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Philippe Bich, Jean-Pierre Drugeon, Lisa Morhaim

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Philippe Bich[†], Jean-Pierre Drugeon[‡] & Lisa Morhaim[§]

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Abstract

In the tradition of Irving Fisher, the current article advocates an approach to dynamic programming that is based upon elementary aggregating functions where current action and future expected payoff combine to yield overall current payoff. Some regularity properties are provided on the aggregator which allow for establishing the existence, the uniqueness and the computation of the Bellman equation. Some order-theoretic foundations for such aggregators are also established. The aggregator line of argument encompasses and generalizes many previous results based upon additive or non-additive recursive payoff functions.

Keywords: Dynamic Programming, Aggregators, Intertemporal Choice.
JEL Classification: C61, D90.

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†Paris School of Economics and University of Paris 1 Panthéon-Sorbonne.

‡Paris School of Economics and Centre National de la Recherche Scientifique.

§University of Paris 2 Panthéon-Assas, CRED Paris Center for Law and Economics.

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1. Introduction

Dating from Stokey & Lucas with Prescott [19], dynamic programming techniques and the Bellman principle of optimality have become the cornerstone of the analysis of most of intertemporal equilibria, the main rationale for this popularity being the associated possibility of analyzing complicated infinite horizon problems through a sequence of stationary two-period problems, formalized by Bellman equation. Even though this commonly requires some stationary structure on the original optimization problem, the suitability of this approach for numerical tools has made it increasingly popular by the last decade.

Many authors have been interested in the mathematical properties of the Bellman equation, and have in particular provided conditions answering the three main questions related to Bellman equation, *i.e.*, the existence, the uniqueness and the computation of its solutions. To the best knowledge of the authors, the notion of recursive intertemporal payoff have always played a central role in the analysis of such concerns. Consider an agent who is to take some decision z_t in some action set Z at each date $t \in \mathbf{N}$. An *intertemporal payoff function* then associates to every action stream $\underset{\sim}{z}$ in $Z^{\mathbf{N}}$ some payoff $U(\underset{\sim}{z})$, so that the preferences of the agent, defined on $Z^{\mathbf{N}}$, are represented by U. Within such a framework, U is recursive if the intertemporal payoff from today, formally $U(\underset{\sim}{z}) = U(z_0, z_1, \ldots, z_n, \ldots)$, is a function $A(z_0, U(z_1, \ldots, z_n, \ldots))$ of the action today z_0 and the intertemporal payoff from tomorrow $U(z_1, \ldots, z_n, \ldots)$. The function A *aggregates* the current action z and the future payoff into the current payoff, and is commonly labelled as an *aggregating function*, or *aggregator*.

Formally, two classes of approaches have been retained to deal with recursive payoffs: the first, initiated by Koopmans[1][8], provides an axiomatization of preferences which leads to a recursive payoff function. The second is to take the aggregator function as a primitive, and then to provide conditions under which there exists some recursive payoff (*e.g.*, Stokey & Lucas [18], Becker & Boyd [2], Streufert [21], Le Van & Vailakis [10]).[2]

A first objective of our contribution is to initiate a new approach: the preferences of the agents are solely defined on the set of couples (z, v), where z is an action today and v is a future payoff. This contribution introduces an axiomatization of preferences under which some *well-behaved* aggregators represent such preferences. *Well-behaved* means that the aggregator is endowed with some minimal structure—called uniform continuity in v—allowing for answering the three main questions mentioned related to the Bellman equation.

The use of dynamic programming techniques however requires to be able to define an infinite-horizon intertemporal payoff, starting from an aggregator A that it is only defined for two periods. This can be accomplished in two different ways: the first one, that is standard, is to associate to A a recursive payoff, when it exists. A second, natural and currently retained approach, is to extend A to finite-horizon models by iterated backward induction and then to infinite-horizon models by a simple limit process[3].

[1]This approach is presented in Becker & Boyd (see Section 3.3.1. in [2]).

[2]In Yao [22], the aggregator and the recursive payoff are taken together as primitives.

[3]This way of defining the intertemporal payoff is considered in Le Van & Vailakis [10]. The framework adopted by these authors implies that this payoff function is recursive, while this is not necessarily the case in our framework.

A second objective of this contribution is then to compare the two orders defined on $Z^{\mathbf{N}}$ by these two distinct kinds of payoffs. The use of a recursive payoff function may reveal as being inappropriate in some given configurations: even when it exists, the orders that are induced on $Z^{\mathbf{N}}$ by such a recursive payoff and by the aggregator may differ, and the associated optimization programs may have different solutions and different value functions.

In a nutshell, the class of aggregators currently introduced answers to the questions related to the Bellman equation. They are further grounded upon a class of preferences over $Z \times \mathbf{R}$, instead of the usual and complex specification over $Z^{\mathbf{N}}$. This may admittedly be considered as more natural and realistic an answer: there is indeed little doubt that it is extremely demanding to assume that agents are able to optimize over a set of infinite action sequences. Along the same range of ideas, a preference relationship over $Z \times \mathbf{R}$ is simpler and more testable than a preference relationship over $Z^{\mathbf{N}}$.

A third objective of this contribution is to encompass models with possibly unbounded payoffs. The use of dynamic programming tools for additively separable models with unbounded payoffs was first initiated by Boyd [3], Alvarez & Stokey [1], Duran [4] and Le Van & Morhaim [9]. To summarize, four distinct approaches have been proposed in this framework to deal with the issues related to dynamic programming. First, it is possible to use the Banach contraction theorem (or some extension of it) applied to the Bellman operator: for the bounded case in the seminal work by Stokey & Lucas with Prescott [19], and for the possibly unbounded case, Rincon-Zapatero & Rodriguez-Palmero [15] or Martins-da-Rocha & Vailakis [13] have proposed extensions involving local contractions. Second, a direct approach can be used, and requires some lipschitz condition on the aggregator (see Le Van & Morhaim or Le Van & Vailakis [10].). Third, it is possible to replace the lipschitz condition by a topological property (biconvergence assumption) on the recursive payoff function and the feasibility set (Streufert [20, 21]). Last, the order-theoretic fixed-point machinery can be used to replace the Banach fixed-point theorem (Kamihigashi [6]). These approaches are sometimes connected, and each one may have some advantages from some particular point of view: Kamihigashi [6] allows to avoid topological assumptions, and encompasses the existence and uniqueness part of Rincon-Zapatero & Rodriguez-Palmero [15] and Martins-da-Rocha & Vailakis [13]'s results. But it is only valid for additive separable models, and its assumptions may be difficult to check. Streufert's biconvergence assumption is singular in the literature in the sense that it does not imply nor is implied by the other results: a reason is that the biconvergence assumption guarantees that the infinite-horizon payoffs can be approximated by finite-horizon payoffs, this allows to recover some compactness property which is very useful for existence problems in dynamic programming. Usually, the Banach fixed-point theorem is the way to solve such compactness issues.

This contribution unifies most of these papers (in particular Streufert [20, 21], Kamihigashi [7] or Le Van & Vailakis [9]). Such unification requires to introduce a weak continuity assumption on the aggregator (assumption that generalizes uniform continuity in v). This allows, together with a transversality condition, to ensure the existence and the uniqueness of a fixed-point for the Bellman operator. This also provides an algorithm to reach the value function starting from a suitable initial function.

This article is organized as follows. Section 2 introduces aggregators and definitions of the payoffs. Section 3 details comparison between the aggregator approach and the more standard one based upon a recursive payoff function. Section 4 establishes that the value function of the aggregator-based optimization problems is the unique fixed point for the Bellman operator in a given length, and that this value function can be computed by iterating the Bellman operator, starting from some initial condition. The main proofs are gathered in a final appendix.

2. Aggregating Functions

2.1 Aggregators and Definitions of the Payoffs

Time is discrete. Consider some entity, be it an agent, a firm, a player or a decision maker, that can choose to undertake actions z at each date $t \in \mathbf{N}$ in some action set Z and anticipates the aggregate payoff v at the beginning of time $t+1$. Assume $v \in \overline{\mathbf{R}}$, where $\overline{\mathbf{R}}$ denotes the extended real line along $\mathbf{R} \cup \{-\infty, +\infty\}$. The preferences of the entity are modelized through a binary relationship on $Z \times \overline{\mathbf{R}}$: (z, v) is preferred to (z', v') if the entity prefers to choose z today, and receive a future payoff of v, rather than choosing z' today and receiving a future payoff of v'. For convenience, these preferences are modelled through a function,[4] *i.e.*, it is assumed that there exists a function of (z, v) that represents the order relationship. Such an aggregating function, referred to as an *aggregator*, shall henceforth be considered as a *primitive* and is defined as follows.

Definition 2.1. An aggregator is a mapping

$$A : Z \times \overline{\mathbf{R}} \to \overline{\mathbf{R}},$$

where Z is called an action set.

For every current action z and every future payoff $v \in \overline{\mathbf{R}}$, $A(z, v)$ *aggregates* the current action z and the future payoff v into one current payoff. The two following assumptions will be retained through the whole exposition:

Assumption IM.— [Increasing Monotonicity] $\forall z \in Z$, $u \to A(z, u)$ is non-decreasing.

Assumption B.— [Boundary] $A(z, \cdot)$ *satisfies* $A(z, -\infty) = -\infty$ *for every* $z \in Z$ *and* $A(z, +\infty) = +\infty$ *for every* $z \in Z$.

When it makes sense, the following convention shall further be used:

Convention I.— [Infinity] $+\infty - (+\infty) = 0$ and $-\infty - (-\infty) = 0$.

In Stokey & Lucas [18], Streufert [20], [21], Le Van & Vailakis [10], Rincon-Zapatero & Rodriguez-Palmero [16] or Da Rocha & Vailakis [14], the aggregating function A was uniformly defined on $Z \times \mathbf{R}$. Within the current configuration, however, it can be extended to a function on $Z \times \overline{\mathbf{R}} \to \overline{\mathbf{R}}$ by letting $A(z, +\infty) = \lim_{v \to +\infty} A(z, v)$ and $A(z, -\infty) = \lim_{v \to -\infty} A(z, v)$—such a limit exists in $\overline{\mathbf{R}}$ as a result of the retainment of the Increasing Monotonicity assumption IM—. This simple line of argument proves that considering aggregators as functions from $Z \times \overline{\mathbf{R}}$ to $\overline{\mathbf{R}}$ encompasses the case where $A : Z \times \mathbf{R} \to \overline{\mathbf{R}}$.

[4] For example, if Z is a normed vector space, and the preferences are reflexive, complete, transitive and continuous, such a function is insured to exist from Debreu Representation Theorem.

ASSUMPTION UC.— [Uniform continuity] *The mapping A is said to be uniformly continuous in v if for every $\varepsilon > 0$, there exists $\eta > 0$ such that for every $(z, v) \in Z \times \overline{\mathbf{R}}$, for every $\eta' \in [-\eta, \eta]$, $|A(z, v + \eta') - A(z, v)| \leq \varepsilon$.*

Remark 2.1. It may well happen that $A(z, v+\eta) = A(z, v) = +\infty$, especially if $v = +\infty$. For such a conjunction and relying on Convention I, $\big|A(z, v + \eta) - A(z, v)\big|$ is interpreted to be 0 in the above Uniform Continuity property. Thus, whenever $A(z, v) = +\infty$ for $(z, v) \in Z \times \mathbf{R}$, Uniform Continuity in v implies that for η small enough, $A(z, v + \eta) = +\infty$—a similar occurrence is available for $v = -\infty$—.

LEMMA 2.1.— *A mapping A will be uniformly continuous in v if and only if there exists a function $\delta : \mathbf{R} \to \mathbf{R}$ that tends to 0 at 0 and such that, for every $(z, v, v') \in Z \times \overline{\mathbf{R}} \times \overline{\mathbf{R}}$, $|A(z, v) - A(z, v')| \leq \delta(v - v')$.*

In contradistinction with the tradition based upon strong regularity conditions on a benchmark additive aggregator that is linear with respect to the future payoff, the current generalized class of aggregators will refer to some weak form of continuity that does not involve any topological property of Z or X. The aggregator is indeed uniformly continuous in v if, given $z \in Z$ and $\varepsilon > 0$, one can find $\eta > 0$ small enough, *independent of* $v \in \mathbf{R}$, such that modifying the future payoff v of less than η does not change the total payoff $A(z, v)$ of more than ε. Interestingly and from Lemma 2.1, uniform continuity in v clearly encompasses the case where A is λ-lipschitz with respect to v for some $\lambda > 0$ as a special case for which the function $\delta(\cdot)$ simplifies to $\delta(v) = \lambda|v|$.

Example 2.1. (A class of bounded and possibly discontinuous aggregators) For the purpose of this example, temporarily assume that Z is a compact topological space. The aggregator $A : Z \times \overline{\mathbf{R}} \to \mathbf{R}$ being allowed to be discontinuous[5], let $(z, v, v') \in Z \times \overline{\mathbf{R}}^2 \to A(z, v) - A(z, v')$ be continuous under Convention I. Then A is uniformly continuous in v. Indeed, there would otherwise exist $\varepsilon > 0$ and a sequence $(z^n, v^n, w^n) \in Z \times \overline{\mathbf{R}} \times \overline{\mathbf{R}}$ with $|A(z^n, v^n) - A(z^n, w^n)| \geq \varepsilon$ and such that $|v^n - w^n|$ converges to 0 when n tends to $+\infty$. By compactness, it can be assumed that (z^n, v^n, w^n) converges to some $(z, v, v) \in Z \times \overline{\mathbf{R}} \times \overline{\mathbf{R}}$, and by continuity, it derives that $\big|A(z, v) - A(z, v)\big| = 0 \geq \varepsilon$, a contradiction. An example of such an aggregator is $A(z, v) = u(z) + f(v)$, for u any function from Z to $\mathbf{R}$ and $f : \overline{\mathbf{R}} \to \mathbf{R}$ continuous, non-decreasing with bounded values.

Example 2.2. (A class of separable aggregators satisfying a uniform continuity condition) Assume that Z is any set and let $A(z, v) = u(z) + f(v)$ for some function u from Z to $\overline{\mathbf{R}}$, and $f : \mathbf{R} \to \mathbf{R}$ a non-decreasing and uniformly continuous function.[6] Extend A on $Z \times \overline{\mathbf{R}}$ by $A(z, +\infty) = \lim_{v \to +\infty} A(z, v)$, and $A(z, -\infty) = \lim_{v \to -\infty} A(z, v)$. Then A is obviously uniformly continuous in v. Remark that, even though, this class also covers the case where $A(z, v) = u(z) + f(v)$ for some lipschitz mapping f, f above needs not to be lipschitz: consider, *e.g.*, $A(z, v) = \sqrt{v} - z^2$ if $v \geq 0$ and $A(z, v) = -z^2$ otherwise. It is well known that $\sqrt{v}$ is not lipschtiz on $[0, +\infty[$, yet it is uniformly continuous.

[5]The space $\overline{\mathbf{R}}$ is endowed with its standard *compacification*-topology: a neighbourhood of $x \in \mathbf{R}$ is standard, and a neighbourhood of $+\infty$ contains some $]y, +\infty]$ for some $y \in \mathbf{R}$, a similar occurence being available for $-\infty$.

[6]This means that, for every $\varepsilon > 0$ there exists $\eta > 0$ such that for every $(v, v') \in \overline{\mathbf{R}} \times \overline{\mathbf{R}}$ such that $|v - v'| \leq \eta$, it derives that $|f(v) - f(v')| \leq \varepsilon$.

2.2 A Class of Preorders that can be represented by Aggregators satisfying the Building Assumptions

As this is soon to be clarified and thanks to the class of Lipschitz preorders introduced by Levin [11], one can actually exhibit some class of preorders which are representable by aggregators satisfying the just introduced Uniform Continuity property[7].

For the purpose of this subsection and in order to save space on the argument, it is temporarily assumed that A is bounded from below and that the future payoff v cannot take infinite values.

Let $\preceq$ be a complete preorder[8] on $E = Z \times \mathbf{R}$ the set of actions and future payoffs. Recall that, for every $(e, e') \in E \times E$, the strict preference $\prec$ associated with $\preceq$ is defined by $e \prec e'$ if $e \preceq e'$ and $e' \npreceq e$. Moreover, the preorder $\preceq$ on $E = Z \times \mathbf{R}$ is said to be *increasing in* v if for every $z \in Z, v \leq v'$ implies $(z, v) \preceq (z, v')$. Lastly, a representation of $\preceq$ is a function $A : Z \times \mathbf{R} \to \mathbf{R}$ such that for every $e, e' \in E$, $e \preceq e'$ is equivalent to $A(e) \leq A(e')$.

The notion of chain plays a crucial role in the following representation result. For every (z, v) and (z', v') in $Z \times \mathbf{R}$, define $\mathcal{C}((z, v), (z', v'))$ the set of all *chains* from (z, v) to (z', v'), *i.e.*,

$$\mathcal{C}((z,v),(z',v')) = \Big\{((z_i, v_i))_{i=0}^n : n \in \mathbf{N}, (z_0, v_0) = (z, v), (z_n, v_n) = (z', v')\Big\}$$

and for any chain $((z_i, v_i))_{i=0}^n \in \mathcal{C}((z, v), (z', v'))$, let $I_\prec$ be the set of indexes i for which the sequence (z_i, v_i) is strictly decreasing between $i-1$ and i, *i.e.*, $I_\prec = \{i \in \{1, ..., n\} : (z_i, v_i) \prec (z_{i-1}, v_{i-1})\}$.

Theorem 2.1.— *Let (Z, d) be a metric space and $\preceq$ be a complete preorder on $E = Z \times \mathbf{R}$,*

(i) *Assume that, for every (z, v) and (z', v') in $Z \times \mathbf{R}$ such that $(z', v') \prec (z, v)$, there is a $\alpha > 0$ such that for any chain $((z_i, v_i))_{i=0}^n$ from (z, v) to (z', v'), one has*[9] *$\sum_{i \in I_\prec} [d(z_i, z_{i-1}) + |v_i - v_{i-1}|] \geq \alpha$. Then there exists an aggregator A which represents $\preceq$ and is uniformly continuous in v.*

(ii) *Moreover, if $\preceq$ is increasing in v, then A satisfies Increasing Monotonicity.*

The quantity $\sum_{i \in I_\prec} [d(z_i, z_{i-1}) + |v_i - v_{i-1}|] \geq \alpha$ could be seen as a topological *and* order-theoretic measure of decreasingness along the chain. The condition of the theorem says that this measure should be bounded below by some strictly positive constant, for every chain starting at (z, v) and ending at (z', v') with $(z', v') \prec (z, v)$.

Remark 2.2. For example, orders such as the lexicographic one do not fulfill the condition of Theorem 2.1. Consider indeed the case where $Z = \mathbf{R}$ and

[7] This echoes the result of Koopmans, as presented in Becker & Boyd (see Section 3.3.1. in [2]) on the representation of a preorder defined on $Z^{\mathbf{N}}$ by some recursive payoff U (see Definition 3.2) associated to some aggregator. Our representation result is about preorders defined on $Z \times \mathbf{R}$, the domain of the aggregator.

[8] A preorder is a reflexive and transitive binary relation.

[9] A distance δ can be defined on E by $\delta((z, v), (z', v')) = d(z, z') + \mid v - v' \mid$. The assumptions on $\preceq$ in the theorem are those of a δ-Lipschitz preorder (Levin [11]). Remark that this representation result is true whatever is the distance δ on E.

let $\preceq$ denote the lexicographic order on $Z \times \mathbf{R}$. The assumption above would therein fail to be satisfied: if one takes $(z,v) = (0,1)$ and $(z',v') = (0,0)$, then $(z',v') \prec (z,v)$. Taking a chain from (z,v) to (z',v') with an intermediary couple $(\varepsilon,0)$ with $\varepsilon > 0$, and since $(0,1) \prec (\varepsilon,0)$, the measure of decreasingness along this chain is $d((0,0),(\varepsilon,0)) = \varepsilon$ which is arbitrarily small, contradicting the assumption of Theorem 2.1.

3. Aggregators versus Recursive Payoff Functions

3.1 Aggregator-based Payoff Functions

It has been shown that $A(z_0, v)$ *aggregates* the current action z_0 and the future payoff v into one current payoff at the current date $t = 0$. In a similar way, if some action z_i, for $i = 0, 1$, is undertaken at time i, and given a future payoff $v \in \overline{\mathbf{R}}$ at date $t = 2$, the aggregate payoff at $t = 0$ can be written in an overlapped way according to $A(z_0, A(z_1, v))$. Iterating and for every sequence $\underset{\sim}{z} = (z_t)_{t \in \mathbf{N}}$ in $Z^{\mathbf{N}}$, every $T \in \mathbf{N}^*$ and every $v \in \overline{\mathbf{R}}$, define

$$A^T\big(\underset{\sim}{z}, v\big) = A\big(z_0, A\big(z_1, A\big(z_2, \ldots, A\big(z_{T-2}, A\big(z_{T-1}, v\big)\big) \cdots \big)\big)\big)$$

as the aggregate payoff at $t = 0$, given actions $z_0, \ldots, z_{T-1}$ of the entity at dates $t = 0, \ldots, T-1$ and a future payoff v at date $t = T$.

Define $S : Z^{\mathbf{N}} \to Z^{\mathbf{N}}$ as the *shift operator*, *i.e.*, $S(z_0, z_1, \ldots) = (z_1, z_2, \ldots)$. The expression $S(\underset{\sim}{z})$ simply features the sequence of actions given by $\underset{\sim}{z}$ from time $t = 1$ on. More generally, iterating, $S^N(\underset{\sim}{z}) = (z_N, z_{N+1}, \ldots)$ denotes the sequence of actions $\underset{\sim}{z}$ from time $t = N$ on.

For the current unbounded time domain and $T \to +\infty$, a natural guess however emerges as the relevant function that should be maximized over time by the entity. An enlightening hint at this issue is provided by the consideration of a benchmark aggregator that is separably additive between current action and future payoff and further linear with respect to this latter one, *i.e.*, $A(z, v) = u(z) + \beta v$, where $u(\cdot)$ is some instantaneous felicity function and $\beta \in]0, 1[$ some discount factor. It is traditional in economics to maximize the infinite sum $\sum_{t=0}^{+\infty} \beta^t u(z_t)$, for $\underset{\sim}{z} = (z_t)_{t \in \mathbf{N}}$ sequence of actions over time ; from the current perspective, this is nothing but the limit of the felicity truncated after some finite time T, *i.e.*, $\lim_{T \to +\infty} \sum_{t=0}^{T} \beta^t u(z_t)$. Within the current setting, it can admittedly be written as $\lim_{T \to +\infty} A^T(\underset{\sim}{z}, 0)$. It is nonetheless worth emphasizing that such a limit may not exist, in which case it is standard to maximize either the supremum limit—*optimistic* point of view—or the infimum limit—*pessimistic* point of view—. This whole range of considerations motivates the following definitions:

Definition 3.1. Given a sequence of actions $\underset{\sim}{z}$:

(i) the *intertemporal* payoff is $\lim_{T \to +\infty} A^T(\underset{\sim}{z}, 0)$.

(ii) the *upper* payoff is $\overline{w}(\underset{\sim}{z}) = \overline{\lim}_{T \to +\infty} A^T(\underset{\sim}{z}, 0)$.

(iii) the *lower* payoff is $\underline{w}(\underset{\sim}{z}) = \underline{\lim}_{T \to +\infty} A^T(\underset{\sim}{z}, 0)$.

Remark that, even though the payoff from time N may not exist for some $\underset{\sim}{z}$, the *upper* and *lower* payoffs would keep on being well-defined in $\overline{\mathbf{R}}$.

3.2 Recursive Payoff Functions

The previous section introduced aggregator functions as the sole primitive for preferences over $Z \times \overline{\mathbf{R}}$, *i.e.*, on the set of couples that build from current actions and future payoffs. The conventional way of modelizing intertemporal preferences is however to rather consider a preference relationship $\lesssim$ on the set $Z^{\mathbf{N}}$ of action sequences. Generally speaking, this relationship is represented through some function $U : Z^{\mathbf{N}} \to \overline{\mathbf{R}}$, *i.e.*, for every $(\underset{\sim}{z}, \underset{\sim}{z}') \in Z^{\mathbf{N}} \times Z^{\mathbf{N}}$, one has $\underset{\sim}{z} \lesssim \underset{\sim}{z}' \Leftrightarrow U(\underset{\sim}{z}) \le U(\underset{\sim}{z}')$. An articulation between such an approach and the current aggregator one is nonetheless available from the notion of *recursive payoff*:

Definition 3.2. Given an aggregator $A : Z \times \overline{\mathbf{R}} \to \overline{\mathbf{R}}$, a payoff function $U : Z^{\mathbf{N}} \to \overline{\mathbf{R}}$, is said to be *recursive* if

$$\forall \underset{\sim}{z} \in Z^{\mathbf{N}}, U\big(\underset{\sim}{z}\big) = A\big(z_0, U\big(S\big(\underset{\sim}{z}\big)\big)\big).$$

This definition captures the idea of stationarity of preferences over time, U being let unaffected by the passage of time—described by the shift operator S—over a sequence of actions. In light of such an articulation, this whole section will try to circumscribe a core matter: can the consideration of aggregator as the sole primitive be argued to allow for a proficient approach that allows for simpler and neater concepts that the standard one invoking a recursive payoff function?

3.3 Preference Orders associated to the Payoffs

In order to reach a neater understanding of the relationship between an aggregator and a recursive payoff function, consider an iteration of the recursive equation followed by U and A, that delivers

$$U\big(\underset{\sim}{z}\big) = A^T\big(\underset{\sim}{z}, U\big(S^T\big(\underset{\sim}{z}\big)\big)\big).$$

Whence, *potentially*, the satisfaction of

$$U\big(\underset{\sim}{z}\big) = \lim_{T \to +\infty} A^T\big(\underset{\sim}{z}, 0\big) \tag{3.1}$$

on some particular subset of $Z^{\mathbf{N}}$, *e.g.*, in order to insure that $U(S^T(\underset{\sim}{z}))$ tends to 0 when T tends to $+\infty$, and for some particular classes of aggregators A possessing sufficient regularity. Even though this is for example the approach followed by Le Van & Vailakis [10], it appears as being quite restrictive, for this imposes strong regularity conditions—close from Lipschitz ones—on A. From a general perspective, the obtention of the Equality 3.1 for any $\underset{\sim}{z}$ is *plainly* out of reach. Indeed, and as this is now to going to be illustrated, the discrepancy between the two may come to other dimensions, for even the orders on $Z^{\mathbf{N}}$ that are respectively induced by U and A may result in being distinct ones.

Following Convention I when these limits are not finite, an aggregator A defines two distinct preference relationships $\precsim_{\overline{A}}$ and $\precsim_{\underline{A}}$ on $Z^{\mathbf{N}}$ along:

$$\begin{aligned}
&\forall \big(\underset{\sim}{z}, \underset{\sim}{z}'\big) \in Z^{\mathbf{N}} \times Z^{\mathbf{N}}, \underset{\sim}{z} \precsim_{\overline{A}} \underset{\sim}{z}' \Leftrightarrow \overline{\lim}_{T \to +\infty} A^T\big(\underset{\sim}{z}, 0\big) \le \overline{\lim}_{T \to +\infty} A^T\big(\underset{\sim}{z}', 0\big), \\
&\forall \big(\underset{\sim}{z}, \underset{\sim}{z}'\big) \in Z^{\mathbf{N}} \times Z^{\mathbf{N}}, \underset{\sim}{z} \precsim_{\underline{A}} \underset{\sim}{z}' \Leftrightarrow \underline{\lim}_{T \to +\infty} A^T\big(\underset{\sim}{z}, 0\big) \le \underline{\lim}_{T \to +\infty} A^T\big(\underset{\sim}{z}', 0\big).
\end{aligned}$$

Assuming that A is associated with a recursive payoff function U defined on $Z^{\mathbf{N}}$, a preference relationship $\precsim_U$ can parallelly be introduced on $Z^{\mathbf{N}}$ as follows

$$\forall(\underset{\sim}{z}, \underset{\sim}{z}') \in Z^{\mathbf{N}} \times Z^{\mathbf{N}}, \underset{\sim}{z} \precsim_U \underset{\sim}{z}' \Leftrightarrow U(\underset{\sim}{z}) \leq U(\underset{\sim}{z}').$$

Even though, upon the satisfaction of Equation 3.1, the three above orders are identical, they can be distinct in the general case. The following proposition proves that in some situations, working with A, and $\precsim_{\overline{A}}$ or $\precsim_{\underline{A}}$, to represent preferences could be very different than the classical choice of working with U and $\precsim_U$, this latter order failing to rank constant sequences whilst $\precsim_{\overline{A}}$ and $\precsim_{\underline{A}}$ would instead provide decisive criteria.

Proposition 3.1.— *The orders $\precsim_{\overline{A}}$ and $\precsim_{\underline{A}}$ (induced by the aggregator A) and the order $\precsim_U$ (induced by a recursive payoff function U associated to A) can differ.*

Proof. Let $Z = \mathbf{R}$, $A(z, v) = v/2 + 1/2 - (z)^2$ if $v < 1$, $A(z, v) = v + 1 - (z)^2$ otherwise. Moreover, $A(z, +\infty) = +\infty$ and $A(z, -\infty) = -\infty$. First, it is easily derived that:

$$A^n(z_0, ..., z_0, 0) = \sum_{k=1}^{n} \frac{1}{2^k} + z_0 \sum_{k=0}^{n-1} \frac{1}{2^k}$$

for every $z_0 \leq 0$. In particular,

$$\lim_{n\to+\infty} A^n(\mathbf{0}, 0) = 1$$

where $\mathbf{0}$ denotes the null sequence, and

$$\lim_{n\to+\infty} A^n(-\mathbf{1}, 0) = -1$$

where $-\mathbf{1}$ denotes the constant sequence whose terms are all equal to -1.

Similarly:

$$\lim_{n\to+\infty} A^n(\mathbf{1}, 0) = \lim_{n\to+\infty} (1/2 + 2n + 1) = +\infty$$

where $\mathbf{1}$ denotes the constant sequence whose terms are all equal to $+1$.

In particular, if $\precsim_A$ is the order defined by A on $\mathbf{R}^{\mathbf{N}}$, and $\precnsim_A$ is the strict order associated[10] to $\precsim_A$, it is obtained that:

$$-\mathbf{1} \precnsim_A \mathbf{0} \precnsim_A \mathbf{1}.$$

Thus, if a recursive function U induces the same order, one should have

$$U(-\mathbf{1}) < U(\mathbf{0}) < U(\mathbf{1}).$$

But since U is recursive,

$$A(0, U(\mathbf{0})) = U(\mathbf{0}).$$

First assume $U(\mathbf{0})$ finite. Then if $U(\mathbf{0}) \geq 1$, the last equality can be written

$$U(\mathbf{0}) + 1 = U(\mathbf{0}),$$

a contradiction. Otherwise, if $U(\mathbf{0}) < 1$, one gets

$$U(\mathbf{0})/2 + 1/2 = U(\mathbf{0})$$

thus $U(\mathbf{0}) = 1$, a contradiction. Hence, finally, $U(\mathbf{0}) = +\infty$ or $U(\mathbf{0}) = -\infty$, which contradicts $U(-\mathbf{1}) < U(\mathbf{0}) < U(+\mathbf{1})$. QED

[10] that is, $x \precnsim_A y$ if $x \precsim_A y$ is true and $y \precsim_A x$ is false.

3.4 Aggregator-based vs Recursive Payoff-based Maximization Programs

3.4.1 Definitions of maximization programs

In order to embrace standard configurations and occurrences, *e.g.*, Economics, Finance, ..., some extra structure is now going to be added to the set of constraints. Assume that some initial state $x_0 \in X$ is defined, X being the set of the states. The law of evolution of such states is defined through a multivalued mapping Γ, defined from X to X, with nonempty values, meaning that states x_{t+1} at time $t+1$ should satisfy $x_{t+1} \in \Gamma\big(x_t\big)$, where x_t denotes the state at time t. Given two states x_t and x_{t+1}, the set of feasible actions at time t is denoted as $\Omega\big(x_t, x_{t+1}\big)$, Ω being a multivalued mapping defined from $\mathrm{Gr}(\Gamma)$—the graph of Γ—to Z with nonempty values. The set of feasible sequences of actions for a given x_0, $\Sigma\big(x_0\big)$, can thus be written as follows:

$$\Sigma\big(x_0\big) = \Big\{ \big(z_t\big)_{t\in \mathbf{N}} \in Z^{\mathbf{N}} : \exists \big(x_t\big)_{t\in \mathbf{N}^*} \in X^{\mathbf{N}} : \forall t \in \mathbf{N}, x_{t+1} \in \Gamma\big(x_t\big), \\ z_t \in \Omega\big(x_t, x_{t+1}\big) \Big\}.$$

It will also at times be of some convenience to consider the associated set of sequences of actions and states, *i.e.*,

$$\widetilde{\Sigma}\big(x_0\big) = \Big\{ \Big(\big(z_t\big)_{t\in \mathbf{N}}, \big(x_t\big)_{t\in \mathbf{N}^*} \Big) \in Z^{\mathbf{N}} \times X^{\mathbf{N}} : \forall t \in \mathbf{N}, x_{t+1} \in \Gamma\big(x_t\big), \\ z_t \in \Omega\big(x_t, x_{t+1}\big) \Big\}.$$

In the following, a *primitive model* will refer to a triple (A, Γ, Ω), for A an aggregator, and Γ and Ω the feasibility correspondences. Given a model (A, Γ, Ω), and a recursive payoff function $U : Z^{\mathbf{N}} \to \overline{\mathbf{R}}$ associated to A, recalling the aggregator-based payoffs of Definition 3.1, distinct optimization problems can be considered:

$$(\overline{P}) \qquad \sup_{\underset{\sim}{z} \in \Sigma(x_0)} \overline{w}\big(\underset{\sim}{z}\big),$$

$$(\underline{P}) \qquad \sup_{\underset{\sim}{z} \in \Sigma(x_0)} \underline{w}\big(\underset{\sim}{z}\big),$$

$$(P) \qquad \sup_{\underset{\sim}{z} \in \Sigma(x_0)} U\big(\underset{\sim}{z}\big).$$

Let $\overline{v}^* : X \to [-\infty, +\infty]$ be the value function of $\overline{P}$, defined by

$$\forall x_0 \in X, \overline{v}^*(x_0) = \sup_{\underset{\sim}{z} \in \Sigma(x_0)} \overline{w}\big(\underset{\sim}{z}\big),$$

and let $\underline{v}^* : X \to [-\infty, +\infty]$ be the value function of $\underline{P}$, defined by

$$\forall x_0 \in X, \underline{v}^*(x_0) = \sup_{\underset{\sim}{z} \in \Sigma(x_0)} \underline{w}\big(\underset{\sim}{z}\big).$$

The current contribution advocates a focus on $(\underline{\mathrm{P}})$ or $(\overline{\mathrm{P}})$ instead of (P). A first reason comes from the ensued simplicity: Problem (P) is a plain maximization over an *infinite* horizon intertemporal payoff whilst Problems $(\underline{P})$ and $(\overline{P})$ are maximization problems of the mere limits of *finite* horizon intertemporal payoffs. Second, it could be somewhat demanding to assume, as this is pre-supposed by Problem (P), that consumers were able to maximize over a set of infinite action sequences. Along the same range of ideas, a preference relationship over $Z \times \overline{\mathbf{R}}$ is simpler and more testable than a preference relationship on $Z^{\mathbf{N}}$. Lastly, having assumed that preferences are firstly and plainly represented by the aggregator, whilst a recursive payoff function brings about some analytical convenience, as this was illustrated through the subsequent proposition, working with a recursive payoff function may reveal inappropriate in some given configurations.

3.4.2 Values of the Programs may differ

As this is soon going to be clarified, the comparison between these programs will be facilitated by the retainment of the following assumption that will also at times be referred to in the subsequent exposition:

Null Assumption.— [Null Consumption] *There exists* $(\underline{a}, \underline{a}, \ldots, \underline{a}, \ldots) \in \cap_{x \in X} \Sigma(x)$ *with* $U(\underline{a}, \underline{a}, \ldots, \underline{a}, \ldots) = 0$.

Such an assumption is rather weak and appears, *e.g.*, as Assumption B6, p.17 in Streufert [20]. It could be perceived as a free disposal assumption and is generally valid in economic setups. Further remark that if there exists $(\underline{a}, \underline{a}, \ldots, \underline{a}, \ldots) \in \cap_{x \in X} \Sigma(x)$ with $\alpha = U(\underline{a}, \underline{a}, ...)$ finite, then Null Consumption can be assumed without any loss of generality. Consider indeed a new recursive function $\tilde{U} = U - \alpha$ and a new aggregator $\tilde{A}$ defined by $\tilde{A}(z, v) = A(z, v + \alpha) - \alpha$. Then $\tilde{U}$ is a recursive payoff function associated to the aggregator $\tilde{A}$ and $\tilde{U}(\underline{a}, \underline{a}, \ldots) = 0$. Besides, the preferences defined by A or U are identical to those defined by $\tilde{A}$ or $\tilde{U}$.

The following statement then establishes how the two first aggregator-based optimization problems $(\overline{P})$ and $(\overline{P})$ can exhibit distinct solutions and different values from the ones that result from the solving of the recursive function payoff-based Optimization problem (P). As stated in Proposition 3.1, this also proves that the orders that derive from the aggregator may differ from the one induced by a recursive payoff function.

Proposition 3.2.— *Consider the optimization programs $(\overline{P})$, $(\underline{P})$ and (P):*

(i) *The optimization problems $(\overline{P})$ and (P) (and similarly $(\underline{P})$ and (P)) may have different solutions.*

(ii) *Further letting the Null Assumption prevail,* $\mathrm{Val}(P) \geq \mathrm{Val}(\overline{P})$, *an inequality that can be strict.*

Proof. (i) Consider the optimization problem where $Z = \mathbf{R}$, $A(z, v) = v/2 + 1/2 - (z)^2$ if $v < 1$, $A(z, v) = v + 1 - (z)^2$ otherwise. Moreover, $A(z, +\infty) = +\infty$ and $A(z, -\infty) = -\infty$. Lastly, assume that there is no feasibility constraints. Then

$$A^n(\underset{\sim}{z}, 0) = \sum_{k=1}^{n} \frac{1}{2^k} - \sum_{k=0}^{n-1} z_k^2 \frac{1}{2^k}.$$

More specifically, the value of $(\overline{P})$ is 1, and this maximum is reached only at $\mathbf{0}$, the null sequence, *i.e.*, the set of solutions to this optimization problem summarizes to $\mathrm{Sol}(\overline{P}) = \{\mathbf{0}\}$.

In contradistinction with this, the aim is to prove that the set of solutions of the associated recursive payoff-based optimization problem is such that

$$\text{Sol}(P) \neq \{\mathbf{0}\}. \tag{3.2}$$

Indeed, either the set of solutions of (P) is empty, and Equation 3.2 is satisfied in a trivial way. Or the maximum of U is reached at some $\underset{\sim}{z} \in \Sigma(x_0)$. Let then prove that $U(\underset{\sim}{z})$ is to be infinite.

Ad absurdum, assume instead that $U(\underset{\sim}{z})$ is finite. Then either $U(\underset{\sim}{z}) < 1$ and for every $a \in Z$ and for small enough values of a, $U(a, \underset{\sim}{z}) = A(a, U(\underset{\sim}{z})) = U(\underset{\sim}{z})/2 + 1/2 - a^2 > U(\underset{\sim}{z})$ a contradiction. Or for every $a \in Z$ and for small enough values of a, $U(\underset{\sim}{z}) \geq 1$, hence $U(a, \underset{\sim}{z}) = A(a, U(\underset{\sim}{z})) = U(\underset{\sim}{z}) + 1 - a^2 > U(\underset{\sim}{z})$, another contradiction. As a conclusion, $U(\underset{\sim}{z})$ is to be infinite.

Now if $U(\underset{\sim}{z}) = -\infty$ then U is to be constantly equal to $-\infty$, and the the set of solutions of (P) is $Z^{\mathbf{N}}$, whence the satisfaction of Equation 3.2. If $U(\underset{\sim}{z}) = +\infty$, then for every $a \in Z$, $+\infty = A(a, +\infty) = A(a, U(\underset{\sim}{z})) = U(a, \underset{\sim}{z})$, and by iteration, $U(\underset{\sim}{y}) = +\infty$ for every sequence $\underset{\sim}{y}$ equal to $\underset{\sim}{z}$ except for a finite number of terms. In particular, the set of solutions of (P) is infinite, hence the satisfaction of Equation 3.2.

(ii) Additionally assuming Null Assumption, it derives that:

$$\begin{aligned}
&\sup_{\underset{\sim}{z} \in \Sigma(x_0)} \overline{\lim}_{T \to +\infty} A^T\big(\underset{\sim}{z}, 0\big) \\
&\qquad\qquad = \sup_{\underset{\sim}{z} \in \Sigma(x_0)} \overline{\lim}_{T \to +\infty} A^T\big(\underset{\sim}{z}, U(\underline{a}, \underline{a}, \ldots)\big) \\
&\qquad\qquad = \sup_{\underset{\sim}{z} \in \Sigma(x_0)} \overline{\lim}_{T \to +\infty} U\big(z_0, z_1, \ldots, z_{T-1}, \underline{a}, \underline{a}, \ldots\big) \\
&\qquad\qquad \leq \sup_{\underset{\sim}{z} \in \Sigma(x_0)} U\big(\underset{\sim}{z}\big),
\end{aligned}$$

since $\big(z_0, z_1, \ldots, z_{T-1}, \underline{a}, \underline{a}, \ldots\big) \in \Sigma\big(x_0\big)$. To prove that such an inequality can be strict, consider an aggregator A such that $A(z, +\infty) = +\infty$. Remark that for every recursive payoff U, one can construct a new recursive payoff $\overline{U}$ as follows: fix $\underset{\sim}{z}' \in Z^{\mathbf{N}}$, and define $\overline{U}\big(\underset{\sim}{z}\big) = +\infty$ if $\underset{\sim}{z} = \underset{\sim}{z}'$ but for a finite number of terms, and $\overline{U}\big(\underset{\sim}{z}\big) = U\big(\underset{\sim}{z}\big)$ otherwise. Then $\overline{U}$ is a recursive payoff associated to A, and the value of (P) associated to this payoff is $+\infty$, thus is strictly larger than the value of $(\overline{P})$ and completes the argument of the proof. QED

3.4.3 A Condition for the Values of the Programs to coincide

In the following, the function w is the objective function of the problem. It can be equal to $\overline{w}$ or $\underline{w}$.

Define

$$\Sigma^0\big(x_0\big) = \Big\{\underset{\sim}{z} \in \Sigma\big(x_0\big) : w\big(\underset{\sim}{z}\big) > -\infty\Big\},$$

as the set of action sequences feasible from an initial state x_0 and which generate a payoff which keeps on being bounded from below. Similarly, the associated set of sequences of actions and states is available as:

$$\widetilde{\Sigma}^0\big(x_0\big) = \Big\{\big(\big(z_t\big)_{t \in \mathbf{N}}, \big(x_t\big)_{t \in \mathbf{N}^*}\big) \in \widetilde{\Sigma}\big(x_0\big) : w\big(\underset{\sim}{z}\big) > -\infty\Big\}.$$

Definition 3.3. A pair of functions $(f_1, f_2) \in V^2$, where $V = \mathscr{F}(X, [-\infty, +\infty])$, is said to satisfy Transversality if[11]

T1 $\forall x_0 \in X, \forall (\underset{\sim}{z}, \underset{\sim}{x}) \in \widetilde{\Sigma}^0(x_0),\ \underline{\lim}_{T\to+\infty}\big(A^T(\underset{\sim}{z}, f_1(x_T)) - A^T(\underset{\sim}{z}, 0)\big) \geq 0.$

T2 $\forall x_0 \in X, \forall (\underset{\sim}{z}, \underset{\sim}{x}) \in \widetilde{\Sigma}(x_0),\ \overline{\lim}_{T\to+\infty}\big(A^T(\underset{\sim}{z}, f_2(x_T)) - A^T(\underset{\sim}{z}, 0)\big) \leq 0.$

Property T1 says that the payoff from $t = 0$ to time T is greater, asymptotically, if f_1 is used instead of 0 to evaluate payoff at time T. Similarly, Property T2 says that the payoff from $t = 0$ to time T is greater, asymptotically, if 0 is used instead of f_2 to evaluate payoff at time T. Roughly, transversality property of a function f compares asymptotically two intertemporal payoffs, the first one with a final payoff at date T defined by f, the second one with a final payoff at date T equal to 0.

A first important implication of this Transversality concept is to make precise the relationship between optimization problems $(\overline{P})$, $(\underline{P})$ and (P):

PROPOSITION 3.3.— *Under the Increasing Monotonicity Assumption* (IM)*, further let a pair* $(\underline{v}, \overline{v})$ *satisfy the Transversality Assumption and* U *be a recursive payoff function associated to* A*. If, for every* $x_0 \in X$ *and every* $\underset{\sim}{z} \in \Sigma(x_0), U(\underset{\sim}{z}) \in [\underline{v}(x_0), \overline{v}(x_0)]$*, then* $\overline{v}^*(x_0) = \underline{v}^*(x_0) = \sup_{\underset{\sim}{z}\in\Sigma(x_0)} U(\underset{\sim}{z})$*.*

4. Dynamic Programming and Bellman equation

This section will embed the main results of this contribution on the existence, the uniqueness and the computation of the Bellman equation. The standard questions related to the associated Bellman operator and the dynamic programming principle are the following: first, does the value functions correspond to a solution to the Bellman equation? Second, is this the only solution to the Bellman equation? Third and finally, can such a solution be computed through an iteration scheme? A first part of this section will be devoted to the description of the general dynamic programming framework, a second one to the comparison between the aggregator approach and the recursive payoff function within such framework that ought to justify the current choice as the aggregating function as a primitive, the fourth part encompasses a class of aggregators defined from the *biconvergent* recursive payoff functions of Streufert [20], [21], and analyzed under a new weak continuity assumption, the fifth part considers some examples.

4.1 Existence of a solution to the Bellman Equation under a Uniform Continuity Property

Let $V = \mathscr{F}(X, [-\infty, +\infty])$ denote the set of functions from X to $[-\infty, +\infty]$. An element v of V associates to every initial condition $x \in X$ a given level for the payoff, possibly infinite.

Definition 4.1. The Bellman operator $B : V \to \mathscr{F}(X, [-\infty, +\infty])$ is defined, for every $v \in V$ and every $x \in X$, by

$$B(v)(x) = \sup_{y\in\Gamma(x)} \Big\{ \sup_{z\in\Omega(x,y)} A(z, v(y)) \Big\}.$$

[11]Remark that $A^T(\underset{\sim}{z}, 0)$ and $A^T(\underset{\sim}{z}, f_i(x_T)), i \in \{1, 2\}$, could be simultaneously equal to ∞, which could lead to an indeterminacy: to avoid such problem, Convention I is retained in this transversality condition.

Example 4.1. Consider the aggregator defined in Proposition 3.2, with $Z = \overline{\mathbf{R}}$, $A(z,v) = v/2 + 1/2 - z^2$ if $v < 1$ and $A(z,v) = v + 1 - z^2$ otherwise. But $|A(z, 1-\eta) - A(z,1)| \geq 1/2$, hence A is not uniformly continuous in v. For every feasible correspondences Γ and Ω such that $\Sigma\big(x_0\big)$ contains the null sequence for every $x_0 \in X$, the value function associated to A is obtained for $\mathbf{0}$ and equal to 1. Yet, $B(1)\big(x_0\big) = 2$, thus the value function is not a fixed-point of the Bellman operator. Theorem 4.1 below will prove that the value function is not being a fixed-point of B springs from the lack of uniform continuity of A in v.

The following statement clarifies how, resting upon an extra boundary assumption, Uniform Continuity and Increasing Monotonicity of the aggregator imply that the value function is a fixed point of the Bellman operator.

THEOREM 4.1.— *If the aggregator is uniformly continuous in v and satisfies the Increasing Monotonicity and Boundary assumptions, then $B\underline{v}^* = \underline{v}^*$ and $B\overline{v}^* = \overline{v}^*$.*

The assumptions of this theorem are admittedly fairly weak, and cover a large part of the aggregators found in the literature (see Examples 2.2 and 2.1). The next subsection covers two other important issues, uniqueness and computation of a solution (see Theorem 4.2). It requires the introduction of Transversality conditions.

4.2 Existence, Uniqueness and Computation of the Solutions to the Bellman Equation

The aim of this section is to refine Theorem 4.1 in three directions:

(i) To get some uniqueness results.

(ii) To get some computation method of the solution by iteration. Otherwise stated, can such a solution be computed through an iteration scheme $v_{n+1} = B(v_n)$, *i.e.*, is it true that the value functions can be written as limits of the sequence (v_n), for some adequate v_0?

(iii) To encompass the aggregators defined from the biconvergent recursive payoff functions introduced by Streufert [20], [21].

This will be argued to be anchored on two distinct ingredients, *i.e.*, the satisfaction of the Transversality properties T1 and T2 plus Weak Continuity of the aggregator with respect to the future payoff, *i.e.*, a refinement of the Uniform Continuity assumption.

4.2.1 A Weak Continuity Assumption

Definition 4.2. Consider a function $\overline{v}$ from X to $\overline{\mathbf{R}}$. Define the function $f : [0,1]^{\mathbf{N}} \to \overline{\mathbf{R}}$ by

$$f(\varepsilon) = \sup_{(\underset{\sim}{z},\underset{\sim}{x}) \in \widetilde{\Sigma}(x_0)} \inf_{n\in\mathbf{N}} A\big(z_0, A\big(z_1, A\big(z_2, \dots \\ \dots, A\big(z_{n-1}, \overline{v}\big(x_n\big)\big) + \varepsilon_{n-1}\big) + \varepsilon_{n-2} + \cdots\big) + \varepsilon_2\big) + \varepsilon_1\big),$$

where $\varepsilon = (\varepsilon_i)_{i\in\mathbf{N}^*}$. The aggregator A is said to be weakly continuous at $\overline{v}$ if the function f is upper semicontinuous at 0.[12]

The price to pay for the generality of this assumption is its complexity: it is admittedly less testable than Uniform continuity. The following statement nonetheless establishes how Uniform continuity in v implies weak continuity at $\overline{v} : X \to \overline{\mathbf{R}}$, whatever $\overline{v}$:

Proposition 4.1.— *If an aggregator A is uniformly continuous in v, then it is weakly continuous at $\overline{v}$ for every $\overline{v} : X \to \overline{\mathbf{R}}$.*

Yet, this weak continuity assumption shall allow for gathering two important class of results in the dynamic programming literature that had hitherto generally been considered as distinct. For the first class of results where some transversality condition is involved, *vid.* Kamihigashi [7] or Levan-Vailakis [10]. For the second one that involves instead some upper and lower convergence criteria on the intertemporal payoff function, *vid.* Streufert [20], [21].

It is worth emphasizing that weak continuity implies that f is continuous at 0, lower semicontinuity at 0 being a consequence of the Increasing Monotonicity Assumption. The interpretation of the weak continuity assumption is then the following: assume that your terminal payoff at $t = n$ is valued with $\overline{v}$ and imagine a *prudent* entity that tries to maximize the *worst possible payoff*—that the *inf* alludes to in the criterion—from 0 to n, n spanning $\mathbf{N}$, that results in $f(0)$. The satisfaction of Weak Continuity then requires that this optimal *prudent* payoff should not dramatically change if small perturbations $\varepsilon_i > 0$ were added by each period i. In short, it says that the optimal value of a *prudent* entity should vary continuously at 0 with respect to small additive perturbations of payoffs through time.

4.2.2 Existence & Uniqueness of the solution to the Bellman Equation

The following theorem is the main result of this section and equips the analysis with some weak conditions under which the value function v^* is the only fixed point of the Bellman equation on some well behaved classes of functions. It further provides an algorithm to compute the value function.

Theorem 4.2.— *Assume Increasing Monotonicity Assumption. For every $(\underline{v}, \overline{v}) \in V^2$ satisfying Transversality, with $\underline{v}(x_0) \leq \overline{v}(x_0) < +\infty$ for every $x_0 \in X$, and such that A is weakly continuous*[13] *at $\overline{v}$. Let $v \in [\underline{v}, \overline{v}]$ such that $B(v) = v$. Then*

1. *$v = \overline{v}^* = \underline{v}^*$.*

2. *For every $f \leq v$ satisfying Transversality Assumption* (T1), *one has $\varliminf_{n\to+\infty} B^n f = v$.*

3. *If $\underline{v}$ and $\overline{v}$ also satisfy $B(\underline{v}) \geq \underline{v}$ and $B(\overline{v}) \leq \overline{v}$, then there exists a (unique) fixed-point of B in $[\underline{v}, \overline{v}]$.*

[12] For some well-suited topology on $[0,1]^{\mathbf{N}}$ such that every neighborhood of 0 intersect $]0,1]^{\mathbf{N}}$. This technical requirement would then provide a meaning to $\lim_{\varepsilon\to 0, \varepsilon\neq 0} f(\varepsilon)$, *i.e.*, a limit whose existence is required from the proof of the main current Theorem 4.2, $\mathbf{R}$ being currently endowed with the standard metric.

[13] This is true, in particular, when A is uniformly continuous in v

To sum up, Theorem 4.2 and Theorem 4.1 both give conditions to get the existence of a solution to the Bellman equation. These conditions are however not comparable in the general case, for on the one hand weak continuity is weaker than uniform continuity in v, but on the other hand Theorem 4.2 requires to be able to get an interval $[\underline{v}, \overline{v}]$ such that $B([\underline{v}, \overline{v}]) \subset [\underline{v}, \overline{v}]$.

A conceivable difficulty with Theorem 4.2 stems from the actual possibility of identifying a pair of functions $\underline{v}$ and $\overline{v}$ which fullfils the assumptions of this theorem. The following class of examples is useful as it introduces a general method that allows for finding these two candidate values. Broadly speaking, it is based upon the idea of a the value function of aggregators that *dominates* the primitive aggregator A.

Definition 4.3 (Aggregators dominated by well behaved aggregators)**.** A model (A, Γ, Ω) is said to be *dominated* by another model (A', Γ', Ω') (written $(A, \Gamma, \Omega) \lesssim (A', \Gamma', \Omega')$) if $A \leq A'$, $\Gamma \subset \Gamma'$ and $\Omega \subset \Omega'$.

From Theorem 4.1:

PROPOSITION 4.2.— *For every aggregator A, call $v^*_{\underline{A}}$ the upper value associated to A. Assume $(\underline{A}, \underline{\Gamma}, \underline{\Omega}) \lesssim (A, \Gamma, \Omega) \lesssim (\overline{A}, \overline{\Gamma}, \overline{\Omega})$. If $\underline{A}$, $\overline{A}$ and A are uniformly continuous with respect to v, satisfy Increasing Monotonicity and Boundary assumption, and if $(v^*_{\underline{A}}, v^*_{\overline{A}}) \in V^2$ satisfies Transversality then v^*_A is the (unique) fixed-point in $[v^*_{\underline{A}}, v^*_{\overline{A}}]$ of the Bellman operator associated to A.*

Proof. Let B^A, $B^{\overline{A}}$ and $B^{\underline{A}}$ be the Bellman operators associated to the aggregators A, $\overline{A}$ and $\underline{A}$. From the domination assumptions, it is obtained that $B^A(v^*_{\underline{A}}) \geq B^{\underline{A}}(v^*_{\underline{A}}) = v^*_{\underline{A}}$ (the last equality being a consequence of Theorem 4.1), and similarly, $B^A(v^*_{\overline{A}}) \leq v^*_{\overline{A}}$, and last, $v^*_{\overline{A}} \geq v^*_{\underline{A}}$ is obvious from domination assumptions. Thus, from point (iii) of Theorem 4.2, v^*_A is the only fixed-point of B^A in $[v^*_{\underline{A}}, v^*_{\overline{A}}]$. QED

4.2.3 Additive aggregators and Biconvergent aggregators

As an Illustration, consider now the Additive Aggregator case:

Example 4.2. (Additive Aggregator: Kamihigashi [5]). Theorem 4.2 allows for recovering the results of Kamihigashi [5] that relate to the Bellman equation. Consider indeed the case where $A(z, v) = u(z) + \beta v$ with $u : [-\infty, +\infty[\to \mathbf{R}$, $\beta \in]0, 1[$, $Z = X \times X$ and $\Omega(x, y) = \{(x, y), y \in \Gamma(x)\}$. Let L be either the infimum or the supremum limit operator. Consider the two following sets

$$\Pi(x_0) = \Big\{ (x_t)_{t \in \mathbf{N}^*} \in X^{\mathbf{N}} : \forall t \in \mathbf{N}, x_{t+1} \in \Gamma(x_t) \Big\},$$
$$\Pi^0_L(x_0) = \Big\{ (x_t)_{t \in \mathbf{N}^*} \in \Pi(x_0) : L_{T \to \infty} \sum_{t=0}^{T} \beta^t u(x_t, x_{t+1}) > -\infty \Big\},$$

Suppose that, as in T. Kamihigashi [5], Theorem 2.1., that there exist $\underline{v}, \overline{v} \in V$ such that $\underline{v} \leq \overline{v}$, with $B(\underline{v}) \geq \underline{v}$, $B(\overline{v}) \leq \overline{v}$ and

$$\forall (x_t)_{t \in \mathbf{N}} \in \Pi^0_L(x_0), \underline{\lim}_{t \to +\infty} \beta^t \underline{v} \geq 0, \tag{4.1}$$

$$\forall (x_t)_{t \in \mathbf{N}} \in \Pi(x_0), \overline{\lim}_{t \to +\infty} \beta^t \overline{v} \leq 0. \tag{4.2}$$

The Increasing Monotonicity Assumption is naturally satisfied with this additive formulation, A being further uniformly continuous in v. Moreover, the pair $(\underline{v}, \overline{v})$ satisfies the Transversality assumption. Indeed, one has

$$A^T\big(z_0, \ldots, z_T, \underline{v}(x_{T+1})\big) - A^T\big(z_0, \ldots, z_T, 0\big) = \beta^T \underline{v}\big(x_{T+1}\big),$$

a similar equality being available for $\overline{v}$. This establishes the Transversality Assumption, from Equation 4.1 and Equation 4.2. Thus, and from Theorem 4.2, v^* is the unique fixed-point of B. Moreover, taking $f = \underline{v}$ in Theorem 4.2, the sequence $B^n\underline{v}$ is easily seen to be increasing and Theorem 4.2(i) implies that $B^n\underline{v}$ converges to v^* for the pointwise convergence. To sum up, Theorem 2.1. in Kamihigashi [5] is recovered.

Theorem 4.2 will allow for recovering a range of classical results due to Pete Streufert and that relate to the status of the Bellman equation under a Biconvergence assumption on the intertemporal payoff function $U(\cdot)$. In order to rest upon the same formalism as Streufert, let $Z = X$ and $\Omega(x, y) = \{y\}$, this modeling device being retained in the course of this subsection. Under such a specialisation, a model can be written (A, Γ) and the aggregator A summarizes to a mapping from $X \times \overline{\mathbf{R}} \to \overline{\mathbf{R}}$. Within such a framework, the Bellman operator $B : \mathscr{F}(X, [-\infty, +\infty]) \to \mathscr{F}(X, [-\infty, +\infty])$ is defined, for every $v \in \mathscr{F}(X, [-\infty, +\infty])$ and every $x \in X$, by:

$$B(v)(x) = \sup_{y \in \Gamma(x)} A(y, v(y)).$$

Let then U denote a recursive payoff function associated to A. The function U is said to be *upper convergent* over $\prod_{t=0}^{+\infty} \Gamma^t(x_0)$ if

$$\forall \underset{\sim}{x} \in \prod_{t=0}^{+\infty} \Gamma^t\big(x_0\big), \lim_{t \to +\infty} \sup U\Big(x_0, \ldots, x_t, \prod_{s=t+1}^{+\infty} \Gamma^s\big(x_0\big)\Big) = U\big(\underset{\sim}{x}\big)$$

The function U is said to be *lower convergent* over $\prod_{t=0}^{+\infty} \Gamma^t\big(x_0\big)$ if

$$\forall \underset{\sim}{x} \in \prod_{t=0}^{+\infty} \Gamma^t\big(x_0\big), \lim_{t \to +\infty} \inf U\Big(x_0, .., x_t, \prod_{s=t+1}^{+\infty} \Gamma^s\big(x_0\big)\Big) = U(\underset{\sim}{x})$$

The function U is *biconvergent* over $\prod_{t=0}^{+\infty} \Gamma^t\big(x_0\big)$ if it is both *upper* and *lower convergent* over $\prod_{t=0}^{+\infty} \Gamma^t\big(x_0\big)$.

An admittedly convenient reformulation of biconvergence states as follows:

Definition 4.4. Consider the model (A, Γ) and $U : X^{\mathbf{N}} \to [-\infty, +\infty]$, a recursive function associated to A. Let $x_0 \in X$. Then U is said to be biconvergent over $\prod_{t=0}^{+\infty} \Gamma^t(x_0)$ if for every $\varepsilon > 0$, there is $N \in \mathbf{N}$ such that for every $(x_0, x_1, \ldots) \in \prod_{t=0}^{+\infty} \Gamma^t\big(x_0\big)$ and every $\big(x'_{N+1}, x'_{N+2}, \ldots\big) \in \prod_{t=N+1}^{+\infty} \Gamma^t\big(x_0\big)$, one has:

$$\big|U\big(x_1, x_2, \ldots, x_N, x_{N+1}, x_{N+2}, \ldots\big) - U\big(x_1, x_2, \ldots, x_N, x'_{N+1}, x'_{N+2}, \ldots\big)\big)\big| \leq \varepsilon.$$

Otherwise stated, there exists some date N for which the recursive payoff function can be approximated by only considering the N first actions, whatever the other actions that locate in the *tail* of the sequence and from date $N + 1$ on.

The following result, obtained by Streufert [20], is a Corollary of Theorem 4.2:

Corollary.— *Assume X is a topological space, U is a recursive payoff function associated to A, the Increasing Monotonicity and Null Assumptions prevail, A is u.s.c., Γ is a u.s.c. multivalued function from X to X with compact values, and $\max U(\prod_{t=1}^{+\infty}\Gamma^t(x_0))$ exists. Then $v^* : X \to \mathbf{R}$ defined by $v^*(x_0) = \sup_{\underset{\sim}{z}\in\Sigma(x_0)} U(\underset{\sim}{z})$ is a solution to the Bellman equation, and the unique solution whenever U is biconvergent over $\prod_{t=0}^{+\infty}\Gamma^t(x_0)$.*

A. Proof of Lemma 2.1

Proof. Let us show (i) $\Leftrightarrow$ (ii) where

(i) There exists $\delta : \mathbf{R} \to \mathbf{R}$ that tends to 0 at 0 and such that, for every $(z, v, v') \in Z \times \overline{\mathbf{R}} \times \overline{\mathbf{R}}, |A(z, v) - A(z, v')| \leq \delta(v - v')$ and

(ii) A is uniformly continuous in v.

Clearly (i) $\Rightarrow$ (ii). Let us show that (ii) $\Rightarrow$ (i). Assume A is uniformly continuous in v and define for every $x \in \mathbf{R}$:

$$\delta(x) = \sup_{(z,v)\in Z\times\overline{\mathbf{R}}} \big\{|A(z, v + x) - A(z, v)|\big\}.$$

By definition of δ, Condition (i) is true, and continuity of δ at 0 comes from Condition (ii). QED

B. Proof of Theorem 2.1

Proof. This is a direct corollary of Levin 1991 [11]'s Theorem 1. Indeed, for a chain τ, the sum $\sum_{i\in I_{\prec}}[d(z_i, z_{i-1}) + |v_i - v_{i-1}|$ is exactly the so-denoted $S_{(d,||),\preceq}(\tau)$ by Levin and the condition that for any $(z, v), (z', v') \in Z\times\mathbf{R}$ such that $(z', v') \prec (z, v)$, there exists $\alpha > 0$ such that for any chain $((z_i, v_i))_{i=0}^n \in \mathcal{C}((z, v), (z', v'))$,

$$\sum_{i\in I_{\prec}} [d(z_i, z_{i-1}) + |v_i - v_{i-1}|] \geq \alpha$$

is equivalent to condition (2) of Levin [11]'s Theorem 1, *i.e.*, if $e' \prec e$,

$$\inf_{\tau\in\mathcal{C}(e,e')} S_{(d,||),\preceq}(\tau) > 0.$$

So the theorem applies and concludes through the assertion (1) of Levin [11]'s Theorem 1 that $\preceq$ can be represented by an aggregator A that satisfies

$$\forall (z, v), (z', v') \in Z \times \mathbf{R}, |A(z, v) - A(z', v')| \leq d(z, z') + |v - v'|$$

which implies that $\exists\delta : \mathbf{R} \to \mathbf{R}^+$ continuous at 0, such that $\delta(0) = 0$ and $\forall z \in Z, \forall v, v' \in \mathbf{R}, |A(z, v) - A(z, v')| \leq \delta(v - v')$, *i.e.*, A is uniformly continuous in v.

Moreover, if $\preceq$ is increasing in v, then A satisfies (IM) Increasing Monotonicity. QED

C. Proof of Proposition 3.3.

Proof. Define $v^{**}(x_0) = \sup\limits_{\underset{\sim}{z}\in\Sigma(x_0)} U(\underset{\sim}{z})$. By definition,

$$\forall \underset{\sim}{z} \in \Sigma(x_0),\ v^{**}(x_0) \geq U(\underset{\sim}{z}).$$

The functional U being recursively associated to A, this restates as:

$$\forall n \in \mathbf{N}^*, \forall (z_0, \ldots, z_n, \ldots) \in \Sigma(x_0),\ v^{**}(x_0) \geq A^n(\underset{\sim}{z}, U(z_{n+1}, z_{n+2}, \ldots)).$$

Thus, From Increasing Monotonicity Assumption and since $\underline{v}$ "bounds" U from below, for every $(\underset{\sim}{z}, \underset{\sim}{x}) \in \widetilde{\Sigma}(x_0)$:

$$v^{**}(x_0) \geq A^n(\underset{\sim}{z}, \underline{v}(x_{n+1})).$$

Writing

$$A^n(\underset{\sim}{z}, \underline{v}(x_{n+1})) = (A^n(\underset{\sim}{z}, \underline{v}(x_{n+1}) - A^n(\underset{\sim}{z}, 0)) + A^n(\underset{\sim}{z}, 0).$$

Taking the supremum limit and then the supremum:

$$v^{**}(x_0) \geq \overline{v}^*(x_0) \geq \underline{v}^*(x_0)$$

To complete the proof, *i.e.*, to establish that $v^{**}(x_0) \leq \underline{v}^*(x_0)$, assume now that $v^{**}(x_0) > -\infty$ (otherwise $v^{**}(x_0) \leq \underline{v}^* x_0)$ is clear). Then, from the definition of the supremum, for every $\varepsilon > 0$, there is $\underset{\sim}{z} \in \Sigma(x_0)$ such that

$$v^{**}(x_0) \leq U(\underset{\sim}{z}) + \varepsilon.$$

The functional U being recursive:

$$\forall n \in \mathbf{N}^*,\ v^{**}(x_0) \leq A^n(\underset{\sim}{z}, U(z_{n+1}, z_{n+2}, \ldots)) + \varepsilon.$$

Thus, From Increasing Monotonicity Assumption, and since $\overline{v}$ "bounds" U from above, for every $\underset{\sim}{x} \in X^{\mathbf{N}}$ such that $(\underset{\sim}{z}, \underset{\sim}{x}) \in \widetilde{\Sigma}(x_0)$:

$$\forall n \in \mathbf{N}^*,\ v^{**}(x_0) \leq A^n(\underset{\sim}{z}, \overline{v}(x_{n+1})) + \varepsilon.$$

Exactly as in the end of Step 2 of the proof of Theorem 4.2, reformulating

$$A^n(\underset{\sim}{z}, \overline{v}(x_{n+1})) = (A^n(\underset{\sim}{z}, \overline{v}(x_{n+1})) - A^n(\underset{\sim}{z}, 0)) + A^n(\underset{\sim}{z}, 0),$$

taking the infimum limit, then the supremum and taking the limit when ε tends to zero:

$$v^{**}(x_0) \leq \underline{v}^*(x_0) \leq \overline{v}^*(x_0),$$

and finally,

$$v^{**}(x_0) = \underline{v}^*(x_0) = \overline{v}^*(x_0),$$

that completes the argument of the proof. QED

D. Proof of Theorem 4.1.

The proof is given for the case where the objective function, denote w, is equal to $\overline{w}$, and the upper value $\overline{v}^*$ shall be simply denoted v^*. The modifications of the proof required for the infimum limit case ($w = \underline{w}$) being made explicit when necessitated.

Two preparatory lemmas will first be needed for the establishment of the argument of the proof of of Theorem 4.1:

Lemma D.1.— *Letting* $x_0 \in X$ *and* $x_1 \in \Gamma(x_0)$, *any* $z_0 \in \Omega(x_0, x_1)$ *and any* $\underset{\sim}{z}' = (z_n)_{n\in\mathbf{N}^*} \in \Sigma^0(x_1)$, $v^*(x_1) = -\infty$ *implies* $\lim_{T\to+\infty} A(z_0, A^T(\underset{\sim}{z}', 0)) = -\infty$ *(or* $\underline{\lim}_{T\to+\infty} A(z_0, A^T(\underset{\sim}{z}', 0)) = -\infty$ *when* $w(\underset{\sim}{z}) = \underline{\lim}_{T\to+\infty} A^T(\underset{\sim}{z}, 0)$*).*

Proof. If $v^*(x_1) = -\infty$ then $\lim_{T\to+\infty} A^T(\underset{\sim}{z}', 0) = -\infty$ for every $\underset{\sim}{z}' \in \Sigma(x_1)$ (or $\underline{\lim}_{T\to+\infty} A^T(\underset{\sim}{z}', 0) = -\infty$ in the infimum limit case). Boundary assumption implies $\lim_{T\to+\infty} A(z_0, A^T(\underset{\sim}{z}', 0)) = -\infty$ for every $z_0 \in \Omega(x_0, x_1)$ (or $\underline{\lim}_{T\to+\infty} A(z_0, A^T(\underset{\sim}{z}', 0)) = -\infty$ in the infimum limit case). QED

Lemma D.2.— *Letting* $x_0 \in X$, *for every* $x_1 \in \Gamma(x_0)$, $v^*(x_1) = +\infty$ *implies* $v^*(x_0) = +\infty$.

Proof. By definition, x_0 and $x_1 \in \Gamma(x_0)$ being fixed,

$$\begin{aligned} v^*(x_0) &= \sup_{\underset{\sim}{z}\in\Sigma(x_0)} \overline{\lim}_{T\to+\infty} A^T(\underset{\sim}{z}, 0) \\ &\geq \sup_{z_0\in\Omega(x_0,x_1),(z_n)_{n\in\mathbf{N}^*}\in\Sigma(x_1)} \overline{\lim}_{T\to+\infty} A(z_0, A^{T-1}((z_n)_{n\in\mathbf{N}^*}, 0)) \\ &\geq A\Big(z_0, \sup_{(z_n)_{n\in\mathbf{N}^*}\in\Sigma(x_1)} \overline{\lim}_{T\to+\infty} A^{T-1}((z_n)_{n\in\mathbf{N}^*}, 0)\Big) \end{aligned}$$

for every $z_0 \in \Omega(x_0, x_1)$, the last inequality being a consequence of uniform continuity of A in v and Increasing Monotonicity Assumption. But $v^*(x_1) = +\infty$, thus

$$\sup_{(z_n)_{n\in\mathbf{N}^*}\in\Sigma(x_1)} \overline{\lim}_{T\to+\infty} A^{T-1}((z_n)_{n\in\mathbf{N}^*}, 0)) = +\infty,$$

and finally Boundary Assumption and the inequality above implies $v^*(x_0) = +\infty$. The proof is identical in the infimum limit case (replacing supremum limit by infimum limit). QED

First show $Bv^* \geq v^*$. Let $x_0 \in X$. If $v^*(x_0) = -\infty$ then $Bv^*(x_0) \geq v^*(x_0)$ is true. Now, assuming $v^*(x_0) > -\infty$.

Let $x_1 \in X$ such that $v^*(x_1) > -\infty$. By definition,

$$\begin{aligned} v^*(x_1) &= \sup_{\underset{\sim}{z}\in\Sigma(x_1)} w(\underset{\sim}{z}) \\ &= \sup_{\underset{\sim}{z}\in\Sigma^0(x_1)} w(\underset{\sim}{z}) \end{aligned}$$

where $\Sigma^0(x_1) = \{\underset{\sim}{z} \in \Sigma(x_1) : w(\underset{\sim}{z}) > -\infty\}$ is nonempty since $v^*(x_1) > -\infty$. Let $\tilde{z} \in \Sigma^0(x_1)$. From the definition of the supremum:

(D.1) $\quad v^*(x_1) \geq w(\underset{\sim}{z}),$

Now, since $w(\underset{\sim}{z}) = \overline{\lim}_{T\to+\infty} A^T(\underset{\sim}{z}, 0)$, and since $\tilde{z} \in \Sigma^0(x_1)$, given $\eta > 0$, there exists $T^{x_1,\underset{\sim}{z},\eta}$ such that for every $T' \geq T^{x_1,\underset{\sim}{z},\eta}$ (or for an infinite number of T when $w(\underset{\sim}{z}) = \underline{\lim}_{T\to+\infty} A^T(\underset{\sim}{z}, 0)$):

(D.2) $\quad w(\underset{\sim}{z}) \geq A^{T'}(\underset{\sim}{z}, 0) - \eta,$

where the two sides ot this inequality can be equal to $+\infty$.

Hence, from Equation D.1 and Equation D.2, for every $x_1 \in \Gamma(x_0)$ with $v^*(x_1) > -\infty$, every $\underset{\sim}{z} \in \Sigma^0(x_1)$, every $\eta > 0$ and every $T' \geq T^{x_1,\underset{\sim}{z},\eta}$ (or for an infinite number of T when $w(\underset{\sim}{z}) = \underline{\lim}_{T\to+\infty} A^T(\underset{\sim}{z}, 0)$), it is derived that:

(D.3) $\quad v^*(x_1) \geq A^{T'}(\underset{\sim}{z}, 0) - \eta.$

Fix $\varepsilon > 0$. From the definition of the uniform continuity of A in v (see Definition 2.1), there exists $\eta > 0$ such that:

$$\text{(D.4)} \quad \forall (z_0, v, v') \in Z \times \overline{\mathbf{R}} \times \overline{\mathbf{R}}, |v - v'| \leq \eta \Rightarrow |A(z_0, v) - A(z_0, v')| \leq \varepsilon.$$

By definition,

$$Bv^*(x_0) = \sup_{x_1 \in \Gamma(x_0)} \Big\{ \sup_{z_0 \in \Omega(x_0, x_1)} A(z_0, v^*(x_1)) \Big\},$$

also equal to

$$Bv^*(x_0) = \sup_{x_1 \in \Gamma(x_0) : v^*(x_1) > -\infty} \Big\{ \sup_{z_0 \in \Omega(x_0, x_1)} A(z_0, v^*(x_1)) \Big\},$$

because $A(z_0, -\infty) = -\infty$ from Boundary Assumption, and since there exists $x_1 \in \Gamma(x_0)$ such that $v^*(x_1) > -\infty$ (otherwise, from Lemma D.1, $v^*(x_0) = -\infty$, a contradiction). This last equality, Equation D.2 and Increasing Monotonicity Assumption imply:

$$\text{(D.5)} \quad \begin{aligned} &Bv^*(x_0) \geq A(z_0, A^T(\underset{\sim}{z}', 0) - \eta), \forall x_1 \in \Gamma(x_0) : v^*(x_1) > -\infty, \\ &\forall z_0 \in \Omega(x_0, x_1), \forall \underset{\sim}{z}' = (z_n)_{n \in \mathbf{N}^*} \in \Sigma^0(x_1), \forall T \geq T_0^{x_1, \underset{\sim}{z}', \eta}. \end{aligned}$$

(this inequality being only true for an infinite number of T in the infimum limit case.) Now, Equation D.5 implies, with Equation D.4:

$$\text{(D.6)} \quad \begin{aligned} &Bv^*(x_0) \geq A(z_0, A^T(\underset{\sim}{z}', 0)) - \varepsilon, \forall x_1 \in \Gamma(x_0) : v^*(x_1) > -\infty, \\ &\forall z_0 \in \Omega(x_0, x_1), \forall \underset{\sim}{z}' = (z_n)_{n \in \mathbf{N}^*} \in \Sigma^0(x_1), \forall T \geq T_0^{x_1, \underset{\sim}{z}', \eta}. \end{aligned}$$

(this inequality being only true for an infinite number of T in the infimum limit case.) If $(z_n)_{n \in \mathbf{N}^*} \notin \Sigma^0(x_1)$ but $(z_n)_{n \in \mathbf{N}^*} \in \Sigma(x_1)$, then $\lim_{T \to +\infty} A^T((z_n)_{n \in \mathbf{N}^*}, 0)) = -\infty$ (or $\underline{\lim}_{T \to +\infty} A^T((z_n)_{n \in \mathbf{N}^*}, 0)) = -\infty$ in the infimum limit case) which implies $\overline{\lim}_{T \to +\infty} A(z_0, A^T((z_n)_{n \in \mathbf{N}^*}, 0)) = -\infty$ (or, in the infimum limit case, $\underline{\lim}_{T \to +\infty} A(z_0, A^T((z_n)_{n \in \mathbf{N}^*}, 0)) = -\infty$) from Boundary Assumption. The same conclusion is true if $v^*(x_1) = -\infty$ and $(z_n)_{n \in \mathbf{N}^*} \in \Sigma^0(x_1)$(from Lemma D.1). Thus, passing to the supremum limit (infimum limit when $w(\underset{\sim}{z}) = \underline{\lim}_{T \to +\infty} A^T(\underset{\sim}{z}, 0)$) in Equation D.6, the conditions $(z_n)_{n \in \mathbf{N}^*} \in \Sigma^0(x_1)$ and $v^*(x_1) > -\infty$ can be removed and one simply gets:

$$Bv^*(x_0) \geq \overline{\lim}_{T \to +\infty} A(z_0, A^T(S(\underset{\sim}{z}), 0)) - \varepsilon, \forall \underset{\sim}{z} \in \Sigma(x_0),$$

The supremum limit being an infimum limit whenever $w(\underset{\sim}{z}) = \underline{\lim}_{T \to +\infty} A^T(\underset{\sim}{z}, 0)$. Passing to the supremum:

$$Bv^*(x_0) \geq v^*(x_0) - \varepsilon.$$

Passing to the limit when $\varepsilon > 0$ tends to 0, one finally obtains

$$Bv^*(x_0) \geq v^*(x_0),$$

which establishes $Bv^* \geq v^*$.

To prove $B(v^*)(x_0) \leq v^*(x_0)$ for every $x_0 \in X$, first assume $B(v^*)(x_0) < +\infty$.

From the definition of the supremum and since $B(v^*)(x_0) < +\infty$, for every $\varepsilon > 0$ (now fixed), there exists $x_1 \in \Gamma(x_0)$ (now fixed) with $z_0 \in \Omega(x_0, x_1)$ such that

$$\text{(D.7)} \quad Bv^*(x_0) \leq A(z_0, v^*(x_1)) + \varepsilon/2$$

If $v^*(x_1) = +\infty$, then from Lemma D.2, $v^*(x_0) = +\infty$, thus $B(v^*)(x_0) \leq v^*(x_0)$ is true. Assume now that $v^*(x_1) < +\infty$. Then, from the definition of $v^*(x_1)$, for every $\eta > 0$, there exists $(z_n)_{n\in\mathbf{N}^*} \in \Sigma(x_1)$ such that

$$v^*(x_1) \leq \overline{\lim}_{T\to+\infty} A^T((z_n)_{n\in\mathbf{N}^*}, 0) + \eta/2,$$

where the supremum limit is an infimum limit in the second case.

The supremum limit being the (greatest) cluster point, for every $\eta > 0$, it is obtained that:

$$\text{(D.8)} \quad v^*(x_1) \leq A^T((z_n)_{n\in\mathbf{N}^*}, 0) + \eta.$$

for an infinite number of $T \in \mathbf{N}$ (or for T large enough in the infimum limit case).

From the definition of uniform continuity of A (see Definition 2.1), there exists $\eta > 0$ (now fixed) such that:

$$\text{(D.9)} \quad \forall (z_0, v, v') \in Z \times \overline{\mathbf{R}} \times \overline{\mathbf{R}}, |v - v'| \leq \eta \Rightarrow |A(z_0, v) - A(z_0, v')| \leq \varepsilon/2.$$

Hence, from Increasing Monotonicity Assumption, from Equations D.7 and D.8, the following condition holds for an infinite number of $T \in \mathbf{N}$ (or for T large enough in the infimum limit case):

$$\begin{aligned} Bv^*(x_0) &\leq A(z_0, A^T((z_n)_{n\in\mathbf{N}^*}, 0) + \eta) + \varepsilon/2 \\ &\leq A(z_0, A^T((z_n)_{n\in\mathbf{N}^*}, 0)) + \varepsilon, \text{ from Equation D.9} \\ &\leq v^*(x_0) + \varepsilon. \end{aligned}$$

passing to the supremum limit (or the infimum limit in the infimum limit case). This prevails for any $\varepsilon > 0$, whence $B(v^*)(x_0) \leq v^*(x_0)$.

Last, to prove $B(v^*)(x_0) \leq v^*(x_0)$ when $B(v^*)(x_0) = +\infty$, simply replace, in all the proof from Equation D.7 up to the end, $B(v^*)(x_0)$ by any constant $L > 0$. The same line of the argument for the proof then yields $L \leq v^*(x_0)$ for every $L > 0$, thus $v^*(x_0) = +\infty = B(v^*)(x_0)$.

E. Proof of Proposition 4.1

Proof. Let δ such that for every $(z, v, v') \in Z \times \overline{\mathbf{R}}^2, \left|A(z, v) - A(z, v')\right| \leq \delta(v - v')$, where $\delta : \mathbf{R} \to \mathbf{R}$ tends to 0 at 0. For every $(\underset{\sim}{z}, \underset{\sim}{x}) \in \widetilde{\Sigma}(x_0)$, iterating the definition of Uniform continuity in v and Monotonicity for every period, one obtains:

$$\begin{aligned} \text{(E.1)} \quad & A(z_0, A(z_1, A(z_2, \ldots, A(z_{n-1}, \overline{v}(x_n)) + \varepsilon_{n-1}) + \varepsilon_{n-2} + \cdots) + \varepsilon_2) + \varepsilon_1) \\ & \leq A(z_0, A(z_1, A(z_2, \ldots, A(z_{n-1}, \overline{v}(x_n))) + \cdots)) \\ & \qquad \cdots + \delta(\varepsilon_1 + \delta(\varepsilon_2 + \cdots + \delta(\varepsilon_{n-2} + \delta(\varepsilon_{n-1})) \cdots). \end{aligned}$$

For every $a > 0$, define

$$V_a = \Big\{(\varepsilon_n)_{n\in\mathbf{N}^*} \in [0,1]^{\mathbf{N}} : \forall(\varepsilon'_n)_{n\in\mathbf{N}^*} \in [0, 2\varepsilon_i]^{\mathbf{N}},$$
$$\delta(\varepsilon'_1) \le a, \delta(\varepsilon'_2) \le \varepsilon_1, \dots, \delta(\varepsilon'_{n-1}) \le \varepsilon_{n-2}, \dots\Big\}.$$

This set is nonempty (it contains 0). Moreover, every V_a intersects $]0,1]^{\mathbf{N}}$ (indeed, since δ is continuous at 0, one can define some $\varepsilon_i > 0$ inductively with $(\varepsilon)_i \in V_a$). Consider on $[0,1]^{\mathbf{N}}$ the topology generated by this family of neighbourhood of 0. It has been shown that every neighborhood of 0 intersects $]0,1]^{\mathbf{N}}$. Moreover, for every $(\varepsilon_n)_{n\in\mathbf{N}^*} \in V_a$ and every integer $n \ge 1$, $\varepsilon_{n-2} + \delta(\varepsilon_{n-1}) \le \varepsilon_{n-2} + \varepsilon_{n-2}$, thus $\delta(\varepsilon_{n-2} + \delta(\varepsilon_{n-1})) \le \varepsilon_{n-3}$. Iterating, one derives that:

$$\delta(\varepsilon_1 + \delta(\varepsilon_2 + \dots + \delta(\varepsilon_{n-2} + \delta(\varepsilon_{n-1}))\cdots) \le a$$

for every integer n, thus from equation (E.1) and the definition of f, passing to infimum with respect n, then to supremum, one obtains:

$$\forall\varepsilon \in V_a, f(\varepsilon) \le f(0) + a,$$

which proves weak continuity of A, that concludes the proof. QED

F. Proof of Theorem 4.2

Proof. Consider $[\underline{v}, \overline{v}]$ such that $(\underline{v}, \overline{v}) \in V^2$ satisfies Transversality and v a fixed point of B on $[\underline{v}, \overline{v}]$, with $\overline{v}(x_0) < +\infty$ for every $x_0 \in X$.

Step 1. First prove that $v \ge \overline{v}^*$.

First, let $x_0 \in X$.
If $\overline{v}^*(x_0) = -\infty$, $v(x_0) \ge \overline{v}^*(x_0)$ is true. Let x_0 such that $\overline{v}^*(x_0) > -\infty$. Then $\overline{v}^*(x_0) = \sup_{\underset{\sim}{z}\in\Sigma(x_0)} \overline{w}(\underset{\sim}{z}) = \sup_{\underset{\sim}{z}\in\Sigma^0(x_0)} \overline{w}(\underset{\sim}{z})$, where $\Sigma^0(x_0) = \{\underset{\sim}{z} \in \Sigma(x_0) : \overline{w}(\underset{\sim}{z}) > -\infty\}$.

By definition of B and since v is a fixed point of B, it is obtained that

$$\text{(F.1)} \quad \forall x_1 \in \Gamma(x_0), \forall z_0 \in \Omega(x_0, x_1), v(x_0) = B(v)(x_0) \ge A(z_0, v(x_1)).$$

Similarly,

$$\text{(F.2)} \quad \forall x_2 \in \Gamma(x_1), \forall z_1 \in \Omega(x_1, x_2), v(x_1) = B(v)(x_1) \ge A(z_1, v(x_2)).$$

Consequently, from Increasing Monotonicity Assumption, reinjecting Equation F.2 into Equation F.1, it derives that, for every $x_1 \in \Gamma(x_0)$, for every $x_2 \in \Gamma(x_1)$, for every $z_0 \in \Omega(x_0, x_1)$ and for every $z_1 \in \Omega(x_1, x_2)$,

$$\text{(F.3)} \quad B(v)(x_0) \ge A(z_0, A(z_1, v(x_2))).$$

Iterating Equation F.3, it is similarly obtained that for every $T \in \mathbf{N}^*$ for every $(\underset{\sim}{z}, \underset{\sim}{x}) \in \widetilde{\Sigma}(x_0)$,

$$\text{(F.4)} \quad B(v)(x_0) \ge A^T(\underset{\sim}{z}, v(x_{T+1})).$$

From $v \geq \underline{v}$ and from Increasing Monotonicity Assumption, Equation F.4 gives

(F.5) $\quad B(v)(x_0) \geq A^T(\underset{\sim}{z}, \underline{v}(x_{T+1}))$

If this last quantity is equal to $+\infty$ for at least one integer T, then $v(x_0) = B(v)(x_0) = +\infty \geq v^*(x_0)$, and the proof of Step 1 is over.

If not, $A^T(\underset{\sim}{z}, \underline{v}(x_{T+1})) < +\infty$ for every integer T. Since $\underline{v}$ satisfies Transversality assumption (T1), this implies $A^T(\underset{\sim}{z}, 0) < +\infty$ for every integer T.

To prove that $B(v)(x_0) \geq v^*(x_0)$ still holds, one will establish that $B(v)(x_0) \geq w(\underset{\sim}{z})$. If $w(\underset{\sim}{z}) = -\infty$, then $B(v)(x_0) \geq w(\underset{\sim}{z})$ is clear. Otherwise, $w(\underset{\sim}{z}) = -\infty$. Then $A^T(\underset{\sim}{z}, 0) > -\infty$ (thus $A^T(\underset{\sim}{z}, 0)$ is real) for an infinite number of T. From the previous equation and for an infinite number of T:

(F.6) $\quad B(v)(x_0) \geq (A^T(\underset{\sim}{z}, \underline{v}(x_{T+1})) - A^T(\underset{\sim}{z}, 0)) + A^T(\underset{\sim}{z}, 0)$

Taking the supremum limit in the above inequality, Transversality Assumption (T) implies, for every $\underset{\sim}{z} \in \Sigma^0(x_0)$

(F.7) $\quad B(v)(x_0) \geq \underline{\lim}_{T\to\infty}(A^T(\underset{\sim}{z}, \underline{v}(x_{T+1}) - A^T(\underset{\sim}{z}, 0)) + \overline{\lim}_{T\to\infty} A^T(\underset{\sim}{z}, 0) = w(\underset{\sim}{z})$

Thus, $B(v)(x_0) \geq w(\underset{\sim}{z})$ is always true.

Now, taking the sup when $\underset{\sim}{z}$ varies in $\Sigma^0(x_0)$, it is finally obtained that:

(F.8) $\quad v(x_0) = B(v)(x_0) \geq \overline{v}^*(x_0)$.

Step 2. The aim is now to prove that $v \leq \underline{v}^*$.

Fix x_0 in X. For $v(x_0) = -\infty$, then $v(x_0) \leq \underline{v}^*(x_0)$ is true. Thus, since $v(x_0) \leq \overline{v}(x_0) < +\infty$, the case $v(x_0) \in \mathbf{R}$ is now to be considered.

For every integer n, let $\varepsilon_n > 0$. From the definition of $B(v)$, there exists $x_1 \in \Gamma(x_0)$ and $z_0 \in \Omega(x_0, x_1)$ (depending on ε_1) such that

(F.9) $\quad v(x_0) = B(v)(x_0) \leq A(z_0, v(x_1)) + \varepsilon_1/2$

where $v(x_1) \leq \overline{v}(x_1) < +\infty$ by assumption. Thus, similarly, there exists $x_2 \in \Gamma(x_1)$ and $z_1 \in \Omega(x_1, x_2)$ (depending on ε_1 and ε_2) such that

(F.10) $\quad v(x_1) \leq A(z_1, v(x_2)) + \varepsilon_2/2.$

Reinjecting Equation F.10 into Equation F.9, and from the Increasing Monotonicity Assumption, it derives that:

$$v(x_0) \leq A(z_0, A(z_1, v(x_2)) + \varepsilon_2/2) + \varepsilon_1/2$$

By induction, for every integer n, one builds $(x_n)_{n\geq 0}$ in X and $(z_n)_{n\geq 0}$ in Z such that

(F.11) $\quad \forall i > 0, x_{i+1} \in \Gamma(x_i), z_i \in \Omega(x_i, x_{i+1}),$

and such that for every integer n,

(F.12)
$$v(x_0) \le A(z_0, A(z_1, A(z_2, \dots, A(z_{n-1}, \overline{v}(x_n)) + \varepsilon_n) + \varepsilon_{n-1}/2 + \cdots) + \varepsilon_2/2) + \varepsilon_1/2.$$

where use has been made of $v \le \overline{v} < +\infty$ and of the Increasing Monotonicity Assumption for the last inequalities. Passing to the infimum with respect to n and taking then the supremum with respect to $\underset{\sim}{z}$ and $\underset{\sim}{x}$, it is obtained that:

(F.13)
$$v(x_0) \le \sup_{(\underset{\sim}{z},\underset{\sim}{x}) \in \widetilde{\Sigma}(x_0)} \inf_{n \ge 0} A(z_0, A(z_1, A(z_2, \dots, A(z_{n-1}, \overline{v}(x_n)) + \varepsilon_n) + \varepsilon_{n-1}/2 + \cdots) + \varepsilon_2/2) + \varepsilon_1/2.$$

Now, from Weak Continuity assumption, the function $f : [0,1]^{\mathbf{N}} \to \mathbf{R}$ define by

$$f(\varepsilon) = \sup_{(\underset{\sim}{z},\underset{\sim}{x}) \in \widetilde{\Sigma}(x_0)} \inf_{n \ge 0} A(z_0, A(z_1, \dots, A(z_{n-1}, \overline{v}(x_n)) + \varepsilon_n) + \varepsilon_{n-1}/2 + \cdots) + \varepsilon_2/2) + \varepsilon_1/2$$

is upper semicontinuous, $[0,1]^{\mathbf{N}}$ being endowed with a good metric.

Thus, passing to the limit when ε tends to 0 in Equation F.13, it derives that:

(F.14)
$$v(x_0) \le \sup_{(\underset{\sim}{z},\underset{\sim}{x}) \in \widetilde{\Sigma}(x_0)} \inf_{n \ge 0} A(z_0, A(z_1, A(z_2, \dots, A(z_{n-1}, \overline{v}(x_n))) + \cdots)).$$

Fix $\varepsilon > 0$. From Equation F.14, there exists $(\underset{\sim}{z}, \underset{\sim}{x}) \in \widetilde{\Sigma}(x_0)$ such that for every integer $n \ge 1$,

(F.15) $v(x_0) \le A^n(\underset{\sim}{z}, \overline{v}(x_n)) + \varepsilon$.

Now, $v(x_0) > -\infty$ implies $A^n(\underset{\sim}{z}, \overline{v}(x_n)) > -\infty$ for every $n \ge 1$, and from Transversality Assumption (T2), one obtains $A^n(\underset{\sim}{z}, 0) > -\infty$ for n large enough.

Also assuming that $A^n(\underset{\sim}{z}, 0) < +\infty$ for an infinite number of n, otherwise $\underline{v}^*(x_0) = +\infty$ and the inequality $v(x_0) \le \underline{v}^*(x_0)$ is proved. Thus $A^n(\underset{\sim}{z}, 0) \in \mathbf{R}$ for an infinite number of n. Equation F.15 can thus be reformulated along:

(F.16) $v(x_0) \le (A^n(\underset{\sim}{z}, \overline{v}(x_n)) - A^n(\underset{\sim}{z}, 0))) + A^n(\underset{\sim}{z}, 0) + \varepsilon$.

for every n such that $A^n(\underset{\sim}{z}, 0)$ is finite. This implies $v(x_0) \le \underline{v}^*(x_0)$, taking the infimum limit with respect to n and from Transversality Assumption ($T2$), then taking supremum with respect to $\underset{\sim}{z} \in \Sigma(x_0)$, and finally taking the limit when $\varepsilon \to 0$.

Step 3. Let us prove the second Assertion in Theorem 4.2. Let $f \in V$ such that $f \le v$ and f satisfies (T1). Prove that $v = \overline{\lim}_{n \to +\infty} B^n(f)$. Recall that $v = \underline{v}^* = \overline{v}^*$.

Since $f \leq v$, composing this inequality n times by B, which is non-decreasing, one derives that $B^n(f) \leq v$ (v being a fixed point of B). Taking the supremum limit and for every $x_0 \in X$,

(F.17) $\overline{\lim}_{n \to +\infty} B^n(f)(x_0) \leq v(x_0)$

Prove then that $\overline{\lim}_{n \to +\infty} B^n(f)(x_0) \geq v(x_0)$ for every $x_0 \in X$. From the definition of the Bellman Operator B and given some $x_0 \in X$:

(F.18) $\forall x_1 \in \Gamma(x_0), \forall z_0 \in \Omega(x_0, x_1),\ B^2(f)(x_0) \geq A(z_0, B(f)(x_1))$.

Similarly, one obtains

(F.19) $\forall x_2 \in \Gamma(x_1), \forall z_1 \in \Omega(x_1, x_2),\ B(f)(x_1) \geq A(z_1, f(x_2))$.

Consequently, from Increasing Monotonicity Assumption, reinjecting Inequation F.19 into Inequation F.18, it is obtained that, for every $x_1 \in \Gamma(x_0)$, for every $x_2 \in \Gamma(x_1)$, for every $z_0 \in \Omega(x_0, x_1)$, for every $z_1 \in \Omega(x_1, x_2)$,

(F.20) $\quad B^2(f)(x_0) \geq A(z_0, A(z_1, f(x_2)))$.

From an immediate induction, it is similarly obtained that:

(F.21) $\forall n \in \mathbf{N}^*, \forall \underset{\sim}{x} \in \Pi(x_0), \forall \underset{\sim}{z} \in \Sigma(x_0),\ B^n(f)(x_0) \geq A^n(\underset{\sim}{z}, f(x_n))$.

If $A^n(\underset{\sim}{z}, f(x_n)) = +\infty$ for a infinite number of n, then passing to the supremum limit, it derives that $\overline{\lim}_{n \to +\infty} B^n(f)(x_0) = +\infty \geq v(x_0)$. Hence assuming now that $A^n(\underset{\sim}{z}, f(x_n)) < +\infty$ for n large enough, and since f satisfies Transversality assumption (T1), it is obtained that $A^n(\underset{\sim}{z}, 0) < +\infty$ for n large enough.

Now, if $v(x_0) = -\infty$, $\overline{\lim}_{n \to +\infty} B^n(f) \geq v(x_0) = -\infty$ is true. Assuming now that $v(x_0) > -\infty$. Since $v(x_0) = \overline{v}^*(x_0) > -\infty$, it derives that $A^n(\underset{\sim}{z}, 0) > -\infty$ for an infinite number of integer n.

Consequently, for every integer n such that $A^n(\underset{\sim}{z}, 0) > -\infty$:

$$B^n(f)(x_0) \geq (A^n(\underset{\sim}{z}, f(x_n)) - A^n(\underset{\sim}{z}, 0))) + A^n(\underset{\sim}{z}, 0).$$

and, taking then the supremum limit and then the supremum for $\underset{\sim}{z} \in \Sigma^0(x_0)$, this finally gives $\overline{\lim}_{n \to +\infty} B^n(f)(x_0) \geq \overline{v}^*(x_0) = v(x_0)$ from (T1) assumption satisfied by f.

Step 4. To prove the last point of Theorem 4.2, assume that $\underline{v}$ and $\overline{v}$ above also satisfy $\underline{v} \leq B(\underline{v})$ and $B(\overline{v}) \leq \overline{v}$, then, from Tarski fixed point theorem on $[\underline{v}, \overline{v}]$, B admits a fixed-point on $[\underline{v}, \overline{v}]$, and from the first part of Theorem 4.2 (i), this fixed-point is equal to the value function.

QED

G. Proof of Corollary 4.2.3

Proof. First prove the following Lemma:

Lemma G.1.— *Assume Assumption (Null) and biconvergence:*

(i) *If A is upper semicontinuous, then* $\underline{v}(x_0) = \inf_{\underset{\sim}{y} \in \prod_{t=0}^{+\infty} \Gamma^t(x_0)} U(\underset{\sim}{y})$ *satisfies* (T_1) *in Transversality Assumption.*

(ii) *If A is lower semicontinuous, then* $\overline{v}(x_0) = \sup_{\underset{\sim}{y} \in \prod_{t=0}^{+\infty} \Gamma^t(x_0)} U(\underset{\sim}{y})$ *satisfies* (T_2) *in Transversality Assumption.*

(iii) *For any aggregator A, then* $\underline{v}(x_0) = \min_{\underset{\sim}{y} \in \prod_{t=0}^{+\infty} \Gamma^t(x_0)} U(\underset{\sim}{y})$ *satisfies* (T_1) *and* $\overline{v}(x_0) = \max_{\underset{\sim}{y} \in \prod_{t=0}^{+\infty} \Gamma^t(x_0)} U(\underset{\sim}{y})$ *satisfies* (T_2)*, whenever the max and min above are reached for every* x_0*.*

Use will be made of the following Claim:

Claim. (i) If $f : \overline{\mathbf{R}} \to \overline{\mathbf{R}}$ is an upper semicontinuous real-valued function and g a real-valued function on a metric compact space M, it derives that

$$f(\inf_{x \in M} g(x)) \geq \inf_{x \in M} f(g(x)).$$

(ii) If $f : \overline{\mathbf{R}} \to \overline{\mathbf{R}}$ is a lower semicontinuous real-valued function and g a real-valued function on a metric compact space M, then

$$f\Big(\sup_{x \in M} g(x)\Big) \leq \sup_{x \in M} f(g(x)).$$

Proof. By definition, $\inf_{x \in M} g(x) = \lim_{n \to +\infty} g(x_n)$ for some sequence (x_n) of M. Without any loss of generality, since M is compact, one can assume that (x_n) converges to some $x \in M$. The function f being upper semicontinuous,

$$f(\inf_{x \in M} g(x)) = f(\lim_{n \to +\infty} g(x_n)) \geq \lim_{n \to +\infty} f(g(x_n)) \geq \inf_{x \in M} f(g(x)).$$

The proof is similar for ii). This ends the proof of the claim. QED

Now, to prove i) of Lemma G.1, assume A upper semicontinuous, and let $\underset{\sim}{x} \in X^{\mathbf{N}}$. By definition of $\underline{v}$,

$$A^T(\underset{\sim}{x}, \underline{v}(x_T)) - A^T(\underset{\sim}{x}, 0) = A^T(\underset{\sim}{x}, \inf_{\underset{\sim}{y} \in \prod_{t=0}^{+\infty} \Gamma^t(x_T)} U(\underset{\sim}{y})) - A^T(\underset{\sim}{x}, 0).$$

Since A is upper semicontinuous, using Claim above, this is larger or equal to

$$\inf_{\underset{\sim}{y} \in \prod_{t=0}^{+\infty} \Gamma^t(x_T)} A^T(\underset{\sim}{x}, U(\underset{\sim}{y})) - A^T(\underset{\sim}{x}, 0),$$

also equal to

$$\inf_{\underset{\sim}{y} \in \prod_{t=0}^{+\infty} \Gamma^t(x_T)} A^T(\underset{\sim}{x}, U(\underset{\sim}{y})) - A^T(\underset{\sim}{x}, U(\underline{a}, \underline{a}, ...)),$$

where the existence of $\underline{a} \in \cap_{x_0 \in X} \Gamma(x_0)$ is given by Null Assumption.

By recursivity, this is also equal to

$$\inf_{\underset{\sim}{y} \in \prod_{t=0}^{+\infty} \Gamma^t(x_T)} U(x_0, \ldots, x_{T-1}, \underset{\sim}{y}) - U(x_0, \ldots, x_{T-1}, \underline{a}, \underline{a}, \ldots)$$

This last quantity can be made as small as one wishes for T large enough (by biconvergence), which proves that $\underline{v}$ satisfies (T_1).

The proof of 2)ii) is symetrical.

Last, to prove for example that if

$$\underline{v}(x_0) = \min_{\underset{\sim}{y} \in \prod_{t=0}^{+\infty} \Gamma^t(x_0)} U(\underset{\sim}{y})$$

for every x_0 (note that now, A may not be u.s.c.), then $\underline{v}$ satisfies (T_1), follow the proof above. Simply note that $\underline{v}(x_T)$ can be written $U(\underset{\sim}{y})$ for some $\underset{\sim}{y}$) (depending on T), and the above proof can be followed without using any infimum. This ends the proof of Lemma G.1.

Proving now Proposition 4.2.3, define, for every $x_0 \in X$,

$$\overline{v}(x_0) = \max U\Big(\prod_{t=1}^{+\infty} \Gamma^t(x_0)\Big)$$

and

$$\underline{v}(x_0) = \inf U\Big(\prod_{t=1}^{+\infty} \Gamma^t(x_0)\Big).$$

Checking first that one can apply Theorem 4.2 to prove that v^* is the unique solution of Bellman equation.

First, $\underline{v} \leq \overline{v} < +\infty$ and clearly, for every $\underset{\sim}{x} \in \Sigma(x_0), U(\underset{\sim}{x}) \in [\underline{v}(x_0), \overline{v}(x_0)]$. Moreover, from Lemma G.1, $(\underline{v}, \overline{v})$ satisfies Transversality Assumption. One can thus apply Proposition 3.3, which gives $\overline{v}^*(x_0) = \underline{v}^*(x_0) = \sup_{\underset{\sim}{x} \in \Sigma(x_0)} U(\underset{\sim}{x})$, simply called $v^*(x_0)$ hereafter.

Secondly proving that $B(\overline{v}) \leq \overline{v}$, and $B(\underline{v}) \geq \underline{v}$.

To show that $B(\overline{v}) \leq \overline{v}$, let $x_0 \in X$.

$$\begin{aligned}
B(\overline{v})(x_0) &= \sup_{x_1 \in \Gamma(x_0)} A(x_1, \overline{v}(x_1)) \\
&= \sup_{x_1 \in \Gamma(x_0)} A\Big(x_1, \max U\Big(\prod_{t=1}^{+\infty} \Gamma^t(x_1)\Big)\Big) \\
&= \sup_{x_1 \in \Gamma(x_0)} \sup_{x_{t+1} \in \Gamma^t(x_1), \forall t \geq 1} A(x_1, U(x_2, x_3, \ldots, x_n, \ldots)) \\
&= \sup_{x_1 \in \Gamma(x_0)} \sup_{x_{t+1} \in \Gamma^t(x_1), \forall t \geq 1} U(x_1, x_2, \ldots, x_n, \ldots) \\
&\leq \sup_{x_t \in \Gamma^t(x_0), \forall t \geq 1} U(x_1, x_2, \ldots, x_n, \ldots) \\
&= \overline{v}(x_0)
\end{aligned}$$

Showing now that $B(\underline{v}) \geq \underline{v}$.

$$\begin{aligned}
B(\underline{v})(x_0) &= \sup_{x_1 \in \Gamma(x_0)} A(x_1, \underline{v}(x_1)) \\
&= \sup_{x_1 \in \Gamma(x_0)} A\Big(x_1, \inf U\Big(\prod_{t=1}^{+\infty} \Gamma^t(x_1)\Big)\Big) \\
&\geq \sup_{x_1 \in \Gamma(x_0)} \inf_{(x_2, \ldots) \in \prod_{t=1}^{+\infty} \Gamma^t(x_1)} A(x_1, U(x_2, x_3, \ldots)) \\
&\geq \sup_{x_1 \in \Gamma(x_0)} \inf_{(x_2, \ldots) \in \prod_{t=1}^{+\infty} \Gamma^t(x_1)} U(x_1, x_2, x_3, \ldots) \\
&\geq \underline{v}(x_0)
\end{aligned}$$

The first inequality being a consequence of the claim above. Lastly, one is only to prove that A is weakly continuous at $\overline{v}$, i.e. that

$$f(\varepsilon) = \sup_{(\underset{\sim}{z},\underset{\sim}{x})\in\widetilde{\Sigma}(x_0)} \inf_{n\geq 0} A(z_0, A(z_1, A(z_2, \dots \\ \dots, A(z_{n-1}, \overline{v}(x_n)) + \varepsilon_{n-1}) + \varepsilon_{n-2} + \cdots) + \varepsilon_2) + \varepsilon_1),$$

is upper semicontinuous (here the topology on $[0,1]^{\mathbf{N}}$ is the standard product topology). Remark that for $n > 1$ fixed, the mapping which associates

$$A(z_0, A(z_1, A(z_2, \dots \\ \dots, A(z_{n-1}, \overline{v}(x_n)) + \varepsilon_{n-1}) + \varepsilon_{n-2} + \cdots) + \varepsilon_2) + \varepsilon_1)$$

to every $(n-1)-$uple $(\varepsilon_1, ..., \varepsilon_{n-1}) \in \mathbf{R}^{n-1}$ is upper-semicontinuous, from upper semicontinuity of A, from upper semicontinuity of $\overline{v}$ (a consequence of Berge theorem), and from Increasing Monotonicity Assumption. Thus, the function which associates

$$A(z_0, A(z_1, A(z_2, \dots \\ \dots, A(z_{n-1}, \overline{v}(x_n)) + \varepsilon_{n-1}) + \varepsilon_{n-2} + \cdots) + \varepsilon_2) + \varepsilon_1)$$

to every sequence $(\varepsilon_k)_{k\geq 1}$ is also upper semicontinuous (by definition of the product topology chosen on $[0,1]^{\mathbf{N}}$). Passing to the infimum with respect to n, one obtains an upper-semicontinuous function. Then from Berge theorem, the feasibility contraint having a closed and compact graph, it is finally proved that f is upper-semicontinuous at 0 (in fact everywwhere).

This concludes the argument of the the proof. QED

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