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ABSTRACT

We prove that under mild conditions individually rational Pareto optima will exist even in presence of non-convex preferences. We consider decision makers dealing with a countable flow of payoffs or choosing among financial assets whose outcomes depend on the realization of a countable set of states of the world. Our conditions for the existence of Pareto optima can be interpreted as a requirement of impatience in the first context and of some pessimism or not unrealistic optimism in the second context. A non-existence example is provided when, in the second context, some decision maker is too optimistic. We furthermore show that at an individually rational Pareto optimum at most one strictly optimistic decision maker will avoid ruin at each state or date. Considering a risky context this entails that even if risk averters will share risk in a comonotonic way as usual, at most one classical strong risk lover will avoid ruin at each state or date. Finally some examples illustrate circumstances when a risk averter could take advantage of sharing risk with a risk lover rather than with a risk averter.

1 Introduction

Our aim is to study how the interaction between optimism and pessimism (and also risk propensity and risk aversion) could affect the allocation of economic resources. In order to do this, we will analyze which are the properties of the socially desired allocations, in the sense of being Pareto optimal (PO) for which any decision maker (DM) of the economy is not worse than with the initial endowments of goods. One should remark that the presence of non-convex preferences can cause technical issues to ensure the existence of general equilibrium for the respective economy, but even in these cases, it is possible to ask about the existence of individually rational Pareto optimal (IRPO) - which is a lesser demanding concept than equilibrium, as it does not require the existence of a price system - and on the properties that should be verified by such allocations.

We will model a pure exchange economy as did Bewley (1972), with a finite number of DMs and an infinite dimensional consumption space. More specifically, the consumption space is the set of non-negative real bounded sequences ℓ^∞_+, and, thus, we can encompass the analysis of a decision problem with infinitely many states or with an infinite flow of payments along the time.

This paper has two main results. The first is about the existence of IRPO allocations for economies without the usual requirements as completeness or convexity over decision makers' preferences. A central assumption for this result is that preferences display weak* sequential upper semi-continuity. Actually, we show that in the particular case of ℓ^∞ the standard result in the literature of Aliprantis and Burkinshaw (2003) ensuring the existence of an IRPO allocation remains valid by replacing their assumption of weak* upper semi-continuity by our weaker one. Furthermore, we show that by merely adding the simple needed upper norm-continuity usual condition our central assumption is equivalent to the important notion of upper semi-myopia which was introduced in Brown and Lewis (1981). It turns out that both in the case of an infinite flow of payments or in the case of assets whose outcomes depend on the realization of a countable set of states, upper semi-myopia prevents cases of unreasonable optimism. We give several examples of preferences (e.g. Choquet, à la Yaari, expected utility) that satisfy this assumption and illustrate that its lack (i.e. the presence of some extreme optimistic DM) may lead to economies with no IRPO allocation. Indeed, Araujo (1985) provides examples of economies for which there is no IRPO due to the lack of impatience[1].

We then consider the presence of reasonably strictly optimistic DM. Our second main result, Theorem 2, shows that at IRPO allocations at most one such strictly optimistic DM will always avoid ruin (in the sense of having an interior outcome plan).

Then specializing in the context of risk, we obtain that as in the standard case with finite states and without non-convexities studied by Chateauneuf, Dana, and Tallon (2000), at an IRPO, the allocations of the risk averters are comonotonic. The main feature of risk-sharing between risk averters and risk lovers as settled in Theorem 3, is that for classical strong risk lovers, i.e. those with strictly concave preferences, again ruin could be avoided by at most one of them. We finally illustrate through some examples that a risk averter under some circumstances could take advantage of sharing risk with a risk lover rather than with a risk averter.

The paper is structured as follows: In Section 2 we present sufficient conditions in order to guarantee that there are IRPO and provide the proof of existence. In Section 3 we relate the condition of weak* sequentially upper semi continuity of preferences and the notions of myopia and impatience. Section 4 gives some examples of weak* sequentially upper-semicontinuous preferences. In section 5 we analyze the optimal allocations of strictly optimistic DM and then specialize to the context of risk. Finally, Section 6 aims to give some intuition about circumstances when it is beneficial for a risk averter to share risk with a risk lover.

[1] It is important to remark that, in this work, as in Brown and Lewis (1981), the term impatience is simply a specialization of the broader concept of myopia. Some authors such as Chateauneuf and Rébillé (2004) propose alternative definitions for these two concepts.

2 Existence of IRPO allocations under weak* sequentially upper-semicontinuous preferences

There are some standard results in the literature on the existence of an IRPO allocation for the economies on ℓ^∞ (and other infinite dimensional consumption spaces). For instance, Yannelis (1991) presents a related result[2] for non-ordered preferences on very general consumption spaces under the requirement of a weak form of convexity on these preferences (in the sense that any plan x can not belong to the convex hull of its strict upper contour set). Aliprantis and Burkinshaw (2003)) shows the existence even when preferences on ℓ^∞ may be non-convex under the assumption of weak* upper semi-continuity. In Theorem 1 below, we show that this assumption can be replaced by the weaker assumption of weak* sequential upper semi-continuity[3]. The motivation for this assumption is given in Proposition 1 at Section 3, where it is proved by merely adding simple usual conditions that it is equivalent to the important notion of upper semi-myopia introduced in Brown and Lewis (1981) [4]. [5]

Some readers might wonder why we do not attempt to prove existence of IRPO allocations for the usually greatest class of Mackey sequentially upper-semicontinuous preferences. In fact, it turns out that in the particular case of ℓ^∞ the weak* $\sigma(\ell^\infty, \ell^1)$ sequential convergence and the Mackey $\tau(\ell^\infty, \ell^1)$ sequential convergence coincide, which is an immediate application of the corollary in appendix of (Hervés-Beloso, Moreno-García, Núñez-Sanz, and Páscoa, 2000) that states that "the Mackey topology and the weak* topology

[2]More precisely, the Theorem 4.1 of this work shows sufficient conditions for the existence of extreme α-core allocations, a notion that in our framework coincides with the definition of IRPO allocations when preferences are complete.

[3]One will find in Appendix A some brief recalls concerning ℓ^∞ the Banach space of real bounded sequences, and also the weak* topology and the Mackey topology, which are respectively the coarsest and the finest topology on ℓ^∞ for which the dual is ℓ^1 the Banach space of absolutely convergent real sequences.

[4]In this work, they refer to the concept of upper semi-myopia as strong myopia.

[5]In Section 4 one will find an example of preferences which are weak* sequentially upper semi-continuous but not weak* upper semi-continuous. This illustrates that our result strengthens Aliprantis and Burkinshaw's one at least in the ℓ^∞ case.

coincide on bounded subsets of ℓ^∞". This justifies our choice of stating our result with the equivalent and more well-known weak* $\sigma(\ell^\infty, \ell^1)$ sequential convergence[6]. For sake of completeness we additionally give in Appendix B a direct and elementary proof of this "Folk Theorem" about equivalence on ℓ^∞ of both sequentially convergences mentioned.

Assumptions

(1) For every $i = 1, \ldots, m$, $\succsim_i$ is a transitive reflexive preorder on ℓ^∞_+.

(2) For every i, every $y_i \in \ell^\infty_+$, $\{x_i \in \ell^\infty_+ \mid x_i \succsim_i y_i\}$ is weak* sequentially closed.

Let $w_i \in \ell^\infty_+$ be the initial endowments of individual $i = 1, ..., m$.

Let $\mathcal{A} = \{(x_1, ..., x_i, ..., x_m) \in (\ell^\infty_+)^m : \sum\limits_{i=1}^{m} x_i = w$ where $w := \sum\limits_{i=1}^{m} w_i\}$ be the set of feasible allocations.

Definition 1. A feasible allocation $(x_1, ..., x_i, ..., x_m)$ is said to be:

1. *Pareto optimal* (PO) if there is no other allocation $(z_1, ..., z_m)$ satisfying $z_i \succsim_i x_i \; \forall i$ and $z_k \succ_k x_k$ for some k.
2. *Individually rational Pareto optimal* (IRPO) if it is PO and $x_i \succsim_i w_i \; \forall i$.
3. The set of all individually rational allocations is denoted $\mathcal{A}_r$, i.e. ,

$$\mathcal{A}_r = \{(x_1, ..., x_m) \in \mathcal{A} : x_i \succsim_i w_i \text{ for each } i\}.$$

By a special case of Theorem 8.12 in Aliprantis and Burkinshaw (2003) (i.e., $L = \ell^\infty, X' = \ell^1$) we have that an IRPO allocation exists if (1) holds and preferences are weak* upper semicontinuous on ℓ^∞_+. One can easily observe that the proof of this result also works if we assume that preferences satisfy the corresponding continuity assumption only on $\mathcal{A}_r$. This observation is of great importance in our model, since due to weak* metrizability of the unit ball in ℓ^∞ it follows that preferences which satisfy assumption (2) are weak* upper semicontinuous on $\mathcal{A}_r$. The details follow in the next proof.

Theorem 1. *Under Assumptions* (1) *and* (2), IRPO *allocations exist.*

[6]Appendix A makes precise that sequential weak* convergence on ℓ^∞ is as tractable as the standard convergence in the Euclidean space $\mathbb{R}^m$ $(m \in \mathbb{N})$.

Proof: By the Banach-Alaoglu Theorem, the Edgeworth box $[0, \omega] = \{x \in \ell^\infty : 0 \leq x \leq \omega\}$ is weak* compact in ℓ^∞. Thus by applying Theorem 8.10 in Aliprantis and Burkinshaw (2003), for $L = \ell^\infty$ and $X' = \ell^1$ we have that $\mathcal{A}$ is $\mathcal{P} = \sigma((\ell^\infty)^m, (\ell^1)^m)$-compact.
Let $\pi = \sup_{n \in \mathbb{N}} w(n)$ and note that $\mathcal{A}_r = \mathcal{A} \cap \prod_{i=1}^{m} (\{x_i \in \ell^\infty_+, x_i \succsim_i w_i\} \cap \bar{B}(0, \pi))$.
Since the weak* topology on $\bar{B}(0, \pi)$ is metrizable from Theorem 6.30 in Aliprantis and Border (2006), it turns out that $\{x_i \in \ell^\infty_+, x_i \succsim_i w_i\} \cap \bar{B}(0, \pi)$ is weak* compact since $\{x_i \in \ell^\infty_+, x_i \succsim_i w_i\}$ is weak* sequentially closed.
Therefore $\prod_{i=1}^{m} (\{x_i \in \ell^\infty_+, x_i \succsim_i w_i\} \cap \bar{B}(0, \pi))$ is $\mathcal{P}$-closed, hence $\mathcal{A}_r$ is $\mathcal{P}$-closed in $\mathcal{A}$. Thus $\mathcal{A}_r$ is $\mathcal{P}$-compact.
The end of the proof is similar to Aliprantis and Burkinshaw's one while using the weaker assumption of weak* sequentially upper semicontinuity. Let us prove now that individually rational Pareto efficient allocations exist.
Let us define a partial preorder $\succsim$ on $\mathcal{A}_r$ by:

$$(x_1, ..., x_i, ..., x_m) \succsim (y_1, ..., y_i, ..., y_m) \text{ if } x_i \succsim_i y_i, \forall\, i = 1, ..., m.$$

So an individually rational Pareto efficient allocation exists if and only if $\mathcal{A}_r$ has a maximal element.
From Zorn's lemma, we know that the preordered set $(\mathcal{A}_r, \succsim)$ will have a maximal element if every totally preordered subset of $(\mathcal{A}_r, \succsim)$ is majorized.
Let (ζ_α) be a totally preordered subset of $(\mathcal{A}_r, \succsim)$ and define, for each α, $C_\alpha = \{t \in \mathcal{A}_r, t \succsim \zeta_\alpha\}$. From weak* sequential upper-semicontinuity of each preference i, it turns out as above that C_α is a $\mathcal{P}$-closed subset of the $\mathcal{P}$-compact set $\mathcal{A}_r$, hence C_α is $\mathcal{P}$-compact. Since (ζ_α) is totally preordered, it comes that the intersection of a finite number of C_α is non-empty. Consequently, $C = \bigcap_\alpha C_\alpha$ is non-empty, and any $t \in \mathcal{C}$ majorizes (ζ_α), so a maximal element does exist and consequently there exist IRPO allocations. ■

3 Existence of individually rational Pareto efficient allocations under upper semi-myopic preferences

We now recall the central notion introduced by Brown and Lewis (1981) (see also Araujo (1985)) under the denomination of strong myopia (or in a sequential context, strong impatience) and quoted also in (Araujo, Novinski, and Páscoa, 2011) under the denomination of upper semi-myopia. In order to do that, let us state a bit of notation: given $x \in \ell_+^\infty$ and $E \subset \mathbb{N}$, x_E denotes the sequence such that $x_E(s) = x(s)$ if $s \in E$ and $x_E(s) = 0$ otherwise. Moreover, E_n is the set of all natural numbers greater than n. Lastly, $\mathbf{1}$ is the real sequence whose terms are all equal to one.

Definition 2. A preference relation $\succsim_i$ on ℓ_+^∞, is upper semi-myopic if $\forall (x, y) \in \ell_+^\infty \times \ell_+^\infty$, $x \succ_i y$, $z \in \ell_+^\infty$, implies that there exists n large enough such that $x \succ_i y + z_{E_n}$.

Note that in case of preferences over a countable flow of payoffs, the previous definition clearly fits to the notion of upper semi-myopia, while in case of preferences over financial assets whose outcomes depend on the realization of a countable set of states of the world the previous definition would model neglecting gains on "small events" i.e. some pessimism or not unreasonable optimism.

In other words upper semi-myopia prevents unreasonable optimism since it prevents overweighting outcomes arising in a very far future, in case of a flow of payments, and arising on "small events", in the case of assets whose outcomes depend on the realization of a countable set of states.

We now make precise the relationship between weak* sequentially upper semi-continuity and upper semi-myopia.

Definition 3. A preference relation $\succsim$ on ℓ_+^∞ is upper norm-continuous if for every $y \in \ell_+^\infty$, $\left\{x \in \ell_+^\infty \mid x \succsim y\right\}$ is norm-closed.

Throughout the paper we will assume that the preferences $\succsim_i$ are complete and transitive (i.e. a total preorder) and monotone (i.e. $x, y \in \ell_+^\infty$, $x \geq y \Rightarrow x \succsim y$). It turns out that:

Lemma 1. *Let $\succsim$ on ℓ_+^∞ satisfying the previous hypotheses then (1) implies (2):*

(1) $\succsim$ is weak sequentially upper semi-continuous.*

(2) $\succsim$ is upper norm-continuous and upper semi-myopic.

Proof: Let $x^m \in \ell_+^\infty$, where x^m tends in norm towards $x \in \ell_+^\infty$, denoted $x^m \xrightarrow{\|\cdot\|} x$, it is known (see for instance Brézis (2011)) that $x^m \xrightarrow{w^*} x$, hence $y \in \ell_+^\infty$ and $x^m \succsim y$ $\forall m$ and (1) implies $x \succsim y$, so $\succsim$ is upper norm-continuous.
Let $x, y \in \ell_+^\infty$ with $x \succ y$ and let $z \in \ell_+^\infty$. Assume that for any n, $y + z_{E_n} \succsim x$. Clearly $z_{E_n} \xrightarrow{w^*} 0$ then, from (1), $y + 0 \succsim x$ i.e. $y \succsim x$ a contradition, which completes the proof. ■

For the converse of Lemma 1, let us introduce:

Definition 4. A preference relation $\succsim$ on ℓ_+^∞ is finitely upper norm-continuous if for every $y \in \ell_+^\infty$ and any sequence (x^m), $x^m \in \ell_+^\infty$ such that $x^m \succsim y$ $\forall m$ where

$$x^m(k) = \begin{cases} x^m(k) & k \in K \text{ finite} \subseteq \mathbb{N} \\ x(k) & k \notin K \end{cases}$$

and $\exists\ \lim_{m\to\infty} x^m(k) = x(k) \in \mathbb{R}_+$ $\forall k \in K$ one has: $x \succsim y$.

One obtains:

Lemma 2. *Let $\succsim$ be a monotone total preorder on ℓ_+^∞ then (ii) $\Rightarrow$ (i):*

(i) $\succsim$ is weak sequentially upper semi-continuous.*

(ii) $\succsim$ is finitely upper norm-continuous and upper semi-myopic.

Proof: Assume $\exists\ x \in \ell_+^\infty$, sequence (y^m) such that $y^m \in \ell_+^\infty$ $\forall m$ with $y^m \xrightarrow{w^*} y$, $y \in \ell_+^\infty$, where $y^m \succsim x$ $\forall m$ and $x \succ y$. Since $y^m \xrightarrow{w^*} y$, the sequence is bounded let us say by $k \in \mathbb{R}_+$, hence $\forall n \in \mathbb{N}$: $y^m \leq (y^m(1), ..., y^m(n), k, k, ...)$, so from monotonicity: $x \precsim (y^m(1), ..., y^m(n), k, k, ...)$, and for any fixed n, finitely upper norm-continuity implies $x \precsim (y_1, ..., y_n, k, k, ...)$. But upper semi-myopia implies that there exists $n \in \mathbb{N}$ such that: $(y_1, ..., y_n, y_{n+1} + k, y_{n+2} + k, ...) \prec (y_1, ..., y_n, k, k, ...)$ a contradition with respect to monotonicity which completes the proof. $\blacksquare$

While it is clear from Lemma 1 and Lemma 2 that weak* sequentially upper semi-continuity is characterized by upper semi-myopia and finitely upper norm-continuity, we nevertheless prefer to state the following proposition which underlines the necessity of upper norm-continuity.

Proposition 1. *Let $\succsim$ be a monotone total preorder on ℓ^∞_+. Then, $\succsim$ is weak* sequentially upper semi-continuous if and only if $\succsim$ is upper norm-continuous and upper semi-myopic.*

Note that even for a monotone total preorder, *upper semi-myopia is not sufficient for weak* sequential upper semi-continuity* as illustrated by the Example 1 below.

Example 1. Let $\succsim$ be the lexicographical order on ℓ^∞_+. For $x, y \in \ell^\infty_+$ with $x \succ y$ if there exist $k_o \in \mathbb{N}$ such that $x(k) = y(k)$ for $1 \leq k \leq k_0$ and $x(k_0 + 1) > y(k_0 + 1)$, $x \sim y$ if $x(k) = y(k)$ $\forall k \in \mathbb{N}$.

$\succsim$ exhibits upper semi-myopia, actually let $x, y \in \ell^\infty_+$ with $x \succ y$ as defined above, clearly if $z \in \ell^\infty_+$ then $x \succ y + z_{E_n}$ for $n \geq k_0 + 2$.

$\succsim$ is not upper norm-continuous, hence not weak* sequentially upper semi-continuous (see Proposition 1). Actually consider $x^m \succ y$, with $x^m, y \in \ell^\infty_+$ where $x^m(1) = y(1) + \frac{1}{m}$, $x^m(2) = x(2) < y(2)$, $x^m(k) = y(k)$ $\forall k \geq 3$, clearly $x^m \xrightarrow{\|\cdot\|} x$ where $x(1) = y(1)$ but $x \prec y$. $\square$

Note also that even for a monotone total preorder *upper norm-continuity does not imply weak* sequential upper semi-continuity* as illustrated by Example 2 below.

Example 2. Let z be a positive sequence of ℓ^∞_+ ($z_k > 0$ for all $k \in \mathbb{N}$) and define the (utility) function U from ℓ^∞_+ to $\mathbb{R}_+$ by $U(x) = \sup_{k \in \mathbb{N}}(z_k \cdot x_k)$. Define $\succsim$ on ℓ^∞_+ by $x, y \in \ell^\infty_+$, $x \succsim y \iff U(x) \geq U(y)$. Observe that $\succsim$ is upper norm-continuous since U is norm-continuous. In fact, let $\varepsilon > 0$ and take $x, y \in \ell^\infty_+$ such that $\|x - y\| \leq \varepsilon$. Therefore $y_k - \varepsilon \leq x_k \leq y_k + \varepsilon$ for all k, implying that $y_k \cdot z_k - \varepsilon \cdot z_k \leq x_k \cdot z_k \leq y_k \cdot z_k + \varepsilon \cdot z_k$ for all k. Hence $|\sup_k(x_k z_k) - \sup_k(y_k z_k)| \leq \varepsilon \|z\|$ i.e., $\|U(x) - U(y)\| \leq \varepsilon \|z\|$.

Let us prove now that if z does not converge to 0, then $\succsim$ is not weak* sequential upper semi-continuous.

Consider $\mathcal{P} = \{x \in \ell^\infty_+ : U(x) \geq 1\}$ and let us prove that $\mathcal{P}$ is not weak*-sequentially closed. Since z does not converge to 0, there exists a subsequence $(z_{\varphi(k)})$ and $\underline{t} > 0$ such that for all k, $z_{\varphi(k)} \geq \underline{t}$. We consider the sequence (x^k) in ℓ^∞_+ defined by $x^k(j) = 0$ if $j \neq \varphi(k)$ and $x^k(\varphi(k)) = \frac{1}{\underline{t}}$. Then for all k, $u(x^k) = \frac{z_{\varphi(k)}}{\underline{t}} \geq 1$, so $x^k \in \mathcal{P}$.

But (x^k) is bounded and $\lim_{k\to\infty} x^k(j) = 0$ for all $j \in \mathbb{N}$, so $x^k \xrightarrow{w^*} 0$, but $0 \notin \mathcal{P}$ which completes the proof. □

In accordance with Proposition 1, we now assume throughout the paper that the preferences satisfy Assumption A.1:

Assumption A.1: Preferences $\succsim_i$ on ℓ^∞_+ are complete, transitive, monotone and upper norm-continuous.

The Corollary 1 below thus straightforwardly characterizes a large class of preferences for which one can guarantee the existence of IRPO allocations.

Corollary 1. *If all the preferences $\succsim_i$ $i = 1, ..., m$ of the decision makers satisfy A.1 then are equivalent:*

1. *$\succsim_i$ is weak* sequentially upper semi-continuous.*

2. *$\succsim_i$ is upper semi-myopic.*

Furthermore if every $\succsim_i$ satisfy (1) or equivalent (2) then there exist individually Pareto efficient allocations.

Proof: Immediate from Proposition 1 and Theorem 1.

We now give in Section 4 some examples of suitable preferences for our purpose i.e. of weak* sequentially upper semi-continuous preferences satisfying A.1, either checking directly the required properties above, or using Corollary 1, or else Proposition 1.

4 Weak* sequentially upper-semicontinuous preferences

4.1 Existence of weak* sequentially upper-semicontinuous preferences on ℓ^∞_+ which are not weak* upper-semicontinuous

We first illustrate the fact that our Theorem 1 is a true improvement of Aliprantis and Burkinshaw (2003) - at least in the case of ℓ^∞ - by exhibiting weak* sequentially upper semi-continuous preferences on ℓ^∞_+ which are not weak* upper-semicontinuous.

Example 3. Consider again as in the Example 2 the (utility) function U from ℓ^∞_+ to $\mathbb{R}_+$ defined by $U(x) = \sup_{k\in\mathbb{N}}(z_k x_k)$ where z is a positive sequence of ℓ^∞ which now converges to 0, but is assumed not to be in ℓ^1 (that is, $\sum_{k=1}^\infty z_k = +\infty$).
Note that such a utility function is meaningful particularly when valuing flow of payoffs, since it models a decision-maker which discounts "severely" the future (since $\lim_k z_k = 0$), while expressing some minimal requirement of continuous behaviour ($\succsim$ is upper norm-continuous, from Example 2).
Let us prove first that the monotone total preorder $\succsim$ induced by U on ℓ^∞_+ is weak* sequentially upper semi-continuous. It is enough to show that for $a > 0$, the set $\mathcal{P} = \{x \in \ell^\infty_+ : U(x) \geq a\}$ is sequentially weak*-closed. Let (x^m) be a sequence of elements of $\mathcal{P}$ weak* converging to $x \in \ell^\infty_+$. Then, the sequence (x^m) is bounded and for all $k \in \mathbb{N}$, $\lim_m x^m_k = x_k$.
Let $r > 0$ be such that $x^m \in \overline{B}_\infty(0, r)$ for all m, which means that for all m and for all k, $x^m(k) \leq r$. Let $\underline{k} \in \mathbb{N}$ be such that $r < \frac{a}{2z_k}$ for all $k \geq \underline{k}+1$. So for all $k \geq \underline{k}+1$, for all m we have $z_k x^m_k < \frac{a}{2}$. For all m, since $x^m \in \mathcal{P}$, there exists $k \leq \underline{k}$ such that $z_k \cdot x^m_k \geq a$. So, there exists a subsequence $(x^{\varphi(m)})$ of (x^m) and $k_0 \leq \underline{k}$ such that for all m, $\ z_{k_0} \cdot x^{\varphi(m)}_{k_0} \geq a$. Consequently, since $x_{k_0} = \lim_m x^{\varphi(m)}_{k_0}$, one concludes that $z_{k_0} \cdot x_{k_0} \geq a$, hence $x \in \mathcal{P}$, which means that $\mathcal{P}$ is sequentially weak*-closed.
We now prove that $\succsim$ is not weak* upper semi-continuous. It is enough to show that taking $a = 1$, $\mathcal{P}$ is not weak*-closed. This will be performed by showing that 0 belongs to the weak* closure of $\mathcal{P}$ whereas $0 \notin \mathcal{P}$. Let G be a weak* neighborhood of 0. There

exists $\varepsilon > 0$ and a finite family $(\alpha^i)_{i=1}^I$ of elements of ℓ^1 such that $V = \{x \in \ell^\infty \mid \forall i = 1, ..., I, |\alpha^i(x)| < \varepsilon\} \subset G$.
Let us prove by contraposition that $V \cap \mathcal{P} \neq \emptyset$, which implies $G \cap \mathcal{P} \neq \emptyset$. For all $m \in \mathbb{N}$, let x^m defined by $x_k^m = 0$ if $k \neq m$ and $x_k^m = \frac{1}{z_m}$. Note that $x^m \in \mathcal{P}$. If $V \cap \mathcal{P} = \emptyset$, x^m does not belong to V, so, there exists $i \in \{1, ..., I\}$ such that $\dfrac{|\alpha_m^i|}{z_m} \geq \varepsilon$. In others words, for all $m \in \mathbb{N}$, $\sup\limits_{i=1,...,I} \{|\alpha_m^i|\} \geq \varepsilon z_m$.
So, the sequence $\alpha = \left(\sup_{i=1,...,I}\{\alpha_m^i\}\right)_m$ is not in ℓ^1 since it is larger for the pointwise order than εz, which is not in ℓ^1. This is in contradiction with the fact that the pointwise supremum of a finite family of sequences in ℓ^1 is also in ℓ^1. □

4.2 Some others examples of weak* sequentially upper-semi-continuous preferences

In order to model preferences under uncertainty or even preferences on flow of payments it has been suggested (see Schmeidler (1989) for the first circumstance and Gilboa (1989) or else Chateauneuf and Rébillé (2004) in the second case) to use the Choquet integral (Choquet (1953)).

In our framework every $x \in \ell_+^\infty$ would be valued through $I(x) = \int x \, d\vartheta$ where ϑ is a capacity on $\mathcal{A} = 2^{\mathbb{N}}$ i.e. a set-function $\vartheta \colon A \in \mathcal{A} \to \vartheta(A) \in [0,1]$ such that $\vartheta(\emptyset) = 0$, $\vartheta(\mathbb{N}) = 1$ and $A, B \in \mathcal{A}$. $A \subset B \Rightarrow \vartheta(A) \leq \vartheta(B)$ [7] and the Choquet integral of x w.r.t. to ϑ is defined as $\int x \, d\vartheta = \int_0^{+\infty} \vartheta(x \geq t) \, dt$.

Proposition 2. *A Choquet preference on ℓ_+^∞ is weak* sequentially upper semi-continuous if and only if ϑ is outer-continuous i.e. $\forall A_n$, $A \in \mathcal{A}$, $A_n \downarrow A \Rightarrow \vartheta(A_n) \downarrow \vartheta(A)$.*

Proof: It is well-known that preferences represented by the Choquet integral satisfy Assumption 1, therefore, from Proposition 1, Choquet preferences $\succsim_i$ on ℓ_+^∞ are weak* sequentially u.s.c. iff $\succsim_i$ is upper semi-myopic. So, let us prove that a Choquet preference on ℓ_+^∞ is upper semi-myopic if and only if ϑ is outer-continuous, i.e. $\forall A_n$, $A \in \mathcal{A}$, $A_n \downarrow A \Rightarrow \vartheta(A_n) \downarrow \vartheta(A)$.

[7] Under uncertainty $\vartheta(A)$ is the subjective evaluation of the likelihood of the event A, while when valuing flow of payments, $\vartheta(A)$ is the weight given by the DM to the time period A.

Outer-continuity is necessary. Actually let $A_n,\ A \in \mathcal{A}$ be such that $A_n \downarrow A$ and assume that $\vartheta(A_n) \downarrow k$ with $k > \vartheta(A)$. Therefore we get $k\,\mathbf{1}_{\mathbb{N}} \succ_i \mathbf{1}_A$.

Let $B_n = A_n \backslash A$, one gets $B_n \downarrow \phi$. So one can find an increasing sequence of integers $n(m)$, increasing with $m \in \mathbb{N}$ such that $\mathbf{1}_{E_m} \geq \mathbf{1}_{B_{n(m)}}$ where $E_m = \{p \in \mathbb{N},\, p \geq m\}$. Under upper semi-myopia $\exists\, m_0$ such that $k\,\mathbf{1}_{\mathbb{N}} \succ_i \mathbf{1}_A + \mathbf{1}_{E_{m_0}}$ but $\mathbf{1}_{E_{m_0}} \geq \mathbf{1}_{B_{n(m_0)}} \Rightarrow \mathbf{1}_A + \mathbf{1}_{E_{m_0}} \geq \mathbf{1}_A + \mathbf{1}_{A_{n(m_0)} \backslash A} = \mathbf{1}_{A_{n(m_0)}}$. Hence by monotonicity of the Choquet integral one would get $k\,\mathbf{1}_{\mathbb{N}} \succ_i \mathbf{1}_{A_{n(m_0)}}$ but $I\big(\mathbf{1}_{A_{n(m_0)}}\big) = \vartheta(A_{n(m_0)}) \geq k$ and $I\big(k\,\mathbf{1}_{\mathbb{N}}\big) = k$, a contradiction.

Let us prove now that outer-continuity of ϑ is a sufficient condition.

It is enough to see that $\forall\, y, z \in \ell^\infty_+$, one gets $I(y + z_{E_n}) \to I(y)$ when $n \to +\infty$ one has:

$$I(y + z_{E_n}) = \int_0^{+\infty} \vartheta(y + z_{E_n} \geq t)\, dt.$$

Let $A_n(t) = \{y + z_{E_n} \geq t\}$ and $A(t) = \{y \geq t\}$. It is straightforward to see that $A_n(t) \downarrow A(t)\ \ \forall\, t \in \mathbb{R}_+$. From outer-continuity of ϑ, it turns out that $\forall\, t \in \mathbb{R}_+$, $f_n(t) = \vartheta(A_n(t)) \downarrow f(t) = \vartheta(A(t))$.

Thus by the Dominated Convergence Theorem $I(y + z_{E_n}) \to I(y)$. ■

Imagine now that we are in situation of risk, i.e., there is a given σ-additive probability P on $(\mathbb{N}, 2^{\mathbb{N}})$, with $P(\{n\}) > 0\ \ \forall\, n$. Let us first consider Yaari's preference (Yaari (1987)), that is any $x \in \ell^\infty_+$ is valued through $I(x) = \int x\, d\vartheta$ where $\vartheta = f \circ P$ and f the distortion function: $f\colon [0,1] \to [0,1]$ satisfies $f(0) = 0$, $f(1) = 1$ is strictly increasing, and continuous.

From Proposition 2, one gets immediately:

Proposition 3. *A preference à la Yaari on ℓ^∞_+ is weak* sequentially upper semi-continuous.*

Let us finally consider the classical expected utility preferences under risk.

In such a case $I(x) = \sum\limits_{n=0}^{\infty} P(\{n\}) u(x(n))$ where u is assumed to be strictly increasing and continuous. It is immediate to check that such preferences satisfy Assumption 1.

In fact we obtain:

Proposition 4. *A EU (expected utility) preference on ℓ^∞_+ is weak* sequentially upper semi-continuous.*

Proof: From Proposition 1, it is enough to see that $\forall\, y, z \in \ell_+^\infty$ one gets $I(y+z_{E_n}) \to I(y)$ when $n \to +\infty$

$$I(y+z_{E_n}) = \sum_{m=0}^{n} P(\{m\})u(y(m)) + \sum_{m=n+1}^{+\infty} P(\{m\})u(y(m)+z(m)).$$

Since $y+z$ is norm bounded and u is monotone, it turns out that $u(y+z)$ is bounded, and since $\lim_{n\to\infty} \sum\limits_{n+1}^{\infty} P(\{m\}) = 0$ one gets $I(y+z_{E_n}) \to \sum\limits_{n=0}^{\infty} P(\{n\})u(y(n)) = I(y)$. ■

Let us study now the important case of optimistic and pessimistic DMs.

5 Optimistic and pessimistic DMs

The concepts of pessimism and optimism presented in Definitions 5 and 6 below are mainly motivated by the evidence that such opposite behaviours can be observed specially under uncertainty, hence by the application of Theorem 1 more specifically to the case in which $x \in \ell_+^\infty$ is interpreted as a plan of outcomes contingent to the realization of one of the (infinitely countable) possible states of nature.

Definition 5. A DM will be said pessimistic if her preferences are convex i.e. $\forall\, x, y \in \ell_+^\infty$, $\alpha \in (0,1)$ $x \succsim_i y \Rightarrow \alpha x + (1-\alpha)y \succsim_i y$.

Definition 6. A DM will be said optimistic if her preferences are concave i.e. $\forall\, x, y \in \ell_+^\infty$, $\alpha \in (0,1)$ $x \succsim_i y \Rightarrow x \succsim_i \alpha x + (1-\alpha)y$.

Note that pessimism expresses that smoothing payoffs can only make the DM better off, while for an optimistic DM this will never be the case. This feature associates the optimism to a willingness to bet. For instance, if x^{o} is an alternative that delivers two units of outcome if the state is odd (and zero otherwise) and x^{e} is the analogous alternative that delivers on even states, an optimistic DM would prefer to bet on x^{o} or x^{e} than receive a certain amount of 1 unit of outcome.

Under uncertainty convexity of preferences is usually called uncertainty aversion (Schmeidler (1989), Gilboa and Schmeidler (1989)) while concavity is called uncertainty loving (see also Wakker (1990) and the seminal survey of Gilboa and Marinacci (2013)). It is well-known that, for Choquet preferences, optimism is equivalent to a convex capacity and pessimism is equivalent to a concave capacity (see Definition 7 below), for Yaari preferences these two notion are respectively equivalent to f convex and f concave, and that for EU DM this amounts to respectively a concave or a convex VNM utility function u.

Definition 7. A capacity ϑ is convex (resp. concave) if $\forall\, A, B \in \mathcal{A}$ $\vartheta(A \cup B) + \vartheta(A \cap B) \geq \vartheta(A) + \vartheta(B)$ (resp: $\vartheta(A \cup B) + \vartheta(A \cap B) \leq \vartheta(A) + \vartheta(B)$).

Let us recall that for Choquet preferences optimism and pessimism are well motivated by the following properties (Schmeidler (1986, 1989)):

$$\text{If } \vartheta \text{ convex } \int x d\vartheta = \min\{E_P(x) | P \text{ additive probability } \geq \vartheta\};$$

$$\text{If } \vartheta \text{ concave } \int x d\vartheta = \max\{E_P(x) | P \text{ additive probability } \leq \vartheta\},$$

where $E_P(x)$ denotes the mathematical expectation of x with respect to P.

5.1 Existence of IRPO in presence of upper semi myopic pessimistic and reasonably optimistic DMs

Definition 8. An optimistic DM is said to be reasonably optimistic if her preference satisfies A1 and upper semi-myopia.

Thus the reasonably optimistic DMs will not want to exchange all their wealth for a larger income on events with arbitrarily small likelihood or dates arbitrarily far. That prevents excessive optimism, avoiding the difficulties that will be presented in Example 1 of Subsection 5.2.

From the previous developments, the following DMs belong to the class of reasonable optimists:

1) Any optimistic EU (see Proposition 4)

2) Any optimistic Choquet DM such that $A_n,\ A \in \mathcal{A},\ A_n \downarrow A \Rightarrow \vartheta(A_n) \downarrow \vartheta(A)$ (see Proposition 2).

   An example of such preferences may be obtained by defining $\vartheta = f \circ P$ where P σ-additive on $(\mathbb{N}, 2^{\mathbb{N}})$, f concave and right-continuous. Many other less specific examples of such capacities could be exhibited

3) Any optimistic Yaari's DM.

The terminology reasonably optimistic can be justified for a Choquet optimistic DM by the fact that it merely imposes that if an event decreases towards the empty set she reasonably appreciates this event less and less and is valued 0 at the limit as seen in the next lemma.

Lemma 3. *For a concave capacity the following assertions are equivalent:*

(i) $A_n,\ A \in \mathcal{A},\ A_n \downarrow A \Rightarrow \vartheta(A_n) \downarrow \vartheta(A)$

(ii) $A_n \in \mathcal{A}\ \ A_n \downarrow \emptyset \Rightarrow \vartheta(A_n) \downarrow 0$

Proof: Such a property is a direct consequence of a dual similar property proved by Rosenmüller (1971) for the dual case of convex capacities. ∎

So trivially, from Corollary 1, one gets:

Proposition 5. *If all decision makers have preferences satisfying* A1 *and are either upper semi-myopic pessimistic or reasonably optimistic, then individually rational Pareto optimum exists.*

Thus, it is possible to ensure the existence of IRPO when the optimistic DMs are also upper semi-myopic. Indeed, as can be seen in Example 4 of the next section, some boundedness on the optimism is required in order to guarantee such existence.

5.2 IRPO can fail to exist in the presence of a too optimistic DM

The next example shows an economy for which the unique PO is the trivial one that assigns the whole aggregate endowment for the DM 1.

In this example the nonexistence of an IRPO comes from the presence of an unreasonably optimistic DM. Namely DM_2 exhibits preferences represented by the Choquet integral $U^2(x) = \int x \, d\vartheta$ where ϑ is the concave capacity defined by $\vartheta(A) = 1 \ \forall A \neq \phi$, so clearly condition (ii) of Lemma 3 is not satisfied. Indeed since $U^2(x) = \sup_{s \in \mathbb{N}} x(s)$, such a DM is extremely optimistic.

Example 4. Nonexistence of IRPO

Consider the consumption space $X = \ell^\infty_+$ and the probability measure p on $2^{\mathbb{N}}$ given by $p_s = \frac{1}{2^s} \quad \forall \ s \in \mathbb{N}$.

- The DM 1 is characterized by the utility function $U^1(x) = \sum_{s \in \mathbb{N}} x_s p_s$ and the initial endowments $\omega^1 = (2 - \frac{1}{s})_{s \in \mathbb{N}}$;
- The DM 2 is characterized by the utility function $U^2(x) = \sup_{s \in \mathbb{N}} x_s$ and the initial endowments $\omega^2 = (2 - \frac{1}{s})_{s \in \mathbb{N}}$;

Affirmation: The unique Pareto optimal allocation is the couple $(\overline{x}^1, \overline{x}^2)$ that associates $\overline{x}^1 = \omega^1 + \omega^2$ to DM 1 and $\overline{x}^2 = 0$ to DM 2. Thus, there is no IRPO.

In fact, consider a feasible allocation (x^1, x^2) with $x^2 > 0$. If $\sup_{s \in \mathbb{N}} x^2_s < \sup_{s \in \mathbb{N}} (\omega^1_s + \omega^2_s) = 4$, take $s_0 \in \mathbb{N}$ be such that $x^2_{s_0} > 0$. There is $s_1 \in \mathbb{N}$ large enough such that $U^1(x^1 + x^2_{s_0} e_{s_0} - x^1_{E_{s_1}}) > U^1(x^1)$. As $U^2(x^2 - x^2_{s_0} e_{s_0} + x^1_{E_{s_1}}) = 4 > U^2(x^2)$, we have that (x^1, x^2) is not a PO allocation.

If $\sup_{s \in \mathbb{N}} x^2_s = 4$, there is $n_0 \in \mathbb{N}$ such that $0 < x^2_{n_0} < 4$, so we would get $U^1(x^1 + x^2_{n_0} e_{n_0}) > U^1(x^1)$ and $U^2(x^2 - x^2_{n_0} e_{n_0}) = U^2(x^2)$, so, again, (x^1, x^2) is not a PO allocation.

Thus, the unique PO allocation assigns $x^2 = 0$. □

5.3 At an IRPO at most one strict optimistic will avoid ruin

The results in this section will require a strengthen notion of optimism that is provided in Definition 9.

Definition 9. A DM is strictly optimistic if $\forall x, y \in \ell_+^\infty$, $x \neq y$ and $\forall \alpha \in (0,1)$ $x \succsim_i y \Rightarrow \alpha x + (1-\alpha) y \prec_i x$ [8].

As before (Definition 8), we will prevent extreme forms of optimism:

Definition 10. A strictly optimistic DM is said to be reasonably strictly optimistic if her preference satisfies A1 and upper semi-myopia.

In this section as in Section 5.4, when we consider a strictly optimistic DM, we will assume that her preference $\succsim_i$ satisfies A1 but with the requirement of strict monotonicity that is $x, y \in \ell_+^\infty$ $x > y$ (i.e. $x \geq y, x \neq y$) $\Rightarrow x \succ_i y$. It is easy to find examples of such strictly optimistic DM, who would be reasonably optimistic.

Indeed an EU DM as defined above in Section 4 with a strictly convex VNM utility would be such an example. Equally a Yaari's DM (see Section 4) with a strictly concave distortion function f would satisfy such a requirement.

The next lemma characterizes strict optimism for Choquet preferences from properties of the respective capacity. Its proof is provided in Appendix C.

Lemma 4. *Choquet preferences on ℓ_+^∞ are strictly monotone, reasonably strictly optimistic if and only if*

(a) ϑ is strictly monotone, i.e., $A \supsetneq B \Rightarrow \vartheta(A) > \vartheta(B)$

(b) ϑ is strictly concave, i.e., concave and such that:
$A \cup B \supsetneq A \supsetneq B \supsetneq A \cap B$ $\vartheta(A \cup B) + \vartheta(A \cap B) < \vartheta(A) + \vartheta(B)$

(c) ϑ is outer continuous.

Theorem 2 below shows that, when there are two or more strictly optimistic DMs, "corner" plans are a quite general feature of IRPO allocations.

Theorem 2. *Consider an individually rational Pareto optimum where two* DM *i and j satisfy* A1, *have reasonably strictly optimistic and strictly monotonic preferences. Then, their allocations x^i and x^j satisfy for any $(s,t) \in \mathbb{N}^2$ $s \neq t$:*

$$x^i(s) \cdot x^i(t) \cdot x^j(s) \cdot x^j(t) = 0.$$

[8] Indeed such a DM has non-convex preferences, but actually concave preferences.

Proof: Let us define $x^i_{-(s,t)}$ by $x^i_{-(s,t)}(u) = \begin{cases} x^i(u) \text{ if } u \in \mathbb{N}\backslash\{s,t\} \\ 0 \text{ if } u \in \{s,t\} \end{cases}$, $e_s \in \ell^\infty_+$ by $e_s(u) = \begin{cases} 1 \text{ if } u = s \\ 0 \text{ otherwise} \end{cases}$ and consider $\forall\, (a,b) \in \mathbb{R}^2_+ \quad x^i_{(a,b)} \equiv a\, e(s) + b\, e(t) + x^i_{-(s,t)}$.

Denote $\nu(s) \equiv x^i(s) + x^j(s)$ and $\nu(t) \equiv x^i(t) + x^j(t)$. We will define $x^j_{(a,b)} \equiv (\nu(s) - a)e_s + (\nu(t) - b)e_t + x^j_{-(s,t)}$. Let us show that $x^i(s) > 0$, $x^i(t) > 0$, $x^j(s) > 0$, $x^j(t) > 0$ is impossible.

For any $a \in [0, x^i(s))$ by continuity of $\succsim_i$ and strict monotonicity of $\succsim_i$ $\exists\, b(a) > x^i(t)$ such that $x^i_{(a,b(a))} \sim_i x^i$. It is straightforward to check that when $a \uparrow x^i(s)$, then $b(a) \downarrow x^i(t)$. Therefore for $a = \bar{a}$ sufficiently close to $x^i(s)$, $\bar{b} = b(a)$ is such that $x^j_{(\bar{a},\bar{b})} \geq 0$ i.e. $x^j_{(\bar{a},\bar{b})} \in \ell^\infty_+$. That $x^i_{(\bar{a},\bar{b})} \in \ell^\infty_+$ is immediate.

Let us set $\bar{y}_i = x^i_{(\bar{a},\bar{b})}$ and $\bar{y}_j = x^j_{(\bar{a},\bar{b})}$. Clearly $\bar{y}_i + \bar{y}_j = x^i + x^j$. Since $\bar{y}_i \sim_i x^i$ if $\bar{y}_j \succ_j x^j$ IRPO would be contradicted. So necessarily $x^j \succsim_j \bar{y}_j$.

Let us choose $\beta \in (1,2)$ close enough to 1 in order to guarantee that $y^i \equiv \beta x^i + (1-\beta)\bar{y}_i$ and $y^j \equiv \beta x^j + (1-\beta)\bar{y}_j$ are such that $y^i, y^j \geq 0$ (this is possible since by hypothesis $x^i(s) > 0$, $x^i(t) > 0$, $x^i(s) > 0$, $x^j(t) > 0$). So y^i and y^j belong to ℓ^∞_+ and clearly $y^i + y^j = x^i + x^j$.

It is enough to prove that $y^i \succsim_i x^i$ and $y^j \succ_j x^j$ which contradicts again IRPO, and then completes the proof. One has

$$\begin{aligned} x^i &= \frac{1}{\beta}\, y^i + \frac{\beta - 1}{\beta}\, \bar{y}_i \text{ with } y^i \text{ and } \bar{y}_i \in \ell^\infty_+ \\ x^j &= \frac{1}{\beta}\, y^j + \frac{\beta - 1}{\beta}\, \bar{y}_j \text{ with } y^j \text{ and } \bar{y}_j \in \ell^\infty_+ \end{aligned}$$

Let us show first that $y^i \succsim_i x^i$. Otherwise $x^i \succ_i y^i$ hence $x^i \sim_i \bar{y}_i$ implies $\bar{y}_i \succ_i y_i$ and strict optimism implies $\bar{y}_i \succ_i x_i$ i.e. $x^i \succ_i x^i$, a contradiction.

Now, suppose $x^j \succsim_j y^j$. Note that $\bar{y}_j \neq y_j$ otherwise $y^j = \beta x^j + (1-\beta)\bar{y}_j$ would imply $\bar{y}_j = x^j$ and this is not the case. So, $y^j \succ_j x^j$.

Thus, either $y^j \succsim_j \bar{y}_j$ and $y^j \neq \bar{y}_j \Rightarrow y^j \succ_j x^j$ by strict optimism hence $x^j \succ_j x^j$, which is impossible. Or $\bar{y}^j \succ_j y^j$ and strict optimism would imply $\bar{y}_j \succ_j x^j$ which contradicts $x^j \succsim_j \bar{y}_j$. Henceforth $y^j \succ_j x^j$ which completes this part of proof. ∎

Now we will introduce a notion related to the possible non-interiority of some of the plans designed by an IRPO allocation to DMs.

Definition 11. Given an IRPO allocation $(x^i)_{i=1}^I$, we say that the DM i is going to ruin in state $s \in \mathbb{N}$ (according to this allocation) when $x^i(s) = 0$; When $x^i(s) > 0$ we will say that the DM i will avoid ruin in state s.

From Definition 11 and Theorem 2, we obtain the second main result of this paper.

Corollary 2. *At an* IRPO *at most one reasonably strictly optimistic will avoid ruin in any state* (*or equally at any time period*).

Proof: Assume that at least two DM i and j are strictly optimistic and satisfy the natural hypotheses of Theorem 2.
Imagine that one of them let us say i has an interior allocation, then at most in one state s the DM j can get $x^j(s) > 0$. ■

Remark 1. A further conclusion from Theorem 2 is that, under the same hypotheses, if there is an optimistic DM that will avoid ruin in each state, then all the other optimistic DM are going to ruin except in one state. In other words, this extremal willingness to bet could induce an ex post social problem (may be with a very large likelihood).

5.4 Optimal risk sharing for risk lovers and risk averters decision makers

Interpret ℓ^∞ as the set of all bounded $\mathcal{A}$-measurable mappings x from $\mathbb{N}$ to $\mathbb{R}$ where $\mathcal{A} = \mathcal{P}(\mathbb{N})$ and assume that a σ-additive probability P is given on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$. [9] So any x can now be interpreted as a random variable.

Note that under risk, rational DM are assumed to strictly agree with first order stochastic dominance (FSD), which is the natural extension of monotone dominance under risk.

[9] For sake of simplicity we will assume that $P(\{n\}) > 0 \ \forall n \in \mathbb{N}$.

Definition 12. Given $x, y \in \ell^\infty$, we say that:

(i) x dominates y in the FSD sense, which will be denoted by $x \succsim_{\text{FSD}} y$, if $P(x \leq u) \leq P(y \leq u)\ \ \forall u \in \mathbb{R}$;

(ii) x strictly dominates y in the FSD sense if furthermore one of the previous inequalities is strict.

Definition 13. Let $\succsim_i$ be the preference relation of a DM. We say that $\succsim_i$ strictly agrees with FSD if $x \succsim_{\text{FSD}} y \Rightarrow x \succsim_i y$ and $x \succ_{\text{FSD}} y \Rightarrow x \succ_i y$.

So, considering in all this section only DM strictly agreeing with FSD, it turns out that the usual definition of strict strong risk aversion coincides with strictly agreeing with second order stochastic dominance in the case of alternatives with the same mean.

Definition 14. Let $x, y \in \ell^\infty$ be such that $E_P(x) = E_P(y)$.

(i) x is said to be less risky than y for the second order stochastic dominance or else y is a mean-preserving spread of x, which will be denoted by $x \succsim_{\text{MPS}} y$, if:

$$\int_{-\infty}^{t} P(x \leq u)\, du \leq \int_{-\infty}^{t} P(y \leq u)\, du \quad \forall\ t \in \mathbb{R}.$$

(ii) x is strictly less risky than y, which will be denoted by $x \succ_{\text{MPS}} y$, if furthermore one of the previous inequalities is strict.

Hence the definitions of strict (strong) risk aversion and of strict (strong) risk loving.

Definition 15. Let $\succsim_i$ be the preference relation of a DM.

(i) $\succsim_i$ is a strict (strong) risk averter if $x \succsim_{\text{MPS}} y \Rightarrow x \succsim_i y$ and $x \succ_{\text{MPS}} y \Rightarrow x \succ_i y$.

(ii) $\succsim_i$ is a strict (strong) risk lover if $x \succsim_{\text{MPS}} y \Rightarrow x \precsim_i y$ and $x \succ_{\text{MPS}} y \Rightarrow x \prec_i y$.

As examples of weak* sequentially usc strict risk lovers or strict risk averters we can mention:

(a) EU DM are such strict risk lovers or strict risk averters if and only if they are strictly optimistic or strictly pessimistic.

(b) Yaari's DM are such strict risk lovers or strict risk averters if and only if they are strictly optimistic or strictly pessimistic.

The following Theorem 3 offers two important features of risk sharing between strict risk averters and strict risk lovers.

Feature 1: As in the classical case of risk sharing between only risk averters, the Pareto optimal allocations of risk averters will be comonotonic, but note that this does not entail any longer that these allocations will be necessarily ranked as the aggregate endowment, because nothing in the Theorem guarantees global comonotonicity between risk averters and risk lovers. Example 5 illustrates this last point.

Example 5. IRPO without comonotonicity between risk averters and risk lovers
Let $S = \{1, 2\}$ and p the probability measure given by $p_1 = p_2 = \frac{1}{2}$. The DM 1 is a strict strong risk averter characterized by the utility function $U^1(x_1, x_2) = \frac{1}{2}(-\exp(-2x_1)) + \frac{1}{2}(-\exp(-2x_2))$ and the initial endowment $\omega^1 = (8, 4)$.
The DM 2 is a strict strong risk lover characterized by the utility function $U^2(y_1, y_2) = \frac{1}{2}\exp(y_1) + \frac{1}{2}\exp(y_2)$ and the initial endowment $\omega^2 = (4, 5)$.
So, the aggregate endowment is $\omega = (12, 9)$. We intend to show that if $\overline{x} = (4.5, 7.5)$ and $\overline{y} = (7.5, 1.5)$, then the couple $(\overline{x}, \overline{y})$ is an IRPO, which will imply that global comonotonicity fails.
It is immediate that $(\overline{x}, \overline{y})$ is a feasible allocation. Furthermore it is straightforward to verify that $\overline{x} \succsim_1 \omega^1$ and $\overline{y} \succsim_2 \omega^2$. So, it suffices to show that $\overline{x}$ is the solution of the problem:

$$\begin{aligned} \max \quad & U^1(x_1, x_2) \\ \text{s.t.} \quad & U^2(12 - x_1, 9 - x_2) = U^2(\overline{y}) \\ & (x_1, x_2) \in [0, 12] \times [0, 9] \end{aligned}$$

Since the constraint on U^2 can be written as $e^{-x_2} = e^{-1.5} + e^{-7.5} - e^{3-x_1}$, to solve the above problem it is enough to find a solution $z^* \in \operatorname{argmax}\{h(z) : 0 \le z \le 12\}$ for $h(z) = -e^{-2z} - (k - e^{3-z})^2$ and $k = e^{-1.5} + e^{-7.5}$ and make $x_1^* = z^*$, $x_2^* = -\ln(e^{-1.5} + e^{-7.5} + e^{3-z^*})$.
It is straightforward to verify that $z^* = 4.5$ (and that this solution is unique), which implies $(x_1^*, x_2^*) = (\overline{x}_1, \overline{x}_2)$ as desired.

Feature 2: As soon as the strict risk lovers are also strictly optimistic (note that while this is the case for EU or Yaari's DM, this might not be the case for some particular class of strict risk lovers, see the Example 6 below adapted from Chateauneuf and Lakhnati (2007) who were dealing with strict risk averters) then at most one of these strict risk lovers will avoid ruin.

Example 6. A strict risk lover with weak* sequentially usc-preferences who is not strictly optimistic

We just exhibit here such a DM. Assume that any $x \in \ell_+^\infty$ is valued through $I(x) = \int u(x)dP + f(\int xdP)$, where $u : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, strictly increasing and strictly convex, $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and strictly increasing.

Since the preference $\succsim_i$ represented by I clearly satisfies A1 from Proposition 1, it will be weak* sequentially usc as soon as we prove that $I(y + z_{E_n}) \to I(y)$ when $n \to \infty$. But, denoting $I_1(x) = \int u(x)dP$ and $I_2(x) = f(\int xdP)$. From Proposition 4, we have that $I_1(x)$ satisfies this property and clearly this is also the case for $I_2(x)$.

Since u and f are strictly increasing it is immediate that $\succsim_i$ strictly agrees with FSD. From $E_P(x) = E_P(y)$, it turns out that $x \succsim_{\text{MPS}} y \Rightarrow I_2(x) = I_2(y)$. Moreover, u strictly increasing and strictly convex implies that $x \succsim_{\text{MPS}} y \Rightarrow I_1(x) \geq I_1(y)$ and $x \succ_{\text{MPS}} y \Rightarrow I_1(x) > I_1(y)$. Consequently $\succsim_i$ exhibits strict risk loving.

Assume now that $u(x) = x^2$ and f be such that $f(2) = 1, f(\frac{5}{2}) = \frac{5}{2}$ and $f'(2) = 2$.

Let B_1 and B_2 be a partition of $\mathbb{N}$ such that $P(B_1) = P(B_2) = \frac{1}{2}$ and consider $x = 4\,\mathbf{1}_{B_2}$ and $y = 2\,\mathbf{1}_{B_1} + 3\,\mathbf{1}_{B_2}$. Simple computations show that $I(x) = I(y)$, that $g(\alpha) = I(\alpha x + (1-\alpha)y)$ is continuously differentiable on $[0, 1]$, and that $g'(1) = 3 > 0$.

Thus, there exists $\alpha^* \in (0, 1)$ such that $I(\alpha x + (1-\alpha)y) > I(y)$ for $\alpha \in (\alpha^*, 1)$, hence the DM is there locally strictly pessimistic. So $\succsim_i$ does not exhibit strictly optimism (indeed, does not even exhibit optimism). $\square$

Theorem 3 makes precise the statement about comonotonicity of plans of risk averters and emphasizes the tendency of strict risk lovers DMs to go to ruin in some states in IRPO allocations.

Theorem 3. *Consider m strict risk averters $i = 1, \ldots, m$ satisfying A.1 with weak**

sequentially u.s.c. preferences $\succsim_i$ and n strict risk lovers $j = 1, \ldots, n$ satisfying A.1 with weak sequentially u.s.c. preferences $\succsim_i$ but exhibiting strict optimism.*
Assume initial endowments $w_i \in \ell^\infty_{++}$, $w_j \in \ell^\infty_{++}$.
Then individual rational Pareto efficient allocations (in $(\ell^\infty_+)^{m+n}$) exist. For such PO $(x_i, i = 1, \ldots, m; y_j, j = 1, \ldots n)$ *we have:*

1) *The allocations of risk averters are pairwise comonotonic, i.e.,* $\big(x_{i_1}(s) - x_{i_1}(t)\big)\big(x_{i_2}(s) - x_{i_2}(t)\big) \geq 0 \; \forall (s,t) \in \mathbb{N}^2, \forall (i_1, i_2) \in \{1, \ldots, m\}^2$

2) *At most one risk lover will avoid ruin in any state.*

Proof:

1) A proof similar as the one of Proposition 4.1 of Chateauneuf, Dana, and Tallon (2000) applies. Actually if $(x_1, x_2, ..., x_m)$ is the allocation of the risk averters at an IRPO, the proof of Proposition 4.1 shows that without loss of generality if x_1 and x_2 would not be comonotonic, one could find $\overline{x}_1$ and $\overline{x}_2$ in ℓ^∞_+ such that $\overline{x}_1 \succ_1 x_1$, $\overline{x}_2 \succ_2 x_2$ and $\overline{x}_1 + \overline{x}_2 + \sum_{i=3}^m x_i = \sum_{i=1}^m x_i$, thus contradicting the fact that $(x_1, x_2, ..., x_m, y_1, ..., y_n)$ is an IRPO.

2) This is just Corollary 2.

■

6 Is it beneficial to share risk with a risk lover?

Is it more beneficial for a risk averter to share risk with a risk lover rather than with another risk averter? This section aims at introducing this question through some elementary examples.

We focus on the simple situation of two EU decision makers $i = 1, 2$ sharing risk in the case of a two-states space $S = \{s_1, s_2\}$ endowed with a probability measure P such that $P \gg 0$. Let us assume that the initial endowments ω_1 and ω_2 satisfy $\omega_1 \gg 0, \omega_2 \gg 0$ and denote by ω the aggregate endowment.

In order to get tractable formulas, we specialize to the situation where DM1 is a constant absolute risk averse (CARA) DM, i.e., with an EU utility index $u_1(x) = -e^{-\rho_1 x}$, $\rho_1 > 0$, while DM2 is either a CARA DM with a constant coefficient of absolute risk aversion $\rho_2 > 0$ or a constant absolute risk lover (CARL) with coefficient $\rho_2 > 0$ of absolute risk loving, i.e., with EU utility index u_2 given by $u_2(x) = e^{\rho_2 x}$.

Let us recall that if both DM1 and DM2 are CARA then the interior PO allocations are given by $x_1 = a + \frac{\rho}{\rho_1}\omega$ and $x_2 = -a + \frac{\rho}{\rho_2}\omega$, where $a \in \mathbb{R}$ is chosen in order that $x_1 \gg 0, x_2 \gg 0$ and where ρ is the aggregate coefficient of risk aversion defined by $\frac{1}{\rho} = \frac{1}{\rho_1} + \frac{1}{\rho_2}$.

The Proposition 6 (whose proof is available upon request) gives sufficient conditions for interior PO allocations and describes the allocations when DM2 is a CARL DM.

Proposition 6. *Let $r = \frac{\rho_1}{\rho_2}$ be the relative level of risk aversion and $a_r = \frac{\max_s \omega(s)}{\min_s \omega(s)}$ be the relative level of aggregate risk. If $r > a_r$ then there exist interior Pareto optimal allocations and they are given by the formulas $x_1 = a + \frac{\rho}{\rho_1}\omega$ and $x_2 = -a - \frac{\rho}{\rho_2}\omega$ where $a \in \mathbb{R}$ is chosen in order that $x_1 \gg 0, x_2 \gg 0$ and where ρ defined by $\frac{1}{\rho} = \frac{1}{\rho_1} + \frac{1}{-\rho_2}$ extends the usual aggregate coefficient of risk aversion.*

Remark 2. Note that in the Proposition 6 ρ is negative so here the allocation of risk averter is anticomonotonic with the aggregate endowment, while the risk lover's one proves to be comonotonic with ω. This indeed contrasts with the situation in which both DM are risk averters, since their PO allocations are comonotonic with ω. Such a fact will matter in the Examples 7 and 8 (in which we assume that $\rho_1 = 2$).

Example 7. No aggregate risk

Consider the endowments described by Table 1 and assume that $P(s_1) = P(s_2) = \frac{1}{2}$.

Table 1: Endowments of Example 7

	s_1	s_2
ω_1	5	4
ω_2	4	5
ω	9	9

Case 7.a: DM2 is CARL with $\rho_2 = 1$: It is straightforward to see that Proposition 6 applies, so the interior PO allocations which are full insurance satisfy:

$x_1(s) = a - 9, \ x_2(s) = -a + 18$ and $9 < a < 18$.

Simple computations show that there exists $k_1 \in (0, 4.5)$ such that at any interior IRPO one has $x_1(s) \leq k_1 < 4.5 \leq x_2(s)$, hence DM1 will get a full insurance always strictly smaller than DM2.

Case 7.b: DM2 is CARA with $\rho_2 = 1$. In such a case interior PO allocations are indeed again full insurance and satisfy:

$x_1(s) = a + 3, x_2(s) = -a + 6$ and $-3 < a < 6$.

Here, simple computations show that if a is close enough of $\frac{3}{2}$ then the previous PO are IRPO with either $x_1 > x_2$ or $x_1 < x_2$ (in fact, $x_1 = x_2$ if $a = \frac{3}{2}$), so DM1 could get a better allocation than DM2.

In conclusion in this example of no aggregate risk economy, it appears it would be more beneficial to share risk with a risk averter.

Example 8. Aggregate risk

Consider again $P(s_1) = P(s_2) = \frac{1}{2}$. Now the endowments are described by Table 2.

Table 2: Endowments of Example 8

	s_1	s_2
ω_1	8	4
ω_2	4	5
ω	12	9

Case 8.a: DM2 is CARL with $\rho_2 = 1$. By Proposition 6, the interior PO allocations are given by:

$x_1(s_1) = a - 12, x_1(s_2) = a - 9, x_2(s_1) = -a + 24, x_2(s_2) = -a + 18$ with $a \in (12, 18)$.

Note that since PO allocations of DM1 are anticomonotonic with ω, if this DM would like to transfer wealth from state s_1 to state s_2 "in an anticomonotonic way" for some "strategical" reason, this is a priori possible. It turns out in fact that it is possible in an efficient way, since one can check that if $a = 16.5$ then the resulting PO is an IRPO, and moreover $E_P(x_1) = E_P(\omega_1)$, $VAR_P(x_1) < VAR_P(\omega_1)$.

Case 8.b: DM2 is CARA with $\rho_2 = 1$ In this case, the interior PO allocations are given by:

$x_1(s_1) = a + 4, x_1(s_2) = a + 3, x_2(s_1) = -a + 8, x_2(s_2) = -a + 6$ with $a \in (-3, 6)$.

Since the PO allocations are comonotonic with ω, there is no hope for DM1 to transfer her wealth from state s_1 to state s_2 in an "anticomonotonic way" through an IRPO.

In conclusion, in case of aggregate risk this example illustrates that for a risk averter who would like to transfer substantially her wealth from state s_1 to state s_2 and in an efficient way, this would only be possible when sharing risk with a risk lover.

7 Conclusions and Further Research

As the Theorem 1 has shown, the existence of IRPO allocation is a quite general feature of GE models when the space of alternatives is ℓ^∞ (or $\mathbb{R}^S$), even in the presence of non-complete preferences or optimistic agents. One of the sufficient requirements is that the DMs should be myopic or impatient for gains, what can be interpreted as a bound on optimism, since, in that case, even the optimistic DMs will not want to exchange all their wealth for a larger income on events with arbitrarily small likelihood or dates arbitrarily far (which avoids the difficulties present in Example 1).

So, one perceives that the concept of IRPO accommodates well the notions of risk and ambiguity propensity (both relevant for Applied Microeconomics and Finance). A further research program is to find the sufficient conditions to ensure the existence of a barter equilibrium (i.e., the non-emptiness of the core of the economy) or an equilibrium with non-linear prices even in the presence of optimistic agents.

At an IRPO allocation of resources, the risk averters will be sharing the individual risk (in the sense that their plans will be 2×2 comonotonic), but not necessarily the aggregate risk when there exist risk lovers DMs too. However, when the aggregate risk is very large, maybe the existence of optimistic agents turns the risk averters better than they would be with an aggregate risk sharing as it becomes possible to smooth their consumption plans.

About the properties of IRPO allocation, Theorems 2 and 3 say that is quite likely to observe plans of optimistic DMs leading to ruin at some states. In fact, if there were two optimistic DMs with consumption plans that are bounded away from zero in two states of nature, there would be room for a Pareto improvement. Under the theorems hypotheses, in the case that the consumption plan is interior for one of the optimistic DMs, all the others will concentrate the future wealth in just one of the possible future states, characterizing a very speculative pattern. Thus, a question that naturally arises for future studies is if a policy maker concerned with the ex post aggregate welfare should or not deviate the output allocation of the economy from ex ante IRPO allocations in order to avoid this extreme gambling. And, if the policy maker decides to intervene, which type of instrument should be chosen. Or even, if the incompleteness of markets could weaken the effects of this willingness to gamble future wealth, thereby preventing any need of intervention.

APPENDIX

A The space ℓ^∞

$\mathbb{N}$ denotes the set of natural numbers $\{1, 2, 3, ...\}$.

The space ℓ^∞ is the Banach space of real bounded sequences equipped with the norm defined by $||x|| = \sup_t |x_t|$. The space ℓ^1 is the Banach space set of absolutely convergent real sequences equipped with the norm defined by $||x||_1 = \sum_{t=1}^{\infty} |x_t|$.

When ℓ^∞ is endowed with the norm topology, the dual is denoted by $(\ell^\infty)^*$. A coarser topology is the *Mackey topology*, defined as the finest topology on ℓ^∞ for which the dual is ℓ^1. Now, (x^n) converges to x in this topology if and only if, for any weakly compact subset A of ℓ^1, $\langle x^n, y\rangle \to \langle x, y\rangle$ uniformly on $y \in A$.

The weak* topology is defined as the coarsest topology on ℓ^∞ for which the dual is ℓ^1.

By definition a sequence (x^n) converges to x in the weak* topology, denoted $x^n \xrightarrow{w^*} x$, if we have (x^n) in ℓ^∞, $x \in \ell^\infty$ and $\langle x^n, y\rangle \to \langle x, y\rangle$ for any $y \in \ell^1$. Furthermore (see for instance Brézis (2011)) $x^n \xrightarrow{w^*} x$ implies $(\|x^n\|)$ is bounded.
It turns out that, in the particular case of ℓ^∞, the simple following tractable property - which looks like the standard convergence in the Euclidian space $\mathbb{R}^m$ $(m \in \mathbb{N})$ - characterizes sequential weak* convergence:

Proposition 7. *Let (x^n), $x^n \in \ell^\infty$ then $x^n \xrightarrow{w^*} x$ where $x \in \ell^\infty$ if and only if $\lim_{n\to\infty} x^n(p) = x(p) \in \mathbb{R}$ for all $p \in \mathbb{N}$ and $(\|x^n\|)_n$ is bounded.*

Notice that in all this paper when $x \in \ell^\infty$ we denote indifferently its p-th component as $x(p)$ or x_p.

Proof: Since the necessary condition is immediate, we just prove for sake of completeness the sufficiency property.
So let (x^n) with $x^n \in \ell^\infty \quad \forall n$, be such that $x^n(p) \to x(p) \in \mathbb{R}$ $\forall p \in \mathbb{N}$ and $(\|x^n\|)$ is bounded.

Let us first prove that $x \in \ell^\infty$. Let $K \geq 0$ be such that $\|x^n\| \leq K \quad \forall n$. Let $\varepsilon > 0$ be given, $\lim_{n\to\infty} x^n(p) = x(p)$, then there exist $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies $|x(p)| \leq |x^n(p)| + \varepsilon$ hence $|x(p)| \leq K + \varepsilon$ and so $x \in \ell^\infty$ since $\|x\| \leq K + \varepsilon$.

It remains to show that for all $y \in \ell^1$ we have $\langle x^n, y\rangle \to \langle x, y\rangle$ i.e. $S_n \to S$ where $S_n = \sum_{p=1}^\infty x^n(p)y(p)$ and $S = \sum_{p=1}^\infty x(p)y(p)$. Let $m \in \mathbb{N}$, then:

$$|S_n - S| \leq \sum_{p=1}^{m} |x^n(p) - x(p)| \cdot |y(p)| + \sum_{p=m+1}^{\infty} |x^n(p) - x(p)| \cdot |y(p)| \,.$$

Hence,

$$|S_n - S| \leq \|y\|_1 \cdot \left(\sum_{p=1}^{m} |x^n(p) - x(p)| \right) + (K + \|x\|) \cdot \sum_{p=m+1}^{\infty} |y(p)| \,.$$

Since $y \in \ell^1$, there exist m_0 such that $\sum_{p=m_0+1}^\infty |y(p)| \leq \varepsilon$. For such m_0, $x^n(p) \to x(p)$ for all $1 \leq p \leq m_0$. Then there is $n_0(\varepsilon)$ such that $n \geq n_0(\varepsilon)$ implies $\sum_{p=1}^{m_0} |x^n(p) - x(p)| \leq \varepsilon$. Therefore $\forall \varepsilon > 0$, $\exists n_0(\varepsilon)$ such that $|S_n - S| \leq (\|y\|_1 + K + \|x\|)\varepsilon$. Hence $\langle x^n, y\rangle \to \langle x, y\rangle$ which completes the proof. ∎

B Mackey and weak* sequential convergence coincide on ℓ^∞

For sake of completeness we additionally give a direct and elementary simple proof of Proposition 8 below.

Proposition 8. *On ℓ^∞ the weak* $\sigma(\ell^\infty, \ell^1)$ sequential convergence and the Mackey $\tau(\ell^\infty, \ell^1)$ sequential convergence coincide.*

Proof: Since $\sigma(\ell^\infty, \ell^1) \subset \tau(\ell^\infty, \ell^1)$ we just need to prove that for (x^n), $x^n \in \ell^\infty$, $x^n \xrightarrow{\omega^*} x$ implies $x^n \xrightarrow{\tau} x$.

From Proposition 3.13 in Brézis (2011) it comes that $x^n \xrightarrow{\omega^*} x$ if and only if the sequence (x^n) is bounded and $\forall\, i \in \mathbb{N}$ $\lim_n x^n(i) = x(i)$, therefore we may assume without loss of generality that (x^n) is a sequence in the closed unit ball $\overline{B}(0,1)$ of ℓ^∞ and that x belongs to $\overline{B}(0,1)$. Recall that on $\overline{B}(0,1)$ the metric $d(x,y) = \sum_{i=0}^{\infty} \frac{1}{2^i} |x(i) - y(i)|$ defines the weak star topology (see e.g. Exercise 4 p.136 in Conway (1994)).

Let us recall furthermore that a basis of neighborhoods of 0 for the Mackey topology on ℓ^∞ is given by the sets: $\vartheta\big((a_i),0\big) = \left\{ z \in \ell^\infty, |z(i)| \leq \frac{1}{a_i} \ \forall\, i \in \mathbb{N} \right\}$ where (a_i) is a sequence of real numbers strictly decreasing towards 0 (see e.g. Brown and Lewis (1981) p.34). Therefore we need to show, letting $z^n = |x^n - x|$ that $z^n \xrightarrow{\omega^*} 0$ in $\overline{B}(0,1)$ implies $z^n \xrightarrow{\tau} 0$.

Let $\vartheta\big((a_i),0\big)$ be a neighborhood of 0 for the Mackey topology τ. From $\frac{1}{a_i} \uparrow +\infty$ there exists $i_0 \in \mathbb{N}^*$ such that $i \geq i_0 \Rightarrow 2 \leq \frac{1}{a_i}$ hence $|z^n(i)| \leq \frac{1}{a_i} \ \forall\, n$ since $|z^n(i)| \leq 2 \ \forall\, n, \forall\, i$.

So it remains to prove that if $z^n \xrightarrow{\omega^*} 0$, there exists $n_0 \in \mathbb{N}$ such that $n \geq n_0 \Rightarrow z^n \in \vartheta\big((a_i),0\big)$.

Let $\varepsilon = \frac{1}{2^{i_0-1} \cdot a_1}$, $z^n \xrightarrow{\omega^*} 0 \Rightarrow \exists\, n_0(\varepsilon)$ such that $n \geq n_0(\varepsilon) \Rightarrow \sum_{i=0}^{\infty} \frac{1}{2^i} |z^n(i)| \leq \varepsilon$ hence $\frac{1}{2^i} |z^n(i)| \leq \frac{1}{2^{i_0-1}} \times \frac{1}{a_1} \ \forall\, i \leq i_0 - 1$, therefore $|z^n(i)| \leq \frac{2^i}{2^{i_0-1}} \cdot \frac{1}{a_1} \ \forall\, i \leq i_0 - 1$ so $|z^n(i)| \leq \frac{1}{a_i}$ $\forall\, i \leq i_0 - 1$ since $\frac{1}{a_i}$ is strictly increasing; from $|z^n(i)| \leq \frac{1}{a_i} \ \forall\, i \geq i_0$ as obtained previously, it turns out that $z^n \in \vartheta\big((a_i),0\big) \ \forall\, n \geq n_0 = n_0(\varepsilon)$, which completes the proof. ■

C Strictly optimistic Choquet preferences

Proof of Lemma 4

(b) can be found in Schmeidler (1989)

(c) comes from Proposition 2.

The necessity of (a) is immediate.

It remains to prove that under (a) and (c) the preferences are strictly monotone.

So take $x, x' \in \ell_+^\infty$ with $x > x'$ and let us show that $x \succ_i x'$.

$x > x'$ implies there exists $s_0 \in \mathbb{N}$ s.t. $x(s_0) > x'(s_0)$. We need to show that

$$\int_0^{+\infty} \vartheta(x \geq t)\, dt > \int_0^{+\infty} \vartheta(x' \geq t)\, dt$$

or equally $d > 0$ where

$$d = \int_0^{+\infty} \big(\vartheta(x \geq t) - \vartheta(x' \geq t)\big)\, dt.$$

Let $t_0 = x(s_0)$, therefore $\{x \geq t_0\} \supsetneq \{x' \geq t_0\}$ and (a) implies $\vartheta(x \geq t_0) > \vartheta(x' \geq t_0)$.

Let $y \in \ell_+^\infty$, and let us see that $f(t) = \vartheta(y \geq t)$ is left-continuous.

Take $A_n = \{s \in \mathbb{N}, y(s) \geq t_n\}$ with $t_n \in \mathbb{R}^+$, $t_n \uparrow t$. It is straightforward to see that $A_n \downarrow A = \{y \geq t\}$, so since from (c) $\vartheta(A_n) \downarrow \vartheta(A)$, f is actually left-continuous. So since $t_0 = x(s_0) > 0$, there exists an interval $[t_0 - \varepsilon, t_0) \subseteq \mathbb{R}^+$ such that $\forall\, t \in [t_0 - \varepsilon, t_0]$, $\vartheta(x \geq t) - \vartheta(x' \geq t) > 0$, since indeed $x \geq x'$ implies $\vartheta(x \geq u) \geq \vartheta(x' \geq u)$ $\forall\, u \in \mathbb{R}^+$ one gets $d > 0$. ■

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