *Human)*

From the point of view of Lorenz's Dialogical Constructivism, me might say that the colon in both claims separates, using his words, the significative, particular, part of

---

[192] In the previous section (XI.6) we briefly present the discussion of the *Cratylus* found in (Lorenz & Mittelstrass, 1967).

the expression from the communicative, universal, side of it, placed at the right side of the colon.

These considerations deserve further investigation, though this conclusion is not their place. But the point here is to stress that, according to the dialogical principle, pragmatization and semantization are two different aspects: $a{:}B$ and $B(a)$ are not the result of an ambiguity of some sort, but are simply two aspects, the semantic aspect and the pragmatic aspect.

The dialogical nature of meaning and the I-You-perspective

From the point of view of Dialogical Constructivism however, it is not sufficient to simply observe utterances and to conceptually distinguish their two functions. Beyond descriptive observation, what it is needed is to be aware of the necessarily *social* character of the *acquisition process* by the means of which these expressions are used and some distinctions are drawn. Meaning is dialogical by nature and it emerges from *participation* and *collective interaction*.

In other words, Dialogical Constructivism—and immanent reasoning after it—is based on the idea that the acquisition of meaning is intimately linked to both, the accomplishment of acting, and living together and participating in the actions the utterances express:

> […]*the use of predicators as passed from one generation to the another is not usually accomplished by mere deictic activities, that is, in distantiated pointing, but rather "empractically" (Bühler), i.e. in very accomplishment of acting and living together. What "walking" or "eating" is* […] *we learn these things linguistically only along with the activities themselves, at the same time. In living with one another over a long period of time we acquired the use of such predicators as "father", "brother"* […]. *(Kamlah & Lorenzen, 1984, p. 36)*

Lorenz (2010a, pp. 142-146) develops further this *empractical* approach, based on the work of Johan Gottfried Herder (1960 [1772]) Part II, who uses a dialogue model of teaching and learning to identify items of cultural process, inasmuch as both "doing" and "suffering", terms that can be used to characterize the two roles which always occur together: that of teaching and that of learning. Indeed, from the point of view of the teacher, there is a predominant *active* feature of "doing", while the learner, in its *passive* perspective, is taking the doing of the teacher as the way to actualize the action-schema in question. This duality also applies to performances by the means of which singulars are brought forward, and holds likewise of actualizations by the means of which particulars are brought forward.

In other words, within the model of an elementary dialogue situation in which two agents are engaged in the process of acquiring an action competence, the activities of actualizing and schematizing should not be understood as carrying out two separate actions; rather one acquires the compentence of one and the same action by learning to play both the active and the passive role. Active actualization makes the action appear in *I*-perspective, passive schematization lets it appear in *You*-perspective. Performing an action-schema by *myself*, that has been experienced as a schema through *your* actualization of the schema, amounts to learning by means of an *I-You*-perspective.

In order to switch from an *I-You*-perspective to a *He-She*-perspective, the constitution of a *third-man perspective* is required. The perspective of the "third-man" results from the nesting of *I-You*-perspectives— (Lorenz, 2010a, p. 48). In such a framework,[193]

---

[193] See (Lorenz, 2010a, pp. 144-146).

- *individuality*, i.e. a difference between individuals on the level of reflection, will be recognized only within some common activity where *I*- and *You*-perspectives are put into action.
- *sociality*, i.e. an equality of individuals on the level of reflection, can be exercised only by being conscious of the different approaches within the same common activity. This amounts to cooperation by means of individual contributions.

XI.8 Final Words

The play level is the level where meaning is forged: it provides the material with which we reason.[194] It reduces neither to the (singular) performances that actualize the interaction-types of the play level, nor to the "tactics" for the constitution of the schema that yields a winning strategy.

We call our dialogues involving rational argumentation *dialogues for immanent reasoning* precisely because *reasons* backing a statement, that are now *explicit* denizens of the object-language of plays, are *internal* to the development of the dialogical interaction itself.

More generally, the emergence of concepts, so we claim, are not only games of giving and asking for reasons (games involving *Why*-questions) they are also games that include moves establishing *how is it that the reason brought forward accomplishes the explicative task.* Dialogues for immanent reasoning are dialogical games of *Why* and *How*.

More generally, Lorenz (2010a, pp. 140-147) associates *competition* with the process of *individualization* and cooperation with the process of *socialization*, so that both are different ways to display the *I* and the *You* perspectives. In this sense, the dialogical teaching-learning situation is where competition and cooperation interact: both intertwine in collective forms of dialogical interaction that take place at the play level.

The point is that within the dialogical framework actualizing and schematizing should not be understood as performing two separate actions: through these actions we acquire the competence that is associated to the meaning of an expression by *learning* to play both, *the active* and *the passive* role. This feature of Dialogical Constructivism stems from Herder's view[195] that the cultural process is a process of education, in which teaching and learning always occur together: dialogues display this double nature of the cultural process in which concepts emerge from a complex interplay of *why* and *how* questions.

If the reader allows us to condense our proposal once more, we might say that the perspective we are trying to bring to the fore is rooted in the intimate conviction that meaning and knowledge are something we do together; our perspective is thus an invitation to participate in the open-ended dialogue that is the human pursuit of knowledge and collective understanding, since philosophy's endeavour is immanent to the kind of dialogical interaction that makes reason happen.

---

[194] To use Peregrin's (2014, pp. 228-229) words.

[195] See (Herder, 1960 [1772]), Part II.

APPENDIX: MAIN NOTATION FOR CTT

Equality and Identity

- Judgemental equality
    - $a = b : B$
    - $A = B$: **set**
- Identity-Predicate: **Id**(A, x, y**),** alternatively $x =_A y$

Judgement

- Categorical
    - $a : A$
    - $a = b : A$
    - A : **set**, alternatively A : **prop**
    - $A = B$: **set**, alternatively , $A = B$: **prop**

- Hypothetical
    - $x : A \vdash b : B$, alternatively $b : B$ ($x : A$)
    - $x : A \vdash b = c : B$, alternatively $b = c : B$ ($x : A$)
    - $x : A \vdash B$: **set**, alternatively B : **set** ($x : A$)
    - $x : A \vdash B = C$: **set**, alternatively $B = C$: **set** ($x : A$)

Types

- proposition: **prop**
- sets: **set**
- natural numbers: ℕ
- propositional-function of prime numbers : **Pr**

Operators over a family of sets

- Σ-operator ($\Sigma\, x : A)B$
    - Proof object: $<a, b> : (\Sigma\, x : A)B$
    - Projectors for c : $(\Sigma x : A)B$ **fst**(c) : $(\Sigma x : A)B$ **snd**(c) : $(\Sigma x : A)B$[**fst**(c)]

- Π-operator $(\Pi x : A)B$
    - Proof object for Π $\lambda x.b : (\Pi x : A)B$:
    - Application for $c : (\Pi x : A)B$ **,** $a : A$ **ap**($\lambda x.b, a$) : $B[a]$

- Disjoint union $A + B$
    - Proof-object given $a : A$ **i**(a) : $A + B$
    - Proof-object given $b : B$ **j**(b) : $A + B$
    - Selector given $c : A + B,, x : A \vdash d : C$[**i**($x$)], $y : B \vdash e : C$[**j**(y)] **D**(c, $x.d$, $y.e$) : $C[c]$

BIBLIOGRAPHY

Aczel, P. (1978). The Type Theoretic Interpretation of Constructive Set Theory. *Logic Colloquium 77*, (pp. 55-66). Amsterdam: Norht-Holland.

Almog, J. (1991). The What and the How. *Journal of Philosophy, 88*, 225-244.

Armgardt, M. (2001). *Das rechtslogische System der Doctrina Conditionum von G. W. Leibniz.* Marburg: Elwert Verlag.

Armgardt, M. (2008). Zur Bedingungsdogmatik im klassischen römischen Recht und zu ihren Grundlagen in der stoischen Logik. *Tijdschrift voor Rechtsgeschiedenis, 76*, 219-235.

Armgardt, M. (2010). Zur Rückwirkung der Bedingung im klassischen römischen Recht und zum stoischen Determinismus. *Tijdschrift voor Rechtsgeschiedenis, 78,* 341-349.

Austin, J. L. (1946). Other Minds. *The Aristotelian Society Supplementary Volume, 20,* 148-187.

Barnes, J. (1984). *The Complete Works of Aristotle. The Revised Oxford Translation.* Princeton NJ: Princeton University Press.

Barrio, E., Clerbout, N., & Rahman, S. (2017). Introducing Consistency in a Dialogical Framework. *submitted*.

Batens, D. (1980). Paraconsistent Extensional Propositional Logics. *Logique et analyse, 90/91*, 195-234.

Beirlaen, M., & Fontaine, M. (2016). Inconsistency-Adaptive Dialogical Logic. *Logica Universalis*.

Bell, J. (2009). *The Axiom of Choice.* London: College Publications.

Bishop, E. (1967). *Foundations of Constructive Mathematics.* New York, Toronto, London: McGraw-Hill.

Bonbenrieth Miserda, A. (1996). *Inconsistencias : Por Qué No? Un Estudio Filosófico Sobre la Lógica Paraconsistente.* Santafé de Bogotá: Tercer Mundo Editores.

Brandom, R. (1994). *Making it Explicit.* Cambridge: Harvard University Press.

Brandom, R. (1997). A Study Guide. In W. Sellars, *Empiricism and the Philosophy of Mind* (pp. 119-189). Cambridge-Mass: Harvard University Press.

Brandom, R. (2000). *Articulating Reasons.* Cambridge: Harvard University Press.

Brandom, R. (2002). The Centrality of Sellars's Two-Ply Account of Observation to the Arguments of 'Empiricism and the Philosophy of Mind'. In R. Brandom, *Tales of the Mighty Dead: Historical Essays in the Metaphysics of Intentionality.* Cambridge MA: Harvard University Press.

Brandom, R. (2008). *Between Saying and Doing.* New York: Oxford University Press.

Cardascia, P. (2016). Dialogique des matrices. *Revista de Humanidades de Valparaíso, 6*.

Carnap, R. (1934). *Logische Syntax der Sprache.* Vienna: Julius Springer.

Carnielli, W., & Coniglio, M. (2016). *Paraconsistent Logic: Consistency, Contradiction and Negation.* Springer International Publishing.

Carnielli, W., Coniglio, M., & Marcos, J. (2007). Logics of Formal Inconsistency. In *Handbook of Philosophical Logic* (pp. 1-93). Springer Netherlands.

Clerbout, N. (2014a). First-Order Dialogical Games and Tableaux. *Journal of Philosophical Logic, 43*(4), 785-801.

Clerbout, N. (2014b). *Étude sur quelques sémantiques dialogiques : Concepts fondamentaux et éléments de métathéorie.* London: College Publications.

Clerbout, N. (2014c). Finiteness of Plays and the Dialogical Problem of Decidability. *IfCoLog Journal of Logics and their Applications, 1*(1), 115-140.

Clerbout, N., & Rahman, S. (2015). *Linking Game-Theoretical Approaches with Constructive Type Theory: Dialogical Strategies as CTT-Demonstrations.* Dordrecht: Springer.

Clerbout, N., Gorisse, M.-H., & Rahman, S. (2011). Context-sensitivity in Jain Philosophy: A Dialogical Study of /Siddharsiganis Commentary on the Handbook of Logic/. *Journal of Philosophical Logic, 40*(5), 633-662.

Coniglio, M., & Silvestrini, L. (2014). An Alternative Approach for Quasi-Truth. *Logic Journal of the IGPL, 22*(2), 387-410.

Corcoran, J. (1974). Aristotle's Natural Deduction System. In J. (. Corcoran, *Ancient Logic and its Modern Interpretations* (pp. 85-131). Dordrecht: Synthese Historical Library.

Crubellier, M. (2008). The Programme of Aristotelian Analytics. (C. Dégremont, L. Keiff, & H. Rückert, Eds.) *Dialogues, Logics and Other Strange Things. Essays in Honour of Shahid Rahman*, 103-129.

Crubellier, M. (2014). *Aristote, Premiers analytiques. Traduction, introduction et commentaire.* Garnier-Flammarion.

Crubellier, M., Marion, M., McConaughey, Z., & Rahman, S. (2018). Dialectic, The Dictum de Omni and Ecthesis. *forthcoming*.

Curry, H. B. (1951). *Outlines of a Formalist Philosophy of Mathematics.* Amsterdam: North-Holland.

Curry, H. B. (1956).

Dango, A. B. (2014). Des dialogues aux tableaux dans le contexte de révision des croyances : De l'oralité à l'écriture. (C.-Z. Bowao, & S. Rahman, Eds.) *Entre l'orature et l'écriture : Relations croisées*, 175-192.

Dango, A. B. (2015). Interaction et révision de croyances. *Revista de Humanidades de Valparaíson, 5*, 75-98.

Dango, A. B. (2016). *Approche dialogique de la révision des croyances dans le contexte de la Théorie Constructive des Types.* London: College Publications.

Diaconescu, R. (1975). Axiom of Choice and Complementation. *Proc. Amer. Math. Soc., 51*, 176-178.

Diller, J., & Troelstra, A. (1984). Realizability and Intuitionistic Logic. *Synthese, 60*, 253-282.

d'Ottaviano, I. M., & da Costa, N. C. (1970). Sur un problèle de Jáskowski. *Comptes Rendus de l'Académie de Science Paris (A-B)*, 1349-1353.

Dummett, M. (1973). *Frege. Philosophy of Language* (2nd (1981) ed.). London: Duckworth.

Dummett, M. (1978). *Truth and Other Enigmas.* Harvard University Press.

Duthil Novaes, C. (2015). A Dialogical, Multiagent Account of the Normativity of Logic. *Dialectica, 69*(4), 587-609.

Dybjer, P. (1994, July). Inductive families. Form Asp Comput vol. 6, pp. 440–465. *Formal Aspects of Computing, 6*, 440-465.

Ebbinghaus, K. (1964). *Ein formales Modell der Syllogistik des Aristoteles.* Göttingen: Vandenhoeck & Ruprecht GmbH.

Felscher. (1985). Dialogues as a Foundation for Intuitionistic Logic. (D. Gabbay, & G. F, Eds.) *Handbook of Philosophical Logic, 3*, 341-372.

Fischer, M. (1989). Phases and phase diagrams: Gibbs' legacy today. In G. D. Mostow, & D. G. Caldi, *Proceedings of the Gibbs Symposium: Yale University, May 15-17, 1989.* American Mathematical Society.

Fontaine, M. (2013). *Argumentation et engagement ontologique. Être, c'est être choisi.* London: College Publications.

Fraenkel, A., Bar-Hillel, Y., & Levy, A. (1973). *Foundations of Set Theory* (2nd ed.). Dordrecht: North-Holland.

Frege, G. (1892). Über Sinn und Bedeutung. *Zeitschrift für Philosophie und philosophische Kritik, 100*, 25-50.

Frege, G. (1893). *Grundgesetze der Arithmetik.* Jena: Hermann Pohle.

Frege, G. (1979). Boole's Logical Calculus and the Concept Script [1880/81]. In Hermes, Kambartel, & K. (eds), *Gottlob Frege Posthumous Writtings* (pp. 9-52). Oxford: Basil Blackwell.

Garner, R. (2009). On the Strength of Dependent Products in the Type Theory of Martin-Löf. *Annals of Pure and Applied Logic, 160*, 1-12.

Gentzen, G. (1933). Untersuchungen über das logische Schliessen. *Mathematische Zeitschrift, 39*, 176-210.

Gentzen, G. (1969). *The Collected Papers of Gerhard Gentzen.* (M. Szabo, Ed.) Amsterdam: North-Holland.

Ginzburg, J. (2012). *The Interactive Stance.* Oxford: Oxford University Press.

Girard, J.-Y. (1999). On the meaning of logical rules I: syntax vs. semantics. In U. Berger, & H. (. Schwichtenberg, *Computational Logic. Proceedings of the NATO Advanced Study Institute Computational Logic Held in Marktoberdorf (Germany) July / August 1997* (pp. 215-272). Heidelberg: Springer.

Goodman, N. D., & John, M. (1978). Choice Implies Excluded Middle. *Zeitschrigt für mathematische Logik und Grundlagen der Mathematik, 24*, 461.

Granström, J. G. (2011). *Treatise on Intuitionistic Type Theory.* Dordrecht: Springer.

Hale, B., & Wright, C. (2001). To bury Caesar... In *The Reason's Proper Study* (pp. 335-396). Oxford: Oxford University Press.

Heinzmann, G. (2006). Naturalizing Dialogic Pragmatics. In J. van Benthem, G. Heinzman, M. Rebushi, & H. (. Visser, *The Age of Alternative Logics* (pp. 285-297). Springer.

Herder, J. G. (1960 [1772]). Abhandulung über der Ursprung der Sprache. In E. (. Heintel, *Johann Gottfried Herder. Spachphilosophische Schriften.* (pp. 3-87). Hamburg: Felix Meiner.

Hintikka, J. (1969). *Models for Modalities.* Dordrecht: Reidel.

Hintikka, J. (1973). *Logic, Language-Games and Information: Kantian Themes in the Philosophy of Logic.* Oxford: Clarendon Press.

Hintikka, J. (1983). Game-Theoretical Semantics: Insights and Prospects. *Notre Dame Journal of Formal Logic, 23*(2), 219-249.

Hintikka, J. (1996a). *Ludwig Wittgenstein: Half Truths and One-and-a-Half Truths.* Dordrecht: Kluwer.

Hintikka, J. (1996b). *The Principles of Mathematics Revisited.* Cambridge: Cambridge University Press.

Hintikka, J. (1997). *Lingua Universalus vs. Calculus Rationcinator: An Ultimate Presupposition of Twentieth-Century Philosophy.* Dordrecht: Kluwer.

Hintikka, J. (2001). Intuitionistic Logic as Epistemic Logic. *Synthese, 127*, 7-19.

Hodges, W. (2001). Dialogue foundations: A sceptical look. *Aristotelian Society Supplementary, 75*(1), 17-32.

Hodges, W. (2008). Logic and games. *Stanford Encyclopedia of Philosophy.*

Hofmann, M. (1995). Extensional Concepts in Intensional Type Theory. University of Edinburgh.

Hofmann, M., & Streicher, T. (1998). The Groupoid Interpretation of Type Theory. (G. Sambin, & S. J. M, Eds.) *Twenty-Five Years of Constructive Type Theory,* 83-111.

Howard, W. A. (1980). The Formulae-as-Types Notion of Construction. In J. P. Seldin, & J. R. Hindley (Eds.), *To H. B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism* (pp. 479-490). London: Academic Press.

Ilievski, V. (2013). Language and Knowledge in Plato's Cratylus and Beyond it. *Филозофија / Filozofija - A Journal of Philosophical Inquiry, 2*(35).

Jovanovic, R. (2013). Hintikka's Take on the Axiom of Choice and the Constructive Challenge. *Revista de Humanidades de Valparaíso, 2*, 135-152.

Jovanovic, R. (2015). *Hintikka's Take on Realism and the Constructive Challenge.* London: College Publications.

Kamlah, W., & Lorenzen, P. (1972). *Logische Propädeutik. Vorschule des vernünftigen Redens* (Second Edition ed.). Stuttgart/Weimar: Metzler.

Kamlah, W., & Lorenzen, P. (1984). *Logical Propaedeutic: Pre-School of Reasonable Discourse.* University Press of America.

Kapp, E. (1942). *Greek Foundations of Traditional Logic.* AMS Press.

Keiff, L. (2004). Heuristique formelle et logiques modales non normales. *Philosophia Scientiae, 8*(2), 39-59.

Keiff, L. (2007). *Le Pluralisme dialogique : Approches dynamiques de l'argumentation formelle*. Lille: PhD.

Keiff, L. (2009). *Dialogical Logic.* (E. N. Zalta, Ed.) Retrieved from The Stanford Encyclopedia of Philosophy: http://plato.stanford.edu/entries/logic-dialogical

Klev, A. (2014a). *Categories and Logical Syntax*. Leiden: PhD.

Klev, A. (2014b). Protagoras and his Pupil. *Unpublished manuscript*, 6.

Klev, A. (2017). The Justification of Identity Elimination in Martin-Löf's Type Theory. *Topoi*, 1-25.

Krabbe, E. C. (1982). *Studies in Dialogical Logic*. Rijksuniversiteit, Gröningen: PhD.

Krabbe, E. C. (1985). Formal Systems of Dialogue Rules. *Synthese, 63,* 295-328.

Krabbe, E. C. (2006). Dialogue Logic. In D. Gabbay, & J. Woods (Eds.), *Handbook of the History of Logic* (Vol. 7, pp. 665-704). Amsterdam: Elsevier.

Lecomte, A. (2011). *Meaning, Logic and Ludics.* Imperial College Press.

Lecomte, A., & Quatrini, M. (2010). Pour une étude du langage via 'interaction : dialogues et sémantique en ludique. *Mathématiques et sciences humaines, 189,* 37-67.

Leibniz, G. W. (1964). *Sämtliche Schriften und Briefe.* Berlin: Akademie.

Linant de Bellefonds, Y. (1965). *Traité de droit musulman comparé.* Mouton.

Lloyd, G. E. (1996). Science in Antiquity: the Greek and Chinese cases andtheir relevance to the problem of culture and cognition. In D. Olson, & N. (. Torrance, *Modes of Thought: Explorations in Culture and Cognition* (pp. 15-33). Cambridge: Cambridge University Press.

Lorenz, K. (1970). *Elemente der Sprachkritik. Eine Alternative zum Dogmatismus und Skeptizismus in der Analytischen Philosophie.* Frankfurt: Suhrkamp.

Lorenz, K. (1981). Dialogical logic. In W. Marciszewsku, *Dictionary of Logic as Applied in the Study of Language* (pp. 117-125). The Hague: Martinus Nijhoff.

Lorenz, K. (1994). Competition and Cooperation: Are They Antagonistic or Complementary? In W. de Graaf, & R. (. Maier, *Sociogenesis Reexamined.* New York: Springer.

Lorenz, K. (2001). Basic Objectives of Dialogue Logic in Historical Perspective. (S. Rahman, & H. Rückert, Eds.) *127*(1-2), 225-263.

Lorenz, K. (2009). *Dialogischer Konstruktivismus.* Berlin and New York: De Gruyter.

Lorenz, K. (2010a). *Logic, Language and Method: On Polarities in Human Experiences.* Berlin / New York: De Gruyter.

Lorenz, K. (2010b). *Philosophische Variationen: Gesammelte Aufsätze unter Einschluss gemeinsam mit Jürgen Mittelstrass greschrievener Arbeiten zu Platon und Leibniz.* Berlin / New York: De Gruyter.

Lorenz, K., & Mittelstrass, J. (1967). On Rational Philosophy of Language. The Programme in Plato's Cratylus Reconsidered. *Mind, 76*(301), 1-20.

Lorenzen, P. (1955). *Einführung in die operative Logik und Mathematik.* Berlin: Springer.

Lorenzen, P. (1958). Logik und Agon. *Arti del XII Congresso Internationale de Filosofia*, 187-194.

Lorenzen, P. (1969). *Normative Logic and Ethics.* Mannheim / Zürich: Bibliographisches Institut.

Lorenzen, P., & Lorenz, K. (1978). *Dialogische Logik.* Damstadt: Wissenschaftliche Buchgesellschaft.

Lorenzen, P., & Schwemmer, O. (1975). *Konstruktive Logik, Ethik und Wissenschaftstheorie* (2nd ed.). Mannheim: Bibliographisches Institut.

Luce, J. V. (1969). Plato on Truth and Falsity in Names. *The Classical Quaterly, 19*(2), 222-232.

Lukasiewicz, J. (1957). *Aristotle's Syllogistik from the Standpoint of Modern Formal Logic* (2nd ed.). Oxford: Clarendon Press.

Magnier, S. (2013). *Approche dialogique de la dynamique épistémique et de la condition juridique.* London: College Publications.

Magnier, S. (2015). Suspensive Condition and Dynamic Epistemic Logic: A Leibnizian Survey. In M. Armgardt, P. Canivez, & S. Chassagnard-Pinet, *Past and Present Interactions in Legal Reasoning and Logic* (pp. 71-94). Dordrecht: Springer.

Marion, M. (2006). Hintikka on Wittgenstein: from Language Games to Game Semantics. In T. Aho, & A.-V. (. Pietarinen, *Truth and Games: Essays in Honour of Gabriel Sandu* (pp. 237-256). Helsinki: Acta Philosophica Fennica.

Marion, M. (2009). Why Play Logical Games? In L. a. Games: Unifying Logic, O. Majer, A. V. Pietarinen, & T. Tulenheimo (Eds.). Dordrecht: Springer.

Marion, M. (2010). Between saying and Doing: From Lorenzen to Brandom and Back. In P. E. Bour, M. Rebuschi, & L. Rollet (Eds.), *Constructions: Essays in Honour of Gerhard Heinzmann* (pp. 489-497). London: College Publications.

Marion, M., & Rückert, H. (2015). Aristotle on universal quantification: a study from the perspective of game semantics. *History and Philosophy of Logic, 37*(3), 201-209.

Martin-Löf, P. (1971). Hauptsatz for the Intuitionistic Theory of Iterated Inductive Definitions. In J. E. Fenstad (Ed.), *Proceedings of the Second Scandinavian Logic Symposium* (pp. 179-216). Amsterdam: North-Holland.

Martin-Löf, P. (1975a). About Models for Intuitionistic Type Theories and the Notion of Definitional Equality. In S. Kanger (Ed.), *Proceedings of the Third Scandinavian Logic Symposium* (pp. 81-109). Amsterdam: North-Holland.

Martin-Löf, P. (1975b). An Intuitionistic Theory of Types: Predicative Part. In H. E. Rose, & J. C. Shepherdson (Eds.), *Logic Colloquium '73* (pp. 73-118). Amsterdam: North-Holland.

Martin-Löf, P. (1982). Constructive Mathematics and Computer Programming. In J. L. Cohen, & J. Los (Eds.), *Logic, Methodology and Philosophy of Science VI, 1979* (pp. 153-175). Amsterdam: North-Holland.

Martin-Löf, P. (1984). *Intuitionistic Type Theory. Notes by Giovanni Sambin of a Series of Lectures given in Padua, June 1980.* Naples: Bibliopolis.

Martin-Löf, P. (1992). Substitution Calculus. *http://archive-pml.github.io/martin-lof/pdfs/Substitution-calculus-1992.pdf*. Lecture notes.

Martin-Löf, P. (1996). On the Meanings of the Logical Constants and the Justifications of the Logical Laws. *Nordic Journal of Philosophical Logic, 1*, 11-60.

Martin-Löf, P. (2006). 100 Years of Zermelo's Axiom of Choice: What was the Problem with it? *The Computer Journal, 49*(3), 345-350.

Martin-Löf, P. (2011). When did 'judgement' come to be a term of logic? *Lecture held at Ecole Normale Supérieure 14 October 2011.* Video recording available: http://savoirs.ens.SR//expose.php?id=481.

Martin-Löf, P. (2014). Truth of Empirical Propositions. *Lecture held at the University of Leiden, February 2014.* Transcription by: Amsten Klev.

Martin-Löf, P. (2015). Is Logic Part of Normative Ethics? *Lecture Held at the research Unity Sciences, Normes, Décisions (FRE 3593), Paris, May 2015.* Transcription by:: Amsten Klev.

Martin-Löf, P. (2017a). Assertion and Request. *Lecture held at Oslo, 2017.* Transcription by Ansten Klev.

Martin-Löf, P. (2017b). Assertion and Request. *Lecture held at Stockholm.* Transcription by Ansten Klev.

McDowell, J. (2004). Sellars and the Space of Reasons. *Talk in Cape-town.* http://www.pitt.edu/~brandom/me-core/downloads/McD%20Cape%20Town%20talk--Sellars%20EPM.doc.

McDowell, J. (2009). *Having the World in View: Essays on Kant, Hegel, and Sellars.* Harvard: Harvard University Press.

Netz, R. (1999). *The Shaping of Deduction in Greek Mathematics: A study incognitive history.* Cambridge: Cambridge University Press.

Netz, R. (2005). The aesthetics of mathematics: A study. In P. Mancosu, K. F. JØrgensen, & S. A. Pedersen, *Visualization, Explanation and Reasoning Styles in Mathematics* (pp. 251-293). Dordrecht: Springer.

Netz, R. (2009). *Ludic Proof: Greek Mathematics and the Alexandrian Aesthetic.* Cambridge: Cambridge University Press.

Nordström, B., Petersson, K., & Smith, J. M. (1990). *Programming in Martin-Löf's Type Theory: An Introduction.* Oxford: Oxford University Press.

Nordström, B., Petersson, K., & Smith, J. M. (2000). Martin-Löf's Type Theory. In S. Abramsky, D. Gabbay, & T. S. Maibaum (Eds.), *Handbook of Logic in Computer Science* (Vol. 5 : Logic and Algebraic Methods, pp. 1-37). Oxford: Oxford University Press.

Nzokou, G. (2013). *Logique de l'argumentation dans les traditions orales africaines.* London: College Publications.

Paseau, A. (2011). Proofs of the Compactness Theorem. *History and Philosophy of Logic, 31*(1), 73-98.

Peregrin, J. (2014). *Inferentialism. Why Rules Matter.* New York: Plagrave MacMillan.

Piecha, T. (2012). *Formal Dialogue Semantics for Definitional Reasoning and Implications as Rules*. Faculty of Science, University of Tübingen, Phd thesis: avalable online: http://nbn-resolving.de/urn:nbn:de:bsz:21-opus-63563.

Piecha, T., & Schröder-Heister, P. (2011). Implications as Rules in Dialogical Semantics. In M. Pelis, & V. Puncochar (Eds.), *The Logica Yearbook 2011* (pp. 211-225). London: College Publications.

Plato. (1997). *Plato. Complete Works.* (J. M. Cooper, Trans.) Indianapolis IN: Hackett.

Poincaré, H. (1905). *La valeur de la science.* Paris: Flammarion.

Poincaré, H. (1907). *The Value of Science.* (G. B. Halstead, Trans.) New York: The Science Press.

Popek, A. (2012). Logical Dialogues from Middle Ages. In C. Barés Gómez, S. Magnier, & F. J. Salguero (Eds.), *Logic of Knowledge. Theory and Applications.* (pp. 223-244). London: College Publications.

Prawitz, D. (1965). *Natural Deduction.* Stockholm: Almqvist & Wiksell.

Primiero, G. (2008). *Information and Knowledge.* Dordrecht: Springer.

Prior, A. (1949). *Logic and the Basis of Ethics.* Oxford: Clarendon Press.

Prior, A. (1960). The Runabout Inference-ticket. *Analysis, 21*, 38-39.

Quine, W. V. (1969). *Ontological Relativity and Other Essays.* New York: Columbia University Press.

Rahman, S. (1993). *Über Dialogue, Prologische Kategorien und andere Seltenheiten.* Frankfurt, Paris, New York: P. Lang.

Rahman, S. (2012). Negation in the Logic of First Degree Entailment and Tonk. A Dialogical Study. In S. Rahman, G. Primiero, & M. (. Marion, *The Realism-Antirealism Debate in the Age of Alternative Logics* (pp. 213-250). Springer Netherlands.

Rahman, S. (2015). On Hypothetical Judgements and Leibniz's Notion of Conditional Right. In M. Armgardt, P. Canivez, & S. (. Chassagnard-Pinet, *G. W. Leibniz: Past and Present Interactions in Legal Reasoning* (pp. 109-167). Dordrecht: Springer.

Rahman, S. (2015). On Hypothetical Judgements and Leibniz's Notion of Conditional Right. In M. Armgardt, P. Canivez, & S. Chassagnard-Pinet, *G. W. Leibniz: Past and Present Interactions in Legal Reasoning* (pp. 109-168). Dordrecht: Springer.

Rahman, S. (2017). Boole Algebra in a Contemporary Setting. Boole-Operations, Types. *Talk in the workshop in Honour of Bachir Diagne, Dakar, 20-12-2017.* https://www.academia.edu/34728996/Boole_Algebra_in_a_Contemporary_Setting._Boole-Operations_Types_as_Propositions_and_Immanent_Reasoning._Draft_of_a_contribution_for_a_conference_in_honour_of_Souleyman_Bachir_Diagne._Dec._2017.

Rahman, S., & Clerbout, N. (2013). Constructive Type Theory and the Dialogical Approach to Meaning. *The Baltic International Yearbook of Cognition, Logic and Communication: Games, Game Theory and Game Semantics, 11*, 1-72.

Rahman, S., & Clerbout, N. (2015). Constructive Type Theory and the Dialogical Turn : A New Start for the Erlanger Konstruktivismus. In J. Mittelstrass, & C. von Bülow (Eds.), *Dialogische Logik* (pp. 91-148). Münster: Mentis.

Rahman, S., & Iqbal, M. (2018). Unfolding parallel reasoning in islamic jurisprudence (I). Epistemic and Dialectical Meaning within Abū Isḥāq al-Shīrāzī's System of Co-Relational Inferences of the Occasioning Factor. *Cambridge Journal of Arabic Sciences and Philosophy, 28*, 67-132.

Rahman, S., & Keiff, L. (2005). On How to be a Dialogician. In D. Vanderveken (Ed.), *Logic, Thought and Action* (pp. 359-408). Dordrecht: Kluwer.

Rahman, S., & Keiff, L. (2010). La Dialectique entre logique et rhétorique. *Revue de métaphysique et de morale, 66*(2), 149-178.

Rahman, S., & Lion, C. (2018). Aristote et la question de la complétude. Le modèle formel de Kurt Ebbinghaus. *Philosophie Antique*, accepted: in print.

Rahman, S., & Redmond, J. (2015). A Dialogical Frame for Fictions as Hypothetical Objects. *Filosofia Unisinos, 16*(1), 2-21.

Rahman, S., & Redmond, J. (2016). Armonía Dialógica: tonk, Teoría Constructiva de Tipos y Reglas para Jugadores Anónimos. (Dialogical Harmony: Tonk, Constructive Type Theory and Rules for Anonymous Players). *Theoria. An International Journal for Theory, History and Foundations of Science, 31*(1), 27-53.

Rahman, S., & Rückert, H. (Eds.). (2001). Special Volume Synthese 127. *New Perspectives in Dialogical Logic*. Dordrecht: Springer.

Rahman, S., & Tulenheimo, T. (2009). From Games to Dialogues and Back: Towards a General Frame for Validity. In O. Majer, A. Pietarinen, & T. Tulenheimo (Eds.), *Games: Unifying Logic, Language and Philosophy* (pp. 153-208). Dordrecht: Springer.

Rahman, S., Clerbout, N., & Jovanovic, R. (2015). The Dialogical Take on Martin-Löf's Proof of the Axiom of Choice. *South American Journal of Logic, 1*(1), 179-208.

Rahman, S., Clerbout, N., & Keiff, L. (2009). On Dialogues and Natural Deduction. In G. Primiero, & S. Rahman (Eds.), *Acts of Knowledge: History, Philosophy and Logic: Essays Dedicated to Göran Sundholm* (pp. 301-336). London: College Publications.

Rahman, S., Clerbout, N., & McConaughey, Z. (2014). On Play Objects in Dialogical Games. Towards a Dialogical Approach to Constructive Type Theory. In P. Allo, & V. v. Kerkhove (Eds.), *Modestly Radical or Radically Modest. Festschrift for Jean-Paul van Bendegem* (pp. 127-154). London: College Publications.

Rahman, S., Granström, J., & Salloum, Z. (2014). Ibn Sina's Approach to Equality and Unity. *Cambridge Journal for Arabic Sciences and Philosophy, 4*(2), 297-307.

Rahman, S., Redmond, J., & Clerbout, N. (2017).

Ranta, A. (1988). Propositions as Games as Types. *Syntese, 76*, 377-395.

Ranta, A. (1994). *Type-Theoretical Grammar.* Oxford: Clarendon Press.

Read, S. (1995). *Thinking about Logic. An Introduction to the Philosophy of Logic.* Oxford University Press.

Read, S. (2008). Harmony and Modality. In C. Dégremont, L. Kieff, & H. (. Rückert, *Dialogues, Logics and Other Strange Things: Essays in Honour of Shahid Rahman* (pp. 285-303). London: College Publications.

Read, S. (2010). General Elimination Harmony and the Meaning of the Logical Constants. *Journal of Philosophical Logic, 39*, 557-576.

Redmond, J. (2010). *Logique dynamique de la fiction. Pour une approche dialogique.* London: College Publications.

Redmond, J., & Fontaine, M. (2011). *How to Play Dialogues: An Introduction to Dialogical Logic.* London: College Publications.

Redmond, J., & Rahman, S. (2016). Armonía Dialógica: tonk Teoría Constructiva de Tipos y Reglas para Jugadores Anónimos. *Theoria, 31*(1), 27-53.

Rückert, H. (2011a). *Dialogues as a Dynamic Framework for Logic.* London: College Publications.

Schröder-Heister, P. (1984). A Natural Extension of Natural Deduction. *Journal of Symbolic Logic, 49*, 1284-1300.

Schröder-Heister, P. (2008). Lorenzen's Operative Justification of Intuitionistic Logic. In M. van Atten, P. Boldini, M. Bourdeau, & G. Heinzmann (Eds.), *One Hundred Years of Intuitionism (1907-2007)* (pp. 214-240). Basel: Birkhäuser.

Sellars, W. (1956). Empiricism and the Philosophy of Mind. In H. Feigl, & M. (. Scriven, *Minnesota Studies in the Philosophy of Science* (pp. 253-329). Minneapolis, MN: University of Minnesota Press.

Sellars, W. (1991). *Science Perception and Reality.* Atascadero-California: Ridgeview Publishing Company.

Smith, J. (1988). The Independence of Peano's Fourth Axiom from Martin-Löfs Type Theory Without Universes. *Journal of Symbolic Logic, 53*(3), 840-845.

Smith, R. (1982). What is Aristotelian Ecthesis? *History and Philosophy of Logic, 3,* 113-127.

Smith, R. (1989). *Aristotle's Prior Analytics.* Indianapolis: Hackett.

Smullyan, R. (1968). *First-Order Logic.* New York: Springer-Verlag.

Sundholm, G. (1986). Proof Theory and Meaning. In D. Gabbay, & F. Guenthner.

Sundholm, G. (1997). Implicit Epistemic Aspects of Constructive Logic. *Journal of Logic, Language and Information, 6*(2), 191-212.

Sundholm, G. (1998). Inference versus Consequence. In T. Childers (Ed.), *The Logica Yearbook 1997* (pp. 26-36). Prague: Filosofía.

Sundholm, G. (2001). A Plea for Logical Atavism. In O. Majer (Ed.), *The Logica Yearbook 2000* (pp. 151-162). Prague: Filosofía.

Sundholm, G. (2006). Semantic Values for Natural Deduction Derivations. *Synthese, 148,* 623-638.

Sundholm, G. (2009). A century of judgement and inference, 1837-1936: Some strands in the development of logic. In L. (. Haaparanta, *The Development of Modern Logic* (pp. 264-317). Oxford: Oxford University Press.

Sundholm, G. (2012). Inference versus Consequence Revisited: Inference, Conditional, Implication. *Syntese, 187*, 943-956.

Sundholm, G. (2013a, December 2-3). Inference and Consequence as an Interpreted Language.

Sundholm, G. (2013b). Containment and Variation. Two Strands in the Development of Analyticity from Aristotle to Martin-Löf. In M. van der Schaar (Ed.), *Judgement and Epistemic Foundation of Logic* (pp. 23-35). Dordrecht: Springer Netherlands.

Sundholm, G. (2016). Independence Friendly Language is First Order After All? *Logique et Analyse,* forthcoming.

Sundholm, G. (2017). The Neglect of Epistemic Considerations in Logic: The Case of Epistemic Assumptions. *forthcoming*.

Tasistro, A. (1993). *Formulation of Martin-Löf's Theory of Types with Explicit Substitutions*. Gothenburg: Master's thesis, Chalmers University of Technology.

The Univalent Foundations Program. (2013). *Homotopy Type Theory: Univalent Foundations of Mathematics.* Retrieved from http://homotopytypetheory.org/book

Thiercelin, A. (2009). *La théorie juridique leibnizienne des conditions : ce que la logique fait au droit (ce que le droit fait à la logique).* Université de Lille: Ph. D. thesis.

Thiercelin, A. (2010). Epistemic and Practical Aspects of Conditionals in Leibnizs Legal Theory of Conditions. In D. M. Gabbay, P. Canivez, S. Rahman, & A. Thiercelin, *Approaches to Legal Rationality* (pp. 202-217). Springer.

Thompson, S. (1991). *Type Theory and Functional Programming.* Boston: Addison-Wesley.

Trafford, J. (2017). *Meaning in Dialogue. An Interactive Approach to Logic and Reasoning.* Dordrecht: Springer.

Troelstra, A., & van Dalen, D. (1988). *Constructivism in Mathematics.* Amsterdam: North-Holland.

Tulenheimo, T. (2017). Intentional Objects as World Lines. In T. Tulenheimo, *Objects and Modalities. A Study in the Semantics of Modal Logic* (pp. 85-112). Springer International.

van Heijenoort, J. (1967). Logic as Calculus and Logic as Language. *Synthese, 17,* 324-330.

Wittgenstein, L. (1922). *Tractatus Logico-Philosophicus.* Lille: Kegan Paul.

Zermelo, E. (1904). Neuer Beweis, dass jede Menge Wohlordnung werden kan (Aus einem an Hern Hilbert gerichteten Briefe). *Mathematische Annalen, 59,* 514-516.

Zermelo, E. (1908a). Neuer Beweis für die Möglichkeit einer Wohlordnung. *Mathematische Annalen*, 107-128.

Zermelo, E. (1908b). Untersuchungen über die Grundlagen der Mengenlehre, I. *Mathematische Annalen*, *65*, 261-281.

Zermelo, E. (1930). Über Grenzzahlen und Mengenbereiche. *Fundamenta Mathematicae, 16*, 29-47.

INDEX OF TABLES

List of Names

Aczel
Almog
Aristotle
Armagardt
Austin
Baire
Bar-Hillel
Barrio
Barnes
Batens
Beirlaen
Bell
Bishop
Bobenrieth
Borel
Brandom
Breckenridge
Brouwer
Cardascia
Carnap
Carnielli
Church
Clerbout
Conigilio
Cooper
Corcoran
Cratylus
Crubellier
Curry
Dango
Diaconescu
D'Ottaviano
Dummett
Dutilh Novaes
Dybjer
Ebbinghaus
Eckoubili
Felscher
Fine
Fischer
Fontaine
Fraenkel
Frege
Garner
Gentzen
Ginzurg
Girard
Goodman

List of Subjects

LIST OF EXERCISES AND EXAMPLES

	Negation ¬	**Negation ⊃⊥**	**Sections**
1	$(A \vee B) \supset (B \vee A)$		III.4
2	$(A \wedge B) \supset A$		III.4
3	$A \vee \neg A$	$A \vee (A \supset\bot)$	III.4; IV.5; XI.1
4	$\neg\neg A \supset A$	$\big((A \supset\bot) \supset\bot\big) \supset A$	III.4; IV.5
5	$\big((A \vee B) \wedge \neg A\big) \supset B$	$\big((A \vee B) \wedge (A \supset\bot)\big) \supset B$	V.3
6	$\Big(\big((A \vee (B \wedge C)) \wedge \neg A\big) \supset (B \wedge C)\Big) \wedge \big((D \vee E) \supset (D \vee E)\big)$	$\Big(\big((A \vee (B \wedge C)) \wedge (\mathrm{A} \supset\bot)\big) \supset (B \wedge C)\Big) \wedge \big((D \vee E) \supset (D \vee E)\big)$	V.3
7	$(A \wedge B) \supset (B \wedge A)$		VI.4
8	$(\forall x{:}\, A)\big(B(x) \supset C(x)\big){:}\, prop\big[A{:}\, set\,; B(x){:}\, prop\,; C(x){:}\, prop\ [x{:}\, A]\big]$		VII.1
9	$(\forall x{:}\, D)Q(x) \supset Q(x)$		VII.3; VII.5
10	$B \vee A\ [c{:}\, A \vee B]$		VII.6.1
11	$\neg\neg(A \vee \neg A)$	$\big((A \vee (A \supset\bot)) \supset\bot\big) \supset\bot$	III.4; IV.5; VII.6.2
12	$\neg(A \wedge \neg A)$	$\big(A \wedge (A \supset\bot)\big) \supset\bot$	VII.6.3
13	$\big((A \supset B) \supset A\big) \supset A$		V.2.1; VII.6.4
14	$(A \wedge \neg B) \supset \neg(A \supset B)$	$\big(A \wedge (B \supset\bot)\big) \supset \big((A \supset B) \supset\bot\big)$	VII.6.5
15	$(A \wedge B) \wedge C\ [c{:}\, A \wedge (B \wedge C)]$		VII.6.6; VII.7.4; IX.5
16	$(B \wedge A) \supset C\ [c{:}\, (A \wedge B) \supset C]$		VII.6.7; VII.7.4; IX.5
17	$\neg(\forall x{:}\, D)A(x)[(\exists x{:}\, D)\neg A(x)]$	$(\forall x{:}\, D)A(x) \supset\bot\ [(\exists x{:}\, D)(A(x) \supset\bot)$	VII.6.8
18	$(\exists x{:}\, D)A(x) \wedge (\exists x{:}\, D)B(x)\big[(\exists x{:}\, D)\big(A(x) \wedge B(x)\big)\big]$		VII.6.9
19	$(\exists x{:}\, D)(\exists y{:}\, D)\big(A(x) \supset B(y)\big)\ [A(a) \supset B(b); a{:}\, D\,; b{:}\, D]$		VII.6.10
20	$(\exists x{:}\, D)B(x) \wedge (\exists x{:}\, D)P(x)\ [B(a) \wedge P(b); a{:}\, D\,; b{:}\, D]$		VII.6.11
21	$(\exists x{:}\, D)(\exists y{:}\, D)A(x,y)[(\exists x{:}\, D)A(x,x)]$		VII.6.12
22	$(\forall x{:}\, D)(\forall y{:}\, D)\big(A(x,y) \wedge A(y,x)\big)\ [(\forall x{:}\, D)(\forall y{:}\, D)A(x,y)]$		VII.6.13VII.4
23	$(\exists x{:}\, D)\big(A(x) \supset (\forall x{:}\, D)A(x)\big)$ $\big[\big((\exists x{:}\, D)\neg A(x)\big) \vee \big((\forall x{:}\, D)A(x)\big)\big]$	$(\exists x{:}\, D)\big(A(x) \supset (\forall x{:}\, D)A(x)\big)$ $\big[\big((\exists x{:}\, D)(A(x) \supset\bot)\big) \vee \big((\forall x{:}\, D)A(x)\big)\big]$	VII.6.14
24	$(\forall x{:}\, A)\big(\exists y{:}\, B(x)\big)C(x,y) \supset \big(\exists f{:}\, (\forall x{:}\, A)B(x)\big)C\big(x, f(x)\big)$		VIII.2
25	$\big(\forall x : \mathbf{Bool}\big)\big(\boldsymbol{Id}(\mathbf{Bool}, x, \boldsymbol{yes}) \vee\ \boldsymbol{Id}(\mathbf{Bool}, x, \boldsymbol{no})\big)$		X.2.3
26	$\boldsymbol{Id}(\mathbf{Bool}, \boldsymbol{yes}, \boldsymbol{no}) \supset\bot \Big[\boldsymbol{n}^{\mathbf{0}}, \boldsymbol{n}^{\mathbf{1}}{:}\, \boldsymbol{\mathcal{U}};\ \mathbb{N}^{\mathbf{1}};\ \boldsymbol{yes}, \boldsymbol{no} :\ \mathbf{Bool};\ \boldsymbol{G}(x){:}\, \boldsymbol{\mathcal{U}}\ [x :\ \mathbf{Bool}]\Big]$		X.3.3
27	$A \wedge \neg A$	$A \wedge (A \supset\bot)$	XI.1