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Xavier Raurich[*]and Thomas Seegmuller[†‡]

October 12, 2015

Abstract

The aim of this paper is to study the interplay between long term productive investments and more short term and liquid speculative ones. A three-period lived overlapping generations model allows us to make this distinction. Agents have two investment decisions. When young, they can invest in productive capital that provides a return during the following two periods. When young or in the middle age, they can also invest in a bubble. Assuming, in accordance with the empirical evidence, that the bubbleless economy is dynamically efficient, the existence of a stationary bubble raises productive investment and production. Indeed, young agents sell short the bubble to increase productive investments, whereas traders at middle age transfer wealth to the old age. We outline that a technological change inducing either a larger return of capital in the short term or a similar increase in the return of capital in both periods raises productive capital, production and the bubble size. This framework also allows us to discuss several economic applications: the effects of both regulation on limited borrowing and fiscal policy on the occurrence of bubbles, the introduction of a probability of market crash and the effect of bubbles on income inequality.

JEL classification: E22; E44; G12.

Keywords: Bubble; Efficiency; Vintage capital; Short sellers; Overlapping generations.

[*]Departament de Teoria Econòmica and CREB, Universitat de Barcelona.

[†]Corresponding author. Aix-Marseille University (Aix-Marseille School of Economics), CNRS-GREQAM and EHESS. Centre de la Vieille Charité, 2 rue de la Charité, 13236 Marseille CEDEX 02, France. E-mail: thomas.seegmuller@univ-amu.fr.

1 Introduction

In the recent years, there is a renew of interest in studying the link between productive and purely speculative investments. Of course, this research agenda has also been motivated by the last crisis. Some questions that naturally emerge are of course whether speculative investments are good or rather bad for capital accumulation and production, whether bubbles are compatible with dynamic efficiency, and what is the role played by speculative assets. To address these issues, the literature mainly focuses on the polar case where one may invest either in productive capital or in an asset without fundamental value, which is a pure bubble when its price is positive (Tirole 1985), Weil (1987), Bosi and Seegmuller (2010), Fahri and Tirole (2012), Martin and Ventura (2012), Hirano and Yanagawa (2013)).[1] These two investments have returns and can be traded at the same terms or periods. This also means that these assets have the same liquidity. Despite the fact that following an idea already developed in Woodford (1990), some of these contributions are concerned with the liquidity role of bubbles, no difference is introduced between liquid and more illiquid assets. Such a differential in the liquidity of assets is however underlined by some empirical findings (Longstaff (2004), Amihud et al. (2005)).

In this paper, we take into account such a differential. We make a temporal distinction between speculation and investments in productive assets. Speculative investments give returns in the short run, whereas productive ones also give returns in the longer term. Obviously, this distinction implies that the speculative asset is more liquid than the productive one. As examples of productive investments, we think about real estate, infrastructure, equipment and software, education, research and development. The significance of long term productive investments has recently been emphasized in some reports. Following Group of Thirty (2013), long term investments represent 25-30% of GDP, even more in some emerging countries, and will grow significantly by 2020.

Taking into account this difference between the two types of investments, we show that the existence of a bubble is beneficial for production and is consistent with dynamic efficiency. Indeed, the speculative asset will be sold short by some traders to finance production. This can be an equilibrium because other traders will buy the speculative asset to transfer their wealth in the next period. This allows to have a bubble with a finite value in equilibrium. The second main issue concerns the effect of technological change on the productive investment and the bubble. We show that technological improvements raise capital investment, production and the bubble size, provided they do not increase productivity in the longer term more than in the short term. This result is documented by several contributions (Caballero et al. (2006), Lansing (2008, 2012), Scheinkman (2013)) explaining that episodes of bubbles are often associated to new innovations.

In the literature on rational bubbles, first results have shown that investment in a speculative bubble reduces capital accumulation in the long run to reach

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[1]There are however some exceptions. See for instance Kamihigashi (2008).

the golden rule (Tirole (1985)). This is the so-called crowding-out effect, which requires that the steady state without bubble is dynamically inefficient[2] and implies that there is less production in the long run with a speculative bubble than without. Dynamic inefficiency has been criticized by Abel et al. (1989) for its lack of empirical relevance and, moreover, the lower level of production when there is a bubble is not observed in data, as it is for instance illustrated by Caballero et al. (2006) and Martin and Ventura (2012). More precisely, these authors show that bubbles occur in periods of large GDP growth.

Recently, several papers have provided answers to these two criticisms. One of the closest contribution to ours is surely Martin and Ventura (2012), based on the existence of heterogeneous returns of capital investments and bubble creation. Other close explanations are based on the existence of a financial constraint generating an investment multiplier that boosts the wealth effect associated to bubble holding (Fahri and Tirole (2012), Hirano and Yanagawa (2013)). Some financial constraints also attribute to the bubble the role of collateral (Kocherlakota (2009), Miao and Wang (2011)), meaning that it allows to borrow larger amounts.[3] We differ from the first paper because our analysis does not need bubble shocks and from the last ones because our result does not require any binding finance constraint. In fact, our mechanism is based neither on some wealth effect, nor on a collateral role of bubble. Instead, we highlight that traders investing in the productive investment sell short the speculative asset to raise capital accumulation.

The model we examine is an overlapping generations model with three-period lived households. When young, households can invest in two assets: capital, which is used in production and give returns in the middle and old ages, and an asset without fundamental value, which is traded also in the middle age. This asset is a bubble when it is positively valued. In the middle age, agents can invest only in the bubble to transfer purchasing power to the old age.[4] It is important to underline that this framework introduces the notion that capital investment engages the investor and gives returns in a longer term than the more liquid speculative investment.

When the speculative asset is not valued, i.e. there is no bubble, the economy converges, with oscillations, to a unique steady state which is dynamically efficient if the saving rate of young agents is low enough. In all the paper, we maintain this assumption since it seems to be the more credible one (Abel et al. (1989)) and it is already well-known that bubbles may exist under dynamic inefficiency (Samuelson (1958), Tirole (1985)).

Focusing now on equilibria where the speculative asset is positively valued, there exists a steady state with a positive bubble that reaches the golden rule. The significance of this result is twofold. First, the existence of a bubble is in

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[2]As it is well-known, this idea was already emphasized in the model without capital accumulation by Samuelson (1958).

[3]See Miao (2014) for a short survey of this recent literature.

[4]In the middle age, agents do not invest in the productive asset because, at the equilibrium with bubble we focus on, they will prefer to reallocate income across generations using the speculative bubble.

accordance with a dynamically efficient bubbleless steady state. Second, the bubble is productive as it allows to have larger levels of capital accumulation and production than at the bubbleless steady state. This result is consistent with empirical evidence showing that bubbles occur in periods of large GDP growth. The underlying mechanism is based on the interaction between two effects, a credit effect and a crowding-out effect. The first one corresponds to the fact that young agents sell short the speculative asset to finance a raise of productive capital investment. In the middle age, agents reimburse the amount borrowed and also buy the speculative asset to transfer purchasing power to the old age. This saving in middle age corresponds to the crowding-out effect, which allows to reach the golden rule and also to give a positive value to the bubble.

The bubble reallocates the income of middle age households to young and old ones. This means that the demographic structure of overlapping generations play a key role because first, some agents are able to be short sellers of the bubble asset and second, there are heterogeneous investors in the market, young and middle age agents.[5] It is already known that heterogeneous investment opportunities play a key role to generate bubbles enhancing production. In overlapping generations economies, this has been emphasized by Martin and Ventura (2012). In their model, the bubble reallocates resources towards more productive agents. We contribute to this literature by showing that a productive bubble also arises if it reallocates resources towards young agents, by generating liquidities used to raise productive investment.

Since capital investment, which is a long term one that gives returns at the following two periods, is a key element of our story, we study the effect of different types of technological changes on the level of capital and bubble in the long-run. We show that a biased technological shock implying a larger return in the longer term may reduce capital. This happens because the bubble may disappear. Indeed, in this case, the bubble can no more be used to finance productive investment. On the contrary, if the technological shock is biased toward short term return of capital or increases the returns of capital in the short and long terms in like manner, we observe a raise of capital, production and bubble size. This seems to be in accordance with episodes of bubbles that are associated to new innovations, as it is documented and discussed in several contributions (see for instance Caballero et al. (2006), Lansing (2008, 2012), Scheinkman (2013)).

We further apply our framework to several economic concerns. We first focus on the consequences of regulation on limited borrowing. To this end, we introduce a credit constraint that prevents young agents from selling short more than an amount collateralized by the capital investment. When the constraint is binding, there exists again a bubbly steady state. The levels of production and capital investment are larger than at the bubbleless steady state, but they are lower than at the unconstrained bubbly steady state. Indeed, to ensure

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[5]Note that in models with infinitely lived agents, short sale positions cannot sustain the existence of a bubble. See Kocherlakota (1992) for more details.

a binding finance constraint, the return of capital needs to be larger than in the unconstrained case, which implies that capital is lower than the golden rule level. This allows us to discuss the effect of financial regulation on the long-run allocation. As a main insight with clear policy implications, we show that deregulation is associated to larger levels of production and capital and a lower price of the speculative bubble.

In a second extension, we contribute to the debate on the design of the fiscal policy that aims to promote long term investments and GDP (Wehinger (2011)). We show that the difference between capital and labor income taxes determines both the existence and the nature of the bubble, productive or unproductive. First, a reduction in the capital tax rate reduces the incentives to invest in the speculative asset, which may rule out the bubble. If this occurs, this fiscal policy reduces productive investment and production, as the bubble provided liquidity to finance productive investment. Second, a reduction in the labor income tax rate may decrease production. Indeed, such a fiscal policy reduces the liquidity role of the bubble and, for a sufficiently large reduction in the labor income tax rate, makes the bubble unproductive, meaning that it does no more enhance capital and production.

Following Weil (1987), we introduce in a third extension a probability of bubble crash. This stochastic environment introduces a risk premium between the return of the risky bubble and the return of capital. In comparison to the deterministic economy, this pushes up the level of capital at the stochastic bubbly steady state. In particular, the riskier the bubble is, the larger the level of capital. This also means that if the market crash occurs, the loss of capital will be greater.

As a last application, we compare income inequalities at the bubbly and bubbleless steady states. As we have seen, income reallocation across ages plays a key role in our story of bubble enhancing production. This also means that the bubble causes a reallocation of income across generations coexisting at the same period: a young one, a middle age one and an old one. This is an important issue because it allows us to study whether after a market crash, income inequalities are raising. Comparing the distribution of incomes at the bubbly and bubbleless steady states, we conclude that there are less income inequalities at the bubbly steady state. This means that if a market crash occurs, the economy may converge to an equilibrium characterized by more income inequalities. This finding is in accordance with recent empirical studies, like OECD (2013), that show an increase in income inequality after bubble crashes.

This paper is organized as follows. In the next section, we present our framework. In Section 3, we study our benchmark economy without bubble. In Section 4, we show the existence and provide a characterization of the steady state with a bubble. In Section 5, we discuss the effect of technological change. Section 6 is devoted to our four extensions on regulation on limited borrowing, fiscal policy, stochastic bubbles, and income inequality. Section 7 concludes, while many technical details are relegated to the Appendix.

2 Model

Time is discrete $(t = 0, 1, ..., +\infty)$ and there are two types of agents, households and firms.

2.1 Households

We consider an overlapping generations economy with constant population size. Each generation is populated by a continuum of mass one of agents that live for three periods. Each household has utility for consumption at each period of time. Preferences of an individual born in period t are represented by the following utility function:

$$u(c_{1,t}) + \beta u(c_{2,t+1}) + \beta^2 u(c_{3,t+2}) \tag{1}$$

where $\beta \in (0,1)$ is the subjective discount rate and $c_{j,t}$ amounts for consumption when young $(j = 1)$, in the middle age $(j = 2)$, and when old $(j = 3)$. For tractability, we assume that $u(c_{j,t}) = \ln c_{j,t}$.

The household supplies one unit of labor when young. She shares her labor income, given by the wage w_t, between consumption $c_{1,t}$ and a portfolio of two assets: productive capital k_{t+1} and a speculative asset $b_{1,t}$. To capture the idea that, in contrast to investing in the liquid speculative asset, productive investment is less liquid and engages the household in the long term, we assume that k_{t+1} provides a return $\phi_1 q_{t+1}$ in the second period of life and a return $\phi_2 q_{t+2}$ in the third period. Taking into account that the return on the speculative asset is R_{t+1} in the middle age, the household shares her income coming from the returns of his portfolio choice between consumption $c_{2,t+1}$ and a new investment in the speculative asset $b_{2,t+1}$. When old, the household's consumption $c_{3,t+1}$ is equal to her income coming from the return of the productive investment done in the young age and the investment in the speculative asset done in the middle age and remunerated at the return R_{t+2}.

Accordingly, the budget constraints in each period of life of an agent born in period t are:

$$c_{1,t} + k_{t+1} + b_{1,t} = w_t \tag{2}$$
$$c_{2,t+1} + b_{2,t+1} = \phi_1 q_{t+1} k_{t+1} + R_{t+1} b_{1,t} \tag{3}$$
$$c_{3,t+2} = \phi_2 q_{t+2} k_{t+1} + R_{t+2} b_{2,t+1} \tag{4}$$

Since capital investment provides returns in the following two periods, this introduces the difference between short term speculative investments and long term productive investments, the first ones being more liquid than the second ones, which is the key ingredient of our story. In order to clarify this, in Appendix B, we assume that capital fully depreciates after one period and the household can invest in capital in the middle age too. We show that, in such a model, there are no productive bubbles which foster production.

Note also that there is no productive investment in the middle age. This assumption is, in fact, an equilibrium result from a steady state with bubbles.

At this steady state a middle age household has no incentive to invest in long term investment. She will prefer to reallocate her income across ages using short term asset holdings, $b_{1,t}$ and $b_{2,t+1}$, and invest in the two-period lived productive asset when young. This is formally shown in Appendix C.

The speculative asset is supplied in one unit at a price p_t at period t. New investments in this asset by young and middle age agents are in quantities ϵ_t and $1 - \epsilon_t$, respectively. Therefore, the values of this asset bought or sold by these agents are $b_{1,t} = p_t \epsilon_t$ and $b_{2,t} = p_t(1 - \epsilon_t)$. Of course, if either $b_{1,t} < 0$ or $b_{2,t} < 0$, one type of trader is a short seller of this asset. It corresponds to equilibria where either $\epsilon_t < 0$ or $\epsilon_t > 1$. Finally, since this asset has no fundamental value, it is a bubble if $p_t = b_{1,t} + b_{2,t} > 0$, whereas there is no bubble if $p_t = b_{1,t} + b_{2,t} = 0$ and $b_{1,t} = b_{2,t} = 0$. Taking into account the definition of the variables $b_{1,t}$ and $b_{2,t}$,

$$b_{1,t+1} + b_{2,t+1} = R_{t+1} \left(b_{1,t} + b_{2,t} \right), \tag{5}$$

where the left-hand side of (5) measures the value of the bubble net purchasings in period $t+1$ and the right-hand side the value of the net sales in that period. It follows that the return R_{t+1} is the growth factor of the bubble price.

2.2   Firms

Firms produce with the following technology:

$$Y_t = K_t^\alpha L^{1-\alpha}, \quad \text{with } \alpha \in (0, 1/2)$$

where L is the number of workers and K_t is aggregate productive capital composed by investment of generations born at period $t-1$ and $t-2$. Since households, that live three periods, invest in capital when young and this investment has returns in the middle and old ages, the investment is two-period lived, and completely depreciates after. We assume that the different investments are perfect substitutes in the production

$$K_t = \phi_1 k_t + \phi_2 k_{t-1} \tag{6}$$

where $\phi_1 > 0$ and $\phi_2 > 0$ measure the productivities of the investments of the generations born in $t-1$ and $t-2$ respectively.[6] In a sense, k_t corresponds to vintage capital of the One-Hoss Shay type with a lifetime of two periods, but with different productivities during the lifetime (see Benhabib and Rustichini (1991) and Boucekkine et al. (2005)).

Since $L = 1$, profit maximization under perfect competition implies that the wage w_t and the return q_t from aggregate productive capital K_t are given by:

$$w_t = (1 - \alpha) K_t^\alpha \tag{7}$$

$$q_t = \alpha K_t^{\alpha-1} \tag{8}$$

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[6] We do not allow to have $\phi_2 = 0$. Indeed, in this case, capital is no more a long term asset and households have an incentive to invest in capital in middle age. This can be deduced from Appendix C.

3 Equilibrium without bubble

We first analyze the model without bubble, i.e. $b_{1,t} = b_{2,t} = 0$. It corresponds to our benchmark case and it will be used in the following sections to compare the properties of steady states with and without bubble. The household's budget constraints rewrite:

$$c_{1,t} + k_{t+1} = w_t \tag{9}$$
$$c_{2,t+1} = \phi_1 q_{t+1} k_{t+1} \tag{10}$$
$$c_{3,t+2} = \phi_2 q_{t+2} k_{t+1} \tag{11}$$

Maximizing the utility (1) under the budget constraints (9)-(11), we get the following arbitrage condition:

$$u'(c_{1,t}) = q_{t+1}\phi_1\beta u'(c_{2,t+1}) + q_{t+2}\phi_2\beta^2 u'(c_{3,t+2}) \tag{12}$$

Using $u(c_{j,t}) = \ln c_{j,t}$, we deduce that the dynamics are given by:

$$k_{t+1} = Bw_t = B(1-\alpha)(\phi_1 k_t + \phi_2 k_{t-1})^\alpha \tag{13}$$

with $B \equiv \beta(1+\beta)/(1+\beta+\beta^2) \in (0,1)$ being the savings rate when young.

At an interior steady state, we have $w/k = q(\phi_1 + \phi_2)(1-\alpha)/\alpha$. Using equation (13), we deduce that:

$$q = \frac{\alpha}{B(1-\alpha)(\phi_1 + \phi_2)} \equiv \underline{q} \tag{14}$$
$$k = [B(1-\alpha)]^{\frac{1}{1-\alpha}}(\phi_1 + \phi_2)^{\frac{\alpha}{1-\alpha}} \equiv \underline{k} \tag{15}$$

Note that, at a steady state, the resource constraint on the good market writes as $c_1 + c_2 + c_3 = [(\phi_1 + \phi_2)k]^\alpha - k$. Therefore, there is dynamic efficiency if the net return of investment is positive, i.e. $(\phi_1 + \phi_2)q > 1$. Using equation (14), this is satisfied under the following assumption:

Assumption 1 $B < \alpha/(1-\alpha)$.

In the rest of the paper, we restrict our analysis to the configuration where the steady state without bubble is dynamically efficient, which seems to be the more realistic assumption (see Abel et al. (1989)).

To address the convergence to this steady state, we can rewrite equation (13) as follows:

$$k_{t+1} = B(1-\alpha)(\phi_1 k_t + \phi_2 X_t)^\alpha \tag{16}$$
$$X_{t+1} = k_t \tag{17}$$

Using this dynamic system, we can construct the phase diagram of Figure 1 and analyze local dynamics to establish that the steady state is stable and the equilibrium converges to this steady state with oscillations. All these results are summarized in the following proposition, while details are given in Appendix A.1:
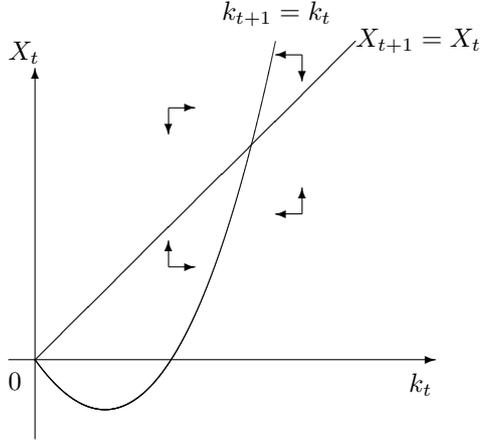
Figure 1: Dynamics without bubble

Proposition 1 *Under Assumption 1, there is a unique steady state without bubble $(\underline{q}, \underline{k}) \in \mathbb{R}^2_{++}$, which is dynamically efficient. This steady state is stable and the equilibrium converges to this steady state with dampened oscillations.*

4 Equilibrium with a bubble

We study now the equilibrium with a positive bubble, i.e. $b_{1,t} + b_{2,t} > 0$. Consumers maximize the utility subject to (2)-(4). Their optimal choices satisfy:

$$R_{t+1} = q_{t+1}\phi_1 + \frac{q_{t+2}\phi_2}{R_{t+2}} \tag{18}$$

$$R_{t+1} = \frac{u'(c_{1,t})}{\beta u'(c_{2,t+1})} \tag{19}$$

$$R_{t+2} = \frac{u'(c_{2,t+1})}{\beta u'(c_{3,t+2})} \tag{20}$$

The first equation is the no-arbitrage condition between the investment in the bubble and in productive capital. The second and third equations define the intertemporal choices in the first and second periods of life.

Bubbly steady state

From (5), a bubbly steady state is such that $R = 1$ and $b_1 + b_2 > 0$.[7] From

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[7]When there is a bubble, we restrict our attention to the analysis of the steady state. Dynamics are of a high order, strictly larger than three. Therefore, it is not possible to derive analytical results as it has been done in the bubbleless case.

(8) and (18), we obtain:

$$q = 1/\left(\phi_1 + \phi_2\right) \equiv \overline{q} \tag{21}$$

$$k = \alpha^{\frac{1}{1-\alpha}}\left(\phi_1 + \phi_2\right)^{\frac{\alpha}{1-\alpha}} \equiv \overline{k} \tag{22}$$

Note that (21) implies that this bubble steady state attains the golden rule. We then use (19) and (20) to obtain $c_2 = \beta c_1$ and $c_3 = \beta^2 c_1$. We substitute these two relationships and $w/k = q\left(\phi_1 + \phi_2\right)(1-\alpha)/\alpha = (1-\alpha)/\alpha$ in (2)-(4) to obtain:

$$b_2 = \left(\frac{\beta^2}{1+\beta+\beta^2} - \frac{\phi_2}{\phi_2 + \phi_1}\frac{\alpha}{1-\alpha}\right)\overline{w} \equiv \overline{b}_2 \tag{23}$$

$$b_1 = \left(\frac{\beta\left(1+\beta\right)}{1+\beta+\beta^2} - \frac{\alpha}{1-\alpha}\right)\overline{w} \equiv \overline{b}_1 \tag{24}$$

$$\overline{b}_1 + \overline{b}_2 = \left(\frac{\beta + 2\beta^2}{1+\beta+\beta^2} - \frac{2\phi_2 + \phi_1}{\phi_2 + \phi_1}\frac{\alpha}{1-\alpha}\right)\overline{w} \tag{25}$$

where $\overline{w}$ is the equilibrium value of the wage. The following proposition establishes the existence of a bubbly steady state:

Proposition 2 *Under Assumption 1, there is a bubbly steady state $(\overline{q}, \overline{k})$ if the following condition is satisfied:*

$$\frac{\beta + 2\beta^2}{1+\beta+\beta^2}\frac{\phi_2 + \phi_1}{2\phi_2 + \phi_1} > \frac{\alpha}{1-\alpha} \tag{26}$$

In addition, we have $\overline{q} < \underline{q}$ and $\overline{k} > \underline{k}$.

This proposition establishes an important result: without any binding finance constraint, there exists a bubbly steady state characterized by a higher level of production than at the efficient bubbleless steady state. The underlying mechanism is therefore not based on an investment multiplier associated to a credit constraint.

We note that inequality (26) is consistent with Assumption 1, if ϕ_2 is sufficiently low, in any case $\phi_2 < \beta\phi_1$. It follows that in the standard One-Hoss Shay case where $\phi_1 = \phi_2$, inequality (26) cannot be satisfied under Assumption 1.[8]

We also observe that Assumption 1 implies that $\overline{b}_1 < 0$. Since $\overline{b}_1 + \overline{b}_2 > 0$, it means that $\overline{b}_2 > 0$. On the one hand, $\overline{b}_1 < 0$ implies that the bubble allows young agents to invest more in productive asset by having a short position on the speculative asset. As a consequence, the bubble is beneficial for output, since output is larger than in the bubbleless (efficient) steady state. This corresponds to the credit effect of the bubble. This mechanism can also be understood noting that without bubble, we have $k = Bw$ at the steady state (see equation (13)). Here, we rather have $k + b_1 = Bw$. Strictly speaking, we have not a usual

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[8]We further discuss the role of ϕ_2 on the existence of a bubbly steady state in Section 5.

leverage effect coming from a binding finance constraint (see for instance Fahri and Tirole (2012)). But, being a short seller of the bubble, the young agent increases investment in productive capital. In fact, using (21), we easily deduce that $k = w\alpha/(1-\alpha) > Bw$ under Assumption 1. Therefore, at a bubbly steady state, the productive investment is larger than savings when young.

On the other hand, $\bar{b}_2 > 0$ implies that middle age households finance the amount borrowed by the young and also generate the positive value of the bubble. This induces a crowding-out effect that fosters the economy to reach the golden rule, whose capital stock is $\bar{k}$. This last effect, already identified in the seminal paper by Tirole (1985), has a negative impact on production but improves consumption since it allows to reach the golden rule. In fact, both the credit and the crowding-out effects imply a reallocation from the middle age to young and old ages, since at the middle age, the household reimburses the amount borrowed when young and invests in the bubble to raise consumption when old.

The demographic structure of overlapping generations play a key role because it introduces heterogeneous traders in the market, young and middle age agents. This heterogeneity explains that the bubble increases production. As young agents are short sellers of the speculative asset, they obtain liquidity needed to raise productive investment.

It is worth mentioning that if Assumption 1 does not hold, we have $\bar{b}_1 \geqslant 0$. Young agents do not borrow to raise the production. Since the bubbleless steady state $(\underline{q}, \underline{k})$ is characterized by dynamic inefficiency in this case, we have $\bar{q} > \underline{q}$ and $\bar{k} < \underline{k}$. In other words, the existence of the bubble reduces productive investment and production. The only role played by the bubble is to crowd-out productive investment, as highlighted in Tirole (1985).

It is also relevant to note the importance of our long term investment. Indeed, as we already mentioned, we develop in Appendix B the same model but capital fully depreciates after one period and agents invest in capital in the middle age too. We show that, in this case, it is not possible to have a bubbly steady state if the bubbleless one is dynamically efficient. In other words, the existence of a bubbly steady state requires that the bubbleless one is characterized by over-accumulation. As in Tirole (1985), the bubble reduces production in the long run. Regarding the two effects we just highlight, this can be explained as follows. The credit effect has a smaller impact if capital depreciates after one period of use and the crowding-out effect is larger because, in the middle age, the agent makes a portfolio choice between capital and bubble holding.

In this paper, we identify a new mechanism through which the bubble raises production, which is in accordance with the empirical evidence. This also means that if the bubble suddenly crashes, production will reduce, which is also in accordance with what we observe.[9] These different facts are well documented by Caballero et al. (2006) and Martin and Ventura (2012), among others.

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[9]Bubble crash is accurately studied in Section 6.3.

5 Technological change

Since illiquid capital is a key ingredient of our results, we study in this section the effects of technological changes on both the bubbly and bubbleless steady states. These technological changes consist on modifying the productivities ϕ_1 or ϕ_2. In this way, we can study the effect of a technological change biased toward either short term return (ϕ_1 relative to ϕ_2 increases) or a long term return (ϕ_2 relative to ϕ_1 increases). Increasing ϕ_1 and ϕ_2 proportionally, we also investigate the case of a technological change neutral with respect to the short versus long term returns of capital. We use this analysis to highlight the effect of technological change on long run capital accumulation in our model, where the bubble provides liquidities to finance the productive investment.

Starting with the steady state without bubble, that exists and is efficient for all parameter configurations in accordance with Assumption 1, the effect of technological changes on the allocation is quite obvious. By direct inspection of equations (14) and (15), we easily show the following result:

Proposition 3 *(Bubbleless steady state) Under Assumption 1, productive investment $\underline{k}$ increases and the return $\underline{q}$ decreases if there is an increase of either ϕ_1, or ϕ_2, or of both, ϕ_1 and ϕ_2, in the same proportion.*

The explanation is quite simple. Without bubble, capital is given by savings of young and, therefore, it depends on the level of the wage. A larger level of ϕ_1, or ϕ_2, or ϕ_1 and ϕ_2 taking ϕ_1/ϕ_2 as constant pushes up the wage, which raises productive investment and decreases q.

We now focus on the steady state with a positive bubble:

Proposition 4 *(Bubbly steady state) Considering that Assumption 1 and inequality (26) are satisfied, we have the following:*

1. *Productive investment $\overline{k}$ increases, while the return $\overline{q}$ decreases if there is an increase of either ϕ_1, or ϕ_2 or of both, ϕ_1 and ϕ_2, in the same proportion;*

2. *$\overline{b}_1$ decreases if there is an increase of either ϕ_1, or ϕ_2 or of both, ϕ_1 and ϕ_2, in the same proportion;*

3. *$\overline{b}_2$ and $\overline{b}_1 + \overline{b}_2$ increase if there is an increase of either ϕ_1 or of both, ϕ_1 and ϕ_2, in the same proportion, but they decrease if there is an increase of ϕ_2.*

Proof. See Appendix A.2. ∎

An increase of ϕ_1 increases the incentive to invest in productive capital when young. As a consequence, the wage increases, which reinforces the investment in productive capital and, hence, the amount borrowed by the young households increases (lower $\overline{b}_1$). The middle age household has a larger income, even relatively to what she expects to have when old. Hence, she transfers a larger

amount to the old age, increasing her holding of the bubble $\bar{b}_2$. Despite the more significant short position of the young on the speculative asset, the value of the bubble, $\bar{b}_1 + \bar{b}_2$, increases.

Following a raise of ϕ_2, the difference between the returns of the productive asset and the bubble when old increases. Therefore, there is a substitution effect that pushes up productive investment, but pushes down the purchasing of the bubble $\bar{b}_2$ in the middle age. At a steady state, the wage increases following a general equilibrium effect. This reinforces the raise of $\bar{k}$, because young increase their savings. Under Assumption 1, the cost of investment is lower than its benefit, meaning that a young agent has an incentive to borrow more (lower $\bar{b}_1$) to still increase her productive investment. Finally, the income effect associated to the larger wage also has a positive impact on bubble purchasing $\bar{b}_2$, but is dominated by the negative substitution effect. This implies that the value of the bubble $\bar{b}_1 + \bar{b}_2$ decreases.

When there is a proportional increase of ϕ_1 and ϕ_2, i.e. keeping ϕ_2/ϕ_1 constant, the technological change is neutral. Of course, capital investment raises because its return through the life-cycle is greater. Because of the general equilibrium effect mentioned earlier, the wage and the capital incomes in the middle and old ages increase. Therefore, there is no reallocation effect in the portfolio choices, but more significant positions of bubble holdings, meaning a lower $\bar{b}_1$ and a higher $\bar{b}_2$. The value of the bubble $\bar{b}_1 + \bar{b}_2$ increases, because it linearly depends on income.

Note that aggregate production is given by $qk\left(\phi_1 + \phi_2\right)/\alpha = \left(q/\alpha\right)^{\frac{\alpha}{\alpha-1}}$. Since at each steady state, bubbly or bubbleless, q decreases following a raise of either ϕ_1, or ϕ_2, or both proportionally, aggregate production evaluated at these different steady states raises in each case.

In Proposition 4, we have enlightened how the bubbly steady state evolves according to various technological changes. We have shown that whatever the type of technological change considered, productive capital increases. However, the results on the value of the bubble may be the opposite, depending if the technological change is biased toward long term, or rather toward short term or neutral. In addition, by direct inspection of Proposition 2, we see that the level of the relative return of investment between the long and short terms ϕ_2/ϕ_1 affects the existence of the bubbly steady state and, therefore, its coexistence with the bubbleless one. We study now more deeply what is the effect of the different technological changes on the existence of the bubbly steady state.

We first note that inequality (26) is equivalent to $\phi_2/\phi_1 < \Phi$, with:

$$\Phi \equiv \frac{\frac{\beta+2\beta^2}{1+\beta+\beta^2} - \frac{\alpha}{1-\alpha}}{2\frac{\alpha}{1-\alpha} - \frac{\beta+2\beta^2}{1+\beta+\beta^2}} \tag{27}$$

Using Assumption 1, we note that the denominator of (27) is strictly positive. Hence, Φ is strictly positive if and only if:

Assumption 2 $\alpha/(1-\alpha) < \frac{\beta+2\beta^2}{1+\beta+\beta^2}$.

13

We can also show that under Assumption 1, $\Phi < \beta$. We then deduce the following proposition:

Proposition 5 *Under Assumptions 1 and 2, the bubbly steady state coexists with the bubbleless steady state for all $\phi_2/\phi_1 < \Phi$, the bubble collapses for $\phi_2/\phi_1 = \Phi$, and there is only the bubbleless steady state for $\phi_2/\phi_1 > \Phi$. This means that:*

1. *Any neutral technological change that increases both ϕ_1 and ϕ_2 proportionally does not affect the existence of the bubbly steady state;*

2. *A technological change biased toward short term return that increases ϕ_1 making $\phi_1 > \phi_2/\Phi$ causes the existence of the bubbly steady state;*

3. *A technological change biased toward long term returns that increases ϕ_2 making $\phi_2 < \phi_1\Phi$ rules out the existence of the bubbly steady state.*

There is a level of ϕ_2/ϕ_1 above which the bubbly steady state does no more exist and the only steady state is the bubbleless one. In contrast, for ϕ_2/ϕ_1 low enough, there is a multiplicity of steady states, the bubbleless one coexists with the bubbly one. Indeed, if ϕ_2 is low regarding ϕ_1, k is more a short term investment because its return is larger in the first period than in the second one. In this case, middle age agents transfer wealth to the future by using the speculative asset. This explains the existence of the bubble, which allows to have a steady state where productive investment benefits from the bubble liquidities. On the contrary, if ϕ_2 is large regarding ϕ_1, k is more a long term investment because its return is larger in the second period. In this case, middle age agents do not use the bubble to transfer wealth to the future and there is no bubble liquidities to raise productive investment. Since there is no bubble, there will be dynamic paths that converge to the bubbleless steady state with low accumulation of capital.[10]

Propositions 3-5 allow to have a general picture of long-run capital following a technological change, biased toward short or long term returns, or which is neutral. Figure 2 illustrates the different cases. Panel (b) shows that if agents start by coordinating their expectations on the bubbly steady state and there is a technological change that raises ϕ_2, productive investment k increases if ϕ_2 is low enough. Then, there is a level of ϕ_2 above which the bubbly steady state disappears and productive investment may converge to the stable bubbleless steady state characterized by a lower level of k. When ϕ_2 crosses the value such that the bubbly steady state disappears, we may observe a decrease of productive investment k even if the technological change increases its return in the long term. This means that a larger return in the long term does not

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[10]Indeed, even we are not able to address analytically the dynamic issues, we can see using equation (5), that the dynamic path of the bubble is given by $\prod_{i=t}^{+\infty} R_i \equiv \mathcal{R}_t$. In the long run, we can exclude $\mathcal{R}_t > 1$ because the bubble tends to $+\infty$ and could no more be bought by the households. $\mathcal{R}_t = 1$ mainly corresponds to the bubbly steady state. When $\mathcal{R}_t < 1$, the bubble collapses in the long run, i.e. $b_{1,t}$ and $b_{2,t}$ tend to 0.

necessarily imply higher levels of investment. In this paper, the bubble provides liquidities to finance the productive investment, but a too significant return of productive investment in the long-term rules out the bubble, and is therefore damaging for productive investment itself.
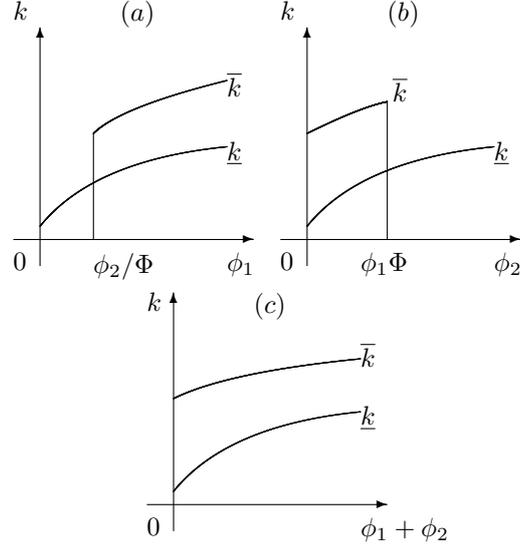

Figure 2: Effect of technological change

Panel (a) of Figure 2 illustrates the opposite configuration, which happens if there is a technological change that raises ϕ_1. There is a threshold level above which the bubbly steady state exists. Then, if agents coordinate their expectations on the bubbly steady state, any further raise of the return ϕ_1 implies a raise of capital. In addition, Proposition 4 allows us to argue that the bubble also enlarges under this type of technological change.

Finally, if $\phi_2/\phi_1 < \Phi$, a positive technological change which is neutral has a similar effect than the previous one. This is shown in Panel (c) of Figure 2. If agents coordinate their expectations on the bubbly steady state, such a technological change induces a raise of capital and of the bubble size.

The last two results are especially interesting because they illustrate than in our model, an innovation biased toward the short term or neutral between the short and the long terms is associated to larger levels of capital, production and bubble size. The evidence that episodes of bubbles are often associated to new innovations has been documented and illustrated in several contributions (see for instance Caballero et al. (2006), Lansing (2008, 2012) and Scheinkman (2013)). This paper contributes to this literature by showing that this occurs if the innovation does not mainly increase the productivity in the long term.

15

6 Economic applications

In this section, we apply and extend our framework to several economic concerns. First, we introduce in our framework a credit constraint that limits borrowing. This allows us to discuss the effect of regulation on the financial markets on the level of productive investment in the long-run. Second, we highlight the role of the fiscal policy on the existence of bubbles. Third, we extend our framework to consider a positive probability of bubble crash. We are therefore able to discuss the interplay between the riskiness of the speculative asset and productive investment. Finally, we further compare the properties of the bubbly and bubbleless steady states analyzing income inequalities evaluated at each steady state.

6.1 Regulation on limited borrowing

We extend our basic framework to introduce a credit constraint that limits the amount borrowed using short positions on the speculative asset. This allows us to study the implications of credit constraints on productive investment and also the effect of regulation of the financial sphere on the level of capital.

As argued by Scheinkman (2013), borrowers are often forced to cover their short positions and loans are often collateralized. Accordingly, since young agents are those who have an incentive to borrow, we limit their possibilities to borrow introducing the following credit constraint:

$$R_{t+1}b_{1,t} \geq -\theta q_{t+1}\phi_1 k_{t+1}, \text{ with } \theta \in [0,1) \tag{28}$$

This constraint, which is extensively used in the literature (Hirano and Yanagawa (2013), Le Van and Pham (2015)), stipulates that if one household borrows when young having a short position on bubble holdings, the reimbursement of this debt in the middle age is limited by the return of the productive investment at that period. If this constraint is binding, productive investment acts as a collateral. The parameter θ measures the borrowing limit. If $\theta = 0$, credit is not possible. As θ increases, the credit constraint relaxes as the borrowing limit increases.

Note first that equation (28) also implies that the income in the middle age is strictly positive. Second, we do not introduce such a constraint on the investment decision in the middle age, because the equilibria we consider are always characterized by $b_{2,t+1} > 0$, meaning that the introduction of such a constraint is irrelevant.

In the following, we call a constrained equilibrium / steady state, an equilibrium / steady state where the credit constraint (28) is binding.

6.1.1 Constrained equilibrium

In the constrained equilibrium, the optimal choice of consumers satisfies:[11]

$$R_{t+1}b_{1,t} + \theta q_{t+1}\phi_1 k_{t+1} = 0 \tag{29}$$

$$R_{t+2} = \frac{u'(c_{2,t+1})}{\beta u'(c_{3,t+2})} \tag{30}$$

$$\beta u'(c_{2,t+1}) \left((1-\theta)q_{t+1}\phi_1 + \frac{q_{t+2}\phi_2}{R_{t+2}}\right) = \left(1 - \frac{\theta q_{t+1}\phi_1}{R_{t+1}}\right) u'(c_{1,t}) \tag{31}$$

The first equation is the binding credit constraint, the second and third equations correspond to the intertemporal choices in the first and second periods of life.

For further reference, we rewrite equations (29)-(31). First, using (2), (3) and (29), we rewrite (31) as:

$$\beta \left[w_t - \left(1 - \theta \frac{q_{t+1}\phi_1}{R_{t+1}}\right) k_{t+1}\right] \left((1-\theta)q_{t+1}\phi_1 + \frac{q_{t+2}\phi_2}{R_{t+2}}\right)$$
$$= \left(1 - \frac{\theta q_{t+1}\phi_1}{R_{t+1}}\right) [(1-\theta)q_{t+1}\phi_1 k_{t+1} - b_{2,t+1}] \tag{32}$$

Finally, we use (3), (4) and (29) to rewrite (30) as

$$\beta R_{t+2} [(1-\theta)q_{t+1}\phi_1 k_{t+1} - b_{2,t+1}] = q_{t+2}\phi_2 k_{t+1} + R_{t+2}b_{2,t+1} \tag{33}$$

6.1.2 Constrained bubbly steady state

From equation (5), a bubbly steady state is obtained when $b_1 + b_2 > 0$ and $R = 1$. From (29) and (33), we obtain:

$$b_2 = \frac{[\beta(1-\theta)\phi_1 - \phi_2]qk}{(1+\beta)} \tag{34}$$

$$b_1 = -\theta\phi_1 qk < 0 \tag{35}$$

$$b_1 + b_2 = \left(\frac{[\beta - \theta(1+2\beta)]\phi_1 - \phi_2}{1+\beta}\right) qk \tag{36}$$

Combining (32) and (34), we deduce that

$$k = \frac{Bw^*}{1 - \theta q^*\phi_1} \equiv k^* \tag{37}$$

and, using (7) and (8), we obtain:

$$q = \frac{1}{B\left(\frac{1-\alpha}{\alpha}\right)(\phi_1 + \phi_2) + \theta\phi_1} \equiv q^* \tag{38}$$

where w^* is the equilibrium value of the wage. Of course, substituting k^* and q^* into (34) and (35), we get the equilibrium values b_1^* and b_2^*.

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[11]The consumers' problem is precisely solved in Appendix A.3.

Proposition 6 *Assuming that the credit constraint is binding and Assumption 1 holds, there is a constrained bubbly steady state (q^*, k^*) if the following two conditions are satisfied:*

$$\phi_2/\phi_1 < \beta \ \text{ and } \ \theta < \frac{\beta\phi_1 - \phi_2}{(1 + 2\beta)\phi_1} \equiv \widehat{\theta} \tag{39}$$

In addition, we have $q^* \leqslant \underline{q}$ and $k^* \geqslant \underline{k}$.

When the credit constraint is binding, a bubble exists if (i) ϕ_1 is large in comparison to ϕ_2, which means that households are not able to transfer to much wealth to the old age without the bubble; (ii) θ is low, implying that the credit constraint restricts the possibility of credit when young. In particular, in the standard One-Hoss Shay case where $\phi_1 = \phi_2$, inequalities (39) cannot be satisfied and there is no bubbly steady state with a binding constraint.

As in the unconstrained case,[12] the important finding is that production is larger at a steady state with bubble than in the steady state without bubble. Hence, the bubble is beneficial for the level of output, in accordance with the empirical evidence. This result is again due to the coexistence of a credit effect and a crowding-out effect. However, in contrast to the unconstrained case, there is now a leverage effect that pushes up productive investment. Indeed, by direct inspection of equation (37), we see that productive investment raises with income and savings when young Bw^*. However, there is also an investment multiplier $1/(1 - \theta q^* \phi_1)$ which is a measure of leverage. The larger is the return from capital, the more a young agent can invest being a short seller of the bubble.

6.1.3 Co-existence of steady states and effect of deregulation

On the one hand, there always exists a steady state without bubble (Proposition 1). On the other hand, Proposition 6 shows that there exists a bubbly steady state when the credit constraint is binding, whereas the unconstrained bubbly steady state may, of course, exist if the constraint is not binding (Proposition 2). The question that arises is then to know which type of bubbly steady state will coexist with the bubbleless one. We now investigate more deeply this question, relating it to the value of the parameter θ, which can also be seen as a measure of financial deregulation.

As shown in Appendix A.3, the credit constraint is binding along the dynamic path if $c_{2,t+1} > R_{t+1}\beta c_{1,t}$. At a steady state, this means that:

$$(1 - \theta)\phi_1 q k - b_2 > \beta R\left(w - k + \frac{\theta q \phi_1 k}{R}\right)$$

At a constrained bubbly steady state, we have $R = 1$, $b_2 = b_2^*$ and $q = q^*$. Hence, the constrained bubbly steady state exists if:

$$\theta < \left[1 - B\left(\frac{1-\alpha}{\alpha}\right)\right]\frac{(\phi_1 + \phi_2)}{\phi_1} \equiv \theta^*. \tag{40}$$

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[12]In this section, we call the unconstrained bubbly steady state, the steady state obtained in Proposition 2.

We note that $\theta^* < \widehat{\theta}$ if and only if inequality (26) is satisfied.[13] This means that for $\theta \geqslant \theta^*$, Proposition 2 applies, the credit constraint is not binding when the bubble is stationary, and the unconstrained bubbly steady state may exist. We then deduce the following proposition:

Proposition 7 *Under Assumption 1, there always exists a unique steady state $(\underline{q}, \underline{k})$ without bubble.*

1. *If inequality (26) holds, it coexists with the constrained bubbly steady state (q^*, k^*) for $\theta < \theta^*$ and with the unconstrained bubbly steady state $(\overline{q}, \overline{k})$ for $\theta > \theta^*$.*

2. *If inequality (26) does not hold, it coexists with the constrained bubbly steady state (q^*, k^*) for $\theta < \widehat{\theta}$. If $\theta \geqslant \widehat{\theta}$, there is no bubbly steady state.*

This proposition establishes that a constrained bubbly steady state exists if θ is sufficiently low. Indeed, if θ is large enough, the credit constraint is not so stringent. Young agents are allowed to borrow significant amounts. Either the credit constraint is not binding and the bubbly steady state is the unconstrained one (case 1), or the supply of speculative asset from young $-b_1$ is too important to have an equilibrium with $b_1 + b_2 > 0$ (case 2).

Strictly speaking, the parameter θ can be seen as a policy parameter that regulates the credit market.[14] The larger θ is, the more important is the deregulation of the financial sector, since a young trader can raise her short position on the asset market in accordance with the credit constraint. On the contrary, if θ is quite small, there is more regulation of the financial sphere, because borrowing and short positions are more limited.

An interesting insight is that when inequality (26) holds, i.e. case 1 of Proposition 7 applies, deregulation cannot prevent the existence of bubbles. However, both the nature of the bubble and the amount of productive investment change. We study now more deeply these effects of θ. The results are summarized in the following proposition:

Proposition 8 *Under Assumption 1 and inequality (26), the following holds for all $\theta \in [0, \theta^*)$.*

1. *When θ increases, k^* increases, while q^*, b_1^* and $b_1^* + b_2^*$ decreases.*

2. *If $\theta = 0$, $k^* = \underline{k}$, $q^* = \underline{q}$, $b_1^* = 0$, and $b_2^* > 0$.*

3. *$\lim_{\theta \to \theta^*} k^* = \overline{k}$, $\lim_{\theta \to \theta^*} q^* = \overline{q}$, $\lim_{\theta \to \theta^*} b_1^* = \overline{b}_1$ and $\lim_{\theta \to \theta^*} b_2^* = \overline{b}_2$.*

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[13]Note that if Assumption 1 is not satisfied, i.e. the steady state without bubble is dynamically inefficient, θ^* is negative. This means that a constrained bubbly steady state cannot exist.

[14]Note that Hirano and Yanagawa (2013) interpret this parameter as the degree of market imperfection, the larger θ is, the lower the imperfection.

Proof. See Appendix A.4. ∎

This proposition shows that when θ increases from 0 to θ^*, k^* continuously increases from $\underline{k}$ to $\overline{k}$ (see also Figure 3) and q^* continuously decreases from $\underline{q}$ to $\overline{q}$. This means that when the deregulation is more important (θ larger), the production raises. The mechanism goes of course through the credit effect, which is reinforced when θ becomes larger. In particular, when $\theta = 0$, young agents cannot borrow by selling the speculative asset. There is no credit effect. Therefore, the production is exactly the same than at a bubbleless steady state. When θ tends to its upper bound θ^*, the credit effect is the largest one at a constrained steady state and the production tends to the level obtained in the unconstrained case.

This result can be connected to some findings of Fahri and Tirole (2012), who rather interpret θ as the degree of pledgeability. In contrast to them, a bubble may exist if the degree of pledgeability is high enough, because it may exist when the credit constraint is no more binding. Our result also differs from Hirano and Yanagawa (2013) where an intermediate degree of pledgeability is required for the existence of a bubble.
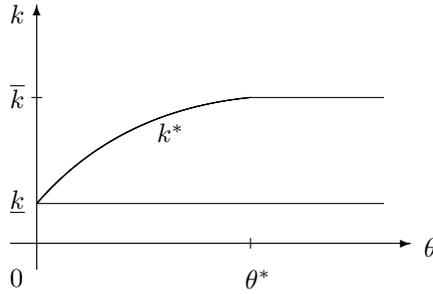

Figure 3: Effect of deregulation

As a corollary, this means that the difference between k^* and $\underline{k}$ raises with θ. It follows that, even if it is not formally introduced, a crash of the bubble that fosters the economy to converge to a bubbleless steady state with $k = \underline{k}$ is more costly for an economy more deregulated, because the economy experiences a larger decrease of capital and production.[15]

A relevant insight that follows from this analysis is that under more deregulation, the short position becomes larger (b_1 more negative), explaining the more significant credit effect, but the value of the bubble asset $b_1 + b_2$ reduces. The intuition is as follows. Deregulation, by raising the borrowing facilities, increases the net supply of speculative asset, which causes the reduction in the price of this asset. Hence, more deregulation implies a lower level of speculative bubble. It follows that if inequality (26) holds, then regulation of the financial

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[15]Indeed, when the bubble crashes ($b_{1,t} = b_{2,t} = 0$), the economy behaves as in Section 3.

system cannot prevent the existence of bubbles and, by constraining the supply of speculative assets, it increases the size of the bubble and reduces production.

6.2 Fiscal policy

There is a debate on the design of the fiscal policy that aims to promote long term investments (Wehinger (2011)). The purpose of this section is to contribute to this debate by studying the effects of taxes on both the existence of bubbles and on the stationary values of capital and GDP. To this end, we consider a government that collects taxes on labor income, $\tau_w < 1$, and on capital income, $\tau_k < 1$.[16] We do not introduce a tax on the return of the speculative asset. Its net return is zero at a bubbly steady state and, thus, such a tax would be irrelevant in the long run.

We assume that government revenues are employed to finance non-productive government spending G_t. In this way, government spending does not distort individuals decisions. At each period, the government budget is balanced:

$$G_t = \tau_w w_t + \tau_k q_t (\phi_1 k_t + \phi_2 k_{t-1})$$

Note that if either $\tau_w < 0$ or $\tau_k < 0$, there is a subsidy on either labor or capital. Such a policy scheme is in accordance with the government budget constraint, as long as $G_t \geqslant 0$.

The introduction of this fiscal policy modifies the individual budget constraints as follows:

$$c_{1,t} + k_{t+1} + b_{1,t} = (1 - \tau_w) w_t \tag{41}$$

$$c_{2,t+1} + b_{2,t+1} = (1 - \tau_k) q_{t+1} \phi_1 k_{t+1} + R_{t+1} b_{1,t} \tag{42}$$

$$c_{3,t+2} = (1 - \tau_k) q_{t+2} \phi_2 k_{t+1} + R_{t+2} b_{2,t+1} \tag{43}$$

We study now how taxes modify the long run equilibrium with and without bubble. We examine in particular which type of fiscal policy promotes capital accumulation.

6.2.1 Bubbleless steady state with taxation

Without bubble ($b_{1,t} = b_{2,t} = 0$), solving the consumer's problem when the budget constraints are (41), (42) and (43), it can be easily shown that capital accumulation is only modified by the labor income tax as follows:

$$k_{t+1} = (1 - \tau_w) B w_t = (1 - \tau_w) B (1 - \alpha)(\phi_1 k_t + \phi_2 k_{t-1})^\alpha$$

From the firm's problem, we obtain that, at a steady state, $w/k = q(\phi_1 + \phi_2)(1 - \alpha)/\alpha$. We easily deduce that:

$$q = \frac{\alpha}{(1 - \tau_w) B (1 - \alpha)(\phi_1 + \phi_2)} \equiv \underline{q}^T$$

$$k = [(1 - \tau_w) B (1 - \alpha)]^{\frac{1}{1-\alpha}} (\phi_1 + \phi_2)^{\frac{\alpha}{1-\alpha}} \equiv \underline{k}^T$$

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[16]We do not introduce lower bounds on the tax rates to allow to subsidize either capital or labor income.

Since capital accumulation comes from savings when young, labor income and labor taxation play a key role in determining the stationary level of capital. In contrast, because we have log-linear preferences, the capital return, and therefore the tax rate on capital, do not affect the steady state. Of course, if savings would depend on the return on capital, but would be not too sensitive, the key role of labor taxation regarding the tax rate on capital would still hold.

6.2.2 Bubbly steady state with taxation

Maximizing the utility (1) under the budget constraints (41)-(43), we show that, at the bubbly equilibrium $(b_{1,t} + b_{2,t} > 0)$, the consumer's optimal choices satisfy:

$$R_{t+1} = (1 - \tau_k) \left(q_{t+1}\phi_1 + \frac{q_{t+2}\phi_2}{R_{t+2}} \right)$$

$$R_{t+1} = \frac{u'(c_{1,t})}{\beta u'(c_{2,t+1})}$$

$$R_{t+2} = \frac{u'(c_{2,t+1})}{\beta u'(c_{3,t+2})}$$

At the bubbly steady state, $R = 1$. Then, the previous equations simplify to $c_2 = \beta c_1$, $c_3 = \beta^2 c_1$ and

$$q = \frac{1}{(1 - \tau_k)(\phi_1 + \phi_2)} \equiv \bar{q}^T$$

Using (7) and (8), we obtain that:

$$
\begin{aligned}
k &= [(1 - \tau_k)\alpha]^{\frac{1}{1-\alpha}} (\phi_1 + \phi_2)^{\frac{\alpha}{1-\alpha}} \equiv \bar{k}^T \\
w &= (1 - \alpha)[\alpha(1 - \tau_k)(\phi_1 + \phi_2)]^{\frac{\alpha}{1-\alpha}} \equiv \bar{w}^T
\end{aligned}
$$

and using (41)-(43),

$$b_2 = \left(\frac{\beta^2}{1 + \beta + \beta^2} - \frac{\Delta\phi_2}{\phi_2 + \phi_1}\frac{\alpha}{1 - \alpha} \right)(1 - \tau_w)\bar{w}^T \equiv \bar{b}_2^T$$

$$b_1 = \left(B - \frac{\alpha\Delta}{1 - \alpha} \right)(1 - \tau_w)\bar{w}^T \equiv \bar{b}_1^T$$

$$\bar{b}_1^T + \bar{b}_2^T = \left(\frac{\beta + 2\beta^2}{1 + \beta + \beta^2} - \frac{\alpha\Delta}{1 - \alpha}\frac{\phi_1 + 2\phi_2}{\phi_1 + \phi_2} \right)(1 - \tau_w)\bar{w}^T$$

where $\Delta \equiv \frac{1-\tau_k}{1-\tau_w}$, measures the asymmetric effect of taxes on the two sources of income. When $\Delta = 1$, the two tax rates are identical. In this case, they do not affect the existence of the bubble, as follows from the expression of $\bar{b}_1^T + \bar{b}_2^T$. Thus, the existence of a bubble does not depend on the level of the two tax rates, but on the difference between them. More precisely, a bubble $(\bar{b}_1^T + \bar{b}_2^T > 0)$ exists if:

$$\Delta < \left(\frac{\beta + 2\beta^2}{1 + \beta + \beta^2} \right)\left(\frac{\phi_1 + \phi_2}{\phi_1 + 2\phi_2} \right)\left(\frac{1 - \alpha}{\alpha} \right) \equiv \overline{\Delta}$$

22

The parameter Δ also determines if the bubble is capital enhancing or productive. More precisely, $\bar{q}^T < \underline{q}^T$ and $\bar{k}^T > \underline{k}^T$ if and only if $\Delta > \underline{\Delta}$, where $\underline{\Delta} \equiv \left(\frac{1-\alpha}{\alpha}\right) B$. In this case, the bubble is productive and is characterized by $\bar{b}_1^T < 0$. Therefore, the productive role of the bubble is determined by the relative significance of the capital tax rate τ_k and the labor one τ_w. The reason is quite immediate. As we have seen above, capital at the bubbleless steady state is mainly determined by the wage income, and therefore, decreases with the labor tax rate. In contrast, the level of capital at the bubbly steady comes from an arbitrage condition between the speculative asset and capital. This explains that capital at the bubbly steady state is only affected by the capital tax rate, through a decreasing relationship. We understand from these two observations that the bubble is capital enhancing if Δ is large enough, i.e. the capital tax rate is low enough and/or the labor tax rate is sufficiently large.

Since $\underline{\Delta} < \overline{\Delta}$ if and only if $\phi_2/\phi_1 < \beta$, we deduce that:

Proposition 9 *Under Assumption 1, the following holds:*

1. *For $\phi_2/\phi_1 < \beta$, (i) if $\Delta \leq \underline{\Delta}$, there is a stationary non-productive bubble with $\bar{k}^T \leq \underline{k}^T$ and $\bar{b}_1^T \geq 0$; (ii) if $\Delta \in \left(\underline{\Delta}, \overline{\Delta}\right)$, there is a stationary productive bubble with $\bar{k}^T > \underline{k}^T$ and $\bar{b}_1^T < 0$; and (iii) if $\Delta \geq \overline{\Delta}$ there is no bubbly steady state.*

2. *For $\phi_2/\phi_1 \gtrless \beta$, (i) if $\Delta < \overline{\Delta}$, there is a stationary non-productive bubble with $\bar{k}^T < \underline{k}^T$ and $\bar{b}_1^T > 0$; and (ii) if $\Delta \geq \overline{\Delta}$, there is no bubbly steady state.*

This proposition shows that the fiscal policy parameter Δ determines the effect of fiscal policy on both the existence of bubbles and its nature, productive or unproductive. As explained above, if this parameter is sufficiently small (τ_k large with respect to τ_w), the bubble is non-productive. In this case, the young agents have a long position on the speculative asset ($\bar{b}_1^T \geq 0$). Since a large tax on capital income makes the investment in productive capital less attractive, the bubble is used to transfer wealth to the future, implying that the bubble is not productive. The same rationality explains that if Δ is sufficiently large, then there is no bubble in the economy. In this case, the tax on capital income is low in comparison to the taxes on labor income. With respect to capital, there is less incentive to invest in the speculative asset because the fiscal policy is in favor of the return on capital and the after-tax income when young is quite low. Finally, for intermediate values of Δ, a productive bubble may arise because the after-tax return on capital is not too large to rule out bubble holding in the middle age and the after-tax wage is large enough to sustain a large enough productive investment when young and the purchase of the bubble in the middle age.[17] In this case, the bubble is used by the young to finance productive capital and by the middle age to transfer wealth to the future.

---

[17]We further note that the condition $\phi_2/\phi_1 < \beta$ required to get a productive bubble was already highlighted in Section 5.

We can now advocate which type of fiscal policy is the most adequate to promote high levels of capital and GDP. Let us focus on the most interesting case where $\phi_2/\phi_1 < \beta$ (Proposition 9.1) and a productive bubble may exist. As we have already discussed, $\overline{k}^T$ is decreasing in τ_k and $\underline{k}^T$ is decreasing in τ_w, meaning that $\overline{k}^T/\underline{k}^T$ is increasing in Δ while a positive bubble exists. This is precisely what is described in Proposition 9.1. Let us assume that agents coordinate their expectations on the bubbly steady state. If Δ is sufficiently low (lower than $\underline{\Delta}$), this steady state has an unproductive bubble ($\overline{k}^T < \underline{k}^T$), because the too large tax on capital depresses productive investment and/or a too significant after-tax wage engages agents to use the bubble to transfer purchasing power in the middle and old ages. When Δ raises, the difference between the levels of capital at the bubbly and bubbleless steady states goes up, becoming positive for $\Delta > \underline{\Delta}$. However, when Δ attains and becomes higher than $\overline{\Delta}$, the bubbly steady state disappears and the economy may converge to the bulbleless one with lower levels of capital and GDP. From the previous analysis, we obtain two interesting insights on the fiscal policy design. First, decreasing capital taxation is not always an appropriate policy to boost productive investment. If the policy is already accommodating, it may rule out the bubble and eliminate a mean to finance productive investment. Second, reducing the labor income tax may not be growth enhancing when there is already a bubble. In this case, this policy may change the nature of the bubble making it unproductive.

6.3   Stochastic bubbles

In the previous sections, when the economy is at a bubbly steady state, there is a permanent bubble. This is clearly not consistent with the recurrent evidence that shows that bubbles eventually crash. In this section, we aim to understand whether a probability of market crash will affect the existence and the level of capital at the bubbly steady state. To this end, we extend our framework to the case where the bubble is stochastic. When there is a bubble, agents may coordinate their expectations on an equilibrium without bubble because of a sunspot process which associates a positive probability to a market crash. In such an economy, we study the existence of a stochastic bubbly steady state, i.e. a steady state with positive bubble that takes into account that the bubble may crash with a positive probability. Of course, such a steady state will coexist with the bubbleless one examined in Section 3.

Following Weil (1987), we consider a Markov process of a bubble crash. If there is no bubble at period t, there is no bubble at period $t + 1$ with a probability 1. If there is a bubble at period t, there is a probability $\pi \in (0, 1]$ such that the bubble persists at the next period and a probability $1 - \pi$ such that the bubble crashes at period $t + 1$. Note that a market crash in period $t + 1$ means that the price p_{t+1} of the asset without fundamental value is zero, i.e. $b_{1,t+1} = b_{2,t+1} = R_{t+1} = 0$ using our notations.

Let us examine the household's behavior in such a stochastic environment. To fix ideas, we focus on an household born at period t and we assume that

there is a bubble at this date.[18] In the following, we denote $c_{2,t+1}^+$ ($c_{2,t+1}^0$) the consumption in middle age when the bubble persists (crashes) in $t+1$. If the bubble crashes in $t+1$, the consumption when old is $c_{3,t+2}^{00}$. If the bubble does not crash in $t+1$, the consumption when old is $c_{3,t+2}^{++}$ if it does not crash in $t+2$ too, while it is $c_{3,t+2}^{+0}$ if it crashes in $t+2$ (see also Figure 4).
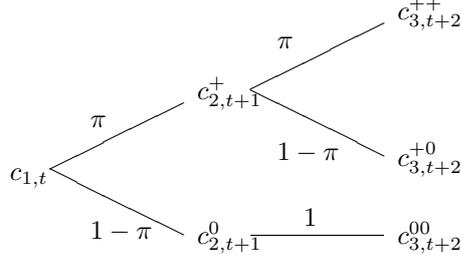

Figure 4: Consumption profile when the bubble has a probability to crash $1 - \pi$

Using (2)-(4), we deduce that the different household's consumptions are given by:

$$c_{1,t} = w_t - k_{t+1} - b_{1,t}, \tag{44}$$

$$c_{2,t+1}^+ = q_{t+1}\phi_1 k_{t+1} + R_{t+1}b_{1,t} - b_{2,t+1}, \ c_{2,t+1}^0 = q_{t+1}\phi_1 k_{t+1}, \tag{45}$$

$$c_{3,t+2}^{++} = q_{t+2}\phi_2 k_{t+1} + R_{t+2}b_{2,t+1}, \ c_{3,t+2}^{+0} = q_{t+2}\phi_2 k_{t+1} = c_{3,t+2}^{00} \tag{46}$$

where $c_{3,t+2}^{+0} = c_{3,t+2}^{00}$ because capital investment is done when young only. The household determines her choices maximizing her expected utility $\mathbb{E}_t[u(c_{1,t}) + \beta u(c_{2,t+1}) + \beta^2 u(c_{3,t+2})]$, that rewrites:

$$u(c_{1,t}) + \beta\left[\pi u(c_{2,t+1}^+) + (1-\pi)u(c_{2,t+1}^0)\right] + \beta^2\left[\pi^2 u(c_{3,t+2}^{++}) + (1-\pi^2)u(c_{3,t+2}^{00})\right] \tag{47}$$

because $c_{3,t+2}^{+0} = c_{3,t+2}^{00}$, $c_{3,t+2}^{+0}$ occurs with probability $\pi(1-\pi)$ and $c_{3,t+2}^{00}$ with probability $1-\pi$. Maximizing (47) under the constraints (44)-(46), we obtain:

$$c_{2,t+1}^+ = \beta\pi R_{t+1}c_{1,t} \tag{48}$$

$$c_{3,t+2}^{++} = \beta\pi R_{t+2}c_{2,t+1}^+ \tag{49}$$

$$1 = \frac{\phi_1 q_{t+1}}{R_{t+1}} + \frac{\phi_2 q_{t+2}}{R_{t+2}R_{t+1}} + \beta(1-\pi)[1 + \beta(1+\pi)]\frac{c_{1t}}{k_{t+1}} \tag{50}$$

At a stochastic bubbly steady state, we have by definition $R = 1$. Substituting (44)-(46) into (48)-(50) and using $w/k = q(\phi_1 + \phi_2)(1-\alpha)/\alpha$, we deduce

---

[18]If there is no bubble at this period, one get the bubbleless economy.

after some computations that:

$$q = \frac{\mathcal{R}^+}{\phi_1 + \phi_2} \equiv q^+, \tag{51}$$

$$k = \frac{\alpha^{\frac{1}{1-\alpha}}(\phi_1 + \phi_2)^{\frac{\alpha}{1-\alpha}}}{(\mathcal{R}^+)^{\frac{1}{1-\alpha}}} \equiv k^+, \tag{52}$$

$$\text{with } \mathcal{R}^+ \equiv \frac{1}{1 + \frac{1-\alpha}{\alpha}\frac{\beta(1-\pi)(1+\beta(1+\pi))}{1+\beta+\beta^2}} \leqslant 1 \tag{53}$$

In addition, the bubble size is given by:

$$b_1 = \left(\frac{\beta\pi(1+\beta\pi)}{1+\beta+\beta^2} - \frac{\alpha}{1-\alpha}\right)w^+ \equiv b_1^+ \tag{54}$$

$$b_2 = \left(\frac{\beta^2\pi^2}{1+\beta+\beta^2} - \frac{\phi_2}{\phi_1+\phi_2}\frac{\alpha}{1-\alpha}\right)w^+ \equiv b_2^+ \tag{55}$$

$$b_1^+ + b_2^+ = \left(\frac{\beta\pi(1+2\beta\pi)}{1+\beta+\beta^2} - \frac{\alpha}{1-\alpha}\frac{\phi_1+2\phi_2}{\phi_1+\phi_2}\right)w^+ \tag{56}$$

where w^+ is the wage evaluated at the stochastic bubbly steady state. Using these expressions, we deduce the following proposition:

Proposition 10 *Under Assumption 1, there is a stochastic bubbly steady state (q^+, k^+) if the following condition is satisfied:*

$$\frac{\beta\pi(1+2\beta\pi)}{1+\beta+\beta^2} > \frac{\alpha}{1-\alpha}\frac{\phi_1+2\phi_2}{\phi_1+\phi_2} \tag{57}$$

In addition, we have $q^+ \leqslant \bar{q}$ and $k^+ \geqslant \bar{k}$.

To understand this proposition, it is relevant to compare it with Proposition 2, which establishes the existence of a bubbly steady state in the deterministic economy. By direct inspection of inequalities (26) and (57), we immediately see that the condition for the existence of a stochastic bubble (57) requires the existence of a deterministic bubble (26). In fact, the left-hand side of inequality (57) is increasing in π and is equal to 0 when π is equal to 0. This means that a stationary stochastic bubble exists only if the probability of a bubble crash $1 - \pi$ is not too close to 1, which is of course in accordance with the seminal paper by Weil (1987).

Proposition 10 also establishes that the return q is smaller in the stochastic case. Therefore, the level of capital is larger at the stochastic bubbly steady state than at the deterministic one. This can be explained as follows. When the probability of crash is zero ($\pi = 1$), the global return of capital investment $(\phi_1 + \phi_2)q$ is equal to the return of the bubble $R = 1$. In contrast, when the probability of crash is strictly positive ($\pi < 1$), the bubble asset becomes risky. Therefore, there is a risk premium between the returns of bubble and capital, which is measured by the inverse of R^+. In fact, the global return of capital

$(\phi_1 + \phi_2)q$ is smaller than 1 and equal to R^+. One can easily show that the lower π is, the more risky the bubble asset and the smaller $\mathcal{R}^+$. The effect of π on the stochastic bubbly steady state is accurately investigated in the next proposition:

Proposition 11 *Consider that Assumption 1 and inequality (57) hold. At the stochastic bubbly steady state, an increase of π implies the following:*

1. *Productive investment k^+ decreases, while the return q^+ increases;*

2. *Both b_1^+ and b_2^+ increase, implying that $b_1^+ + b_2^+$ also increases.*

Proof. See Appendix A.5. ∎

Taking into account that there is a stochastic bubbly steady state, i.e. inequality (57) holds, this proposition proves what happens when the bubble asset is more risky, i.e. π becomes lower. The risk premium of the bubble is raising, which implies a larger level of capital at the stochastic bubbly steady state, because of the decreasing marginal productivity. Young households finance capital by a more significant short position on the speculative asset (lower b_1^+). Because it becomes more risky, risk averse middle age agents buy a weaker amount of the bubble (lower b_2^+).

We can also deduce from this proposition that there is a level of the probability of market crash $1 - \pi$ above which any stochastic bubble does no more exist. Another interesting implication is that the level of capital increases with the probability of market crash $1 - \pi$. As long as inequality (57) is satisfied, the riskier the bubble asset, the larger the difference between the capital levels before and after the crash ($k^+ - \underline{k}$). The presence of a riskier speculative asset is beneficial for capital accumulation, but generates a larger cost, in terms of capital loss, when the market crash occurs.

6.4 Income inequalities

At each steady state (bubbly or bubbleless), households have the same utility over the life-cycle. There is however a sort of heterogeneity among households because of the demographic structure of overlapping generations. Taking into account that the population size of each generation is constant and normalized to 1, three agents coexist at each period of time: a young with an income $I_1 \equiv w$, a middle age one born one period before with an income $I_2 \equiv \phi_1 q k + R b_1$, and an old one born two periods before with an income $I_3 \equiv \phi_2 q k + R b_2$. This means that, because of the demographic structure, there is an income profile associated to each steady state, with an associated degree of inequality between age groups. Obviously, aggregate measures of inequality depend on both between age groups inequality and on within age groups inequality. The former type of inequality is directly related to the ability of agents to transfer wealth across periods and, therefore, it will be affected by the existence of bubbles. The aim of the following analysis is to compare the income profiles evaluated at the bubbleless

and bubbly steady states. The underlying question is of course to know whether a speculative bubble is associated to more or less income inequalities and, as a corollary, whether an income distribution is more or less unequal after a market crash.

To address this issue, we begin by computing the income profile $(\underline{I}_1, \underline{I}_2, \underline{I}_3)$ evaluated at the bubbleless steady state. Using the results obtained in Section 3, we get:

$$\underline{I}_1 = \underline{w}, \ \underline{I}_2 = \frac{\phi_1}{\phi_1 + \phi_2} \frac{\alpha}{1 - \alpha} \underline{w}, \ \underline{I}_3 = \frac{\phi_2}{\phi_1 + \phi_2} \frac{\alpha}{1 - \alpha} \underline{w} \tag{58}$$

Using equations (21), (23) and (24), and $w/k = (1 - \alpha)/\alpha$, we determine the income profile $(\overline{I}_1, \overline{I}_2, \overline{I}_3)$ evaluated at the bubbly steady state:

$$\overline{I}_1 = \overline{w}, \ \overline{I}_2 = \left(B - \frac{\alpha}{1 - \alpha} \frac{\phi_2}{\phi_1 + \phi_2} \right) \overline{w}, \ \overline{I}_3 = \frac{\beta^2}{1 + \beta + \beta^2} \overline{w} \tag{59}$$

Using these two income profiles, we first show the following:

Lemma 1 *Under Assumption 1 and inequality (26), we have $\underline{I}_1 < \overline{I}_1$, $\underline{I}_2 > \overline{I}_2$ and $\underline{I}_3 < \overline{I}_3$. Moreover, noting $\underline{I}_a \equiv (\underline{I}_1 + \underline{I}_2 + \underline{I}_3)/3$ and $\overline{I}_a \equiv (\overline{I}_1 + \overline{I}_2 + \overline{I}_3)/3$ the average incomes at the bubbly and bubbleless steady states respectively, we further get $\underline{I}_a < \overline{I}_a$.*

Proof. See Appendix A.6. ∎

This lemma confirms what we highlight in previous sections. The existence of the bubble has a redistribution effect of income among the different periods of life, i.e. young, middle age and old. This also means that there is a reallocation of income across the different generations living at the same period. When there is a bubble, the generation living at the middle age reimburses her borrowing, reducing her income with respect to the bubbleless case. The generation which lives in the young age has a larger income in the bubbly steady state because the larger level of capital increases the wage. The generation living in the old age has also more income in the bubbly steady state because of the larger level of production and the positive level of the bubble that allows to transfer purchasing power in the last period of life. Finally, the average income is larger at the bubbly steady state because of the redistributive effect across generations allowed by the bubble and the larger wage.

Despite these preliminary results, we need to discuss now more deeply the issue of income inequality. We first define the index of inequalities we choose to work with. We consider the coefficient of variation, which is one of the index that has good properties, as weak principle of transfers, income scale independence, principle of population and decomposability (see Cowell (1995)). Let the variance V of income be defined by $V = (I_1^2 + I_2^2 + I_3^2)/3 - I_a^2$. The coefficient of variation is defined by $CV = \sqrt{V}/I_a$. Evaluating it at the bubbly steady state and the bubbleless one, we compare the income inequality at both steady states:

Proposition 12 *Let $\underline{\alpha} \equiv \frac{\beta + \beta^2}{1 + 2\beta + 2\beta^2}$. Under Assumption 1 and inequality (26), according to the level of the coefficient of variation, there is a lower income inequality at the bubbly steady state than at the bubbleless one if ϕ_2 is low enough and α larger but sufficiently close to $\underline{\alpha}$.*

Proof. See Appendix A.7. ∎

The lower level of income inequality at the bubbly steady state is clearly explained by the redistributive effect of the bubble across generations. This result means that the bubbly steady state that coexists with the bubbleless one is more desirable according to several criteria. It is not only characterized by a larger level of capital, but also lower inequalities.

7 Concluding remarks

This paper analyzes the interplay between liquid speculative bubbles with returns in the short term and less liquid productive investments with returns in a longer term. We introduce this temporal distinction using an overlapping generations model with three-period lived agents. Agents make a portfolio choice between one investment in capital that gives returns during two periods and a bubble traded at each period of time. Considering that the bubbleless economy is dynamically efficient, the bubble enhances production because young traders, who invest in productive capital, are short sellers of the speculative asset, while middle age traders buy the bubble.

Our framework also allows us to discuss the effect of technological change. Productivity shocks either biased toward the short term return or neutral with respect to the relative productivities of capital in the short and longer terms push up capital investment, production and the bubble size. As it is documented in the literature, this illustrates the idea that innovations are often associated to the raise of both production and bubbles. However, we show that this does not happen if the technological shock mainly increases the productivity of capital in the long run.

Our framework allows us to discuss many economic applications. Introducing a credit constraint, we discuss the effect of financial regulation on the levels of capital and bubble. Introducing fiscal policy, we also argue that a low fiscal pressure on capital income relative to labor income can be bad for productive investments. When the bubble becomes stochastic, we show that the riskier the bubble is, the larger the level of capital, but the larger the loss in the case of a market crash. Finally, our model also allows us to conclude that income inequality among generations is lower in the presence of a speculative bubble.

A   Appendix A

A.1   Stability of the steady state without bubble

To construct the phase diagram, we consider the dynamic system (16)-(17). We use equation (17) to determine the regions where X_t is growing. Indeed, $X_{t+1} \geqslant X_t$ is equivalent to $k_t \geqslant X_t$. Using equation (16), we can determine the regions where k_t is growing. Indeed, $k_{t+1} \geqslant k_t$ is equivalent to:

$$X_t \geqslant \frac{1}{\phi_2}\left[\left(\frac{k_t}{B(1-\alpha)}\right)^{\frac{1}{\alpha}} - \phi_1 k_t\right]$$

Using these different ingredients, we can easily deduce Figure 1. To address the stability properties of the steady state, we can also differentiate the dynamic system (16)-(17) around the steady state with $k > 0$. We easily get the characteristic polynomial $P(\lambda) = \lambda^2 - T\lambda + D = 0$, with:

$$
\begin{aligned}
T &= \alpha\frac{\phi_1}{\phi_1 + \phi_2} > 0 \\
D &= -\alpha\frac{\phi_2}{\phi_1 + \phi_2} < 0
\end{aligned}
$$

This implies that $P(-1) > 0$, $P(0) < 0$ and $P(1) > 0$. Therefore, one eigenvalue λ_1 belongs to $(0,1)$ and the other one λ_2 to $(-1,0)$, which proves local convergence with oscillations.

A.2   Proof of Proposition 4

Since we have $\overline{q} = 1/(\phi_1 + \phi_2)$ and $\overline{k} = \alpha^{\frac{1}{1-\alpha}}(\phi_1 + \phi_2)^{\frac{\alpha}{1-\alpha}}$, we obviously deduce that $\partial\overline{q}/\partial\phi_1 < 0$, $\partial\overline{k}/\partial\phi_1 > 0$, $\partial\overline{q}/\partial\phi_2 < 0$, $\partial\overline{k}/\partial\phi_2 > 0$ and, taking ϕ_2/ϕ_1 constant, $\partial\overline{q}/\partial(\phi_1 + \phi_2) < 0$, $\partial\overline{k}/\partial(\phi_1 + \phi_2) > 0$.

We now focus on the bubble size and speculative asset holdings. The wage is given by $\overline{w} = (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}(\phi_1 + \phi_2)^{\frac{\alpha}{1-\alpha}}$. Therefore, using (24) and Assumption 1, we easily deduce that $\partial\overline{b}_1/\partial\phi_1 < 0$ $\partial\overline{b}_1/\partial\phi_2 < 0$ and, taking ϕ_2/ϕ_1 constant, $\partial\overline{b}_1/\partial(\phi_1 + \phi_2) < 0$.

Note that $\phi_2/(\phi_2 + \phi_1)$ is decreasing in ϕ_2 and does not vary when ϕ_2/ϕ_1 stays constant. By direct inspection of (23), we get $\partial\overline{b}_2/\partial\phi_1 > 0$ and, taking ϕ_2/ϕ_1 constant, $\partial\overline{b}_2/\partial(\phi_1 + \phi_2) > 0$. Substituting the expression of the wage in (23), we also get:

$$\frac{\partial\overline{b}_2}{\partial\phi_2} = \alpha^{\frac{1}{1-\alpha}}(\phi_1 + \phi_2)^{\frac{2\alpha-1}{1-\alpha}}\left(\frac{\phi_2}{\phi_1 + \phi_2}\frac{1-2\alpha}{1-\alpha} - \frac{1+\beta}{1+\beta+\beta^2}\right)$$

This is strictly negative if and only if $(1-\alpha)(1+\beta) > \left[(1-2\alpha)\beta^2 - \alpha(1+\beta)\right]\phi_2/\phi_1$. This last inequality is satisfied for all $\phi_2/\phi_1 > 0$ if its right-hand side is negative. It requires:

$$\frac{\alpha}{1-\alpha} > \frac{\beta^2}{1+\beta+\beta^2}$$

Since this is always satisfied under Assumption 1, we have $\partial \bar{b}_2 / \partial \phi_2 < 0$.

Using all these results, we obviously have $\partial(\bar{b}_1 + \bar{b}_2)/\partial \phi_2 < 0$. Let us note that $(2\phi_2 + \phi_1)/(\phi_2 + \phi_1)$ decreases with respect to ϕ_1 and does not vary when ϕ_2/ϕ_1 stays constant. Using (25), we deduce that $\partial(\bar{b}_1 + \bar{b}_2)/\partial \phi_1 > 0$ and, taking ϕ_2/ϕ_1 constant, $\partial(\bar{b}_1 + \bar{b}_2)/\partial(\phi_1 + \phi_2) > 0$.

A.3 Consumers' problem when there is a credit constraint

Consumers maximize the utility subject to (2)-(4) and (28). Consider the following Lagrangian function:

$$
\begin{aligned}
\pounds \;=\;\; & u\left(c_{1,t}\right) + \beta u\left(c_{2,t+1}\right) + \beta^2 u\left(c_{3,t+2}\right) + \lambda_{1,t}\left(w_t - c_{1,t} - k_{t+1} - b_{1,t}\right) + \\
& \lambda_{2,t}\left(q_{t+1}\phi_1 k_{t+1} + R_{t+1}b_{1,t} - c_{2,t+1} - b_{2,t+1}\right) + \lambda_{3,t}(q_{t+2}\phi_2 k_{t+1} \\
& + R_{t+2}b_{2,t+1} - c_{3,t+2}) + \lambda_{4,t}\left(R_{t+1}b_{1,t} + \theta q_{t+1}\phi_1 k_{t+1}\right).
\end{aligned}
$$

The first order conditions with respect to $c_{1,t}$, $c_{2,t+1}$, $c_{3,t+2}$, k_{t+1}, $b_{1,t}$, $b_{2,t+1}$ are, respectively,

$$
\lambda_{1,t} = u'\left(c_{1,t}\right), \;\; \lambda_{2,t} = \beta u'\left(c_{2,t+1}\right), \;\; \lambda_{3,t} = \beta^2 u'\left(c_{3,t+2}\right) \tag{60}
$$

$$
-\lambda_{1,t} + \lambda_{2,t}q_{t+1}\phi_1 + \lambda_{3,t}q_{t+2}\phi_2 + \lambda_{4,t}\theta q_{t+1}\phi_1 = 0 \tag{61}
$$

$$
-\lambda_{1,t} + \lambda_{2,t}R_{t+1} + \lambda_{4,t}R_{t+1} = 0 \tag{62}
$$

$$
-\lambda_{2,t} + \lambda_{3,t}R_{t+2} = 0 \tag{63}
$$

From (60) and (62), we obtain that $\lambda_{4,t} = u'\left(c_{1,t}\right)/R_{t+1} - \beta u'\left(c_{2,t+1}\right)$. The credit constraint is binding if $u'\left(c_{1,t}\right) > R_{t+1}\beta u'\left(c_{2,t+1}\right)$, which implies that $c_{2,t+1} > R_{t+1}\beta c_{1,t}$. In this case, $\lambda_{4,t} > 0$ and we obtain (29) in the main text. Combining (60)-(63), we obtain (30) and (31) in the main text.

A.4 Proof of Proposition 8

Using (38), we easily get $\partial q^*/\partial \theta < 0$. Since $k = (q/\alpha)^{\frac{1}{\alpha-1}}(1/(\phi_1 + \phi_2))$, we have $\partial k^*/\partial \theta > 0$. Moreover,

$$
b_1 = -\theta q \phi_1 k = -\theta q^{\frac{\alpha}{\alpha-1}}\phi_1 \frac{\alpha^{\frac{1}{1-\alpha}}}{\phi_1 + \phi_2}
$$

which implies that $\frac{\partial b_1^*}{\partial \theta} < 0$. Using now (36), we have:

$$
b_1^* + b_2^* = \left(\frac{[\beta - \theta\left(1 + 2\beta\right)]\phi_1 - \phi_2}{1 + \beta}\right)\frac{\alpha^{\frac{1}{1-\alpha}}}{\phi_1 + \phi_2}(q^*)^{\frac{\alpha}{\alpha-1}}
$$

After some computations, we obtain:

$$
\frac{\partial(b_1^* + b_2^*)}{\partial \theta} = (b_1^* + b_2^*)\phi_1 q^*\left[\frac{\alpha}{1 - \alpha} - (1 + 2\beta)\frac{B^{\frac{1-\alpha}{\alpha}}(\phi_1 + \phi_2) + \theta\phi_1}{(\beta - \theta(1 + 2\beta))\phi_1 - \phi_2}\right]
$$

31

Using (39), we deduce that:

$$\frac{B^{\frac{1-\alpha}{\alpha}}(\phi_1 + \phi_2)}{\beta\phi_1 - \phi_2}(1 + 2\beta) > \frac{B(1 + 2\beta)}{\beta}\frac{1-\alpha}{\alpha} > \frac{1-\alpha}{\alpha} > 1 > \frac{\alpha}{1-\alpha},$$

where the last two inequalities follow because we assume $\alpha < 0.5$. This implies that $\partial(b_1^* + b_2^*)/\partial\theta < 0$.

The other assessments of the proposition are established substituting $\theta = 0$ and $\theta = \theta^*$ in equations (34), (35), (37) and (38).

A.5 Proof of Proposition 11

Using (53), we derive:

$$\frac{\partial \mathcal{R}^+}{\partial \pi} = \frac{1-\alpha}{\alpha}\frac{\beta(1 + 2\beta\pi)}{1 + \beta + \beta^2}(\mathcal{R}^+)^2 > 0$$

Using (51) and (52), we easily deduce that q^+ is increasing in π and k^+ is decreasing in π. Using now (7), (8) and (51), the wage w^+ rewrites:

$$w^+ = (1 - \alpha)\alpha^{\frac{\alpha}{1-\alpha}}(\phi_1 + \phi_2)^{\frac{\alpha}{1-\alpha}}(\mathcal{R}^+)^{-\frac{\alpha}{1-\alpha}}$$

Using this expression and (54), we can show that:

$$\frac{\partial b_1^+}{\partial \pi} = \frac{\beta(1 + 2\beta\pi)}{1 + \beta + \beta^2}w^+\left[1 - \left(\frac{\beta\pi(1 + \beta\pi)}{1 + \beta + \beta^2} - \frac{\alpha}{1-\alpha}\right)\mathcal{R}^+\right]$$

This is strictly positive because $\mathcal{R}^+ \leqslant 1$ and $\frac{\beta\pi(1+\beta\pi)}{1+\beta+\beta^2} < 1$. In a similar way, using (55), we derive:

$$\frac{\partial b_2^+}{\partial \pi} = w^+\left[\frac{2\beta^2\pi}{1 + \beta + \beta^2} - \left(\frac{\beta^2\pi^2}{1 + \beta + \beta^2} - \frac{\phi_2}{\phi_1 + \phi_2}\frac{\alpha}{1-\alpha}\right)\mathcal{R}^+\frac{\beta(1 + 2\beta\pi)}{1 + \beta + \beta^2}\right]$$

Since $\frac{\beta(1+2\beta\pi)}{1+\beta+\beta^2} < 1$, $\mathcal{R}^+ \leqslant 1$ and $\frac{2\beta^2\pi}{1+\beta+\beta^2} > \frac{\beta^2\pi^2}{1+\beta+\beta^2}$, we deduce that b_2^+ is increasing in π.

A.6 Proof of Lemma 1

Using (58) and (59), we have $\overline{I}_1/\underline{I}_1 = \overline{w}/\underline{w} > 1$, because $\overline{k} > \underline{k}$. We also have:

$$\frac{\overline{I}_3}{\underline{I}_3} = \frac{\frac{\beta^2}{1+\beta+\beta^2}\overline{w}}{\frac{\phi_2}{\phi_1+\phi_2}\frac{\alpha}{1-\alpha}\underline{w}} > \frac{\overline{w}}{\underline{w}} > 1$$

where the first inequality comes from Assumption 1 and inequality (26), which also ensure $\overline{b}_2 > 0$. We further compute:

$$\frac{\overline{I}_2}{\underline{I}_2} = \frac{\left(B - \frac{\alpha}{1-\alpha}\frac{\phi_2}{\phi_1+\phi_2}\right)\overline{w}}{\frac{\phi_1}{\phi_1+\phi_2}\frac{\alpha}{1-\alpha}\underline{w}}$$

Since $\underline{w} = \underline{k}/B = (1-\alpha)^{\frac{1}{1-\alpha}}(\phi_1+\phi_2)^{\frac{\alpha}{1-\alpha}}B^{\frac{\alpha}{1-\alpha}}$ and $\overline{w} = \overline{k}(1-\alpha)/\alpha = (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}(\phi_1+\phi_2)^{\frac{\alpha}{1-\alpha}}$, we obtain:

$$\frac{\overline{I}_2}{\underline{I}_2} = \left[\left(1+\frac{\phi_2}{\phi_1}\right)B - \frac{\alpha}{1-\alpha}\frac{\phi_2}{\phi_1}\right]\left(\frac{\alpha}{1-\alpha}\right)^{\frac{2\alpha-1}{1-\alpha}}B^{\frac{\alpha}{\alpha-1}}$$

Under Assumption 1, we note that $\overline{I}_2/\underline{I}_2$ decreases with respect to ϕ_2 and, when $\phi_2 = 0$, we have $\overline{I}_2/\underline{I}_2 = (\overline{I}_{2,0}/\underline{I}_{2,0})$, where:

$$\frac{\overline{I}_{2,0}}{\underline{I}_{2,0}} \equiv \left(\frac{\alpha}{(1-\alpha)B}\right)^{\frac{2\alpha-1}{1-\alpha}} < 1$$

under Assumption 1 and $\alpha < 1/2$. We deduce that $\overline{I}_2/\underline{I}_2 < 1$ for all $\phi_2 \geqslant 0$. Finally, we compute the average incomes evaluated at the bubbleless and bubbly steady states. Using (58) and (59), we get:

$$\underline{I}_a = \frac{1}{3}\frac{\underline{w}}{1-\alpha} \tag{64}$$

$$\overline{I}_a = \frac{\overline{w}}{3}\left[1 + \frac{\beta+2\beta^2}{1+\beta+\beta^2} - \frac{\alpha}{1-\alpha}\frac{\phi_2}{\phi_1+\phi_2}\right] \tag{65}$$

One can easily show that the term into brackets in the last equation is larger than $1/(1-\alpha)$ because inequality (26) is satisfied. Since $\overline{w} > \underline{w}$, we deduce that $\overline{I}_a > \underline{I}_a$.

A.7 Proof of Proposition 12

Let $\underline{V}$ and $\overline{V}$ be the variance of income evaluated at the bubbleless and bubbly steady states, respectively. Using (58), (59), (64) and (65), we get:

$$\underline{V} = \frac{\underline{w}^2}{3}\underline{v}_{\phi_2} \text{ and } \overline{V} = \frac{\overline{w}^2}{3}\overline{v}_{\phi_2} \tag{66}$$

with:

$$\underline{v}_{\phi_2} = 1 + \left(\frac{\alpha}{1-\alpha}\right)^2\frac{\phi_1^2+\phi_2^2}{(\phi_1+\phi_2)^2} - \frac{1}{3(1-\alpha)^2}$$

$$\overline{v}_{\phi_2} = 1 + \frac{\beta^4}{(1+\beta+\beta^2)^2} + \left(B - \frac{\alpha}{1-\alpha}\frac{\phi_2}{\phi_1+\phi_2}\right)^2$$
$$- \frac{1}{3}\left(\frac{1+2\beta+3\beta^2}{1+\beta+\beta^2} - \frac{\alpha}{1-\alpha}\frac{\phi_2}{\phi_1+\phi_2}\right)^2$$

We start by considering the case $\phi_2 = 0$ where $\underline{v}_{\phi_2} = \underline{v}_0$ and $\overline{v}_{\phi_2} = \overline{v}_0$ are given by:

$$\underline{v}_0 = 1 + \left(\frac{\alpha}{1-\alpha}\right)^2 - \frac{1}{3(1-\alpha)^2} \tag{67}$$

$$\overline{v}_0 = 1 + \frac{\beta^4}{(1+\beta+\beta^2)^2} + B^2 - \frac{1}{3}\left(\frac{1+2\beta+3\beta^2}{1+\beta+\beta^2}\right)^2 \tag{68}$$

33

According to Assumptions 1 and inequality (26) under $\phi_2 = 0$, α belongs to the interval $(\underline{\alpha}, \overline{\alpha})$, with:

$$\underline{\alpha} \equiv \frac{\beta + \beta^2}{1 + 2\beta + 2\beta^2} \text{ and } \overline{\alpha} \equiv \frac{\beta + 2\beta^2}{1 + 2\beta + 3\beta^2}$$

where $\underline{\alpha} < 1/2$. When α tends to $\underline{\alpha}$,

$$\overline{v}_0 - \underline{v}_0 = -\frac{\beta^2(2 + 4\beta + 2\beta^2)}{3(1 + \beta + \beta^2)^2} < 0$$

This means that $\overline{v}_0 - \underline{v}_0 < 0$ for α larger but sufficiently close to $\underline{\alpha}$. Using (64), (65), (66) and the proof of Lemma 1, we deduce that, for $\phi_2 = 0$ and α larger but sufficiently close to $\underline{\alpha}$, $\overline{V}/\overline{I}_a^2 < \underline{V}/\underline{I}_a^2$. We conclude that, for ϕ_2 sufficiently low and α larger but sufficiently close to $\underline{\alpha}$, $\overline{CV} < \underline{CV}$, where $\overline{CV}$ and $\underline{CV}$ denote the coefficient of variation evaluated at the bubbly and bubbleless steady states, respectively.

B   Appendix B

It is shown in this paper that if the bubbleless steady state is dynamically efficient, there exists a bubbly steady state with a larger level of capital. One could ask whether this could not happen in an economy without our assumption of vintage capital that has returns during two periods. In the following, we consider the same model, but we assume that capital completely depreciates after one period of use and that the household can invest in capital not only when young, but also in the middle age.[19] To keep things as short as possible, we mainly underline the differences with respect to the model considered in the main text.

B.1   Bubbleless steady state with capital length of one period and efficiency

We first consider the economy without bubble. The household invests k_{t+1} in productive capital when young and $\widetilde{k}_{t+2}$ in the middle age. The aggregate capital used for the production at period t is $K_t = k_t + \widetilde{k}_t$ and the budget constraints write[20]:

$$
\begin{aligned}
c_{1,t} + k_{t+1} &= w_t \\
c_{2,t+1} + \widetilde{k}_{t+2} &= q_{t+1}k_{t+1} \\
c_{3,t+2} &= q_{t+2}\widetilde{k}_{t+2}
\end{aligned}
$$

---

[19]In contrast to our framework, it is optimal for the household to also invest in productive capital in the middle age.

[20]We note that aggregate capital does no more involve some heterogenous productivity parameters ϕ_1 and ϕ_2. Indeed, we would like that the bubble can be traded in the youth and the middle age. This means that the return of the bubble should be equal to the return of capital in middle age and when old. This requires that $\phi_1 = \phi_2$. To simplify the notations, we further consider that $\phi_1 = \phi_2 = 1$.

Maximizing the utility function (1) under these three budget constraints, we obtain:

$$c_{2,t+1} = \beta q_{t+1} c_{1,t} \quad \text{and} \quad c_{3,t+2} = \beta q_{t+2} c_{2,t+1} \tag{69}$$

Substituting these expressions into the budget constraints, we get:

$$k_{t+1} = Bw_t \quad \text{and} \quad \widetilde{k}_{t+1} = \frac{\beta}{1+\beta} q_t k_t \tag{70}$$

Using $w_t = (1-\alpha)(k_t + \widetilde{k}_t)^\alpha$ and $q_t = \alpha(k_t + \widetilde{k}_t)^{\alpha-1}$, we get the dynamic system:

$$
\begin{aligned}
k_{t+1} &= B(1-\alpha)(k_t + \widetilde{k}_t)^\alpha \\
\widetilde{k}_{t+1} &= \frac{\beta}{1+\beta} \alpha (k_t + \widetilde{k}_t)^{\alpha-1} k_t
\end{aligned}
$$

Therefore, a steady state is a solution $(k, \widetilde{k})$ satisfying:

$$k = B(1-\alpha)(k + \widetilde{k})^\alpha \quad \text{and} \quad \widetilde{k} = \frac{\beta}{1+\beta}\alpha[B(1-\alpha)]^{\frac{1-\alpha}{\alpha}} k^{\frac{2\alpha-1}{\alpha}}$$

Using these two last equations, we obtain:

$$[B(1-\alpha)]^{\frac{1}{\alpha}} \left[k^{\frac{\alpha-1}{\alpha}} + \frac{\beta}{1+\beta}\alpha(B(1-\alpha))^{\frac{1-\alpha}{\alpha}} k^{\frac{2\alpha-2}{\alpha}} \right] = 1$$

Since the left-hand side is decreasing in k from $+\infty$ to 0 when k increases from 0 to $+\infty$, there is a unique solution k to this equation. We deduce that there is a unique steady state. Analyzing local dynamics in the neighborhood of this steady state, it is possible to show that it is stable, with one eigenvalue that belongs to $(0,1)$ and the other one to $(-1,0)$.

It is more important for our aim to examine under which condition this steady state is dynamically efficient. Using the same methodology than in Section 3, we see that the resource constraint on the good market writes now $c_1 + c_2 + c_3 = (k + \widetilde{k})^\alpha - (k + \widetilde{k})$. We deduce that there is dynamic efficiency if $q = \alpha(k + \widetilde{k})^{\alpha-1} > 1$. Because of the Cobb-Douglas technology, we have:

$$q\left(1 + \frac{\widetilde{k}}{k}\right) = \frac{\alpha}{1-\alpha}\frac{w}{k}$$

Using (70), this equation is equivalent to:

$$q\left(1 + \frac{\beta}{1+\beta}q\right) = \frac{\alpha}{(1-\alpha)B}$$

Therefore, there is a unique solution $q > 0$ given by:

$$q = \frac{1+\beta}{2\beta}\left[\sqrt{1 + 4\frac{\alpha}{1-\alpha}\frac{1+\beta+\beta^2}{(1+\beta)^2}} - 1\right]$$

35

We deduce that $q > 1$ and the steady state is dynamically efficient if and only if the following inequality is satisfied:

$$\frac{\alpha}{1-\alpha} > \frac{\beta(1+2\beta)}{1+\beta+\beta^2} \tag{71}$$

B.2 Bubbly steady state with capital length of one period

We consider the economy with a bubble. This means that the household faces the following budget constraints:

$$
\begin{aligned}
c_{1,t} + k_{t+1} + b_{1,t} &= w_t \\
c_{2,t+1} + \widetilde{k}_{t+2} + b_{2,t+1} &= q_{t+1}k_{t+1} + R_{t+1}b_{1,t} \\
c_{3,t+2} &= q_{t+2}\widetilde{k}_{t+2} + R_{t+2}b_{2,t+1}
\end{aligned}
$$

Maximizing the utility (1) under these three budget constraints, we get the first order conditions (69) and $R_{t+i} = q_{t+i}$ for $i = 1, 2$. Using them and the budget constraints, we get:

$$
\begin{aligned}
k_{t+1} + b_{1,t} &= \frac{\beta + \beta^2}{1+\beta+\beta^2} w_t \\
\widetilde{k}_{t+2} + b_{2,t+1} &= \frac{\beta}{1+\beta} q_{t+1}(k_{t+1} + b_{1,t}) = \frac{\beta^2}{1+\beta+\beta^2} q_{t+1} w_t
\end{aligned}
$$

At a bubbly steady state, $R = q = 1$ and:

$$k + b_1 = \frac{\beta + \beta^2}{1+\beta+\beta^2} w, \quad \widetilde{k} + b_2 = \frac{\beta^2}{1+\beta+\beta^2} w$$

Taking the sum of these two expressions and using $w = (k + \widetilde{k})(1-\alpha)/\alpha$, we get:

$$\frac{b_1 + b_2}{k + \widetilde{k}} = \frac{\beta + 2\beta^2}{1+\beta+\beta^2} \frac{1-\alpha}{\alpha} - 1$$

When there is dynamic efficiency at the bubbleless steady state, we note, using (71), that the right-hand side of this equality is negative. Therefore, there is no steady state with a positive bubble, which is in contrast to our framework where there are returns on capital investment during two periods.

This also means that bubbles may exit when the bubbleless steady state is inefficient. Therefore, bubbles that foster production do not exist when investment in capital is not a long run project and there is the possibility of reinvesting in capital in the middle age. As explained in Section 4, the reallocation effect that we highlight is dampened because the credit effect is lower and bubbles are aimed to transfer more wealth to the older generation.

C Appendix C

We extend our model to a framework where middle age households may also invest in capital with two-period length. This extension allows us to show whether it is not optimal to invest in capital in the middle age at a bubbly steady state. Let $\widetilde{k}_{t+2}$ the amount of capital investment in the middle age and ϕ_3 its productivity. The budget constraints rewrite:

$$c_{1,t} + k_{t+1} + b_{1,t} = w_t \tag{72}$$

$$c_{2,t+1} + \widetilde{k}_{t+2} + b_{2,t+1} = \phi_1 q_{t+1} k_{t+1} + R_{t+1} b_{1,t} \tag{73}$$

$$c_{3,t+2} = \phi_2 q_{t+2} k_{t+1} + \phi_3 q_{t+2} \widetilde{k}_{t+2} + R_{t+2} b_{2,t+1} \tag{74}$$

The household maximizes the utility (1) under the budget constraints (72)-(74) and $\widetilde{k}_{t+2} \geqslant 0$. Defining the Lagrangian function:

$$
\begin{aligned}
\pounds =\ & u\left(c_{1,t}\right) + \beta u\left(c_{2,t+1}\right) + \beta^2 u\left(c_{3,t+2}\right) + \lambda_{1,t}\left(w_t - c_{1,t} - k_{t+1} - b_{1,t}\right) + \\
& \lambda_{2,t}\left(\phi_1 q_{t+1} k_{t+1} + R_{t+1} b_{1,t} - c_{2,t+1} - b_{2,t+1} - \widetilde{k}_{t+2}\right) + \lambda_{3,t}(\phi_2 q_{t+2} k_{t+1} \\
& + \phi_3 q_{t+2} \widetilde{k}_{t+2} + R_{t+2} b_{2,t+1} - c_{3,t+2}) + \lambda_{4,t} \widetilde{k}_{t+2}
\end{aligned}
$$

we get:

$$\lambda_{1,t} = u'\left(c_{1,t}\right), \lambda_{2,t} = \beta u'\left(c_{2,t+1}\right), \lambda_{3,t} = \beta^2 u'\left(c_{3,t+2}\right) \tag{75}$$

$$\lambda_{1,t} = \lambda_{2,t}\phi_1 q_{t+1} + \lambda_{3,t}\phi_2 q_{t+2} \tag{76}$$

$$\lambda_{2,t} = \lambda_{3,t}\phi_3 q_{t+2} + \lambda_{4,t} \tag{77}$$

$$\lambda_{1,t} = \lambda_{2,t} R_{t+1}, \ \lambda_{2,t} = \lambda_{3,t} R_{t+2} \tag{78}$$

where $\lambda_{4,t} > 0 (= 0)$ if and only if $\widetilde{k}_{t+2} = 0 (> 0)$. Using (78), the arbitrage conditions (76) and (77) rewrite:

$$\frac{\phi_1 q_{t+1}}{R_{t+1}} + \frac{\phi_2 q_{t+2}}{R_{t+1} R_{t+2}} = 1 \tag{79}$$

$$\frac{\phi_3 q_{t+2}}{R_{t+2}} \leqslant 1 \tag{80}$$

where the last condition holds with a strict inequality if $\widetilde{k}_{t+2} = 0$. At a bubbly steady state, we have $R_{t+1} = R_{t+2} = 1$ and $q_{t+1} = q_{t+2} = q$. Therefore, (79) and (80) become:

$$(\phi_1 + \phi_2)q = 1 \text{ and } \phi_3 q \leqslant 1 \tag{81}$$

where the last condition holds with a strict inequality if $\widetilde{k} = 0$. We immediately deduce that $\widetilde{k} > 0$ if and only if $\phi_3 = \phi_1 + \phi_2$ and $\widetilde{k} = 0$ if and only if $\phi_3 < \phi_1 + \phi_2$.

Let us assume that at the first period after investment, the productivity of $\widetilde{k}$ is ϕ_1 and, at the second period, ϕ_2 as for the investment k done when young. Then, two configurations may happen depending if the old household can sell or not to another trader capital $\widetilde{k}$, which still gives returns at the following period.

If $\widetilde{k}$ cannot be sold by the old household and bought by another agent, its return is $\phi_1 q$. This means that $\phi_3 = \phi_1$ and $\widetilde{k} = 0$.

Assume now that there is a stock market where $\widetilde{k}$ can be sold at a price p_k. The return perceived by the old household who holds $\widetilde{k}$ is equal to $\phi_1 q + p_k$. Therefore, $\widetilde{k} > 0$ if and only if $p_k = \phi_2 q$. Indeed, in such a case, we have $\phi_3 = \phi_1 + \phi_2$. However, no agent (young or middle age) has an incentive to buy this asset at this price p_k. Indeed, such an agent would pay an amount $p_k \widetilde{k} = \phi_2 q \widetilde{k}$, whereas its discounted gain holding this asset during one period would be $\phi_2 q \widetilde{k}$, because $R = 1$ at a bubbly steady state. This means that $\widetilde{k} = 0$ in this configuration too.

Therefore we can conclude that at a bubbly steady state, we always have $\widetilde{k} = 0$.

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