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Abstract: *We study a two-sector OLG economy in which a share of old age consumption expenditures must be paid out of money balances and we appraise its dynamic features. We first show that competitive equilibrium is dynamically efficient if and only if the share of capital on total income is large enough while a steady state capital per capita above its Golden Rule level is not consistent with a binding liquidity constraint. We thus focus on the gross substitutability in consumption and on dynamic efficiency assumptions and show that, gathered together, they ensure the local determinacy of equilibrium and, as a consequence, rule out sunspot fluctuations. In addition, we prove that the unique steady state may change its stability from a saddle configuration to a source one (undergoing a flip bifurcation) for a capital intensive investment good as well as for a capital intensive consumption good, when the elasticity of the interest rate is set low enough. However, when the investment good is not too capital intensive, the flip bifurcation turns out to be compatible with high elasticities of the interest rate too. Analogous results within dynamic efficiency are found in the non-monetary model, the existence of a flip bifurcation requiring now a capital intensive investment good. Eventually, under dynamic inefficiency, in the non-monetary economy local indeterminacy may instead appear, either through a Hopf bifurcation or through a flip one, and its scope improves as soon as the consumption good becomes more and more capital intensive*.

Keywords: *Overlapping Generations, Two-Sector, Money Demand, Dynamic Efficiency, Equilibrium Dynamics*.

JEL Classification Numbers: D24, E30, E32, E41, E50.

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1 Introduction

In this paper we study a two-sector *OLG* model with capital accumulation, in which agents when young supply labor elastically and consume when old. However, an exogenously given share of old age consumption must be financed out of outside fiat money, accumulated therefore by young people beside physical capital. We carry out a complete qualitative analysis of the local stability and appraise the conditions under which endogenous fluctuations and sunspots equilibria may or may not arise. We also provide an accurate bifurcation analysis in order to appreciate the changes in stability of the steady state giving rise, nearby, to close orbits as well as deterministic cycles, whose stability, in turn, depends upon the direction of the bifurcation studied.

Analyzing a two-sector OLG model with cash-in-advance on consumption expenditures is by far more than a mere theoretical curiosity. It is not a mystery, indeed, that the production of consumption goods is usually made possible by means of technologies which are rather different with respect to the ones employed to produce the investment goods, as many empirical studies suggest. As an example, according to Takashi et al. [34] and Baxter [3] empirical estimates, the consumption sector appears often to be more capital intensive that the investment one. In addition, the choice of focusing on a two-sector model allows to enrich the equilibrium dynamics: some phenomena are indeed exclusive pertinence of such class of models as, for example, the cyclical behavior arising in the optimal infinite horizon models studied, among the others, in Benhabib and Nishimura [7] and Venditti [36]. By extending the hypothesis of different technologies to an OLG model, one is able to further enrich the dynamic features since, in addition to the mechanism relying on the different factor intensities, one faces the typical instability linked to the limited market participation as stressed, e.g., by Gale [21], Azariadis [1], Grandmont [23], Azariadis and Guesnerie [2] and Reichlin [31].

The choice of focusing on a two-sector OLG model is even more motivated, in addition to standard arguments for valuing positively *fiat* money, in the light of the specific assumption of a cash-in-advance constraint on consumption expenditures. As pointed out by Bosi et al. [11], indeed, within a one-sector model, and thus a unique representative firm, the requirement of a fraction of the consumption good to be paid could be easily avoided (and along with it the loss represented by the nominal interest rate) since customers could always formulate their demand in terms of the investment good even though in concrete, after the purchase, they consume it. Actually, in view of the perfect substitutability of the two goods, the firm is unable to discern to which end the amount of the good purchased is employed. Conversely, under the assumption of two distinct representative firms, each of them producing and selling only one good and thus with its own market, the distinction between the consumption good (subject to the CIA constraint) and the investment one (which could be bought "credit") does not give rise to any kind of ambiguity and the CIA constraint cannot thus be avoided.

Since the seminal studies by, among the others, Gale [21], Azariadis [1], Grandmont [23] and Azariadis and Guesnerie [2], it is well established that the limited market participation characterizing *OLG* models may be a source of expectations-driven fluctuations in a framework of competitive economy where prices flexibility ensures equilibrium simultaneously in all markets.

Such fluctuations occur even in the absence of any exogenous shock affecting economic fundamentals (tastes, endowments, technology) but are generated by the volatility of agents' state of expectations and/or by the non-linearities of the dynamics describing intertemporal equilibrium. Such kinds of fluctuations are often labeled as "sunspot" equilibria, in homage to the early neoclassical economist Stanley Jevons who postulated the existence of a close relationship between the occurrence of sunspots and the harvest outcome. As proved, among others, in Woodford [37], Grandmont *et al.* [24] and Bloise [8], a sufficient condition in order to get sunspot equilibria is to be found in the existence of an indeterminate steady state of the intertemporal equilibrium, i.e. a steady state reached in correspondence to an infinite set of choices for the non-predetermined variables describing the economy. In fact, in such a case, in each period agents are faced with quite a large number of choices, all of them compatible with long-run equilibrium (namely the limit conditions): it follows that the unique available selective device rests upon individual expectations - no matter how they are formed, but simply reflecting some shared beliefs - of the future "state of affairs" of the whole dynamic process.

The aforementioned contributions, however, rest upon the assumption of pure-exchange or productive economies without capital accumulation in which the unique asset to invest savings in is to be found in an exogenously given amount of fiat money. Under such an hypothesis, local indeterminacy and sunspot fluctuations (and also deterministic cycles and maybe chaotic dynamics, in the spirit of Ruelle [33]) require strong enough income effects, namely a saving function reacting negatively with respect to its rate of return which, in a purely monetary economy, boils down to the deflation rate. The requirement of strong income effects for the emergence of sunspot fluctuations has been the object of several criticisms based on empirical grounds: for example, Eichenbaum *et al.* [19] and Hall [26] find that the estimated value of the elasticity of intertemporal substitution in consumption falls within the -0.0-10 - range, which means that the saving rate may easily be positively related to its return. Such a criticism, it is worthwhile noticing, can be extended to infinite horizon models too, with cash-in-advance constraint on consumption expenditures in which the steady state is locally indeterminate only under the hypothesis of a strong complementariness in intertemporal consumption (Bloise *et al.* [9]). Things change dramatically when one assumes a fractional liquidity constraint: if the share of consumption to be paid in cash is set low enough, as shown somewhat paradoxically in Bosi *et al.* [11] in an economy with productive capital, indeterminacy is bound to prevail for whatever parameters configuration.

Reichlin [31] repairs the lack of empirical evidence of the early *OLG* models by accounting for a non-monetary one-sector economy with capital accumulation and gross substitutability in intertemporal consumption. He proves that indeterminacy requires dynamic inefficiency and strong inputs complementariness. Cazzavillan and Pintus [14] show that the co-existence of dynamic efficiency and local indeterminacy is not robust to the consideration of any positive elasticity of capital-labor-substitution. Benhabib and Laroque [5] show in an analogous economy that if one introduces money as a bubble, cyclical equilibria require both the quantity of money to be negative at the Golden Rule and inputs to be complementary. Cazzavillan and Pintus [13] prove, on the other hand, that when capital externalities are introduced into the Benhabib and Laroque [5] model, stationary sunspots may occur when the quantity of money is positive and inputs are substitutable enough. Rochon and Polemarchakis [32] extend the Benhabib and Laroque [5] model by considering an *OLG* economy with cash-in-advance constraint and government bonds. The coexistence of dynamic efficiency and local indeterminacy in two-sector models are studied by Drugeon *et al.* [20], Nourry and Venditti [29], Nourry and Venditti [30] and Le Riche *et al.* [27] . Drugeon *et al.* [20] and Nourry and Venditti [29] prove, in a two-sector model with one consumption good and one investment good, that local indeterminacy is ruled out when the steady state is dynamically efficient, provided the sectoral technologies are not too close to the Leontief production function. On the other hand, Nourry and Venditti [30] and Le Riche *et al.* [27] show, in a two-sector model with one pure consumption good and one consumable capital good, that dynamic efficiency together with local indeterminacy is compatible with standard sectorial technologies if the share of the pure consumption good is low enough. Meanwhile *OLG* economies with *CIA* constraint on consumption expenditures are the object of the studies of Crettez *et al.* [15], [16] and Michel and Wigliolle [28]. However they are mostly concerned with the effects of the monetary policy on aggregate welfare. Grandmont *et al.* [24] within an infinite horizon model with heterogeneous agents and cash-in-advance constraint find a picture for the local dynamics very close to that analyzed by Reichlin [31]: sunspot fluctuations occur when inputs are complementary enough. Cazzavillan *et al.* [12] extend such a model by introducing aggregate externalities in production and show that local indeterminacy reappears for inputs high enough substitutable.

In our study we depart from Reichlin [31], Drugeon *et al.* [20] and Crettez *et al.* [15], by accounting at the same time for different sectoral factor intensities in the production of the consumption good and of the investment good and for a fractional liquidity constraint on consumption expenditures. By restricting attention to the case in which the gross substitutability assumption holds, our first task is to ensure the dynamic efficiency of the economy at the Golden Rule level: only under such a feature is money dominated by capital in terms of returns and is the cash-in-advance constraint thus binding. We show that dynamic efficiency requires, at the unique steady state, a share of capital in total income jointly with a share of consumption to be paid cash not too low, features easily falling within standard empirical estimates.

We show that under dynamic efficiency and gross substitutability, local determinacy is bound to prevail and thus sunspot equilibria are ruled out. However, there is still room for a flip bifurcation, even though some additional requirements are needed in terms of the capital intensity in the consumption sector that must be either sufficiently high or low enough, jointly with an elasticity of the real interest rate and an elasticity of the offer curve high enough. As is well known both in the literature on infinite horizon models as well as on *OLG* ones (Benhabib and Nishimura [7], Venditti [36], Galor [22], Nourry and Venditti [29]) a capital intensive investment good favors the occurrence of endogenous fluctuations, while a capital intensive consumption good seems to reduce the scope for such phenomena. These results are confirmed in our study, both in the non-monetary economy as well as in the monetary one.

In the non-monetary economy, which extends the Reichlin [31] model and is characterized by the absence of the financial constraint, under dynamic efficiency and gross substitutability, local indeterminacy and sunspot fluctuations are ruled out. Still, there is room for a flip bifurcation, provided the investment sector is rather capital intensive and the elasticity of the real interest rate large enough. Conversely, local indeterminacy and sunspot fluctuations may occur under capital over-accumulation. Namely, the scope of such phenomena improves as soon as the investment good is made more and more capital intensive: we obtain that the range of the (high) values for the elasticity of the interest rate compatible with a stable stationary solution improves as soon as the relative capital intensive in the sector producing the investment good becomes larger and larger.

In the monetary economy, dynamic efficiency is required to ensure a binding cash-in-advance constraint and local determinacy is then bound to prevail. When the consumption good is capital intensive, however, and differently from the non-monetary economy, deterministic cycles may arise along a flip bifurcation (obtained by increasing continuously the elasticity of the offer curve), provided the elasticity of the interest rate is set low enough. When the investment good is to be capital intensive, the local dynamics becomes richer and different pictures are obtained by varying the parameters configuration. To synthesize the main results, we obtain, for a not very capital intensive investment good, a flip bifurcation in correspondence, first, to low elasticities of the interest rate and, afterwards, to sufficiently large elasticities of this type. On the other hand, in correspondence to a strongly capital intensive investment good, the flip bifurcation is compatible only under the assumption of an elasticity of the real interest rate low enough.

The remainder of the paper is organized as follows. In Section 2 we present the agents' behavior, the technology, and we provide the definition of intertemporal equilibrium. We also calibrate a particular stationary solution. Section 3 is devoted to the analysis of the dynamic efficiency while the main results of the paper, in terms of the local dynamics features, are left to Section 4. Section 5 contains the concluding remarks. Some proofs are left to the Appendix.

2 The model

2.1 Technology

We consider a competitive economy in which there are two sectors producing, respectively, a pure consumption good and a pure investment good. In each sector operates a representative firm. We denote the consumption good, produced in period t, $Y_{0,t}$, and the investment good Y_t. The consumption good is taken as the *numéraire*. Each sector uses two factors, physical capital K_t and labor L_t, and both factors are perfectly mobile across sectors. Capital fully depreciates from one period to another[1] and therefore one has $K_{t+1} = Y_t$, with K_{t+1} being the total amount of capital available in period $t + 1$. A constant returns to scale technology is used in each sector and the two goods are produced according to the technological relationships $Y_{0,t} = F^0\left(K_t^0, L_t^0\right)$ and $Y_t = F^1\left(K_t^1, L_t^1\right)$, with $K_t^0 + K_t^1 \leq K_t$ and $L_t^0 + L_t^1 \leq L_t$, where K_t^j and L_t^j, $j = 0, 1$, denote,

[1] In a two-period OLG model, full depreciation of capital is justified by the fact that the lenght of the period is about thirty years.

respectively, the amount of capital and labor utilized in the sector j and K_t and L_t the total amount of capital and labor available in the economy. The production functions satisfy the following properties:

Assumption 1. *The production function $F^j : \mathbb{R}^2_+ \rightarrow \mathbb{R}^2_+$, $j = 0, 1$ is C^2, increasing, concave, homogeneous of degree one and satisfies the Inada conditions such that, for any $\mu > 0$, $F^j_1(0,\mu) = F^j_2(\mu, 0) = \infty$, $F^j_1(\infty,\mu) = F^j_2(\mu,\infty) = 0$.*[2]

The optimal allocation of factors between sectors is defined by the social production function $T(K_t, Y_t, L_t)$:

$$T(K_t, Y_t, L_t) = \max_{K^j_t, L^j_t} \left\{ Y_{0,t} : | : Y_t \leq F^1(K^1_t, L^1_t), : K^0_t + K^1_t \leq K_t, : L^0_t + L^1_t \leq L_t \right\}. \tag{1}$$

Under Assumption 1, the function $T(K_t, Y_t, L_t)$ is homogeneous of degree one, concave and twice continuously differentiable[3]. Let us denote r_t the rental rate of capital, p_t the price of the investment good and w_t the wage rate, all in terms of the price of the consumption good. Using the envelope theorem we obtain the following three relationships:

$$r(K_t, Y_t, L_t) = T_1(K_t, Y_t, L_t), p(K_t, Y_t, L_t) = -T_2(K_t, Y_t, L_t), w(K_t, Y_t, L_t) = T_3(K_t, Y_t, L_t). \tag{2}$$

The relative capital intensity difference b is derived from the factor-price frontier:

$$b = \frac{L^1}{Y}\left(\frac{K^1}{L^1} - \frac{K^0}{L^0}\right). \tag{3}$$

The sign of b is positive (resp. negative) if and only if the consumption good is labor (resp. capital) intensive. The Stolper-Samuelson effect $(dr/dp, dw/dp)$ and the Rybczynski effect $(dY^0/dK, dY/dK)$ are determined, respectively, by the factor-price frontier and the full employment condition, and are given by:

$$\frac{dr}{dp} = \frac{dY}{dK} = b^{-1}, \frac{dw}{dp} = \frac{dY^0}{dK} = -ab^{-1}. \tag{4}$$

Under a consumption good labor intensive, the Stolper-Samuelson effect states that an increase of the relative price of the investment good decreases the rental rate of capital and increases the wage rate whereas the Rybczynski effect specifies that an increase of the capital-labor ratio decreases the production of the consumption good and increases the production of the investment

[2] $F^j_1(K^j, L^j)$ and $F^j_2(K^j, L^j)$ denote, respectively, the derivatives $\partial F^j(K^j, L^j)/\partial K^j$ and $\partial F^j(K^j, L^j)/\partial L^j$.

[3] See Benhabib and Nishimura [6].

good. Furthermore, from the *GDP* function $T(K_t, Y_t, L_t) + p_t Y_t = w_t L_t + r_t K_t$, we get the share s of capital on total income:

$$s(K_t, Y_t, L_t) = \frac{r_t K_t}{T(K_t, Y_t, L_t) + p_t Y_t} \in (0, 1). \tag{5}$$

2.2 Preferences

We assume an infinite horizon discrete-time economy populated by overlapping generations of agents living for two periods: in the first one they are young, in the second old. There is no population growth and the total size of the population is normalized to one. In the first period, young agents supply elastically L_t units of labor, with $L_t \in (0, \mathscr{L})$, and receive a wage income. They invest this income in capital K_{t+1} and money balances M_{t+1}. In the second period, old agents are retired and purchase the consumption good C_{t+1} out of capital and money income. We assume that agents are subject to a cash-in-advance (*CIA*) constraint on old age consumption purchases, following analogous lines as in Hahn and Solow [25]: $\chi p^C_{t+1} C_{t+1} \leq M_{t+1}$. Such a constraint claims that at least a share $\chi \in (0, 1)$ of old age consumption expenditures C_{t+1} must be financed out of money balances M_{t+1} saved in the first period of life. Agents have preferences defined over old age consumption C_{t+1} and young age labor L_t resumed by the following utility function:

$$u(C_{t+1}) - Bv(L_t).$$

We make the following standard Assumption on the preferences:

Assumption 2. *The functions $u(C)$ and $v(L)$ are defined and continuous for all $c \geq 0$ and $0 \leq L \leq \mathscr{L}$, respectively. Moreover $u(C)$ and $v(L)$ are C^r, for r large enough, with $u'(C) > 0$, $u''(C) < 0$, $v'(L) > 0$ $v''(L) > 0$, for all $C > 0$ and $0 \leq L \leq \mathscr{L}$, with $\lim_{L \to \mathscr{L}} v'(L) = +\infty$.*

A young agent born at period t solves the following dynamic program:

$$\begin{aligned} \max_{C_{t+1}, L_t, K_{t+1}, M_{t+1}} \quad & u(C_{t+1}) - Bv(L_t) \\ s.t. \quad & M_{t+1} + p^I_t K_{t+1} = \omega_t L_t \\ & p^C_{t+1} C_{t+1} = \rho_{t+1} K_{t+1} + M_{t+1} \\ & \chi p^C_{t+1} C_{t+1} \leq M_{t+1} \\ & C_{t+1}, L_t, K_{t+1}, M_{t+1} \geq 0 \\ & L_t \in (0, \mathscr{L}) \end{aligned} \tag{6}$$

where p^C_{t+1} is the price of the consumption good, p^I_t the price of the investment good, ρ_{t+1} the nominal rental rate of capital and ω_t the nominal wage. Choosing the consumption good as the *numéraire* gives the following alternative formulation for the optimization problem:

$$
\begin{aligned}
\max_{C_{t+1},L_t,K_{t+1},M_{t+1}} \quad & u\left(C_{t+1}\right) - Bv\left(L_t\right) \\
s.t. \quad & q_t M_{t+1} + p_t K_{t+1} = w_t L_t \\
& C_{t+1} = r_{t+1} K_{t+1} + q_{t+1} M_{t+1} \\
& \chi C_{t+1} \leq q_{t+1} M_{t+1} \\
& C_{t+1}, l_t, K_{t+1}, M_{t+1} \geq 0 \\
& L_t \in (0, \mathscr{L})
\end{aligned} \tag{7}
$$

where $p_t = p_t^I/p_t^C$, $q_t = 1/p_t^C$, $r_{t+1} = R_{t+1}/p_{t+1}^C$ and the real balances are given by $q_{t+1}M_{t+1}$. We focus on the case where

$$
\frac{r_{t+1}}{p_t} > \frac{q_{t+1}}{q_t} \tag{8}
$$

holds at all dates, which means that the gross rate of return on capital is higher than the profitability of money holding. Under this Assumption, the *CIA* constraint in (7) binds and we obtain the following arbitrage equation (for the first-order conditions see Appendix 6.1):

$$
u'\left(C_{t+1}\right) - \tfrac{Bv'(L_t)}{w_t}\left[\tfrac{\chi q_t}{q_{t+1}} + \tfrac{(1-\chi)p_t}{r_{t+1}}\right] = 0. \tag{9}
$$

Equation (9) states that by increasing of one unit the labor supply (with the associated increase of the effort disutiliy), the corresponding increase of the consumption utility is a weighted average of the capital real return and of the deflation rate.

2.3 Equilibrium

At intertemporal equilibrium all markets clear in each period. Since there are four markets, respectively the investment good one, the consumption good one, the labor one, and the money one. We assume that the Central Bank supplies a constant amount of *fiat* money $\overline{M}$. By exploiting the Walras law, intertemporal equilibrium, beside the fact that the utility maximization problem is to be solved, must satisfy:

Definition 1.

i⌋ Capital accumulation is determined by $p_{t+1}K_{t+1} = w_t L_t - q_t M_{t+1}$*;*

ii⌋ The consumption good satisfies $C_{t+1} = T\left(K_{t+1}, Y_{t+1}, L_{t+1}\right)$*;*

iii⌋ Money dynamics respects $\overline{M} = M_t$ *for all* t*.*

From old age budget constraint one derives $C_{t+1} = r_{t+1}K_{t+1} + q_{t+1}M_{t+1}$ and from the binding *CIA* constraint $\chi C_{t+1} = q_{t+1}M_{t+1}$. Therefore we have $r_{t+1}K_{t+1} - (1-\chi)C_{t+1} = 0$. By exploiting the consumption good market clearing condition $C_{t+1} = T(K_{t+1}, Y_{t+1}, L_{t+1})$, we obtain:

$$T_1(K_{t+1}, Y_{t+1}, L_{t+1})K_{t+1} - (1-\chi)T(K_{t+1}, Y_{t+1}, L_{t+1}) = 0. \tag{10}$$

Applying the Implicit Function Theorem to the static relation (10), we are able to solve locally for the labor supply in order to obtain a smooth function $L_{t+1} = L(K_{t+1}, K_{t+2})$. By differentiating the latter, we get:

$$dL_{t+1} = -\frac{T_1\chi + T_{11}K}{T_{13}K - (1-\chi)T_3}dK_{t+1} - \frac{T_{12}K - (1-\chi)T_2}{T_{13}K - (1-\chi)T_3}dK_{t+2}\text{ , }\forall t. \tag{11}$$

Using the trade-off between consumption and leisure (9), the static relation (10) and the equilibrium condition in the money $q_t/q_{t+1} = C_t/C_{t+1}$, we derive the intertemporal equilibrium with perfect-foresight:

Definition 2. *An intertemporal equilibrium with perfect-foresight is a sequence $\{K_t, L_t\}_{t=0}^{\infty}$, with $K_{t=0}$ given, satisfying the following difference equation:*

$$u'\left[T(K_{t+1}, K_{t+2}, L_{t+1})\right] - \frac{Bv'(L_t)}{T_3(K_t, K_{t+1}, L_t)}\left[\frac{\chi T(K_t, K_{t+1}, L_t)}{T(K_{t+1}, K_{t+2}, L_{t+1})} - \frac{(1-\chi)T_2(K_t, K_{t+1}, L_t)}{T_1(K_{t+1}, K_{t+2}, L_{t+1})}\right] = 0 \tag{12}$$

with $L_t = L(K_t, K_{t+1})$ and $L_{t+1} = L(K_{t+1}, K_{t+2})$.

Let us define $U(C_{t+1}) = u'(C_{t+1})C_{t+1}$ and $V(L_t) = v'(L_t)L_t$. We have therefore $U'(C) = (1-\varepsilon_u)u'(C)$, with $\varepsilon_u = -u''(C)C/u'(C)$, and $V'(L) = (1+\varepsilon_v)v'(L)$, with $\varepsilon_v = v''(L)L/v'(L)$.

Notice that

$$\left(\frac{V'L}{V}\right)\left(\frac{U}{U'C}\right) = \frac{1+\varepsilon_v}{1-\varepsilon_u} = 1 + \frac{1}{\varepsilon_{uv}}$$

and

$$\varepsilon_{uv} = \frac{1-\varepsilon_u}{\varepsilon_v+\varepsilon_u}$$

where ε_{uv} is the average wage elasticity of labor supply (or the interest factor elasticity of saving) that we will call in the reminder of the paper as the elasticity of the offer curve, ε_v the elasticity of labor supply and ε_u the elasticity of intertemporal substitution in consumption. Notice that under gross substitutability we have $\varepsilon_{uv} \in (0, +\infty)$.

2.4 The Normalized Steady State

A steady state is defined as a constant sequence $\{K_t, L_t\}_{t=0}^{\infty} = (K^*, L^*)$ for all t. We now show that is possible to calibrate a particular stationary solution of the dynamic system defined by (10) and

(12) by choosing appropriately the scaling parameter B. To this end, let us fix $K^* = 1$ and let us analyze equation (10). One immediately verifies that

$$\lim_{L^* \to 0} T_1(1,1,L^*) - (1-\chi)T(1,1,L^*) = +\infty$$

and

$$\lim_{L^* \to +\infty} T_1(1,1,L^*) - (1-\chi)T(1,1,L^*) = -\infty.$$

It follows that there exists a unique positive $L^*(1)$ solving

$$T_1(1,1,L^*) - (1-\chi)T(1,1,L^*) = 0.$$

Therefore, the pair $(1, L^*(1))$ is an interior stationary solution of the system defined by (10) and (12) if and only if the scaling parameter $B = B^*(1, L^*(1))$ is set such that

$$B = B^*(1, L(1)) = \frac{U'\left[T(1,1,L^*(1))\right] T_3(1,1,L^*(1))}{V'(L(1))\left[\frac{\chi T(1,1,L^*(1))}{T(1,1,L^*(1))} - \frac{(1-\chi)T_2(1,1,L^*(1))}{T_1(1,1,L^*(1))}\right]}. \tag{13}$$

In the remainder of the paper we make the following Assumption in order to ensure the existence of a normalized steady state (NSS):

Assumption 3. $B = B^*(1, L^*(1))$.

3 Dynamic Efficiency

In this Section we analyze the dynamic efficiency properties of the competitive equilibrium around the NSS. We know that in a one-sector OLG model, competitive equilibrium can be not Pareto optimal (Diamond [17]) since intertemporal exchanges are restricted in view of agents' limited planning horizon (two periods). As matter of fact, if too much capital is accumulated, the economy turns out to be dynamically inefficient. This occurs when the population growth factor (1) exceeds the steady state marginal product of capital (r/p) and the capital-labor ratio exceeds the Golden Rule level. We first characterize the Golden Rule level, i.e. the steady state allocation chosen by a central planner that maximizes the utility of each individual at the steady state. The highest utility is defined as the maximum of the utility function $u(C) - Bv(L)$ subject to the total stationary consumption $C = T(K, K, L)$. The central planner must select non-negative values for capital, labor and consumption in order to solve the following optimization program:

$$\begin{aligned} \max_{\hat{C},\hat{K},\hat{L}} \quad & u(C) - Bv(L) \\ s.t. \quad & C = T(K,K,L). \end{aligned} \tag{14}$$

In the Appendix 6.2 we provide the expressions for the Lagrangian. The first-order conditions are:

$$u'(C) = \frac{Bv'(L)}{T_3}, R\left(\hat{K}, \hat{K}, \hat{L}\right) = -\frac{T_1\left(\hat{K}, \hat{K}, \hat{L}\right)}{T_2\left(\hat{K}, \hat{K}, \hat{L}\right)} = 1. \tag{15}$$

As in the traditional one-sector *OLG* model (Diamond [17]), the Golden Rule level does not depend upon the intertemporal allocation of consumption. We have thus the following Proposition:

Proposition 1. *Under Assumptions 1-3, there exits a unique optimal stationary path* $(\hat{K}, \hat{L})$ *which is characterized by the following conditions:*

$$R\left(\hat{K}, \hat{K}, \hat{L}\right) = 1, \hat{C} = T(\hat{K}, \hat{K}, \hat{L}), u'\left(\hat{C}\right) = \frac{Bv'(\hat{L})}{T_3(\hat{K}, \hat{K}, \hat{L})}$$

with $\hat{K}/\hat{L}$ *the Golden Rule capital-labor ratio.*

Proof: See Appendix 6.3. ■

From Proposition 1, the dynamic efficiency properties of the equilibrium paths are appraised through the comparison of the NSS with respect to the Golden Rule level. The concept of feasible path is therefore defined as:

Definition 3. *A sequence of capital stock* $\{K_t\}_{t=0}^{\infty}$ *is a feasible path if, for all* $t \geq 0$*, the associated total level of consumption is non-negative.*

From Definition 3, we can introduce the property of efficiency of a feasible path:

Definition 4. *A feasible sequence of capital stock* $\{K_t\}_{t=0}^{\infty}$ *is efficient if it is not possible to increase the total consumption at one date without decreasing total consumption at another date, i.e. if there does not exist another feasible path* $\left\{K_t^{'}\right\}_{t=0}^{\infty}$ *with* $K_0^{'} = K_0$*, such that:*
i⌋ $T(K_t^{'}, K_{t+1}^{'}, L_t) \geq T(K_t, K_{t+1}, L_t) \; \forall \; t$;
ii⌋ $T(K_t^{'}, K_{t+1}^{'}, L_t) > T(K_t, K_{t+1}, L_t)$ *for some* $t \geq 0$.

Let us consider the stationary gross rate of return ($R = r/p$). Using the binding CIA constraint $qM = T\chi$, the budget constraint $qM + pK = wL$ and the fact that $wL/rK = (1-s)/s$, we determine the stationary gross rate of return R evaluated at the NSS:

$$R = \frac{(1-\chi)s}{1-\chi-s}. \tag{16}$$

We assume through the paper that the following Assumption holds:

Assumption 4. $\chi < 1 - s \equiv \overline{\chi}$

Then we are able to provide a condition on the share of capital in the economy to get a NSS (K^*, L^*) lower than the Golden-Rule level $(R > 1)$ and therefore ensuring the dynamic efficiency of the intertemporal equilibrium. Following the proof of Proposition 3 in Drugeon *et al.* [20], the following Proposition is immediately proved:

Proposition 2. *Under Assumptions 1-4, let* $\underline{s} = (1-\chi)/(2-\chi)$. *Then, the* NSS (K^*, L^*) *is characterized by an under-accumulation of capital if and only if* $s > \underline{s}$.

Under-accumulation of capital can be attained provided the share of capital in the economy is large enough, namely $s > \underline{s}$.

4 Local Dynamics

In the system describing intertemporal equilibrium, there is one pre-determinate variable, the initial stock of capital, and one forward-looking variables, the labor supply. In such a configuration, the existence of local indeterminacy requires that the two characteristic roots associated with the linearization of the dynamic system (12) around the normalized steady state have modulus less than one. In the opposite case the steady state is locally determinate. It is useful to introduce here the elasticity ε_{rk} of the rental rate of capital evaluated at the normalized steady state:

$$\varepsilon_{rk} = -\frac{T_{11}(K^*, K^*, L^*)K^*}{T_1(K^*, K^*, L^*)} \in (0, +\infty). \tag{17}$$

Drugeon [18] points out that the elasticity of the rental rate of capital is negatively linked to the elasticities of capital-labor substitution:

$$\Sigma = \frac{(Y_0 + pY)\left(pYK^0L^0\sigma_0 + Y_0K^1L^1\sigma_1\right)}{pYKY_0}, \ \varepsilon_{rk} = \left(\frac{L^0}{Y_0}\right)^2 \frac{w(Y_0 + pY)}{\Sigma} \tag{18}$$

with $\sigma_0 \in (0, +\infty)$ and $\sigma_1 \in (0, +\infty)$ being the sectorial elasticities of inputs substitution.

Then the following proposition holds:

Proposition 3. *Under Assumptions 1-3, the characteristic polynomial is defined by* $\mathscr{P}(\lambda) = \lambda^2 - \lambda\mathscr{T} + \mathscr{D}$ *where:*

$$\mathscr{T}(\varepsilon_{uv}) = \frac{1 - \chi - s - s\chi\varepsilon_{uv} + \varepsilon_{rk}\left[bs + \varepsilon_{uv}(bs\chi + 1 - \chi)\right]}{(bs\chi + 1 - \chi - s)\,\varepsilon_{rk}\varepsilon_{uv}} \tag{19}$$

$$\mathscr{D}(\varepsilon_{uv}) = \frac{s(\varepsilon_{rk} - \chi)(1 + \varepsilon_{uv})}{(bs\chi + 1 - \chi - s)\,\varepsilon_{rk}\varepsilon_{uv}} \tag{20}$$

Proof: See Appendix 6.4. ■

In view of the complicated form of the above expressions, it may seem that the study of the local dynamics of system (12) requires long and tedious computations. However, by applying the geometrical method adopted in Grandmont *et al.* [24] and Cazzavillan *et al.* [12], it is possible to analyze qualitatively the (in)stability of the characteristic roots of the Jacobian evaluated at the steady state of system defined by (12) and their bifurcations (changes in stability) by locating the point $(\mathscr{T}, \mathscr{D})$ in the plane and studying how $(\mathscr{T}, \mathscr{D})$ varies when the value of some parameter changes continuously. In our case, the bifurcation parameter is $\varepsilon_{uv} \in (0, +\infty)$. In the Appendix 6.5, we present the geometrical approach adopted to study the local stability.

4.1 The Non-Monetary Economy

Before analyzing the monetary economy, let us consider first the case without cash-in-advance constraint $\chi = 0$. Under such an hypothesis, we are able to appraise the role played on local dynamics by the two technological parameters of the model: b and ε_{rk}. Within such an hypothesis, the Trace $\mathscr{T}$, the Determinant $\mathscr{D}$ and the slope $\mathscr{S}$ defined in (19), (20) and (33) boil down to, respectively:

$$\mathscr{T} = \frac{1 - s + \varepsilon_{rk}(bs + \varepsilon_{uv})}{\varepsilon_{rk}\varepsilon_{uv}(1 - s)};\ \mathscr{D} = \frac{s(1 + \varepsilon_{uv})}{\varepsilon_{uv}(1 - s)};\ \mathscr{S} = \frac{s\varepsilon_{rk}}{\varepsilon_{rk}bs + 1 - s} \tag{21}$$

To analyze the local dynamics of the economy, we consider the properties of the starting point $(\mathscr{T}_\infty, \mathscr{D}_\infty)$ and of the slope $\mathscr{S}$ as a function of the two parameters $b \in (-\infty, 1)$ and $\varepsilon_{rk} \in (0, +\infty)$. Using equation (21), we find that the starting point, obtained setting $\varepsilon_{uv} = +\infty$, has coordinates:

$$\mathscr{D}_\infty = \frac{s}{1 - s};\ \mathscr{T}_\infty = \mathscr{D}_\infty + 1 \tag{22}$$

Notice that the location of the starting point does not depend upon b and ε_{rk}. Moreover, the starting point lies always on the $\mathscr{D} = \mathscr{T} - 1$ line. In the following, we will consider two cases; the first corresponding to the over-accumulation of capital, i.e. $s < 1/2$, and the other to the under-accumulation of capital, i.e. $s > 1/2$, since both configurations are compatible with the non-monetary economy. When $s < 1/2$ and then the economy is characterized by capital over-accumulation, one has $\mathscr{D}_\infty < 1$ and $\mathscr{T}_\infty < 2$ and thus the starting point lies always below the point $\mathscr{C}$ depicted in Figure 1.

To appraise the local stability properties and the occurrence of bifurcations, we must analyze how does the slope $\mathscr{S}$ move as soon as b and ε_{rk} are made to vary. Notice that, for $\varepsilon_{rk} = +\infty$, the slope is $1/b$. When b increases from $-\infty$ to 1, the half-line Δ undergoes a clockwise rotation. Let us now define $b_1 = (2s - 3)/(1 - 2s)$ the relative critical capital intensity difference such that the

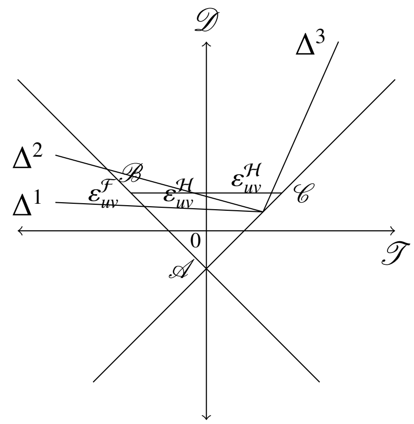

Figure 1: Hopf and flip bifurcations under over-accumulation of capital.

half-line Δ goes through the point $\mathscr{B}$ and $b_2 = -1$ the relative critical capital intensity difference such that the slope of the half-line Δ is -1.

After having fixed b, let us vary ε_{rk}. Consider first the case $b < b_1$. Here we can define three critical values for ε_{rk}: ε_{rk}^1 such that the half-line Δ goes through the point $\mathscr{B}$; ε_{rk}^2 such that the slope $\mathscr{S}$ of the half-line Δ is equal to -1 and ε_{rk}^3 such that the slope $\mathscr{S}$ is equal to one. The critical values for ε_{rk} can be summarized with the help of the following notation:

$$\begin{aligned} \Delta \text{ goes through } \mathscr{B} &\Longleftrightarrow \varepsilon_{rk}^1 = -\tfrac{(1-s)(1-2s)}{s[3-2s+b(1-2s)]}; \\ \mathscr{S} = -1 &\Longleftrightarrow \varepsilon_{rk}^2 = -\tfrac{1-s}{s(1+b)}; \\ \mathscr{S} = 1 &\Longleftrightarrow \varepsilon_{rk}^3 = \tfrac{1-s}{s(1-b)}. \end{aligned}$$

Notice that ε_{rk}^1 exists if and only if $b < b_1$, ε_{rk}^2 requires $b < -1$ and ε_{rk}^3 is well defined for any values of b. It is immediately verifiable that inequalities $\varepsilon_{rk}^1 > \varepsilon_{rk}^2 > \varepsilon_{rk}^3 > 0$ do hold. Let now set $b < b_1$. It follows that when $\varepsilon_{rk} < \varepsilon_{rk}^3$ the steady state is always a saddle. When $\varepsilon_{rk} \in (\varepsilon_{rk}^3, \varepsilon_{rk}^2)$, by relaxing continuously ε_{uv} in the $(0, +\infty)$ interval, we obtain first a source configuration and then, through a Hopf bifurcation, a sink one. The half-line Δ^3 in Figure 1 represents this case. When $\varepsilon_{rk} \in (\varepsilon_{rk}^2, \varepsilon_{rk}^1)$, by increasing ε_{uv}, we have a steady state which is first a saddle, then, through a flip bifurcation, it becomes a source and eventually, through a Hopf bifurcation, a sink. The half-line Δ^2 in Figure 1 corresponds to this case. If $\varepsilon_{rk} > \varepsilon_{rk}^1$, by relaxing continuously ε_{uv}, one obtains first a saddle configuration and then, through a flip bifurcation, a sink one, as depicted in Figure 1.

Let us now consider the case $b \in (b_1, b_2)$. Notice that here ε_{rk}^1 does not more exist since $b > b_1$. Then, when $\varepsilon_{rk} < \varepsilon_{rk}^3$, the steady state is a saddle. When $\varepsilon_{rk} \in (\varepsilon_{rk}^3, \varepsilon_{rk}^2)$, by relaxing continuously ε_{uv}, we obtain first a source configuration and then, through a Hopf bifurcation, a sink one. When $\varepsilon_{rk} > \varepsilon_{rk}^2$ we have a steady state which is first a saddle and then, through a flip bifurcation, it becomes a source and eventually, through a Hopf bifurcation, a sink.

Finally, let us consider the case $b > b_2$. It follows that ε_{rk}^1 and ε_{rk}^2 do not more exist since $b > b_2$. Then, when $\varepsilon_{rk} < \varepsilon_{rk}^3$, the steady state is bound to be a saddle; on the other hand, if

$\varepsilon_{rk} > \varepsilon_{rk}^3$, we obtain first a source configuration and then, through a Hopf bifurcation, a sink one.

All these results are summarized in the following Proposition.

Proposition 4. *Under Assumptions 1-4, let assume $s < 1/2$. Then there exist $b_1 < b_2$, $\varepsilon_{rk}^1 > \varepsilon_{rk}^2 > \varepsilon_{rk}^3 > 0$, $\varepsilon_{uv}^{\mathcal{F}} > 0$ and $\varepsilon_{uv}^{\mathcal{H}} > 0$ such that the following results hold:*

i] Let $b < b_1$. If $\varepsilon_{rk} < \varepsilon_{rk}^3$, the steady state is a saddle, i.e. locally determinate. If $\varepsilon_{rk} \in (\varepsilon_{rk}^3, \varepsilon_{rk}^2)$, the steady state is a source, i.e. locally determinate, for $\varepsilon_{uv} < \varepsilon_{uv}^{\mathcal{H}}$, and a sink, i.e. locally indeterminate, for $\varepsilon_{uv} > \varepsilon_{uv}^{\mathcal{H}}$. If $\varepsilon_{rk} \in (\varepsilon_{rk}^2, \varepsilon_{rk}^1)$, the steady state is a saddle, i.e. locally determinate, for $\varepsilon_{uv} < \varepsilon_{uv}^{\mathcal{F}}$, a source, i.e. locally determinate, for $\varepsilon_{uv} \in (\varepsilon_{uv}^{\mathcal{F}}, \varepsilon_{uv}^{\mathcal{H}})$, and a sink, i.e. locally indeterminate, for $\varepsilon_{uv} > \varepsilon_{uv}^{\mathcal{H}}$. If $\varepsilon_{rk} > \varepsilon_{rk}^1$, the steady state is a saddle, i.e. locally determinate, for $\varepsilon_{uv} < \varepsilon_{uv}^{\mathcal{F}}$ and a sink, i.e. locally indeterminate, for $\varepsilon_{uv} > \varepsilon_{uv}^{\mathcal{F}}$.

ii] Let $b \in (b_1, b_2)$. If $\varepsilon_{rk} < \varepsilon_{rk}^3$, the steady state is a saddle, i.e. locally determinate. If $\varepsilon_{rk} \in (\varepsilon_{rk}^3, \varepsilon_{rk}^2)$, the steady state is a source, i.e. locally determinate, for $\varepsilon_{uv} < \varepsilon_{uv}^{\mathcal{H}}$ and a sink, i.e. locally indeterminate, for $\varepsilon_{uv} > \varepsilon_{uv}^{\mathcal{H}}$. If $\varepsilon_{rk} > \varepsilon_{rk}^2$, the steady state is a saddle, i.e. locally determinate, for $\varepsilon_{uv} < \varepsilon_{uv}^{\mathcal{F}}$, a source, i.e. locally determinate, for $\varepsilon_{uv} \in (\varepsilon_{uv}^{\mathcal{F}}, \varepsilon_{uv}^{\mathcal{H}})$, and a sink, i.e. locally indeterminate, for $\varepsilon_{uv} > \varepsilon_{uv}^{\mathcal{H}}$.

iii] Let $b > b_2$. If $\varepsilon_{rk} < \varepsilon_{rk}^3$, the steady state is a saddle, i.e. locally determinate. If $\varepsilon_{rk} > \varepsilon_{rk}^3$, the steady state is a source, i.e. locally determinate, for $\varepsilon_{uv} < \varepsilon_{uv}^{\mathcal{H}}$, and a sink, i.e. locally indeterminate, for $\varepsilon_{uv} > \varepsilon_{uv}^{\mathcal{H}}$.

When ε_{uv} goes through $\varepsilon_{uv}^{\mathcal{F}}$ and $\varepsilon_{uv}^{\mathcal{H}}$ the steady state undergoes, respectively, a flip and a Hopf bifurcation.

Let us now consider the case of under-accumulation of capital, i.e. $s > 1/2$. From equation (21), we derive that the starting point lies always above the point $\mathscr{C}$ in Figure 2 and thus the NSS is always locally determinate. Then, in the light of the above considerations with the help of Figure 2, the following Proposition is immediately proved:

Proposition 5. *Under Assumptions 1-4, let assume $s > 1/2$. Then there exist $b_2 < 0$, $\varepsilon_{rk}^1 > \varepsilon_{rk}^2 > \varepsilon_{rk}^3 > 0$, $\varepsilon_{uv}^{\mathcal{F}} > 0$ and $\varepsilon_{uv}^{\mathcal{H}} > 0$ such that the following results hold:*

i] let $b < b_2$. If $\varepsilon_{rk} < \varepsilon_{rk}^3$, the steady state is a saddle, i.e. a locally determinate. If $\varepsilon_{rk} \in (\varepsilon_{rk}^3, \varepsilon_{rk}^2)$, the steady state is a source, i.e. locally determinate. If $\varepsilon_{rk} > \varepsilon_{rk}^2$, the steady state is a saddle, i.e. locally determinate, for $\varepsilon_{uv} < \varepsilon_{uv}^{\mathcal{F}}$, and a source, i.e. locally determinate, for $\varepsilon_{uv} > \varepsilon_{uv}^{\mathcal{F}}$.

ii] let $b > b_2$. When $\varepsilon_{rk} < \varepsilon_{rk}^3$, the steady state is a saddle, i.e. locally determinate. If $\varepsilon_{rk} > \varepsilon_{rk}^3$, the steady state is a source, i.e. locally determinate.

When ε_{uv} goes through $\varepsilon_{uv}^{\mathcal{F}}$, the steady state undergoes a flip bifurcation.

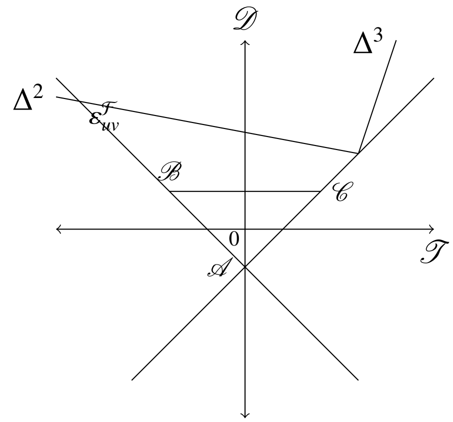

Figure 2: Flip bifurcation with under-accumulation of capital.

It could be interesting, at this point, to compare our results in terms of dynamics features with those found in Drugeon et al. [20] and Nourry and Venditti [29] since their OLG models are similar to our own. In these two papers labor is assumed to be supplied inelastically and agents to consume when young and when old meanwhile, in our framework, labor supply is endogenous and agents consume only when old. Nevertheless, in both models, the dynamical system characterizing the equilibrium is two-dimensional and therefore we can carry out a comparaison of the two models. Under the gross substitutability assumption, Drugeon et al. [20] and Nourry and Venditti [29] show that sunspot fluctuations are ruled out when the steady state is dynamical efficient provided that the sectoral elasticities of capital-labor substitution are not too low. More in details, Drugeon et al. [20] show that, when the steady state is dynamically inefficient, sunspot fluctuations are likely to occur whatever the sectoral elasticities of capital-labor substitution is. On the contrary, in the non-monetary version of our model, under dynamic efficiency, local indeterminacy is bound to be ruled out for whatever parameters configuration. On the other hand, under dynamic efficiency, we obtain local indeterminacy too, even thought our bifurcation parameter is the elasticity of the offer curve meanwhile in Drugeon et al. [20] and Nourry and Venditti [29], it is the elasticity of intertemporal substitution in consumption. It follows that a complete comparison between the two models is rather difficult to carry out although their features are mutually consistent

4.2 The Monetary Economy

We now focus on the monetary economy obtained by setting $\chi > 0$ under the hypothesis of a binding CIA constraint, condition requiring a dynamically efficient NSS. We will consider first the case of a capital intensive investment good, i.e. $b > 0$, and then the case of a capital intensive consumption good, i.e. $b < 0$.

4.2.1 A Capital Intensive Investment Good

Let $b > 0$. From the homogeneity of the social production function $T(K_t, K_{t+1}, L_t)$ and the first-order conditions of the producer (2), we have that, at the NSS (K^*, L^*), $b < 1$. Since $b > 0$, it follows from equations (33)-(34) that the properties of the starting point $(\mathscr{T}_\infty, \mathscr{D}_\infty)$ and of the slope $\mathscr{S}$ depend upon ε_{rk}. Moreover, from (34), one has that $\mathscr{D}_\infty = \mathscr{T}_\infty - 1$. From (34), one has that $\mathscr{D}_\infty$ is greater than one if $\varepsilon_{rk} > (1-\chi)(1-s)/s(1-b\chi) \equiv \varepsilon_{rk}^{5\chi}$. It follows that, when $\varepsilon_{rk} > \varepsilon_{rk}^{5\chi}$, the starting point lies above the point $\mathscr{C}$ depicted in Figure 3.

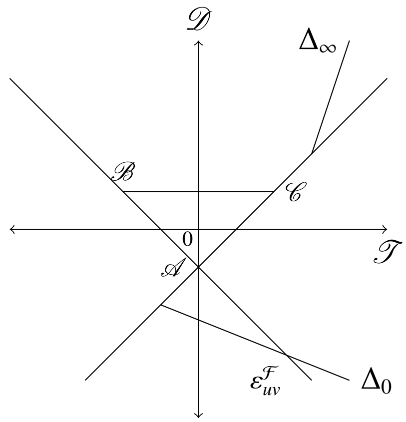

Figure 3: Flip bifurcation.

Let fix $\varepsilon_{rk} = +\infty$. We then obtain:

$$\lim_{\varepsilon_{rk}\to+\infty} \mathscr{D}_\infty(\varepsilon_{rk}) \equiv \mathscr{D}_\infty^\infty = \frac{s}{bs\chi + 1 - \chi - s} > 1; \lim_{\varepsilon_{rk}\to+\infty} \mathscr{T}_\infty(\varepsilon_{rk}) \equiv \mathscr{T}_\infty^\infty = \mathscr{D}_\infty^\infty + 1 < 2; \mathscr{S}^\infty = b^{-1} \tag{23}$$

Since $\mathscr{D}_\infty^\infty$ is greater than one and the starting point lies on the $\mathscr{D} = \mathscr{T} - 1$ line, the pair $(\mathscr{T}_\infty^\infty, \mathscr{D}_\infty^\infty)$ lies above the point $\mathscr{C}$. The steady state is a then source if the half-line Δ has slope greater than one. When $\varepsilon_{rk} = +\infty$, this is true when $b < 1$. By a direct inspection of (33), one has that $\mathscr{S} > 1$ if and only if $\varepsilon_{rk} > (1-\chi)(1-s)/s(1-b) \equiv \varepsilon_{rk}^{3\chi}$. In addition, in view of (20), we have $D'(\varepsilon_{uv}) > 0$. Thus, for $\varepsilon_{rk} \in (\varepsilon_{rk}^{3\chi}, +\infty)$, the steady state is always a source. The corresponding half-line Δ_∞ is depicted in Figure 3.

Let us now decrease ε_{rk} so that the starting point lies now on $\mathscr{A}$. This is the case if $\varepsilon_{rk} = (s\chi)/(1-\chi+bs\chi) \equiv \varepsilon_{rk}^{4\chi}$. Let finally decrease ε_{rk} up to zero. We have:

$$\lim_{\varepsilon_{rk}\to 0} \mathscr{D}_\infty(\varepsilon_{rk}) \equiv \mathscr{D}_\infty^0 = -\infty; \lim_{\varepsilon_{rk}\to 0} \mathscr{T}_\infty(\varepsilon_{rk}) \equiv \mathscr{T}_\infty^0 = \mathscr{D}_\infty^0 + 1 = -\infty; \mathscr{S}^0 = \frac{-s\chi}{1-\chi-s} \tag{24}$$

Notice that, under Assumption 4, the slope is negative and always greater than −1. In addition, from the expression for the Determinant (20), we easily derive $D'(\varepsilon_{uv}) < 0$. Then, for $\varepsilon_{rk} \in (\varepsilon_{rk}^{4\chi}, \varepsilon_{rk}^{3\chi})$, the steady state is a saddle. When ε_{rk} is further reduced below $\varepsilon_{rk}^{4\chi}$, by relaxing continuously ε_{uv} from 0 to $+\infty$, we obtain that the steady state is first a saddle and then, through a flip bifurcation, it becomes a source. In Figure 3 we have depicted the half-line Δ_0 corresponding to the case in which the steady state undergoes a flip bifurcation.

These results are summarized in the following Proposition:

Proposition 6. *Under Assumptions 1-4, there exist $\varepsilon_{rk}^{3\chi} > \varepsilon_{rk}^{4\chi} > 0$ and $\varepsilon_{uv}^{\mathcal{F}} > 0$ such that for $b \in (0, 1)$, the following results hold:*
When $\varepsilon_{rk} > \varepsilon_{rk}^{3\chi}$, the steady state is a source, i.e. locally determinate. When $\varepsilon_{rk} \in (\varepsilon_{rk}^{4\chi}, \varepsilon_{rk}^{3\chi})$, the steady state is a saddle, i.e. locally determinate. When $\varepsilon_{rk} < \varepsilon_{rk}^{4\chi}$, the steady state is a saddle, i.e. locally determinate, for $\varepsilon_{uv} \in (0, \varepsilon_{uv}^{\mathcal{F}})$, and a source, i.e. locally determinate, for $\varepsilon_{uv} > \varepsilon_{uv}^{\mathcal{F}}$.

In addition, when ε_{uv} goes through $\varepsilon_{uv}^{\mathcal{F}}$, the steady state undergoes a flip bifurcation.

4.2.2 A Capital Intensive Consumption Good

Let us now consider the case of a capital intensive consumption good, i.e. $b < 0$, depicted in Figure 4.

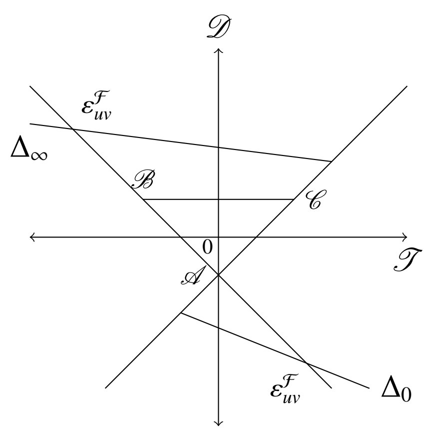

Figure 4: Flip bifurcation.

Since $b < 0$, it follows from equations (33)-(34) that the properties of the starting point $(\mathcal{T}_\infty, \mathcal{D}_\infty)$ and of the slope $\mathcal{S}$ depend upon ε_{rk} and b. From (34), we have $\mathcal{D}_\infty = \mathcal{T}_\infty - 1$. From (34), one has that $\mathcal{D}_\infty$ is greater than one if $\varepsilon_{rk} > (1-\chi)(1-s)/s(1-b\chi) \equiv \varepsilon_{rk}^{5\chi}$. It follows that, when $\varepsilon_{rk} > \varepsilon_{rk}^{5\chi}$, the starting point lies above the point $\mathcal{C}$. Let us now define $b_3 = -(1-\chi-s)/s\chi$ the relative critical capital intensity difference such that the denominator of $\mathcal{D}_\infty$ changes of sign.

Consider first the case $b > b_3$. Here we must introduce three critical values for ε_{rk}: $\varepsilon_{rk}^{2\chi}$ such that the slope $\mathscr{S}$ is equal to -1; $\varepsilon_{rk}^{3\chi}$ such that the slope $\mathscr{S}$ is equal to one; $\varepsilon_{rk}^{4\chi}$ such that the half-line Δ goes through the point $\mathscr{B}$. The expressions for the critical values for ε_{rk} are the following:

$$\mathscr{S} = -1 \Longleftrightarrow \varepsilon_{rk}^{2\chi} = -\tfrac{1-s}{s(1+b)};$$
$$\mathscr{S} = 1 \Longleftrightarrow \varepsilon_{rk}^{3\chi} = \tfrac{1-s}{s(1-b)};$$
$$\Delta \text{ goes through } \mathscr{A} \Longleftrightarrow \varepsilon_{rk}^{4\chi} = (s\chi)/(1-\chi+bs\chi).$$

Notice that $\varepsilon_{rk}^{4\chi}$ exists if and only if $b > -(1-\chi)/s\chi \equiv b_4$ which, as it is immediately verifiable, satisfies inequality $b_4 < b_3$. It is also easy to prove that the inequalities $\varepsilon_{rk}^{2\chi} > \varepsilon_{rk}^{3\chi} > \varepsilon_{rk}^{5} > \chi > \varepsilon_{rk}^{4} > 0$ do hold. Let us now set $b > b_3$. When $\varepsilon_{rk} < \varepsilon_{rk}^{4\chi}$ by relaxing continuously ε_{uv}, we obtain that the steady state is first a saddle and then, through a flip bifurcation, it becomes a source. On the other hand, when $\varepsilon_{rk} \in (\varepsilon_{rk}^{4\chi}, \varepsilon_{rk}^{3\chi})$, the steady state is bound to be a saddle. At the same time, it is immediately verifiable that, when $\varepsilon_{rk} \in (\varepsilon_{rk}^{3\chi}, \varepsilon_{rk}^{2\chi})$, the steady state is a source. Eventually, when $\varepsilon_{rk} > \varepsilon_{rk}^{2\chi}$, by increasing continuously ε_{uv}, one obtains first a saddle configuration and then, through a flip bifurcation, a source one.

Let us now set $b \in (b_4, b_3)$. It is easily verifiable that the inequalities $\varepsilon_{rk}^{2\chi} < \varepsilon_{rk}^{3\chi} < \varepsilon_{rk}^{5\chi} < \chi < \varepsilon_{rk}^{4\chi} > 0$ do hold. It follows that, when $\varepsilon_{rk} < \varepsilon_{rk}^{2\chi}$, by relaxing continuously ε_{uv}, one obtains first a saddle configuration for the steady state and then, through a flip bifurcation, a source one. At the same time, it is immediate to see that, when $\varepsilon_{rk} \in (\varepsilon_{rk}^{2}, \varepsilon_{rk}^{3\chi})$, the steady state is bound to be a source and that, when $\varepsilon_{rk} \in (\varepsilon_{rk}^{3\chi}, \varepsilon_{rk}^{4\chi})$, it is a saddle. Eventually, for $\varepsilon_{rk} > \varepsilon_{rk}^{4\chi}$, by increasing continuously ε_{uv}, one obtains a steady state that is first a saddle and then, through a flip bifurcation, becomes a source.

Let us now suppose $b < b_4$. It follows that the critical value $\varepsilon_{rk}^{4\chi}$ does not more exist. It is easy to prove that, when $\varepsilon_{rk} < \varepsilon_{rk}^{2\chi}$, by increasing ε_{uv}, one obtains first a saddle configuration for the steady state and then, through a flip bifurcation, a source one. It is then immediate to see that, when $\varepsilon_{rk} \in (\varepsilon_{rk}^{2}, \varepsilon_{rk}^{3\chi})$, the steady state is a source and, when $\varepsilon_{rk} > \varepsilon_{rk}^{3\chi}$, a saddle.

All the results are summarized in the following Proposition.

Proposition 7. *Under Assumptions 1-4, there exist $\varepsilon_{rk}^{2\chi} > \varepsilon_{rk}^{3\chi} > \varepsilon_{rk}^{4\chi} > 0$, $b_4 < b_3 < 0$ and $\varepsilon_{uv}^{\mathcal{F}} > 0$ such that the following results hold:*

i⌋ Let $b > b_3$. If $\varepsilon_{rk} < \varepsilon_{rk}^{4\chi}$, the steady state is a saddle, i.e. a locally determinate, for $\varepsilon_{uv} \in (0, \varepsilon_{uv}^{\mathcal{F}})$, and a source, i.e. locally determinate, for $\varepsilon_{uv} > \varepsilon_{uv}^{\mathcal{F}}$. If $\varepsilon_{rk} \in (\varepsilon_{rk}^{4\chi}, \varepsilon_{rk}^{3\chi})$, the steady state is a saddle, i.e. locally determinate. If $\varepsilon_{rk} \in (\varepsilon_{rk}^{3\chi}, \varepsilon_{rk}^{2\chi})$, the steady state is a source, i.e. locally determinate. If $\varepsilon_{rk} > \varepsilon_{rk}^{2\chi}$, the steady state is a saddle, i.e. locally determinate, for $\varepsilon_{uv} \in (0, \varepsilon_{uv}^{\mathcal{F}})$, and a source, i.e. locally determinate, for $\varepsilon_{uv} > \varepsilon_{uv}^{\mathcal{F}}$.

ii⌋ let $b \in (b_4, b_3)$. If $\varepsilon_{rk} < \varepsilon_{rk}^{2\chi}$, the steady state is a saddle, i.e. locally determinate, for $\varepsilon_{uv} \in$

$(0, \varepsilon_{uv}^{\mathcal{F}})$, and a source, i.e. locally determinate, for $\varepsilon_{uv} > \varepsilon_{uv}^{\mathcal{F}}$. If $\varepsilon_{rk} \in (\varepsilon_{rk}^{2\chi}, \varepsilon_{rk}^{3\chi})$, the steady state is a source, i.e. locally determinate. If $\varepsilon_{rk} \in (\varepsilon_{rk}^{3\chi}, \varepsilon_{rk}^{4\chi})$, the steady state is a saddle, i.e. locally determinate. If $\varepsilon_{rk} > \varepsilon_{rk}^{4\chi}$, the steady state is a saddle, i.e. locally determinate, for $\varepsilon_{uv} \in (0, \varepsilon_{uv}^{\mathcal{F}})$, and a source, i.e. locally determinate, for $\varepsilon_{uv} > \varepsilon_{uv}^{\mathcal{F}}$;

iii] let $b < b_4$. If $\varepsilon_{rk} < \varepsilon_{rk}^{2\chi}$, the steady state is a saddle, i.e. locally determinate, for $\varepsilon_{uv} \in (0, \varepsilon_{uv}^{\mathcal{F}})$ and a source, i.e. locally determinate, for $\varepsilon_{uv} > \varepsilon_{uv}^{\mathcal{F}}$. If $\varepsilon_{rk} \in (\varepsilon_{rk}^{2\chi}, \varepsilon_{rk}^{3\chi})$, the steady state is a source, i.e. locally determinate. If $\varepsilon_{rk} > \varepsilon_{rk}^{3\chi}$, the steady state is a saddle, i.e. locally determinate.

In addition, when ε_{uv} goes through $\varepsilon_{uv}^{\mathcal{F}}$, the steady state undergoes a flip bifurcation.

4.2.3 Interpretation of the results

We have seen that endogenous fluctuations under the hypothesis of dynamic efficiency occur in both non-monetary model and the monetary one when the consumption good is capital intensive, i.e. $b < 0$. As a matter of fact, in such a case persistent cycles arise through a flip bifurcation. The underlying mechanism is to be found in the combinaison of the Rybzcinsky effect and of the Stolper-Samuelson one. Such a mechanism is described, among the others, in Venditti [36].

The intuition is the following. Let us suppose that the capital stock at period t increases. Since the consumption good is the most capital intensive good, there is a raise in the production of the consumption good. Meanwhile, this increase of the capital stock implies, through the Rybzsinsky, a decrease of the output of the investment good. This tends to reduce the capital stock at period $t + 1$. In period $t + 1$, the decrease of the capital stock implies a raise in the investment good. Therefore, this tends to increase the investment in period $t + 1$ and thus the capital stock in period $t + 2$. Notice also that the increase of the production of the investment good in period $t + 1$ implies a decrease of the rental rate of capital in period $t + 1$ and, by the Stolper-Samuelson effect, an increase the relative price of investment in period $t + 1$.

What is new in our paper is the finding that deterministic cycles in the monetary economy occur also when the consumption good is labor intensive, i.e. $b > 0$, meanwhile, in the non-monetary model, persistent fluctuations require a capital intensive consumption good. In the non-monetary case, the intuition is the following. Let us suppose that the capital stock at period t increases. Since the investment good is the most capital intensive good, there is a raise in the production of the investment good. Meanwhile, this increase of the capital stock implies, through the Rybzsinsky effect, an increase of the output of the investment good. This tends to raise the capital stock at period $t + 1$. In period $t + 1$, the increase of the capital stock implies a raise in the investment good. In turn, this tends to increase the investment in period $t + 1$ and thus the capital stock in period $t + 2$. Therefore, oscillations are ruled out.

Consider now the monetary case. Let us suppose that the capital stock at period t increases. In principle, this should be accompanied by an increase in the investment good through the Rybzcinscky effect along with an increase in consumption in period $t + 1$. However, the raise in consumption in period $t+1$ implies, in view of the money market clearing condition, a decrease in the inflation rate which makes money holding less expensive. It follows that, in period t, a larger share of the savings will be devoted to money holdings. Such an increase is large enough to offset the initial increase of physical capital in period $t+1$. For the same reason, investment in period $t+1$, and thus capital stock in period $t+2$, will be larger. This, together with the hypothesis of gross substitutability (and thus a saving function positively correlated with its returns), explains the occurrence of persistent fluctuations.

5 Concluding Remarks

In this paper we have considered a two-sector *OLG* economy with partial cash-in-advance constraint applying on old age consumption expenditures, while young agents supply labor elastically and save their income in physical capital and money balances. We first showed that the capital-labor ratio is above the Golden Rule stationary equilibrium if and only if the share of capital on total income is large enough. As a consequence, we established a general result that shed additional light on the topic: dynamic efficiency under the gross substitutability assumption in consumption, without any additional requirement, is sufficient to rule out local indeterminacy and thus sunspot fluctuations. Such a finding is even more worth emphasizing once one observes that dynamic efficiency must be assumed in order to guarantee that money is dominated by capital in terms of returns and thus the liquidity constraint is binding. Were this not the case, money would have to be interpreted as a bubble whose rate of return (deflation) would adjust in each period to equalize the real interest rate, as in Tirole [35] and Benhabib and Laroque [5]. If, on the one hand, local determinacy is bound to prevail under dynamic efficiency, on the other, the latter is nevertheless compatible with deterministic cycles arising along a flip bifurcation, obtained by varying continuously the elasticity of the offer curve. Such a bifurcation occurs under whatever assumption concerned with the relative sectoral capital intensity: it is just worth noticing that, under a capital intensive consumption good, in order to get a flip bifurcation one needs low enough elasticities of the real interest rate, while under the hypothesis of a capital intensive investment good, such an occurrence is compatible with arbitrarily large elasticities of the interest rate too.

Analogous results are found, under dynamic efficiency, in the non-monetary case. However, here one can put forward the hypothesis of dynamic inefficiency and thus obtain a richer picture of the local dynamics which includes now also the occurence of indeterminacy. Specifically, when the consumption good is capital intensive, local indeterminacy arises through a Hopf bifurcation and for high enough elasticities of the interest rate. On the other hand, within a capital intensive investment good, local indeterminacy occurs through either a Hopf or a flip bifurcation, according to the magnitude of the relative capital intensity.

A fruitful extension of the model could take into account the presence of externalities which, as proved in Cazzavillan and Pintus [13], could restore the compatibility between dynamic efficiency and local indeterminacy. Another line of research might take advantage from the study of Benhabib and Laroque [5] and provide the analysis of a similar economy with rational bubbles within a two-sector technology. Eventually, it should be worthwhile to extend the Rochon and Polemarchakis [32] model with capital, government bonds and liquidity constraint, to a two-sector framework.

6 Appendix

6.1 First-order Conditions of the Maximization Program of the Consumer

The associated Lagrangian of (7) is:

$$L = u\left(C_{t+1}\right) - Bv\left(L_t\right) + \lambda_{0,t}\left[w_t L_t - q_t M_{t+1} - p_t K_{t+1}\right] + \lambda_{1,t}\left[q_{t+1} M_{t+1} + r_{t+1} K_{t+1} - C_{t+1}\right] + \lambda_{2,t}\left[q_{t+1} M_{t+1} - \chi C_{t+1}\right]$$

The associated first-order conditions are:

$$u'\left(C_{t+1}\right) = \lambda_{1,t} + \chi\lambda_{2,t}, \tag{25}$$

$$\frac{Bv'(L_t)}{w_t} = \lambda_{0,t}, \tag{26}$$

$$\frac{r_{t+1}}{p_t} = \frac{\lambda_{0,t}}{\lambda_{1,t}}, \tag{27}$$

and

$$\frac{q_{t+1}}{q_t} = \frac{\lambda_{0,t}}{\lambda_{1,t}+\lambda_{2,t}}. \tag{28}$$

6.2 First-order Conditions for Dynamic Efficiency

The associated Lagrangian of (14) is:

$$\mathscr{L} = u\left(C\right) - Bv\left(L\right) + \lambda\left[T\left(K, K, L\right) - C\right].$$

The first-order conditions are easily obtained:

$$u'\left(C\right) = \lambda, \frac{Bv'\left(L\right)}{T_3} = \lambda, T_1\left(K, K, L\right) + T_2\left(K, K, L\right) = 0. \tag{29}$$

6.3 Proof of Proposition 1

Since $R(K, K, L) = -T_1/T_2$, we have that

$$R^{'}(K, K, L) = -\frac{T_{11}}{T_2}\left(1 - b\right)\left(1 - Rb\right).$$

From the homogeneity of degree one of the social production function $T(K_t, K_{t+1}, L_t)$ and from the first-order conditions of the producer problem (2), one obtains that, at the NSS (K^*, L^*), $b < 1$. Let us define the factor price frontier:

$$\begin{pmatrix} a^{0l} & a^{0k} \\ a^{1l} & a^{1k} \end{pmatrix} \begin{pmatrix} w \\ r \end{pmatrix} = \begin{pmatrix} 1 \\ p \end{pmatrix} \tag{30}$$

where $a^{0l} = \frac{L^0}{Y_0}$, $a^{1l} = \frac{L^1}{Y}$, $a^{0k} = \frac{K^0}{Y_0}$ and $a^{1k} = \frac{K^1}{Y}$ indicate the amount of capital and labor used in each sector. From the definition of the relative capital intensity b 3 we obtain:

$$1 - Rb = -\frac{T_{11}(1-b)(1-Rb)}{T_2}$$

and $R^{'}(K, K, L) < 0$. Let us consider now the first order condition of the consumption maximization problem at the steady state with respect to K: $-T_1(K, K, L)/T_2(K, K, L) = 1$. This is equivalent to the equation defining the stationary capital and labor quantities in a two-sector optimal growth model. Since $R^{'}(K, K, L) < 0$, the proof of Theorem 3.1 in Becker and Tsyganov [4] here applies and there exists a unique solution $\hat{K}$ of (12). Along a stationary path of capital stocks, the highest utility is finally defined as the maximum of $U(C) - BV(L)$ subject to $C = T(K, K, L)$.

6.4 Proof of Proposition 3

Under Assumption 1, the first order conditions of firm's profit maximization problem (1) yield

$$\begin{array}{c} T_{12} = -T_{11}b = -\frac{\partial p}{\partial k}\frac{\partial r}{\partial k}, T_{22} = T_{11}b^2 = -\frac{\partial p}{\partial y}, \\ T_{31} = -T_{11}a = \frac{\partial w}{\partial p}\frac{\partial p}{\partial k}, T_{32} = T_{11}ab = \frac{\partial w}{\partial p}\frac{\partial p}{\partial y} \end{array} \tag{31}$$

where $a \equiv K^0/L^0 > 0$, b is defined by (3) and $T_{11} < 0$[4]. Consider now the expressions for ε_{rk}, ε_v, ε_u and ε_{uv} defined in (17) together with $T_1K^*/T_3L^* = s/(1-s)$, $T = T_1K^*/(1-\chi)$, $Bv'/T_3 = u's/(1-s)(1-\chi)$ and $-T_1/T_2 = (1-\chi)s/(1-\chi-s)$. Keeping in mind that the homogeneity of $T\,(K, Y, L)$ implies $a = (1-b)\,K^*/L^*$, one has that the total differentiation of (12) using (10) and (11) evaluated at the NSS gives the characteristic polynomial $\mathscr{P}(\lambda) = \lambda^2 - \lambda\mathscr{T} + \mathscr{D}$ where $\mathscr{T}$ is the Trace and $\mathscr{D}$ the Determinant. ■

6.5 The Geometrical Method

If $\mathscr{T}$ and $\mathscr{D}$ lie in the interior of the triangle $\mathscr{ABC}$ depicted in Figure 5, the stationary solution is a sink, hence locally indeterminate. In the opposite case, it is locally determinate: it is either a saddle when $|\mathscr{T}| > |1 + \mathscr{D}|$, or a source in the opposite case. If we fix all the parameters of the

[4]See Benhabib and Nishimura [7], Bosi et al. [10] and Venditti [36].

model with exception of ε_{uv} (which we let vary from zero to $+\infty$) we obtain a parametrized curve $\{\mathscr{T}(\varepsilon_{uv}), \mathscr{D}(\varepsilon_{uv})\}$ that describes a half-line Δ starting from the point $(\mathscr{T}_0, \mathscr{D}_0)$ when ε_{uv} is close to zero. The linearity of such locus can be verified by direct inspection of the expressions for $\mathscr{T}$ and $\mathscr{D}$ and from the fact they share the same denominator. This geometrical method makes it possible also to characterize the different bifurcations that may arise when ε_{uv} moves from zero to $+\infty$. In particular, as shown in Figure 5, when the half-line Δ intersects the line $\mathscr{D} = \mathscr{T} - 1$ (at $\varepsilon_{uv} = \varepsilon_{uv}^{\mathcal{T}}$), one eigenvalue goes through unity and a saddle-node bifurcation generically occurs; accordingly, we should expect a change in the number and in the stability of the steady states. When Δ goes through the line $\mathscr{D} = -\mathscr{T} - 1$ (at $\varepsilon_{uv} = \varepsilon_{uv}^{\mathcal{F}}$), one eigenvalue is equal to -1 and we expect a flip bifurcation: it follows that there will arise nearby two-period cycles, stable or unstable, according to the direction of the bifurcation. Eventually, when Δ intersects the interior of the segment $\mathscr{B}\mathscr{C}$ (at $\varepsilon_{uv} = \varepsilon_{uv}^{\mathcal{H}}$), the modulus of the complex conjugate eigenvalues is one and the system undergoes, generically, a Hopf bifurcation. Therefore, around the stationary solution, there will emerge a family of closed orbits, stable or unstable, depending on the nature of the bifurcation (supercritical or subcritical).

Following Grandmont *et al.* [24] and Cazzavillan *et al.* [12], this analysis is also powerful enough to chracterize the occurrence of sunspot equilibria around an indeterminate stationary solution of system (12) as well as along flip and Hopf bifurcations[5]. Actually, as is the case in Grandmont *et al.* [24] and Cazzavillan *et al.* [12], system (12) has at each period t one predetermined variable, the initial stock of capital, and two forward-looking variables, the capital stocks of the two consecutive periods. In such a configuration, the existence of local indeterminacy requires that the two characteristic roots associated with the linearization of the dynamic system (12) around the normalized steady state have modulus less than one. In the opposite case, the steady state is locally determinate. Accordingly, multiple equilibria and sunspot fluctuations occur when the modulus of both eigenvalues is lower than unity, i.e. the steady state is located in the interior of the triangle $\mathscr{A}\mathscr{B}\mathscr{C}$ or along a supercritical flip bifurcation or a supercritical Hopf bifurcation.

The bifurcation parameter we will adopt through our analysis is the elasticity of intertemporal substitution in consumption ε_{uv}. Then the variation of the Trace $\mathscr{T}$ and of the Determinant $\mathscr{D}$ in the $(\mathscr{T}, \mathscr{D})$ plane will be studied as ε_{uv} is made to vary continuously within the $(1, +\infty)$ interval. The relationship between $\mathscr{T}$ and $\mathscr{D}$ is given by a half-line $\Delta(\mathscr{T})$ (Figure 5). $\Delta(\mathscr{T})$ is obtained from (19)-(20) and yields to the following linear relationship:

$$\mathscr{D} = \Delta(\mathscr{T}) = \mathscr{S}\mathscr{T} + \mathscr{Z} \tag{32}$$

[5] In the case of supercritical flip bifurcation and s supercritical Hopf bifurcation, sunspot remain in a compact set containing in its interior, respectively, the stable two-period cycle and the stable closed orbit. Unstable cycles and closed orbits emerge in the opposite case of subcritical bifurcations.

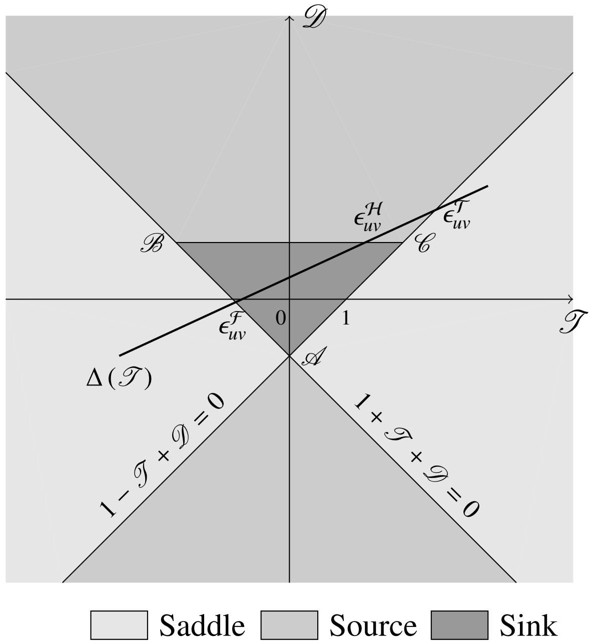

Figure 5: Stability triangle and $\Delta(\mathscr{T})$ segment.

where $\mathscr{X}$ is a constant term. The slope of $\Delta(\mathscr{T})$ is given by:

$$\mathscr{S} = \frac{\mathscr{D}'(\varepsilon_{uv})}{\mathscr{T}'(\varepsilon_{uv})} = \frac{s(\varepsilon_{rk}-\chi)}{\varepsilon_{rk}bs+1-\chi-s}. \tag{33}$$

For a given value $B = B^*$, as ε_{uv} is made to vary in $(0,+\infty)$, $\mathscr{T}(\varepsilon_{uv})$ and $\mathscr{D}(\varepsilon_{uv})$ move linearly along the line $\Delta(\mathscr{T})$. As $\varepsilon_{uv} \in (0,+\infty)$, the properties of the line $\Delta(\mathscr{T})$ are derived from the consideration of its extremities. Actually, the starting point is the couple $(\lim_{\varepsilon_{uv}\to+\infty}\mathscr{T} \equiv \mathscr{T}_\infty, \lim_{\varepsilon_{uv}\to+\infty}\mathscr{D} \equiv \mathscr{D}_\infty)$:

$$\mathscr{T}_\infty(\varepsilon_{rk}) = \frac{\varepsilon_{rk}(bs\chi+1-\chi)-s\chi}{\varepsilon_{rk}(bs\chi+1-\chi-s)}; \mathscr{D}_\infty(\varepsilon_{rk}) = \frac{s(\varepsilon_{rk}-\chi)}{\varepsilon_{rk}(bs\chi+1-\chi-s)}. \tag{34}$$

Using the expressions of $\mathscr{T}_\infty$ and $\mathscr{D}_\infty$, one is able to show that $\mathscr{D}_\infty = \mathscr{T}_\infty - 1$. Finally, the half-line $\Delta(\mathscr{T})$ is pointing upward or downward depending on the sign of $\mathscr{D}'(\varepsilon_{uv})$:

$$\mathscr{D}'(\varepsilon_{uv}) = -\frac{s(\varepsilon_{rk}-\chi)}{(bs\chi+1-\chi-s)\,\varepsilon_{rk}\varepsilon_{uv}^2} \tag{35}$$

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