Newton on Indivisibles

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Though Wallis's *Arithmetica infinitorum* was one of Newton's major sources of inspiration during the first years of his mathematical education, indivisibles were not a central feature of his mathematical production.

To judge from his reading notes, he firstly studied Wallis's treatise at the beginning of 1664 ([13], I, **1**, 3, §§ 1-2, pp. 89-95), and came back to it one year later ([13], I, **1**, 3, § 3, pp. 96-121). In the former occasion, he confined himself to the first part of the treatise, and possibly accompanied his reading with that of the *De sectionibus conicis* and the *De angulo contactus*, also contained in Wallis's *Operum Mathematicarum Pars Altera* ([16] and [17]). At the beginning, his attention was retained by some general remarks, but very often shifted to the elaboration of an original, algebraic version of the method of indivisibles, which he applied to get both a correct quadrature of the parabola and an incorrect quadrature of the hyperbola. In the latter occasion, he rather focused on Wallis's quadrature of the circle—which he deeply transformed ([15], pp. 152-181)—whiteout any particular attention to indivisibles. Only the notes relative to the former reading deserve to be considered here. Section 1 is devoted to them. Sections 2 and 3 document the important but often implicit role indivisibles and infinitesimals play in Newton's mathematics and his reluctance to accept them. The differences between indivisibles and infinitesimals, and their roles in Newton's mathematics are difficult to grasp because Newton himself equivocated about them. As we shall see in sections 2 and 3, he used both indivisibles and infinitesimals (but infinitesimals, particularly) abundantly, but starting in 1671, he explicitly ascribed to them a subordinated status vis-à-vis fluxions and "evanescent" quantities and ratios. This did not prevent him from using in the *Principia* some principles and arguments closely remembering the method of indivisibles. The end of section 3 and section 4 focus on some examples of this use. The latter is, in particular, devoted the proof of two theorems (propositions LXXI and LXXIV of Book I), where this use is quite manifest.

1 Squaring the parabola and the hyperbola: an algebraic version of the method of indivisibles

A trace of Newton's early interest for indivisibles and infinitesimals is provided by the entry "Of Quantity" of the *Questiones quædam Philosophiæ* ([13], I, **1**, 3, § 1, pp. 89-90), a commonplace book dating back to 1664-1665[1]. No personal elaboration seems to be at issue here. Newton merely reports the result of his readings. Some quotes will be enough to illustrate this:

> As finite lines added in an infinite number to finite lines, make an infinite line: so points added twixt points infinitely, are equivalent to a finite line.
>
> All superficies beare the same proportion to a line yet one superficies may bee greater than another (the same may be said of bodys in respect of surfaces) which happens by reason that a surface is infinit in respect of a line [...] Thus $\frac{2}{0}$ is double to $\frac{1}{0}$ & $\frac{0}{1}$ is double to $\frac{0}{2}$ [...] yet if $\frac{2}{0}$ & $\frac{1}{0}$ have respect to 1 they beare the same relation to it that is $1 : \frac{2}{0} :: 1 : \frac{1}{0}$ & ought therefore to bee considered equall in respect of a unite.
>
> The angle of contact is to another angle, as a point to a line, for the crookednes in one circle amounts to 4 right angles & that crookedness may bee conceived to consist of an infinite number of angles of contact, as a line doth of infinite points.

Another, possibly contemporary, note ([13], I, **1**, 3, § 2, pp. 91-95)[2] is not only more focused on indivisibles. It also presents an original, algebraic version of the method of indivisibles.

It begin with the statement of three general lemmas:

> [Lemma 1] All the parallell lines which can be understoode to bee drawne uppon any superficies are equivalent to it [...] [and] may be used instead of the superficies [...].
>
> [Lemma 2] If all the parallell lines drawne uppon any superficies be multiplied by another line they produce a Sollid like that which re[...][s]ults from the superficies drawne into the [...][s]ame line [...] whence All the parallell superficies which can bee understoode to bee in any sollid are equivalent to that Sollid. And If all the lines in any triangle, which are parallell to one of the sides, be squared there results a Pyramid. if those in a square, there results a cube. If those in a crookelined figure there resu[l]ts a sollid with 4 sides terminated & bended according to the fasshion of the crookelined figure[.]
>
> [Lemma 3] If each line in one superficies bee drawne into each correspondent line in another superficies [...] they produce a sollid whose opposite sides are fashioned by one of the superfic [...].

[1]The original Newton's manuscript is conserved at the University Library of Cambridge (now available online on the Cambridge Digital Library): Add. 3996 (the entry "Of Quantity" is at p. 90r) A printed edition, supplemented by a large commentary, is provided in [9].

[2]The original is included in another manuscript conserved at the University Library of Cambridge (and now available online on the Cambridge Digital Library): Add. 4000, p. 82r and 83r-84r. At top of page 82r there is a date: '166$\frac{3}{4}$ January', which refers to January 1663 old style, the month of January 1663 in the Julian calendar still used in Great Britain at that time, equivalent to the month going from January 11th 1664 to February 10th 1664 of our Gregorian calendar, which were then used on the Continent. Still, the blank dividing this date from the beginning of the note suggests that the former could not be that of the latter. A detailed analysis of the note is offered in [15], pp. 134-149. I come back here on the main points that I made there.

Taken as such, these lemmas are far from new and prefigure no original version of the method of indivisibles[3]. What is rather original is the way they are applied, namely by sketching an algebraic version of the method of indivisibles, through which Newton tackles two classical problems: "To square the Parabola" ([13], I, **1**, 3, § 2, p. 93), and "To square the Hyperbola" ([13], I, **1**, 3, § 2, pp. 94-97). The algebraic nature of this version of the method comes from this: Newton considers these curves to be expressed by algebraic equations and takes these equations as a guide for coming into a numbers of geometrical magnitudes, whose construction is governed by the three foregoing lemmas, and which consideration allows him to assign a measure to the surface subtended by the curves to be squared, under the form of an algebraic expression. The reconstruction of his arguments will make this clearer[4].

Let us being with Newton's (correct) solution of the former problem: squaring a parabola. He tackles it in the case of a parabola expressed by two different equations, referred to two distinct system of Cartesian orthogonal co-ordinates. Here is the argument pertaining to the first equation:

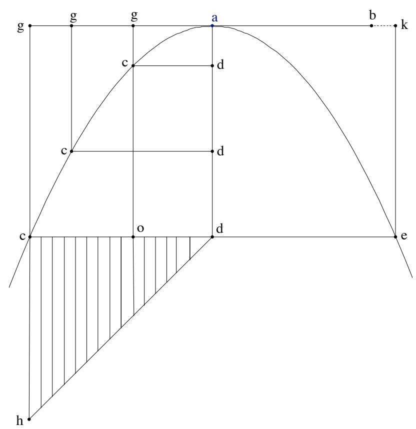


Fig. 1

> In the Parabola cae [fig. 1] suppose the Parameter ab $= r$. ad $= y$. dc $= x$ and $ry = xx$ or $\frac{xx}{r} = y$. Now suppose the lines called x doe increase in arithmetical proportion all the x's taken together make the superficies dch which is halfe a square[.] let every line drawne from cd to hd be square and they produce a Pyramid equall to every $xx = \frac{x^3}{3}$. which if divided by r there remains $\frac{x^3}{3r} = \frac{yx}{3}$ equall to every $\frac{xx}{r}$ equall to every (y) or all the lines drawne from ag to accc equall to the superficies agc equall to a 3^d parte of the superficies adcg & the superficies acd $= \frac{2yx}{3}$.

The difference between Newton's and Wallis's approaches is immediately evident: where the latter saw a curvilinear triangle inscribed in a parallelogram whose chords are in duplicate ratio of arithmetic proportionals, and conceived of the problem as that of inferring from this last ratio that of this triangle and this parallelogram, the former sees a curve referred to an orthogonal system of Cartesian (intrinsic)

[3] As observed by Whiteside ([13], I, **1**, 3, § 2, p. 92, footnote 8), the last remark in lemma 2 concerned with "a crookelined figure" is a generalisation of Wallis's construction of conical Pyramidoides ([16], propp. IX, XIV, and XVIII), and is then adding something new to Newton's readings, though this addition is, as such, quite natural.

[4] In stating these arguments, Newton makes no explicit mention of his three lemmas. My appeal to them in my reconstruction pertains then to my own understanding of these arguments.

co-ordinates of axis ab and origin a, and expressed by a polynomial quadratic equation, and conceives of the problem as that of inferring from this equation an expression of the area of the surface subtended by this curve, in terms of its abscissa, or, more generally speaking, a measure of this surface in term of its basis. According to Newton, no numerical sequence is thus relevant for solving the problem; all the relevant information is included in the equation.

For this information to be easily processed, the equation has, however, to be reduced to an appropriate form. The first stage in Newton's argument is thus the transformation of the equation given in the form '$\mathcal{E}(x, y) = 0$' in an equation of the form '$y = \mathcal{E}(x)$', that is, from $ry = xx$ (or $ry - xx = 0$) to $y = \frac{xx}{r}$. In modern terms, one is passing here from an implicit function to the corresponding explicit function. Still, Newton is not dealing with functions, but rather with a curve. The relevant difference for him is not that of two sorts of functions, but that of two ways of describing this curve. According to the former way, the curve is described as a geometrical locus and then identified through a relation of two linear co-ordinates, which is in turn expressed by an equation in the form '$F(x, y) = 0$'. According to the latter, the curve is described as the graph of a variation and then identified through the law of this variation, which is in turn expressed by an equation in the form '$y = \mathcal{E}(x)$', whose second member directly expresses the ordinate in term of the abscissa, and fixes then the way the former is varying with respect to a parameter provided by the latter. Whereas the former description could also be provided by a system of proportions (in the case at issue, one could replace, for example, the equation $ry = xx$ with the proportion $r : x :: x : y$), this is not so for the latter description (since a system of proportion can express a law of variation only in an indirect, or implicit way). Newton's method takes then start from the understanding that the latter way of describing a curve makes its quadrature easier (which is in fact also suggested by Wallis's method, since the series $\sum_r$ directly express in this method the law of variation of the chords of the relevant surfaces: these chords are taken to vary as the terms of these series).

Newton's basic idea is that of operating on the expression '$\mathcal{E}(x)$' expressing the ordinate of the curve at issue, so as to transform it into a new expression '$\mathcal{E}^*(x)$' expressing the area of the surfaces subtended by this curve. His version of the method of indivisibles reduces to the rules governing his operating on the former expression so as to get the latter. These rules are so conceived to parallel the algebraic operations to be applied to x in order to get $\mathcal{E}(x)$: starting from considering the totality of the values of x included between the origin and the value providing the basis of the surface at issue, one is then supposed to get, through them, an expression of the totality of the corresponding values of y, that is, of the totality of all the ordinates of the curve, which are in turn nothing but all the components of the surface subtended by such a curve. This is justified by lemma 1, according to which a totality of parallel segments is to be identified with the plane figure of which these segments are all the parallel chords (taken in a certain direction).

In general, this procedure can be described as follows. Let n be an integer number, and $\{O_i\}_{i=0}^n$ ($i = 0, 1, \ldots, n$) a sequence of algebraic operations. Let us then denote with '$\langle_\alpha O_i\rangle$' the result got by applying the operation O_i to α, and with and '$\langle_\alpha O_0^n\rangle$' the result got by reiteratively applying to α the operations $O_0, \ldots O_n$. Let us also stipulate that $\langle_\alpha O_0\rangle = \langle_\alpha O_0^0\rangle = \alpha$ (which means that the operation O_0 merely consists in giving the argument on which O_1 applies). Using this notation, an equation $y = \mathcal{E}(x)$ could be written as

$$y = \langle_x O_0^n\rangle\,, \tag{1}$$

provided that $\{O_i\}_{i=0}^n$ be the sequence of operations to be applied to x so as to get $\mathcal{E}(x)$. In the case at issue in Newton's example, n would be 2, and O_0 would consist in giving x, O_1 in getting its square, and O_2 in dividing such a square by r, so that: $\langle_x O_0\rangle = \langle_x O_0^0\rangle = x$; $\langle_x O_1\rangle = \langle_x O_0^1\rangle = x^2$; $\langle_{x^2} O_2\rangle = \langle_x O_0^2\rangle = \frac{x^2}{r}$. What is needed to square the curve expressed by the equation (1) is to come into a measure of the totality of the values of $\langle_x O_0^n\rangle$ corresponding to the values of x between the origin and the value providing the basis of the surface at issue. Ideally, this could be done by coming into a sequence of geometrical magnitudes to be respectively identified with the totalities of the values of $\langle_x O_0^i\rangle$ $(i = 0, 1, \ldots, n)$ corresponding to these same values of x, and into the respective measures of these magnitudes, or at least into the measure of the last of them. As a matter of fact, Newton carries on his method in a slightly different way, however.

Before accounting for his argument, it is important to remark that he does not explicitly distinguish the last value of x from x itself. This suggests denoting the mentioned totalities of the values of $\langle_x O_0^i\rangle$ by '$\sum_0^x [\langle_x O_0^i\rangle]$'. In the case at issue in Newton's example, these are respectively the totality $\sum_0^x [x]$ of the values of x, the totality $\sum_0^x [x^2]$ of the values of x^2, and the totality $\sum_0^x \left[\frac{x^2}{r}\right]$ of the values of $\frac{x^2}{r}$.

In the first stage of the argument (which is to be the same whatever the sequence $\{O_i\}_{i=0}^n$ might be), Newton comes into a geometrical magnitude to be identified with the totality $\sum_0^x [x]$. To this purpose, he understands $\sum_0^x [x]$ as the totality of the values of an orthogonal ordinate equal to x. In other terms, he takes $\sum_0^x [x]$ as the totality $\sum_0^x [y]$ of the values of y for the supposition that $y = x$, provided that the co-ordinates are orthogonal (which in modern terms corresponds to the definite integral of the identity function). According to lemma 1, this brings him to identify $\sum_0^x [x]$ with an isosceles right-angled triangle, namely dch.

In the second stage, Newton comes into a geometrical magnitude to be identified with the totality $\sum_0^x [x^2]$, and into a measure of it. Here lemma 2 comes at play. According to it, if a plane figure is identified with a certain totality of parallel segments, the totality of the respective products of these segments with other segments is to be identified with the solid whose parallel sections are given by the rectangles respectively constructed on each pair of segments which are multiplied to each other[5]. Hence, as Newton makes already clear in stating the lemma, the totality of the squares of the parallel chords of a triangle is to be identified with a pyramids, namely a pyramid whose base and height are respectively the square constructed on the last value of x and this value itself. Insofar as this pyramids is a third of the cube constructed on this very value, Newton draws from this that $\sum_0^x [x^2]$ is equal to, or better measured by $\frac{x^3}{3}$ (by supposing that this cube is in turn equal to, or better measured by x^3).

One could imagine that in the third stage, Newton proceed as in the second—namely by coming into a geometrical magnitude to be identified with the totality $\sum_0^x \left[\frac{x^2}{r}\right]$, and then into a measure of it—, and that to this purpose, he take this totality as the totality of those values whose product with the constant value r are equal to x^2. But, by admitting this, Newton could only have at most concluded that $\sum_0^x \left[\frac{x^2}{r}\right]$ is to be identified with the very surface agc. It is thus not surprising that Newton deals with the division by r differently from the inverse multiplication: instead of conceiving of the former operation as being applied on the very totality $\sum_0^x [x^2]$, he conceives it as being applied to the measure of this totality, *i. e.* to $\frac{x^3}{3}$ (as it were natural to do if r would be taken to be a number). This brings easily him to conclude

[5]For the difference between lemma 2 and lemma 3, cf. footnote 9, below.

that the measure of $\sum_0^x \left[\frac{x^2}{r}\right]$ is $\frac{x^3}{3r}$, without considering any geometrical magnitude to be identified with this totality.

By making the difference between the variable abscissa x and its last value, this result could be rendered as follows:

$$\mathfrak{M}\,[\mathsf{MAP}] = \mathfrak{M}\left[\sum\nolimits_0^{\xi}\left[\frac{x^2}{a}\right]\right] = \frac{\xi^3}{3a}, \tag{2}$$

where '$\mathfrak{M}\,[\alpha]$' denotes the measure of α. This result deeply differs from Wallis's quadrature of the parabola[6]. When applied to the case at issue, the latter merely allows to conclude that the surface agc is a third of the rectangle agcd. Newton provides, instead, a measure of this surface though an appropriate algebraic expression. To this purpose, he is forced to deal with the operations of squaring x and with that of dividing x^2 by r in two different ways. This is because the argument requires to identify a certain totality of values with a geometrical magnitude whose measure is known beforehand, which in the case is the case for $\sum_0^x [x^2]$, but not for $\sum_0^x \left[\frac{x^2}{r}\right]$. This makes quite clear that the argument cannot be generalised to a curve expressed by any equation of the form '$y = \mathcal{E}(x)$'. This is certainly the reason for Newton will abandon his method pretty soon.

Before doing that, he tries however to improve this method, by applying it to other examples.

The first of them ([13], I, **1**, 3, § 2, p. 93) pertains to the same parabola considered above, but referred now to a system of Cartesian (intrinsic) co-ordinates of axis ce and origin c. The argument is as follows. Let $x = \mathsf{co} = \mathsf{cd} - \mathsf{ag}$, $y = \mathsf{oc} = \mathsf{da} - \mathsf{gc}$, and $\mathsf{ce} = 2\,(\mathsf{cd}) = b$, which leads to the new equation $ry = bx - x^2$. Through an argument similar to the previous one, Newton identifies the totality of the values of x between the origin and $\mathsf{ce} = b$, *i. e.* $\sum_0^b [x]$, with the right-angled isosceles triangle with this last segment as base, which, he says, is 4 times dch and is thus equal to, or better measured by, $\frac{b^2}{2}$. According to lemma 2, he next, he identifies the totality $\sum_0^b [bx]$ with the half-cube cut by a diagonal plane in the cube constructed on b, which is in turn measured by $\frac{b^3}{2}$. As showed above, the totality $\sum_0^b [x^2]$ is in turn to be identified with a pyramid whose measure is $\frac{b^3}{3}$. It follows then that the measure of the totality $\sum_0^b [bx - x^2]$ is $\frac{b^3}{2} - \frac{b^3}{3} = \frac{b^3}{6}$, and that of totality $\sum_0^b [y]$ is $\frac{b^3}{6r}$, which is the same "as before", says Newton. Indeed, according to the previous result and supposing that $x = \mathsf{cd} = \frac{b}{2}$, the measure of the surface agc is $\frac{b^3}{24r}$, and, according to lemma 1, the totality $\sum_0^b [y]$ is nothing but the surface eac—which is in turn the difference between the rectangle ekgc of measure $b\,[y]_{x=\frac{b}{2}}$, and the double of the surface agc—, so that $\mathfrak{M}\left[\sum_0^b [y]\right] = b\,[y]_{x=\frac{b}{2}} - \frac{b^3}{12r} = \frac{b^3}{4r} - \frac{b^3}{12r} = \frac{b^3}{6r}$, as just said.

What is new in this second example is the clear distinction between the variables x and y and their values, namely the values $\mathsf{ce} = b$ and $\mathsf{cd} = \frac{b}{2}$ of x, and the value $\mathsf{da} = \frac{b^2}{4r}$ of y, and the admission that $\mathfrak{M}\,[\alpha \pm \beta] = \mathfrak{M}\,[\alpha] \pm \mathfrak{M}\,[\beta]$.

This makes Newton's method appropriate for providing the quadrature of any curve of equation $y = ax^2 + bx + c$ referred to a system of Cartesian (intrinsic) orthogonal co-ordinates (*i. e.* of any parabola referred to such a system), between any pair of values of x. Still at the beginning of his mathematical education, and deluded by this important result, Newton believed for a short time that the same method, if further improved, could have also provided the quadrature of the hyperbola, and

[6] Though formally equivalent to '$\int_0^{\xi} \frac{x^2}{a} dx = \frac{\xi^3}{3a}$', the equality (2) deeply differs also from it both for its justification and its intended interpretation.

even reinforced this illusion with a argument leading to a wrong quadrature. Though fallacious, this argument deserves considerations, since it shows Newton's algebraic method of indivisibles at work on a more difficult (and indeed impossible) case. The basic idea is the following: just as the totality of values of an independent variable can be identified with a right-angled isosceles triangle, the totality of values of any other variable segment can be identified with a surface on which it is possible to construct a solid that can sometimes be identified in turn with the totality of values of another segment depending on the former in an appropriate way; this solid can then be compared with other solids got in the same way, and from this comparison some results to be used for squaring a curve other than a parabola can eventually be drawn.

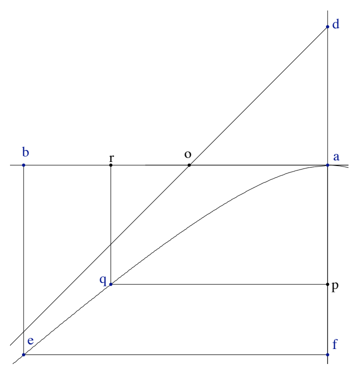


Fig. 2a

Let eqa (fig. 2a)[7] be a branch of hyperbola of vertex a and asymptote od referred to a system of Cartesian (intrinsic) orthogonal co-ordinates of axis ab and origin a, of equation

$$x^2 = dy + xy, \tag{3}$$

where $d =$ oa $=$ ad . Newton's mistake[8] comes by his taking this same hyperbola to be also expressed, with respect to the same system of co-ordinates, by the equation

$$x^2 = \frac{qy + y^2}{5} \tag{4}$$

where $q =$ ad, which entails that $q = d$, and by supposing that in this latter equation y can be taken to be the independent variable, whereas in the former equation the independent variable is x. His argument, works then as follows.

[7]Following Whiteside's suggestion ([13], I, 94-97, footnote 29), I take Newton's main diagram—which seems to represent an equilateral hyperbola of centre d—as inaccurate and I amend it. To this purpose, I only consider the branch of hyperbola which Newton tries to square.

[8]On this mistake, cf. [13], I, **1**, 3, § 2, 94-97, notes 16, 21 et 29.

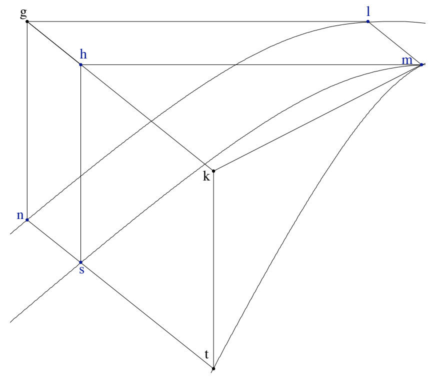


Fig. 2b

By reasoning as before, it is easy to infer that the measure of the totality $\sum_0^a [x^2]$ is $\frac{a^3}{3}$, where $a =$ ab. According to lemma 1, the totality $\sum_0^a [y]$ is to be identified with the surface abe. Hence, according to lemma 2, the totality $\sum_0^a [dy]$ is to be identified with the solid mhsngl (fig. 2b), whose base mhs is congruent with to the surface abe and whose faces mlgh and hgns are two rectangles respectively of sides ml $=$ hg $= d$, mh $=$ lg $= a$, and hg $=$ sn $= d$, hs $=$ gn $= y_{[x=a]} = \frac{a^2}{d+a}$. In the same way, according to lemma 3[9], the totality $\sum_0^a [xy]$ is to be identified with the other solid mhstk, also having mhs as base, but being such that its face mhk is a right-angled isosceles of sides mh $=$ hk $= a$, and its face hstk is a rectangle of sides hk $=$ st $= a$, hs $=$ skt $= y_{[x=a]} = \frac{a^2}{d+a}$. Hence, from equation (3), it follows that the whole solid lgntkm measures $\frac{a^3}{3}$. Take then y as the independent variable and suppose that af $= b$ (fig. 2a). It follows that the totality $\sum_0^b [y^2]$ measures $\frac{b^3}{3}$, and the totality $\sum_0^b [qy]$—which, according to lemma 2, is to be identified with a right-angled prism of height equal to q, having as base a right-angled isosceles triangle whose catheta are equal to b—measures $\frac{qb^2}{2}$. As, according to lemma 1, the totality $\sum_0^b [x]$ is to be identified with the surface aef, according to lemma 2, the totality $\sum_0^b [x^2]$ is in turn to be identified with a solid yxvwz (fig. 2c), whose base xvwz is a square with sides equal to a and whose face yxz is congruent with aef (so that yz $= b$ and zx $= x_{[y=b]} = \sqrt{\frac{qb+b^2}{5}}$). Hence, from equation (4), it follows that this solid measures $\frac{1}{5}\left(\frac{qb^2}{2} + \frac{b^3}{3}\right) = \frac{3qb^2+2b^3}{30}$, and then its half yxvz[10] measures $\frac{3qb^2+2b^3}{60}$. But this last solid can be join to the solid mhstk(fig. 2b), so as to form a right-angled prism of height equal to b having as base a right-angled isosceles triangle whose catheta are equal to a. As this solid measures $\frac{a^2b}{2}$, it follows that the solid mhstk measures $\frac{a^2b}{2} - \frac{3qb^2+2b^3}{60} = \frac{30a^2b-3qb^2-2b^3}{60}$, and therefore the solid mhsngl measures $\frac{a^3}{3} - \frac{30a^2b-3qb^2-2b^3}{60} = \frac{20a^3-30a^2b+3qb^2+2b^3}{60}$. This is then the measure of the totality $\sum_0^a [dy]$. Thus, the measure of $\sum_0^a [y]$, which is also that of the tcredsurface abe, is $\frac{20a^3-30a^2b+3qb^2+2b^3}{60d}$, which, according to Newton, provides the sought our quadrature of the hyperbola at issue.

[9] As it is stated, lemma 3 is reminiscent of the rule of *ductus plani in planum* (cf. [13], I, **1**, 3, § 2, p. 93, footnote 10). But it is still not clear how this lemma is supposed to differ from lemma 2, according to Newton understanding. I take lemma 2 to concern the multiplication of the parallel chords of a certain figure either by themselves or by the same constant segment, respectively, and lemma 3 to concern the multiplication of the parallel chords of a certain figure by the corresponding values of a variable segment (which are taken in turn as the parallel chords of another figure), respectively.

[10] As a matter of fact, Newton does not explicitly mention the whole solid yxvwz and draws no diagram of it, by confining himself to consider its half yxvz.

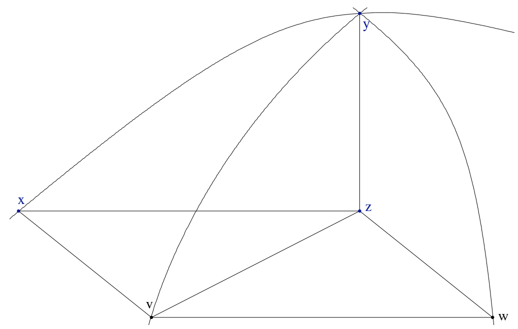


Fig. 2c

In Whiteside's words ([13], I, **1**, 3, § 2, 95, footnote 29, "this is a marvellous tangle". However marvellous it be, it can at most show the genius of the young Newton at work, but not suggest a possible way of generalising his algebraic method of indivisibles. The very idea on which this method is based, namely that of associating solids whose volume can be algebraically expressed, to appropriate components of an algebraic equation expressing a curve, forces it into the tight limits in which it is applied in Newton's first two examples. Newton cannot but abandon it, then. This also coincides with his abandoning the method of indivisibles as a source of inspiration for his search of a general method of quadrature. This did not prevent Newton later on to appeal in different ways to the notion of indivisible, or to discuss it in relation to other connected notions, or to rely on arguments somehow reminiscent of this method. The following sections will offer some examples of this.

2 Infinitesimals and indivisibles in Newton's early tracts (1666-1671)

The differences between indivisibles and infinitesimals, and their roles in Newton's mathematics are difficult to grasp because Newton himself equivocated about them. As we shall see, he usually glossed over the differences between indivisibles and infinitesimals. He used both (but infinitesimals, particularly) abundantly, sometimes separately and sometimes jointly in the same mathematical argument. It is nonetheless true that already in 1671, by the time he wrote his great, unfinished treatise, usually known as *Tractatus de methodis serierum et fluxionum*[11], he was harbouring doubts about the status of "moments" (his preferred word for infinitely or indefinitely small magnitudes, often used in his previous notes on

[11]Unpublished in Newton's lifetime, the first page of Newton's holograph copy of this tract is missing. On the title we use we follow Whiteside's suggestion ([13], vol. III, p. 32, footnotes 2 and 3), based on one of Newton's manuscript in which he mentions the treatise as "Tractatu[s] de methodis serierum et fluxionum anno 1671 composito". When it was printed in Latin in 1779, in Newton's *Opera*, edited by S. Horsley, it was given the title "Artis analyticæ specimina vel Geometria analytica" ([12], vol. 1, pp. 389-518; in the table of contents, *ibid.*, p. XVII, the title becomes: "Geometria analytica sive specimina artis analyticæ"). This Latin edition was preceded by an English translation, edited by J. Colson, in 1736, titled after the mentioned manuscript, namely "The Method of Fluxions and Infinite Series" ([11]). We quote from [13], vol. III, pp. 32-328. For the datation, cf. *ibid.*, Introduction, pp. 3-18. The treatise results from a deep revision, involving many significant changes and additions, of a previous tract, composed in October-November 1666, today known, after Whiteside's edition, as the *October 1666 Tract on Fluxions* ([13], vol. I, pp. 400-448). All translations from these and other Newton's works included in [13] are Whiteside's, unless mention to the contrary.

the same matters[12].) and giving them a subordinated logical status *vis à vis* fluxions[13]. Consequently, he started avoiding appealing to them as much as he could.

De analysi per aequationes numero terminorum infinitas ([13], vol. II, pp. 206-247), probably written in the early summer of 1669, circulated among interested fellows of the Royal Society by early July of the same year. Although not printed until many decades afterwards, it was the only one of the three substantial tracts the young Newton wrote between 1666 and 1671 on his version of the calculus that was meant for public circulation[14]. In the middle of the priority dispute, it was printed in 1712 as a piece of the *Commercium epistolicum* ([1]). In *De analysi*, the basic notion is that of moment, rather than speed or fluxion[15], which is conspicuously absent from the tract. Quantities are generated by motion and moments are introduced by means of motion in a geometrical configuration that is one of the most recurrent items in Newton's accounts of the foundations of his calculus (though it is often presented without any mention of moments).

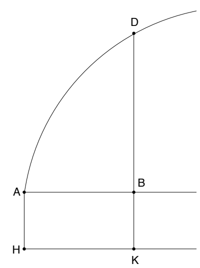


Fig. 3

Newton sets $\mathsf{AB} = x$ (fig. 3), $\mathsf{BD} = y$, $\mathsf{AH} = 1$, and assumes the straight line DBK, perpendicular to AB and parallel to AH, to move uniformly away from AH. He then merely stresses that DBK describes the curvilinear surface ABD and the rectangle AK, and that "BK (1) is the moment by which AK (x) is gradually increased and BD (y) the moment by which ABD is gradually increased" ([13], vol. II, p. 232). Nothing else is said about the nature and properties of those moments, although it seems to be implicit that (the line DBK being in motion) they are something else that mere Cavalierian indivisibles—in any case, whatever Newton had in mind, here, he left to the reader the task of clarifying the matter.

That was not the case in the context of the priority dispute. When this passage and this figure appear on page 14 of the *Commercium epistolicum*, someone (most probably Newton himself) added a footnote to them[16]:

> *Nota bene*. Here is described the method of operating by fluents and their moments. These moments were afterwards called differences by Mr. Leibniz, and hence the name of Differential Method.

[12]For a detailed account of these notes, besides to Whiteside's comments and footnotes to his edition of them ([13], vol. I), we refer to [15]

[13]The term 'fluxion' was introduced by Newton for the first time just in the *De methodis*, to designate (with a significant conceptual stretching) what he had previously identified as punctual speed.

[14]For the composition and circulation of *De analysi*, cf. Whiteside's comments in footnotes 1, 2, pp. 206-207 of ([13], vol. II). The third substantial tract is the *October 1666 Tract on Fluxions*, mentioned in footnote (11).

[15]Cf. footnote 13, above.

[16]Cf. [1], p. 14: "N.B. Hic describitur Methodus per Fluentes et earum Momenta. Haec momenta à D. Leibnitio Differentiae postmodum vocata sunt: Et inde nomen Methodi Differentialis." We slightly modify Whiteside's translation).

This addition was certainly intended to prove Newton's priority on Leibniz also in matter of differentials (or infinitesimal differences). In so doing, however, he made explicit an infinitesimalist understanding of moments that, interestingly enough, was only implicit in the original *De Analysi*.

In *De analysi*, to clarify the introduction of moments, Newton brings up the problem of finding the length of curves, where he shows how to determine the length of an arc of circle. The moment is here the hypotenuse of the "indefinitely small [indefinitè parvo]" triangle whose sides are the moments of the "basis" (abscissa) and the ordinate ([13], vol II, p. 233). He assumes, again, the moment of the principal variable, here x (the abscissa), to be 1, and then makes the following statement mingling together indivisibles and infinitesimals ([13], vol. II, pp. 234-235. We slightly modify Whiteside's translation):

> It must be noted that unit [*unitas*] which is set for the moment[17] is a surface when the question concerns solids, a line when it relates to surfaces and a point when (as in this example) it has to do with lines. Nor am I afraid to talk of a unit in points or infinitely small lines [*infinitè parvis*] inasmuch as geometers now consider proportions in these while using indivisible methods.

Newton certainly knew the basic technique of Cavalierian indivisibles, namely the so-called Cavalieri's principle. As we shall see, in subsequent years it (or a principle referred to first and ultimate ratios structurally analogous to it) gained relevance and came eventually to play a non-negligible role in the *Principia mathematica philosophia naturalis*.

In his early tracts the most important instance of its use seems to be a convoluted proof of equality of areas in the *De methodis*. All pairs of corresponding moments in the two areas, by which here Newton explicitly means infinitesimal rectangles, are shown to keep a constant ratio, and he concludes that the areas themselves will be in the same ratio ([13], vol. III, p. 282). He concludes by stressing the unusual occasion in which he has used that technique and by highlighting his view that the techniques based on motion, and, then, fluxions are "more natural" ([13], vol. III, p. 282):

> I have here used this method of proving that curves are equal or have a given ratio by means of the equality or given ratio of their moments since it has an affinity to the ones usually employed in these cases. However, that based on the genesis of surfaces by their motion or flow appears a more natural approach.

Before we turn to Newton's texts of the 1670s and early 1680s in which he advocates renouncing to infinitesimally small elements, let us look at the role such notions played in Newton's early tracts. In 1665 and 1666 his very first musings on the computation of (punctual) speeds (to be replaced by fluxions later on) are based on the intuition that in each infinitesimal interval of time the motion can be assumed uniform, and therefore when different bodies move, or when a body moves non uniformly, the "infinitely little lines" described are proportional to the instantaneous speeds. As Newton puts it: "[...] though

[17] This specification ("unitas ista quæ pro momento ponitur", in Newton's original Latin) is intended to distinguish the unity "which is set for the moment" from another one, which is rather set for the diameter of the circle (and later for its radius), of which the moment is merely a(n infinitesimal) portion. This double use of the unity makes clear that taking the moment of the abscissa to be equal to 1 is not, for Newton, a way to ascribe a measure to it, but rather a way to identify the abscissa as the principal variable of the problem, and its moment as a sort of (infinitesimal) parameter to which the other relevant moments are to be compared.

they [two moving bodies describing two lines] move not uniformly yet are the infinitely little lines which each moment[18] they describe as their velocities are which they have while they describe them. As if the body A with the velocity p describe the infinitely little line o in one moment. In that moment the body B with the velocity q will describe the line $\frac{oq}{p}$. For $p : q :: o : \frac{oq}{p}$". ([13], vol. I, p. 385).

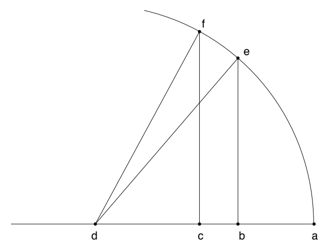


Fig. 4

The quote is drawn from a note dated 13 November 1665 (OS) and comes from Newton's proof of the direct algorithm of the calculus. But the same idea comes back in many occasions, and, though varying for its details from note to note, such a proof invariably relies on this idea, and crucially uses infinitesimals, then. It first appears in a note dated 20 may 1665 (OS) ([13], vol. I, pp. 272-280, esp. 273-274). In this note, the direct algorithm is still conceived, under Descartes's influence, as an algorithm for transforming a polynomial equation $F(x, y) = 0$, expressing a certain curve, in the expression of the subnormal of this same curve given by a ratio of polynomial in x and y, again (which in modern notation is nothing but $-\frac{y\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$). To get this algorithm, Newton takes (fig. 4) $\mathsf{ab} = x$, $\mathsf{be} = y$, $\mathsf{cb} = o$, and $\mathsf{cf} = z$, supposes that de is the normal at e, and that $\mathsf{de} = \mathsf{df}$ and $\mathsf{db} = v$, so as to get the equality $z = \sqrt{y^2 + 2vo}$. Then he replaces x by $x + o$ and y by $\sqrt{y^2 + 2vo}$ in $F(x, y) = 0$, and gets the required expression of v by omitting the terms involving an higher power of o, and operating the appropriate simplifications. In the *October 1666 Tract*, the argument is highly simplified and generalised by the introduction of the punctual speeds p and q ([13], vol. I, pp. 414-415). This allows Newton to replace x and y in the given equation with $x + po$ and $y + qo$—where 'o' is now used to denote an infinitesimal amount of time, which Newton refers to, again, as "one moment"—and to present the direct algorithm as the algorithm of punctual speeds: an algorithm transforming a polynomial equation $F(x, y) = 0$, expressing a certain curve, in an equation involving x, y, p, q linking the rectilinear coordinates x and y of this curve to the punctual speeds p and q of the motion that generate them (in *De Methodis* this will become the algorithm of fluxions, x and y, being conceived as two variables whatever and p, q as their fluxions, but the demonstration will remain unchanged: [13], vol. III, pp. 74-80; we shall come on this new version of the algorithm above, pp. 13).

We find the same notation, 'o', and the same notion of an infinitely little increment of the abscissa as the key ingredient in the demonstration of the most fundamental result in *De analysi*. This tract, which was the first, and for many years the only public testimony of Newton's mathematical achievements, present in fact a number of results on quadratures got in 1665 and 1666. By and large it avoids the

[18]Notice that, though a moment is here indubitably conceived as an infinitesimal amount of time, rather than as a portion of space, as in *De Analysis* and *De Methodis*, once more, no definition or clarification is given for this crucial notion.

language of (punctual) speeds. More precisely, it sets forth a general method for quadratures by series development that rests on three "rules" ([13], vol. II, p. 206).

The second of these rules (for which no proof is deemed necessary and which is tacitly extended to sums of an infinite number of terms) prescribes that the quadrature of a sum of algebraically expressed curves is the sum of their quadratures. The third shows how to obtain a series development for y in power series of x from any algebraic equation $f(x, y) = 0$ either by infinite division, or by root extraction, or by "the resolution of affected equations" (a generalisation of Vi‘ete's method for approximating numerical roots: cf [13], vol. II, p. 219, footnote 121). The crucial rule, which is also the most original statement if the tract, is then the first. It prescribes that the "area" enclosed by the axis x and the curve $y = ax^{\frac{m}{n}}$ between the origin and the abscissa x is $\frac{na}{n+m}x^{\frac{m+n}{n}}$. Newton's proof is indirect: if an unknown curve encloses an area known to be $z = \frac{na}{n+m}x^{\frac{m+n}{n}}$, then the curve's equation is $y = ax^{\frac{m}{n}}$. Exactly in the same style summarised above for the direct algorithm, Newton substitutes x by $x + o$ in the equation expressing z. Assuming the curve is all the way increasing (or decreasing), he also assumes that the increase in the area z can be expressed as $o\nu$ for some quantity ν between $y(x)$ and $y(x + o)$. He then proceeds as above by eventually dismissing the terms with the infinitesimal factor o ([13], vol. II, pp. 242 and 244).

The third, longest, most comprehensive, and most complex of Newton's early tracts, the *Tractatus de methodis serierum et fluxionum* fully adopts, as remarked above the language of fluents and fluxions and the corresponding conceptual frame. It is entirely organised around the solution of two general problems which rephrase, in the new language and frame, propositions 7 and 8 of the *October 1666 Tract*: "Given the relation of the flowing quantities to one another, to determine the relation of the fluxions"; "Given an equation involving the fluxions of the quantities, to determine the relation of the the quantities one to another" ([13], vol. III, pp. 74-75 and 82-83). The relevance of this treatise for our present purpose is that it shows that the new conceptual fluxional frame does not eliminate the appeal to moments and infinitesimally small quantities.

Moments, sometimes defined as "indefinitely small" quantities and sometimes (even in the same argument) as "infinitely small" quantities, are already protagonists in Newton's demonstration of his solution of the first problem. This consists in a new statement of the algorithm of speeds already presented in the *October 1666 Tract*. It takes now the form of Newton's celebrated rule, according to which the terms of any algebraic equation $f(x, y) = 0$ are to be: *i*) arranged according to the powers of x; *ii*) then multiplied by any arithmetical progression and then by $\frac{p}{x}$ (p being the fluxion of x); *iii*) then the terms of $f(x, y) = 0$ are arranged according to the powers of y, multiplied by any arithmetical progression, and then by $\frac{q}{y}$ (q being the fluxion of y); the resulting terms are added and set equal to zero ([13], vol. III, pp. 74-77). In the proof, the moments are defined as "infinitely small additions by which [...] quantities increase during each infinitely small interval of time" ([13], vol. III, pp. 78-79). As in the *October 1666 Tract*, Newton denotes the moments of abscissas and ordinates by po and qo, substitutes x and y by $x + po$ and $y + qo$ in $f(x, y) = 0$, and operates as indicated above. Notice the use of the moment o in the final step of the proof: "since o is supposed to be infinitely small, so that it be able to express the moments of quantities, terms multiplied by it amount to nothing in respect of the others. Therefore I eliminate them" ([13], vol. III, pp. 80-81).

Newton's moments reappear in many and important places. Tangents (problem 4 of *De methodis*) to a curve of equation $f(x, y) = 0$ are determined by the ratio of fluxions of x and y, but the problem is set and solved by explicit appeal to the infinitesimal triangle made by the moments of the basis and the ordinate ([13], vol. III, p. 120). The same happens in finding the length of arcs of curves ([13], vol. III, p. 304). Particularly in Newton's long, sophisticated attacks to the problems of curvature (problem 5) and rectification (problems 10 and 11), infinitesimals play a crucial role.

The centre of curvature of a curve at point D, for instance, is defined by the intuition that it is the meet of the normals to the curve at points located at "infinitely small distances [*infinitè parùm distantium*]" on either side of D ([13], vol. III, pp. 152-153). The centre of curvature is also defined as an instantaneous centre of rotation for the normal to the curve ([13], vol. III, pp. 294-295). Then, in the technically demanding arguments to determine the radius of curvature, Newton needs to "consider as equals quantities [...] whose ratio does not differ from that of equality but by an infinitesimal [nisi infinitè parùm]" ([13], vol. III, pp. 154-155, 170-173; quote on pp. 172-173). He also argues for the equality in length of an evolute (a word Newton did not use yet) and its tangent by decomposing the evolute in infinitely small arcs "which because of their infinite smallness may be taken to be straight lines" and having them applied by orthogonal projection on ' 'all the parts [totidem partibus]" of the tangent ([13], vol. III, p. 296).

It is of course not surprising that Newton employed the notion of moments in solving problems about curves. On the contrary, given what we know about 17th-century mathematics it would be surprising that he did not use infinitesimals in solving those problems[19]. Yet it is true that Newton downplayed indivisibles and/or infinitesimals—notions which he did not strive to differentiate. As we shall see now, in later years to the view that curvilinear figures are made up of moments, Newton forcefully opposed his own view that they are generated by motion. However, even if he took distance from an indivisibilist and/or infinitesimalist understanding of magnitudes, he kept using moments in tackling problems of curves, and he did so even when he tried to set up the method of fluxions on its own set of axioms[20].

3 Infinitesimals and indivisibles in Newton's mature years

Newton presented his critical views on indivisibles and moments in a text of around 1680 which he titled *Geometria curvilinea*, but we already find traces of them in *De methodis*, particularly in a few pages written as an outgrowth of this treatise. D.T. Whiteside, who titled these pages 'Addendum on the Theory of Geometrical Fluxions', dated them as very likely written in the winter 1671-1672 ([13], vol. III, pp. 328-353). Apparently by accident, these pages were not published before the 20th century[21].

De methodis devotes a great many pages (almost a third of the total) to the problem of quadrature, whose solution depends on that of the second of the two general problems mentioned above (which are there listed as problems 1 and 2). This is problem 9: "To determine the area of any proposed curve" ([13], vol. III, pp. 210- 292)[22].

[19] F. De Gandt has stressed the importance that indivisibles, or, more generally infinitely smalls, have in the mathematics of the *Principia*: cf. [4], chap. III, pp. 159-264.

[20] Newton's philosophy of mathematics, with special attention to the problems posed by infinitesimals and indivisibles, is analysed in detail in [6], p. 179-219.

[21] According to Whiteside, the text originated as an augmented replacement for a couple pages in *De methodis*. Apparently this replacement was left behind when copies of the tract were taken in the early 18th century to be never included instead of the two pages it was meant to replace: cf. [13], vol. III, p. 329, notes 1 and 2.

[22] Notice that the whole tract occupies pp. 32-328 of volume III of [13].

As mentioned above, Newton uses in this context a principle similar to Cavalieri's, albeit he stresses that the method based on the generation of surfaces by motion is more natural and to be preferred. Newton's "Addendum" starts from here to offer axiomatic foundations for the fluxional calculus. In fact, Newton opens the "Addendum" by just extending his claim about the "more natural approach" (cf. the quote at p. 11, above) embodied in generation by motion ([13], vol. III, pp. 328-331; we add in square brackets the last sentence in *De methodis* to which Newton's connects his addition):

> [However, that based on the genesis of surfaces by their motion of flow appears a more natural approach,] and it will come to be still more perspicuous and resplendent if certain foundations are, as is customary with the synthetic method, first laid. Such as these.

Newton set the following four axioms for the method of fluxions ([13], vol. III, p. 330: 'simultaneously generated' means that the wholes are generated in the same time):

1. Magnitudes generated simultaneously by equal fluxions are equal.
2. Magnitudes generated simultaneously by fluxions in given ratio are in the ratio of the fluxions.
3. The fluxion of a whole is equal to the fluxions of its parts taken together.
4. Contemporaneous moments are as their fluxions.

The meaning of Axiom 4 is clearer in Newton's first draft: "Fluxions are as the moments generated contemporaneously with these fluxions" ([13], vol. III, p. 330, note 7).

By the time Newton expanded the "Addendum" into the *Geometria curvilinea*, he directly confronted again the view that magnitudes are made up of infinitesimals with his own, "more natural" way of understanding them ([13], vol. IV, p. 423):

> Those who have taken the measure of curvilinear figures have usually viewed them as made up of infinitely many infinitely-small parts. I, in fact, shall consider them as generated by growing [. . .]. I should have believed that this is the natural source for measuring quantities generated by continuous flow according to a precise law, both on account of the clarity and brevity of the reasoning involved and because of the simplicity of the conclusions and the illustrations required.

Newton's assertions notwithstanding, notice that the axiomatic basis of his method focuses attention on a rephrasing of Cavalieri's principle, namely axioms 1 and 2. As usual, Newton uses the notion of moment without precisely specifying what relation a moment keeps with the whole magnitude. Notice, however, that axiom 4 sets the ratios of fluxions equal to the ratios of moments. This is what makes axioms 1 and 2 so similar to Cavalieri's principle. On the other hand, the connexion with the original version of the principle may be directly visualised if one recalls figure 3, above, which represents the generation of magnitude ABD by the motion of the straight line DBK. For the magnitude ABD generated by continuous flow, the moment is the ordinate BD, and the fluxions with which the surfaces ABD and ABKH increase are as the moments BD and BK. In practice, therefore, axioms 1 and 2 are just rendering Cavalieri's principle within the new conceptual fluxional frame.

Notice, secondly, the role of axiom 4. It connects the fluxional calculus to the geometry of the figures. Fluxions, and in general the interpretation of a geometric object in terms of motion, superadd to the object a dimension that is not proper or intrinsic to its geometrical properties, while moments—either defined *à la* Cavalieri as families of parallel lines or understood as infinitesimal strips—are. Axiom 4 provides a way to move from the fluxional calculus to geometrical objects and back. Newton's proof of the fluxional relations between proportional magnitudes exemplifies this point. Theorem 1 of the "Addendum" ([13], vol. III, p. 330) states that if four variable magnitudes are always proportional, $\mathsf{AB} : \mathsf{AD} :: \mathsf{AE} : \mathsf{AC}$, then

$$\mathsf{AB} \cdot fl\,(\mathsf{AC}) + \mathsf{AC} \cdot fl\,(\mathsf{AB}) = \mathsf{AD} \cdot fl\,(\mathsf{AE}) + \mathsf{AE} \cdot fl\,(\mathsf{AD}) \tag{5}$$

(where'$fl\,(\alpha)$' stands for the fluxion of α).

Newton deleted his first demonstration, probably because it hinged on introducing an infinitesimal moment to eliminate in the final equation those terms multiplied by it "because of the infinite smallness of the moment" ([13], vol. III, p. 332-333). In his final demonstration Newton draws the magnitudes and the moments (Bb of AB, Cc of AC, and so on) as in Figure 5. Then, by writing $\mathsf{Ab} = \mathsf{AB} + \mathsf{Bb}$, etc, and since by hypothesis the four incremented magnitudes are proportional, $\mathsf{Ab} : \mathsf{Ad} :: \mathsf{Ae} : \mathsf{Ac}$, from the equality of the cross products, Newton gets

$$\mathsf{Ab} + \mathsf{AC} \cdot \frac{\mathsf{Bb}}{\mathsf{Cc}} = \mathsf{Ad} \cdot \frac{\mathsf{Ee}}{\mathsf{Cc}} + \mathsf{AE} \cdot \frac{\mathsf{Dd}}{\mathsf{Cc}}.$$

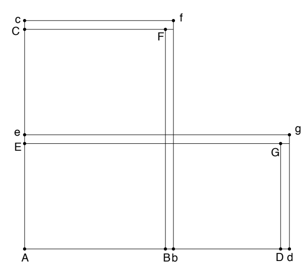


Fig. 5

And now comes the only conceptually tricky step. From ratios of moments $\frac{\mathsf{Bb}}{\mathsf{Cc}}$, $\frac{\mathsf{Ee}}{\mathsf{Cc}}$, and $\frac{\mathsf{Dd}}{\mathsf{Cc}}$, Newton gets, via axioms 4, the ratio of the corresponding fluxions. Then ([13], vol. III, p. 330), he concludes, in order to eventually achieve equation (5):

> Now let the rectangles Af and Ag diminish till they go back into the originary rectangles AF and AG, then Ab will become AB while Ad becomes AD. Hence at the last moment of that infinitely small defluxion, that is at the first moment of the fluxion of the rectangles AF and AG when they start to increase or diminish, there will be
>
> $$\mathsf{AB} \cdot (\mathsf{AC}) + \mathsf{AC} \cdot (\mathsf{AB}) = \mathsf{AD} \cdot (\mathsf{AE}) + \mathsf{AE} \cdot (\mathsf{AD}),$$
>
> as was to be proved.

It is then clear that Newton's axiomatic approach to the calculus of fluxions still goes together with a crucial appeal to infinitesimals, under the form, once again, of moments of quantities.

The *Geometria curvilinea* ([13], vol. IV, pp. 420-484), an unfinished general treatise on the fluxional study of curves written around 1680, is but a revised and expanded version of the "Addendum". Very little is known about its background, the circumstances in which Newton wrote it or his motivations for writing it[23]. In his introductory pages Newton compared his own efforts with Euclid's, who delivered the "foundations of the geometry of straight lines" ([13], vol. IV, pp. 422-423). However, he adds, "since Euclid's elements are scarcely adequate for a work dealing, as this, with curves, I have been forced to frame others" (*ibid.*). The axioms are essentially those of the "Addendum"—the main novelty being the emergence of the notion of "ratio of nascent parts"—but the rendering of Cavalieri's principle in the new fluxional conceptual frame analysed above was keep almost verbatim.

With these antecedents in mind it makes more sense that Newton decided to include a new rendering of Cavalieri's principle among the mathematical lemmas that open the *Principia.* Lemma 4, Section I, states that if two figures divided in an equal number of inscribed parallelograms (such as in figure 6) are such that the "ultimate" ratio of corresponding pairs of parallelograms is constant, then the two figures AacE and PprT are in this ratio ([10], p. 28). The proof, which shows the notion of "ultimate" ratio at work, has no interest except in that it illustrates Newton's discomfort with indivisibles and the lengths he will go to avoid them—for the ultimate ratios of corresponding pairs of parallelograms reduces, in the end to the ratios of corresponding ordinates.

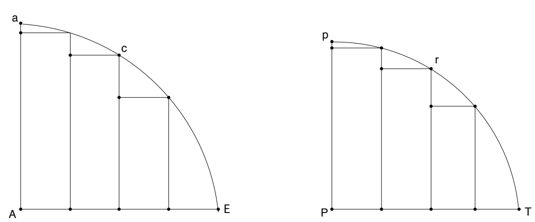


Fig. 6

As for the proof, it is grounded on Lemma 3 ([10], p. 27), where Newton has established that the ultimate ratio of the sum of any series of inscribed parallelograms whose widths are diminished indefinitely and the curvilinear figure (in which they are inscribed) is that of equality. To prove Lemma 4, Newton compares three ultimate ratios. All the parallelograms together (taken in number indefinite) of the first curvilinear figure, say $\mathcal{A}$, have the ultimate ratio of equality with this figure. The same holds for all (in number indefinite) the parallelograms taken together of the second curvilinear figure, $\mathcal{B}$. Finally, by hypothesis, every parallelogram in $\mathcal{A}$ is (ultimately) to its corresponding pair in $\mathcal{B}$ in a constant ratio: for "as the individual parallelograms in the one figure are to the corresponding individual parallelograms in the other, so (by composition) will the sum of all the parallelograms in the one become to the sum of all the parallelograms in the other, and so [for lemma 3] also the one figure to the other", that is $\mathcal{A}$ to $\mathcal{B}$, as it was to be proved ([10], pp. 28-29; [14], pp. 434-435)

We will dedicate our last section to show him tackling a complex physical problem by a sophisticated use of indivisible techniques.

[23]For the scanty information available, cf. Whiteside's introduction and footnote in [13], vol. IV, pp. 411ff.

4 An example from the *Principia*

The example is provided by the proofs of propositions LXXI and LXXIV of book I of the *Principia* ([10], pp. 193-195 and 197; [14], pp. 590-594)[24]. In it, indivisibles and infinitesimals are used in the mathematisation of observable phenomena that were assumed to be the product of the addition of corpuscular micro-effects.

As we shall see, however much Newton avoids the language of infinitesimals and/or indivisibles, the essential object his demonstrations are here dealing with are spherical surfaces that will end up constituting a spherical body. Using these surfaces as spherical indivisibles, Newton cannot eliminate them or disguise them as vanishing entities, since in this case the proof of Proposition LXXIV does not hold—as we shall see presently, the proof depends on conceiving a (material) sphere as decomposed in spherical shells, and then operating a mathematical *compositio*. This conceptual tension blows eventually up in Newton's scholium to Proposition LXXIII, one of the few spots in Newton's mature mathematical writings in which he seems to receive the view that a geometrical object is made up of infinitely many infinitely small elements ([10], p. 196; [14], p. 593):

> The surfaces of which the solids are composed are here not purely mathematical, but orbs so extremely thin that their thickness is as null: namely evanescent orbs of which the sphere ultimately consists when the number of those orbs is increased and their thickness diminished indefinitely. Similarly, when lines, surfaces, and solids are said to be composed of points, such points are to be understood as equal particles of a magnitude so small that it can be ignored.

Proposition LXXI is a theorem asserting that if any point on a spherical shell is the centre of an attractive force inversely proportional to the square of the distance, then any corpuscle outside this shell is attracted towards the shell's centre by a force inversely proportional to the square of its distance to the centre.

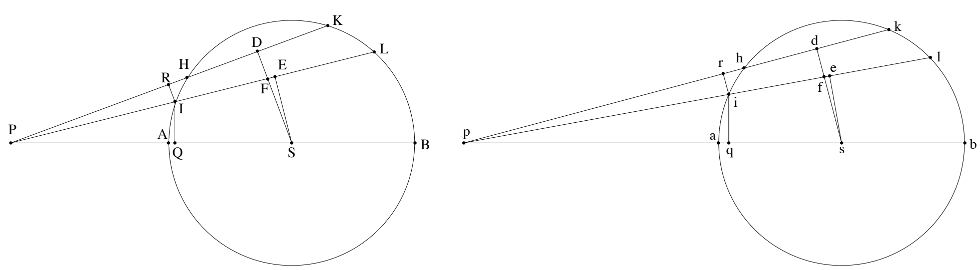


Fig. 7

Let the circle AKB (fig. 7) be a great circle of a spherical shell of centre S, and P a corpuscle located outside the shell. Take IQ to be perpendicular to the diameter AB. Proposition LXXXI states that if any point on the shell is the centre of an attractive force inversely proportional to the square of the distance, then P is attracted towards S by a force inversely proportional to the square of PS. The proof goes as follows.

[24]On these proofs, cf., among others: [20]; [3]; [5], pp. 68-71; [2], pp. 98-99.

Let AHB (fig. 7) and ahb be two great circles of two equal spherical shells of centres S and s. Let also P and p be two corpuscles located outside these shells, on the diameters AB and ab produced, and PHK, PIL and phk, pil four segments joining P and p to the circles AHB and ahb, respectively, such that $\widehat{\mathsf{HK}} = \widehat{\mathsf{hk}}$ and $\widehat{\mathsf{IL}} = \widehat{\mathsf{il}}$. Let DS, ES and ds, es be the perpendiculars through S and s to the chords HK, IL, and hk, il. Notice that the equalities $\widehat{\mathsf{HK}} = \widehat{\mathsf{hk}}$ and $\widehat{\mathsf{IL}} = \widehat{\mathsf{il}}$ implies the other equalities $\mathsf{ES} = \mathsf{es}$ and $\mathsf{DS} = \mathsf{ds}$. Let finally IQ, IR and iq, ir the perpendiculars through I and i to AB, ab and PHK, phk, respectively[25].

Suppose that the angles $\widehat{\mathsf{DPE}}$, $\widehat{\mathsf{dpe}}$ are infinitesimal (or that they are "vanishing [*evanescentes*]", in Newton's idiom: [10], p. 194; [14], p. 591). The segments PE and pe, PF and pf, DF and df, may then "be considered to be [respectively] equal" (*ibid.*). Since $\mathsf{pf} : \mathsf{pi} :: \mathsf{df} : \mathsf{ri}$, it follows that $\mathsf{pf} : \mathsf{pi} :: \mathsf{DF} : \mathsf{ri}$ and, from this and $\mathsf{PI} : \mathsf{PF} :: \mathsf{RI} : \mathsf{DF}$, by multiplying term by term, it follows that $\mathsf{PI} \times \mathsf{pf} : \mathsf{PF} \times \mathsf{pi} :: \mathsf{RI} : \mathsf{ri}$, and then, by the corollary III to lemma VII of the method of first and last ratios (book I, sect. I), $\mathsf{PI} \times \mathsf{pf} : \mathsf{PF} \times \mathsf{pi} :: \widehat{\mathsf{IH}} : \widehat{\mathsf{ih}}$ [26]. But, since $\mathsf{SE} = \mathsf{se}$ and $\mathsf{ps} : \mathsf{pi} :: \mathsf{se} : \mathsf{iq}$, also the proportion $\mathsf{ps} : \mathsf{pi} :: \mathsf{SE} : \mathsf{iq}$ holds, and from this and $\mathsf{PI} : \mathsf{PS} :: \mathsf{IQ} : \mathsf{SE}$, by multiplying term by term again, it follows that $\mathsf{PI} \times \mathsf{ps} : \mathsf{PS} \times \mathsf{pi} :: \mathsf{IQ} : \mathsf{iq}$. A new multiplication term by term allows then to draw that $\mathsf{PI}^2 \times \mathsf{pf} \times \mathsf{ps} : \mathsf{pi}^2 \times \mathsf{PF} \times \mathsf{PS} :: \widehat{\mathsf{IH}} \times \mathsf{IQ} : \widehat{\mathsf{ih}} \times \mathsf{iq}$. But, insofar as the angles $\widehat{\mathsf{DPE}}$, $\widehat{\mathsf{dpe}}$ vanish, the ratio of the products $\widehat{\mathsf{IH}} \times \mathsf{IQ}$ and $\widehat{\mathsf{ih}} \times \mathsf{iq}$ is the same as that of the circular strips $St.\left(\widehat{\mathsf{IH}}\right)$ and $St.\left(\widehat{\mathsf{ih}}\right)$ respectively generated by $\widehat{\mathsf{IH}}$ and $\widehat{\mathsf{ih}}$ under the revolution of the semicircles AHKB and ahkb about the diameters AB and ab[27] Hence: $\mathsf{PI}^2 \times \mathsf{pf} \times \mathsf{ps} : \mathsf{pi}^2 \times \mathsf{PF} \times \mathsf{PS} :: St.\left(\widehat{\mathsf{IH}}\right) : St.\left(\widehat{\mathsf{ih}}\right)$. At this point, Newton argues that the attractive forces exerted by these strips upon the corpuscles P and p are, by hypothesis, directly as these strips themselves and inversely as the squares of their distances PI and pi from these corpuscles, that is, as $\mathsf{pf} \times \mathsf{ps}$ to $\mathsf{PF} \times \mathsf{PS}$. By decomposing these forces into two orthogonal components along the directions of PS, QI and ps, qi, one gets then that their components along the directions of

[25] This geometrical configuration can be constructed by rule and compass even supposing that the positions of corpuscles P and p with respect to the shells is given in advance. Take, indeed, any great circle of the first shell, and draw two segments PHK and PIL from P to it. The chords HK, IL are then given in length. Take then any great circle of the second shell. To get the points h and i such that $\widehat{\mathsf{HK}} = \widehat{\mathsf{hk}}$ and $\widehat{\mathsf{IL}} = \widehat{\mathsf{il}}$, it is enough to insert within this great circles two chords hk, il respectively equal to the chords HK, IL, verging to p. This is a quite simple *neusis* problem tackled by Apollonius in the first book of *On verging constructions* ([7], vol. II, pp. 189-190), which is easily solvable by rule and compass, as shown, for example, by S. Horsley, in solving problem I of his *Apollonii Pergæi Inclinationum* ([8], pp. 1-2).

[26] Corollary III to lemma VII is appealed here in order to warrant the replacement of RI and ri with $\widehat{\mathsf{IH}}$ and $\widehat{\mathsf{ih}}$ in $\mathsf{PI} \times \mathsf{pf} : \mathsf{PF} \times \mathsf{pi} :: \mathsf{RI} : \mathsf{ri}$. On the cogency of this derivation, cf. [20], sect. III (p. 886)—which wrongly denies that this corollary does actually warrant this replacement—and [3]—which rightly argues for the opposite, by showing how the ultimate equalities of RI and $\widehat{\mathsf{IH}}$, and of ri and $\widehat{\mathsf{ih}}$ follows from this corollary.

[27] On the ultimate equality $\frac{\widehat{\mathsf{IH}} \times \mathsf{IQ}}{\widehat{\mathsf{ih}} \times \mathsf{iq}} = \frac{St.\left(\widehat{\mathsf{IH}}\right)}{St.\left(\widehat{\mathsf{ih}}\right)}$, cf. [20], pp. 884 and 889, footnote 15. Weinstock's suggestion is that Newton is here relying on the equalities $St.\left(\widehat{\mathsf{IH}}\right) = 2\pi\left(\widehat{\mathsf{IH}}\right)(\mathsf{IQ})$ and $St.\left(\widehat{\mathsf{ih}}\right) = 2\pi\left(\widehat{\mathsf{ih}}\right)(\mathsf{iq})$, which result, by appropriate replacements (licensed by the supposition that the angles $\widehat{\mathsf{DPE}}$, $\widehat{\mathsf{dpe}}$ vanish), from the equalities $Tr.Cn.(\mathsf{IH}) = 2\pi(\mathsf{IH})(\mathsf{MN})$ and $Tr.Cn.(\mathsf{ih}) = 2\pi(\mathsf{ih})(\mathsf{mn})$, where $Tr.Cn.(\mathsf{IH})$ and $Tr.Cn.(\mathsf{ih})$ are the truncated cones respectively generated by IH and ih under the revolution of the semicircles AHKB and ahkb about the diameters AB and ab, and MN and mn are the radii of the mean circumferences of these truncated cones.

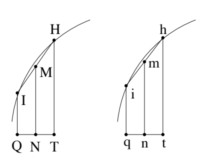
PS and ps—which, by symmetry about the axes PS and ps, are in fact the total forces exerted by these strips upon the corpuscles P and p, respectively ([20] , pp. 885 and 889, footnote 16)—are to each other as $\frac{\mathsf{pf}\times\mathsf{ps}\times\mathsf{PQ}}{\mathsf{PI}}$ to $\frac{\mathsf{PF}\times\mathsf{PS}\times\mathsf{pq}}{\mathsf{pi}}$, that is, as $\frac{\mathsf{pf}\times\mathsf{ps}\times\mathsf{PF}}{\mathsf{PS}}$ to $\frac{\mathsf{PF}\times\mathsf{PS}\times\mathsf{pf}}{\mathsf{ps}}$ (because of the similarity of the triangles PIQ, PSF and piq, psf) or as ps^2 to PS^2. By a similar argument, it follows that this is also the case for the forces respectively exerted upon the corpuscles P and p by the circular strips $St.\left(\widehat{\mathsf{KL}}\right)$ and $St.\left(\widehat{\mathsf{kl}}\right)$.

It is at this point that Newton appeals to an argument openly reminiscent of Cavalieri's principle. He firstly remarks that the same conclusion reached for the strips $St.\left(\widehat{\mathsf{IH}}\right)$, $St.\left(\widehat{\mathsf{ih}}\right)$ and $St.\left(\widehat{\mathsf{KL}}\right)$, $St.\left(\widehat{\mathsf{kl}}\right)$ also holds for all the other pairs of circular strips into which the two spherical shells "can be divided [*distingui potest*]" ([10], p. 195; [14], p. 592), that is: the attractive force exerted upon the corpuscle P by every one of the circular strips of first shell is to the attractive force exerted upon the corpuscle p by the homologous strip of the second shell as ps^2 is to PS^2. Then he infers, "by composition [*per Compositionem*]" (*ibid.*), that this is also the ratio of the attractive force of the whole first shell upon P to the attractive force of the whole second shell upon p, as it was to be proved.[28]

Notice that for this inference by composition to be sound, a dynamic version of Cavalieri's principle has to be admitted. Indeed, Newton is implicitly assuming, that, if the attractive forces exerted by any pair of homologous elements in two bodies are in a certain constant ratio, then also the attractive forces exerted by the whole bodies are one to each other in this same ratio.

Proposition LXXI is nothing but a lemma for the proof of proposition LXXIV, which asserts the same for a whole sphere: if this sphere is such that any point of it is the centre of an attractive force inversely proportional to the square of the distance from this point, then a corpuscle outside this sphere is attracted towards its centre by a force which is, in turn, inversely proportional to the square of the distance between this point and this centre. The proof depends on the same assumption above: let a sphere be "divided into innumerable concentric spherical [...][shells]" ([10], p. 197; [14], p. 594); by proposition LXXI, each of them attracts a corpuscle outside it with a force inversely proportional to the square of distances between this corpuscle and the centre of the sphere; hence, "by composition" again, this is also so for "the sum of the attractions (that is, the attraction of the corpuscle toward the total sphere)" (*ibid.*).

Proposition LXXIV plays a crucial role in Newton's theory, since it legitimates turning an homogenous spherical body into a massive point located in its centre, as far as the attractive power is concerned. This confers in turn a crucial role to the argument reminiscent of Cavalieri's principle, on which its proof is based.



[28] Indeed, as the two shells are equal, this is the same as saying that the attractive forces exerted by these shells upon the two corpuscles are inversely proportional to the square of the distances between these corpuscles and the centres of the shells.

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