On the regularity of smooth production economies with externalities: Competitive equilibrium à la Nash

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Abstract

We consider a general equilibrium model of a private ownership economy with consumption and production externalities. The choices of all agents (households and firms) may affect utility functions and production technologies. The allocation of a competitive equilibrium is a Nash equilibrium. We provide an example showing that, under standard assumptions, competitive equilibria are indeterminate in an open set of the household's endowments. We introduce the model with firms' endowments, in the spirit of Geanakoplos, Magill, Quinzii and Drèze (1990). In our model, firms' endowments impact the technologies and the marginal productivities of the other firms. We then prove that these economies are generically regular in the space of endowments of households and firms.

JEL classification: C62, D51, D62.

Key words: externalities, private ownership economies, competitive equilibrium *à la Nash*, regular economies.

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1 Introduction

The Arrow–Debreu model of general equilibrium has been extended to economies with consumption and production externalities. For such extensions, one has to choose an equilibrium notion. From a normative point of view, markets can be extended in order to obtain the two Fundamental Theorems of Welfare Economics also for such economies, i.e., a *perfect internalization* of the externalities. This is the idea sketched by Arrow (1969) and first analyzed by Laffont (1976). They enlarge the choice sets of the agents and introduce personalized prices that agents face on markets for rights on the consumption and production of any other agent in the economy. Recent contributions by Magill, Quinzii and Rochet (2015), and Crès and Tvede (2013), explore instead corporate governance policies that induce firms to internalize the externalities by maximizing some "social criterion" in order to increase the welfare of stakeholders or shareholders, depending on their respective contribution. Their analysis then focuses on legal systems that allow these goals to be achieved.

On the other hand, the positive theory of competitive equilibrium leads to a definition of equilibrium that combines Arrow–Debreu with Nash, that is, agents (households and firms) maximize their goals by taking as given both the commodity prices and the choices of every other agent in the economy. At equilibrium, agents' optimal choices are mutually consistent and markets clear. This is the notion given in Arrow and Hahn (1971), and Laffont (1988). This notion includes as a special case the classical equilibrium definition without externalities. With such an equilibrium notion, agents cannot choose the consumption and production of other agents, i.e., externalities cannot be internalized, and competitive markets may prevent equilibrium allocations from being Pareto optimal.

We consider a private ownership economy with a finite number of commodities, households and firms. Utility and transformation functions may be affected by the consumption and production activities of all other agents. We take the conventional non-cooperative view of market equilibrium. Our purpose is to provide the genericity of regular economies. We recall that an economy is regular if it has a finite (odd) number of equilibria and every equilibrium locally depends in a differentiable manner on the parameters describing the economy. Therefore, the equilibria of a regular economy are locally unique and persistent under small perturbations of the economy. Furthermore, at a

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regular economy, it is possible to perform classical comparative statics, see Smale (1981), Mas-Colell (1985), and Balasko (1988). Quite a few works deal with the generic regularity of economies with externalities, see Crès (1996), Bonnisseau (2003), Kung (2008), Mandel (2008), Bonnisseau and del Mercato (2010), and Balasko (2015). However, with the exception of Mandel (2008), all these authors focus on economies with externalities only on the consumer side. [2]

Apart from the intrinsic interest in a regularity result, regular economies are also important for the study of Pareto improving policies in terms of taxes and subsidies. In the presence of other sources of market failures, such as incomplete financial markets and public goods, there is a well established methodology for analyzing these policies, see Geanakoplos and Polemarchakis (1986), Geanakoplos, Magill, Quinzii and Drèze (1990), Citanna, Kajii and Villanacci (1998), Citanna, Polemarchakis and Tirelli (2006), Villanacci and Zenginobuz (2006, 2012). Such a methodology applies to the set of regular economies, since it requires equilibria to be differentiable maps of the fundamentals. Therefore, our contribution provides a solid foundation for the analysis of these kinds of policies in the presence of externalities, in the spirit of Greenwald and Stiglitz (1986) and Geanakoplos and Polemarchakis (2008). [3]

We make basic assumptions on utility and transformation functions that are standard in "smooth" equilibrium models without externalities. These assumptions guarantee the non-emptiness and the compactness of the set of equilibria. [4] However, even in the simpler case of consumption externalities, they are not sufficient to establish classical generic regularity. For economies with consumption externalities, establishing generic regularity requires the introduction of an additional assumption on the second order effects of externalities on individual utility functions, see Bonnisseau and del Mercato

[2] Mandel (2008) makes use of the classical regularity result by assuming that the demands and supplies that are mutually consistent are differentiable and their differential is onto. Actually, the purpose of his paper is not the study of regularity, but merely the existence of an equilibrium via the degree theory for correspondences.

[3] The Pareto improving analysis of Greenwald and Stiglitz (1986) mainly focuses on economies with incomplete markets and imperfect information. However, in Section I, the authors consider a general equilibrium model of a private ownership economy with consumption and production externalities. They assume that the aggregate excess demand-supply function is differentiable and its differential is onto. Furthermore, the issue of the existence of such Pareto improving policies is not addressed for the general model provided in Section I. In Geanakoplos and Polemarchakis (2008), the Pareto improving analysis deals with economies with consumption externalities only.

[4] Under these assumptions, the set of competitive equilibria is non-empty and compact. The existence of a competitive equilibrium is demonstrated by Arrow and Hahn (Chapter 6, 1971), Laffont (1977), and del Mercato and Platino (2015).

(2010).[5] However, the analogous assumption on the effects of production externalities on transformation functions is not going to work. We provide an example of a private ownership economy with one household and two firms where, despite well behaved second order external effects, equilibria are indeterminate in an open set of the household's endowments (and the indeterminacy is payoff relevant).

In order to overcome indeterminacy, we incorporate firms' endowments in the general model, in the spirit of Geanakoplos, Magill, Quinzii and Drèze (1990).[6] Firms' endowments consist of amounts of commodities held initially by the firms. In our model, firms' endowments have an impact on the production sets of the other firms. Consequently, perturbing these endowments affects the first and the second order effects of production externalities on transformation functions, thereby allowing us to establish generic regularity.[7] Our main theorem (Theorem 16) states that competitive equilibria are determinate in an open and full measure subset of the endowments of households and firms. Most of the classical properties of the equilibrium manifold then hold true.

In order to prove Theorem 16, we adapt *Smale's approach* to economies with consumption and production externalities. Smale's approach has been used for other economic environments, see for instance Crès (1996), Cass, Siconolfi and Villanacci (2001), Villanacci and Zenginobuz (2005), and Bonnisseau and del Mercato (2010). This approach is an alternative to the aggregate excess demand-excess supply approach. Notice that in our economic environment, the aggregate excess demand-excess supply approach is problematic. This is because the individual demands and supplies are interdependent, making it difficult to define the aggregate excess demand-excess supply out of equilibrium.

The paper is organized as follows. In Section 2, we introduce the classical model, basic assumptions and definitions. In Subsection 2.3, we discuss the difficulties of establishing classical generic regularity. In Section 3, we provide our example of indeterminacy and we analyze the second order external effects in the example. Section 4 is devoted to our regularity result. We present our model with firms' endowments and we prove that these economies are regular in an open and full measure subset of the endowments of households and firms. All the lemmas are proved in Section 5. In Section 6, one finds classical results

[5] In the absence of this assumption, these authors provide an example where equilibria are indeterminate for all initial endowments.

[6] In Geanakoplos, Magill, Quinzii and Drèze (1990), there are no direct externalities in preferences and production sets. However, their model exhibits *pecuniary* externalities arising from the incompleteness of financial markets.

[7] We do not rely on logarithmic or quadratic perturbations of the payoff functions that are commonly used to establish generic regularity in non-cooperative games, see for instance Harsanyi (1973), Ritzberger (1994), and van Damme (2002).

from differential topology used in our analysis and the Jacobian matrix of the equilibrium function.

2 The general model

We consider a private ownership economy. There is a finite number of commodities labeled by the superscript $c \in \mathcal{C} := \{1, \ldots, C\}$. The commodity space is $\mathbb{R}^C$. There are a finite number of firms labeled by the subscript $j \in \mathcal{J} := \{1, \ldots, J\}$ and a finite number of households labeled by the subscript $h \in \mathcal{H} := \{1, \ldots, H\}$.

The production plan of firm j is $y_j := (y_j^1, .., y_j^c, .., y_j^C)$. As usual, if $y_j^c > 0$ then commodity c is produced as an output, if $y_j^\ell < 0$ then commodity ℓ is used as an input. The production plan of firms other than j is $y_{-j} := (y_f)_{f \neq j}$, and let $y := (y_j)_{j \in \mathcal{J}}$. The consumption of household h is $x_h := (x_h^1, .., x_h^c, .., x_h^C)$. The consumption of households other than h is $x_{-h} := (x_k)_{k \neq h}$, and let $x := (x_h)_{h \in \mathcal{H}}$.

The production set of firm j is described by a transformation function t_j.[8] The main innovation of this paper is that the transformation function t_j may depend on the production and consumption activities of all other agents. That is, t_j describes both the technology of firm j and the way in which its technology is affected by the activities of the other agents. More precisely, for a given externality (y_{-j}, x), the production set of the firm j is

$$Y_j(y_{-j}, x) := \left\{ y_j \in \mathbb{R}^C \colon t_j(y_j, y_{-j}, x) \leq 0 \right\}$$

where t_j is a function from $\mathbb{R}^C \times \mathbb{R}^{C(J-1)} \times \mathbb{R}^{CH}_{++}$ to $\mathbb{R}$. Let $t := (t_j)_{j \in \mathcal{J}}$.

The preferences of household h are described by a utility function,

$$u_h : (x_h, x_{-h}, y) \in \mathbb{R}^C_{++} \times \mathbb{R}^{C(H-1)}_{+} \times \mathbb{R}^{CJ} \longrightarrow u_h(x_h, x_{-h}, y) \in \mathbb{R}$$

$u_h(x_h, x_{-h}, y)$ is the utility level of household h associated with (x_h, x_{-h}, y). That is, u_h also describes the way in which the preferences of household h are affected by the activities of the other agents. Let $u := (u_h)_{h \in \mathcal{H}}$.

[8] This is a convenient way to represent a production set using an inequality on a function called the *transformation function*. In the case of a single-output technology, the production set is commonly described by a *production function*. The transformation function is the counterpart of the production function in the case of production processes which involve several outputs, see for instance Mas-Colell et al. (1995) and Villanacci et al. (2002).

The endowment of household h is $e_h := (e_h^1, .., e_h^c, .., e_h^C) \in \mathbb{R}_{++}^C$, and let $e := (e_h)_{h \in \mathcal{H}}$. The total resources are $r := \sum_{h \in \mathcal{H}} e_h \in \mathbb{R}_{++}^C$. The share of firm j owned by household h is $s_{jh} \in [0,1]$, and let $s := (s_{jh})_{j \in \mathcal{J}, h \in \mathcal{H}}$. As usual, $\sum_{h \in \mathcal{H}} s_{jh} = 1$ for every firm $j \in \mathcal{J}$. A private ownership economy is $E := ((u, e, s), t)$.

The price of one unit of commodity c is $p^c \in \mathbb{R}_{++}$, and let $p := (p^1, .., p^c, .., p^C)$. Given $w = (w^1, .., w^c, .., w^C) \in \mathbb{R}^C$, denote $w^\setminus := (w^1, .., w^c, .., w^{C-1}) \in \mathbb{R}^{C-1}$.

2.1 Basic assumptions

In this subsection, we make the following set of assumptions to establish the non-emptiness and the compactness of the equilibrium set.

Assumption 1 *For all $j \in \mathcal{J}$,*

(1) The function t_j is a C^2 function.
(2) For every $(y_{-j}, x) \in \mathbb{R}^{C(J-1)} \times \mathbb{R}_{++}^{CH}$, $t_j(0, y_{-j}, x) = 0$.
(3) For every $(y_{-j}, x) \in \mathbb{R}^{C(J-1)} \times \mathbb{R}_{++}^{CH}$, the function $t_j(\cdot, y_{-j}, x)$ is differentiably strictly quasi-convex, i.e., for all $y'_j \in \mathbb{R}^C$, $D^2_{y_j} t_j(y'_j, y_{-j}, x)$ is positive definite on $\operatorname{Ker} D_{y_j} t_j(y'_j, y_{-j}, x)$. [9]
(4) For every $(y_{-j}, x) \in \mathbb{R}^{C(J-1)} \times \mathbb{R}_{++}^{CH}$, $D_{y_j} t_j(y'_j, y_{-j}, x) \gg 0$ for all $y'_j \in \mathbb{R}^C$.

Fixing the externalities, the assumptions on t_j are standard in smooth general equilibrium models. Points 1 and 4 of Assumption 1 imply that the production set is a C^2 manifold of dimension C and its boundary is a C^2 manifold of dimension $C - 1$. Point 2 of Assumption 1 states that inaction is possible. Consequently, using standard arguments from profit maximization, the wealth of household h associated with his endowment $e_h \in \mathbb{R}_{++}^C$ and his profit shares is strictly positive for every price $p \in \mathbb{R}_{++}^C$. Thus, one deduces the non-emptiness of the interior of the individual budget constraint. Point 3 of Assumption 1 implies that the production set is strictly convex. Furthermore, if the profit maximization problem has a solution then it is unique, because the function $t_j(\cdot, y_{-j}, x)$ is continuous and strictly quasi-convex. We remark that t_j is not required to be quasi-convex with respect to all the variables, i.e., we do not require the production set to be convex with respect to the externalities. Point

[9] Let v and v' be two vectors in $\mathbb{R}^n$, $v \cdot v'$ denotes the *scalar product* of v and v'. Let A be a real matrix with m rows and n columns, and B be a real matrix with n rows and l columns, AB denotes the *matrix product* of A and B. Without loss of generality, vectors are treated as row matrices and A denotes both the matrix and the following linear mapping $A : v \in \mathbb{R}^n \to A(v) := Av^T \in \mathbb{R}^{[m]}$ where v^T denotes the transpose of v and $\mathbb{R}^{[m]} := \{w^T : w \in \mathbb{R}^m\}$. When $m = 1$, $A(v)$ coincides with the scalar product $A \cdot v$, treating A and v as vectors in $\mathbb{R}^n$.

4 of Assumption 1 implies that the function $t_j(\cdot, y_{-j}, x)$ is strictly increasing and so the production set satisfies the classical "free disposal" property.

Remark 2 *Our analysis holds true if some commodities are not involved in the technological process of firm j. In this case, for every firm j, one defines the set $\mathcal{C}_j$ of all the commodities $c \in \mathcal{C}$ that are involved in the technological process of firm j, where C_j denotes the cardinality of the set $\mathcal{C}_j$ with $2 \leq C_j \leq C$. The production plan of firm j is then defined as $y_j := (y_j^c)_{c \in \mathcal{C}_j} \in \mathbb{R}^{C_j}$ and the transformation function t_j is a function from $\mathbb{R}^{C_j} \times \prod_{f \neq j} \mathbb{R}^{C_f} \times \mathbb{R}^{CH}_{++}$ to $\mathbb{R}$. In this case, all the assumptions on the transformation functions are written just replacing $\mathbb{R}^C \times \mathbb{R}^{C(J-1)} \times \mathbb{R}^{CH}_{++}$ by $\mathbb{R}^{C_j} \times \prod_{f \neq j} \mathbb{R}^{C_f} \times \mathbb{R}^{CH}_{++}$. Furthermore, in the definition of a competitive equilibrium, one also adapts the market clearing condition for every commodity c by considering only firms that use commodity c in their technological process. That is, for every commodity c, the sum over $j \in \mathcal{J}$ is replaced by the sum over $j \in \mathcal{J}(c) := \{j \in \mathcal{J} \colon c \in \mathcal{C}_j\}$.*

Given $(x, y) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$, the set of the production plans that are consistent with the externality (x, y) is defined by

$$Y(x, y) := \{y' \in \mathbb{R}^{CJ} \colon t_j(y'_j, y_{-j}, x) \leq 0, \ \forall\, j \in \mathcal{J}\} \tag{1}$$

The next assumption provides a boundedness condition on all the sets of feasible production plans that are consistent with the externalities.[10]

Assumption 3 (Uniform Boundedness) *Given $r \in \mathbb{R}^C_{++}$, there exists a bounded set $C(r) \subseteq \mathbb{R}^{CJ}$ such that for every $(x, y) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$,*

$$Y(x, y) \cap \{y' \in \mathbb{R}^{CJ} \colon \sum_{j \in \mathcal{J}} y'_j + r \gg 0\} \subseteq C(r)$$

The following lemma is an immediate consequence of Assumption 3.

Lemma 4 *Given $r \in \mathbb{R}^C_{++}$, there exists a bounded set $K(r) \subseteq \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$ such that for every $(x, y) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$, the following set is included in $K(r)$.*

$$A(x, y; r) := \{(x', y') \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ} \colon y' \in Y(x, y) \text{ and } \sum_{h \in \mathcal{H}} x'_h - \sum_{j \in \mathcal{J}} y'_j \leq r\}$$

[10] Assumption 3 is analogous to several conditions used to establish the existence of an equilibrium with externalities in production sets, that is, the boundedness condition given in Arrow and Hahn (1971), page 134 of Section 2 in Chapter 6; Assumption UB (Uniform Boundedness) in Bonnisseau and Médecin (2001); Assumption P(3) in Mandel (2008); Assumption 3 in del Mercato and Platino (2015).

It is well known that the boundedness of the set of feasible allocations is a crucial condition for proving the existence of an equilibrium. However, Assumption 3 is a stronger version of the standard boundedness condition used for economies without externalities, because it guarantees that the set of feasible allocations $A(x, y; r)$ is uniformly bounded with respect to the externalities. It means that the bounded set $K(r)$ that includes the set of feasible allocations is independent of the externality effects. In particular, it implies the boundedness of the set of feasible allocations that are *mutually consistent*, i.e., the set $\mathcal{F}(r) = \{(x, y) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ} \colon t_j(y_j, y_{-j}, x) \leq 0,\ \forall\, j \in \mathcal{J} \text{ and } \sum_{h\in\mathcal{H}} x_h - \sum_{j\in\mathcal{J}} y_j \leq r\}$. Notice that in order to prove the existence of an equilibrium it would not be sufficient to assume only the boundedness of the set $\mathcal{F}(r)$.[11]

Assumption 5 *For all* $h \in \mathcal{H}$,

(1) The function u_h *is continuous in its domain and* C^2 *in the interior of its domain.*
(2) For every $(x_{-h}, y) \in \mathbb{R}^{C(H-1)}_{++} \times \mathbb{R}^{CJ}$, *the function* $u_h(\cdot, x_{-h}, y)$ *is differentiably strictly increasing, i.e.,* $D_{x_h} u_h(x'_h, x_{-h}, y) \gg 0$ *for all* $x'_h \in \mathbb{R}^C_{++}$.
(3) For every $(x_{-h}, y) \in \mathbb{R}^{C(H-1)}_{++} \times \mathbb{R}^{CJ}$, *the function* $u_h(\cdot, x_{-h}, y)$ *is differentiably strictly quasi-concave, i.e., for all* $x'_h \in \mathbb{R}^C_{++}$, $D^2_{x_h} u_h(x'_h, x_{-h}, y)$ *is negative definite on* $\operatorname{Ker} D_{x_h} u_h(x'_h, x_{-h}, y)$.
(4) For every $(x_{-h}, y) \in \mathbb{R}^{C(H-1)}_{+} \times \mathbb{R}^{CJ}$ *and for every* $u \in \operatorname{Im} u_h(\cdot, x_{-h}, y)$,

$$\mathrm{cl}_{\mathbb{R}^C}\{x_h \in \mathbb{R}^C_{++} : u_h(x_h, x_{-h}, y) \geq u\} \subseteq \mathbb{R}^C_{++}$$

Fixing the externalities, the assumptions on u_h are standard in smooth general equilibrium models. Point 3 of Assumption 5 implies that the upper contour sets are strictly convex. Consequently, if the utility maximization problem has a solution then it is unique. Notice that u_h is not required to be quasi-concave with respect to all the variables, i.e., we do not require preferences to be convex with respect to the externalities. Point 4 of Assumption 5 implies the classical Boundary Condition (BC), i.e., the closure of the upper counter sets is included in $\mathbb{R}^C_{++}$.

Notice that in Points 1 and 4 of Assumption 5, we allow for consumption externalities x_{-h} on the boundary of the set $\mathbb{R}^{C(H-1)}_{++}$ in order to handle the

[11] In Chapter 6 of Arrow and Hahn (1971), the authors have recognized the need to assume the boundedness of a wider set of feasible production allocations than the ones that are mutually consistent, in order to extend their existence proof to the case of externalities. In Bonnisseau and Médecin (2001), Assumption UB is needed to find the cube to compactify the economy in order to use fixed point arguments. Assumption P(3) in Mandel (2008) or Assumption 3 in del Mercato and Platino (2015) are used to show that the set of feasible allocations is bounded once externalities move along a homotopy arc.

behavior of u_h as consumption externalities approach the boundary. Points 1 and 4 of Assumption 5 imply that BC is still valid whenever consumption externalities converge to zero for some commodities. [12] This property is used to prove properness properties of the equilibrium set (see Step 2 of the proof of Lemma 17 in Section 5). If u_h does not satisfy BC whenever consumption externalities converge to zero for some commodities, we provide below an alternative assumption on u_h from which one still gets Step 2 in the proof of Lemma 17, while maintaining Points 1 and 4 of Assumption 5 only for consumption externalities in $\mathbb{R}^{C(H-1)}_{++}$. [13]

(5) There exists $\delta > 0$ such that for every $(x_{-h}, y) \in \mathbb{R}^{C(H-1)}_{++} \times \mathbb{R}^{CJ}$ and for every $(x_h, x'_h) \in \mathbb{R}^{2C}_{++}$, if $u_h(x_h, x_{-h}, y) > u_h(x'_h, x_{-h}, y)$, then $u_h(x_h, x_{-h} + \delta\mathbf{1}, y) \geq u_h(x'_h, x_{-h} + \delta\mathbf{1}, y)$ where $\mathbf{1} := (1, \ldots, 1) \in \mathbb{R}^C_{++}$. [14]

2.2 Competitive equilibrium and equilibrium function

In this subsection, we provide the definition of competitive equilibrium and the notion of equilibrium function.

We use commodity C as the "numeraire good". Given $p^\setminus \in \mathbb{R}^{C-1}_{++}$, let $p := (p^\setminus, 1) \in \mathbb{R}^C_{++}$.

Definition 6 (Competitive equilibrium) $(x^*, y^*, p^{*\setminus}) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ} \times \mathbb{R}^{C-1}_{++}$ *is a competitive equilibrium for the economy E if for all $j \in \mathcal{J}$, y^*_j solves the following problem*

$$\begin{aligned} &\max_{y_j \in \mathbb{R}^C} \; p^* \cdot y_j \\ &\text{subject to } \; t_j(y_j, y^*_{-j}, x^*) \leq 0 \end{aligned} \tag{2}$$

[12] A simple example of utility function that satisfies this property is given by any additively separable function $u_h(x_h, x_{-h}) = \widetilde{u}_h(x_h) + v_h(x_{-h})$ where $\widetilde{u}_h$ is defined on $\mathbb{R}^C_{++}$ and satisfies the classical BC, and v_h is defined on $\mathbb{R}^{C(H-1)}_{+}$.

[13] For example, consider two commodities and the utility function $u_h(x^1_h, x^2_h, x^1_k) := x^1_h x^2_h x^1_k$ where $(x^1_h, x^2_h) \in \mathbb{R}^2_{++}$ and $x^1_k \in \mathbb{R}_+$. This function does not satisfy BC whenever the externality x^1_k converges to zero, but it satisfies Point 5.

[14] A simple example of utility function that satisfies Point 5 is given by any multiplicatively separable function $u_h(x_h, x_{-h}) = \widetilde{u}_h(x_h) m_h(x_{-h})$ where $\widetilde{u}_h$ is defined on $\mathbb{R}^C_{++}$ and $m_h(x_{-h}) > 0$ for every $x_{-h} \in \mathbb{R}^{C(H-1)}_{++}$.

for all $h \in \mathcal{H}$, x_h^* solves the following problem

$$\begin{aligned} &\max_{x_h \in \mathbb{R}^C_{++}} \; u_h(x_h, x^*_{-h}, y^*) \\ &\text{subject to } \; p^* \cdot x_h \leq p^* \cdot (e_h + \sum_{j \in \mathcal{J}} s_{jh} y_j^*) \end{aligned} \tag{3}$$

and (x^*, y^*) satisfies market clearing conditions, that is

$$\sum_{h \in \mathcal{H}} x_h^* = \sum_{h \in \mathcal{H}} e_h + \sum_{j \in \mathcal{J}} y_j^* \tag{4}$$

The proof of the following proposition is standard, because in problems (2) and (3) each agent takes as given both the price and the choices of every other agent in the economy.

Proposition 7

(1) From Assumption 1, if y_j^* is a solution to problem (2), then it is unique and it is completely characterized by KKT conditions.[15]

(2) From Point 2 of Assumption 1 and Assumption 5, there exists a unique solution x_h^* to problem (3) and it is completely characterized by KKT conditions.

(3) As usual, from Point 2 of Assumption 5, household h's budget constraint holds with an equality. Thus, at equilibrium, due to the Walras law, the market clearing condition for commodity C is "redundant". So, one replaces condition (4) with $\sum_{h \in \mathcal{H}} x_h^{*\setminus} = \sum_{h \in \mathcal{H}} e_h^{\setminus} + \sum_{j \in \mathcal{J}} y_j^{*\setminus}$.

Let $\Xi := (\mathbb{R}^C_{++} \times \mathbb{R}_{++})^H \times (\mathbb{R}^C \times \mathbb{R}_{++})^J \times \mathbb{R}^{C-1}_{++}$ be the set of endogenous variables with generic element $\xi := (x, \lambda, y, \alpha, p^{\setminus}) := ((x_h, \lambda_h)_{h \in \mathcal{H}}, (y_j, \alpha_j)_{j \in \mathcal{J}}, p^{\setminus})$ where λ_h denotes the Lagrange multiplier associated with household h's budget constraint, and α_j denotes the Lagrange multiplier associated with firm j's technological constraint. We describe the competitive equilibria associated with the economy E using the *equilibrium function* $F_E : \Xi \rightarrow \mathbb{R}^{\dim \Xi}$,

$$F_E(\xi) := ((F_E^{h.1}(\xi), F_E^{h.2}(\xi))_{h \in \mathcal{H}}, (F_E^{j.1}(\xi), F_E^{j.2}(\xi))_{j \in \mathcal{J}}, F_E^M(\xi)) \tag{5}$$

where $F_E^{h.1}(\xi) := D_{x_h} u_h(x_h, x_{-h}, y) - \lambda_h p$, $F_E^{h.2}(\xi) := -p \cdot (x_h - e_h - \sum_{j \in \mathcal{J}} s_{jh} y_j)$, $F_E^{j.1}(\xi) := p - \alpha_j D_{y_j} t_j(y_j, y_{-j}, x)$, $F_E^{j.2}(\xi) := -t_j(y_j, y_{-j}, x)$, and $F_E^M(\xi) := \sum_{h \in \mathcal{H}} x_h^{\setminus} - \sum_{j \in \mathcal{J}} y_j^{\setminus} - \sum_{h \in \mathcal{H}} e_h^{\setminus}$.

The vector $\xi^* = (x^*, \lambda^*, y^*, \alpha^*, p^{*\setminus}) \in \Xi$ is an *extended equilibrium* for the economy E if and only if $F_E(\xi^*) = 0$. We simply call ξ^* an equilibrium.

[15] "KKT conditions" means Karush–Kuhn–Tucker conditions.

Theorem 8 (Existence and compactness) *The equilibrium set $F_E^{-1}(0)$ is non-empty and compact.*

del Mercato and Platino (2015) provide a proof of Theorem 8.

2.3 Regular economies

We recall below the formal notion of regular economy.

Definition 9 (Regular economy) *E is a regular economy if F_E is a C^1 function and 0 is a regular value of F, i.e., for every $\xi^* \in F_E^{-1}(0)$, the differential mapping $D_\xi F_E(\xi^*)$ is onto.*

Using Smale's approach, the definition of regular economy becomes a very intelligible notion. The fact that the Jacobian matrix $D_\xi F_E(\xi^*)$ is non-singular simply means that the linear approximation at ξ^* of the non-linear equilibrium system $F_E(\xi) = 0$ has a unique solution. Then, applying the Implicit Function Theorem, around ξ^*, the solution of the equilibrium system is a differentiable mapping of the parameters describing the economy. Furthermore, if the equilibrium set $F_E^{-1}(0)$ is non-empty and compact, as a consequence of the Regular Value Theorem (Corollary 22 in Subsection 6.1), one easily deduces that a regular economy has a finite number of equilibria.

In the presence of externalities, the possibility of equilibrium indeterminacy cannot be excluded by making standard assumptions. Indeed, the equilibrium notion given in Definition 6 has the characteristics described in what follows. All the agents take as given both the price and the choice of every other agent in the economy. Given the price and the choices of the other agents, the individual optimal solutions are completely determined. But, for a given price, the equilibrium allocation $((x_h^*)_{h\in\mathcal{H}}, (y_j^*)_{j\in\mathcal{J}})$ is a Nash equilibrium, and the problem is that, under standard assumptions, one may get indeterminacy in Nash equilibrium. We illustrate the reason below.

Consider the equilibrium function F_E defined in (5). The economy E remains fixed, thus we omit the subscript E. The price $p^\setminus$ is fixed. Consider all the equilibrium equations except the $L-1$ market clearing conditions by defining the following function G.

$$G(q) := ((F^{h.1}(q,p^\setminus), F^{h.2}(q,p^\setminus))_{h\in\mathcal{H}}, (F^{j.1}(q,p^\setminus), F^{j.2}(q,p^\setminus))_{j\in\mathcal{J}})$$

where $q := (x,\lambda,y,\alpha) = ((x_h,\lambda_h)_{h\in\mathcal{H}}, (y_j,\alpha_j)_{j\in\mathcal{J}})$, so that we write ξ as $(q,p^\setminus)$.

Every $q^* = ((x_h^*,\lambda_h^*)_{h\in\mathcal{H}}, (y_j^*,\alpha_j^*)_{j\in\mathcal{J}})$ such that $G(q^*) = 0$ provides the individual demands and supplies $((x_h^*)_{h\in\mathcal{H}}, (y_j^*)_{j\in\mathcal{J}})$ that are *mutually consistent* at the price $p^\backslash$.

Consider the Jacobian matrix of that system, that is,

$$D_qG(q^*) := D_q((F^{h.1}(q^*,p^\backslash), F^{h.2}(q^*,p^\backslash))_{h\in\mathcal{H}}, (F^{j.1}(q^*,p^\backslash), F^{j.2}(q^*,p^\backslash))_{j\in\mathcal{J}}) \tag{6}$$

In the absence of externalities, the Jacobian matrix $D_qG(q^*)$ is non-singular. [16] This is because the transformation and utility functions are respectively differentiably strictly quasi-convex and strictly quasi-concave in the individual choice. Then, the Implicit Function Theorem implies that, locally, the individual demands and supplies are a C^1 mapping of the price.

It turns out that, in the presence of externalities, under standard assumptions, the Jacobian matrix $D_qG(q^*)$ is not necessarily non-singular. Consequently, the individual demands and supplies $((x_h^*)_{h\in\mathcal{H}}, (y_j^*)_{j\in\mathcal{J}})$ that are mutually consistent may not be a differentiable mapping of the price. The matrix $D_qG(q^*)$ may fail to be nonsingular due to the presence of some of the following effects:

(1) the second order external effects on utility functions arising from the derivatives of $F^{h.1}(q,p^\backslash)$ with respect to (x_{-h},y),
(2) the second order external effects on transformation functions arising from the derivatives of $F^{j.1}(q,p^\backslash)$ with respect to (y_{-j},x),
(3) the first order external effects on transformation functions arising from the derivatives of $F^{j.2}(q,p^\backslash)$ with respect to (y_{-j},x).

In economies without externalities, all these effects are equal to zero.

One might believe that if the utility and transformation functions are respectively differentiably strictly quasi-convex and strictly quasi-concave with respect to all the variables (i.e., individual choice and externalities), then the matrix $D_qG(q^*)$ is non-singular. This belief is wrong, since the matrix $D_qG(q^*)$ does not actually involve the whole Hessian matrix of the utility and transformation functions. The matrix $D_qG(q^*)$ involves only a partial block of rows of those Hessian matrices. This is because the first order effects of externalities on utility and transformation functions do not appear in the first order conditions associated with the individual maximization problems, i.e., in the system $G(q) = 0$.

In the case of pure exchange economies with externalities, Bonnisseau and del Mercato (2010) have introduced a specific assumption on the second order external effects on utility and *possibility functions*. [17] We adapt their assumption on utility functions to our framework.

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[16] Then, the matrix $D_\xi F_E(\xi^*)$ is trivially non-singular for almost all initial endowments, and one gets the generic regularity in the space of households endowments.
[17] The *possibility functions* represent general consumption sets with externalities.

Assumption 10 (Bonnisseau and del Mercato, 2010) *Let* $(x, y) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$ *such that the gradients* $(D_{x_h} u_h(x_h, x_{-h}, y))_{h \in \mathcal{H}}$ *are positively collinear. Let* $v \in \mathbb{R}^{CH}$ *such that* $\sum_{h \in \mathcal{H}} v_h = 0$ *and* $v_h \in \operatorname{Ker} D_{x_h} u_h(x_h, x_{-h}, y)$ *for every* $h \in \mathcal{H}$. *Then,* $v_h \sum_{k \in \mathcal{H}} D^2_{x_k x_h} u_h(x_h, x_{-h}, y)(v_k) < 0$ *whenever* $v_h \neq 0$.

We refer to Bonnisseau and del Mercato (2010) for the interpretation of this assumption and an example of utility functions that satisfy this assumption. Notice that Assumption 10 does not require the Hessian matrix of u_h (with respect to all the variables) to be positive definite on $\operatorname{Ker} Du_h(x_h, x_{-h}, y)$. That is, u_h is not required to be differentiably strictly quasi-concave with respect to all the variables. We face positive forms that may induce to think of strict quasi-concavity, but actually in Assumption 10 one takes into account only a partial block of rows of the Hessian matrix of u_h. Assumption 10 is in the same spirit as the assumption of *diagonally strict concavity* introduced in Rosen (1965) on a weighted sum of the payoff functions of the agents. However, compared with the condition of Theorem 6 of Rosen (1965), Assumption 10 is easier to read, it does not involve any vector of weights and it focuses only on directions $(v_h)_{h \in \mathcal{H}}$ that sum to zero where v_h is orthogonal to the gradient $D_{x_h} u_h(x_h, x_{-h}, y)$.

Under Assumption 10, the classical result of generic regularity holds true if there are no externalities in the production sets.

In the presence of external effects on the production side, an assumption on the transformation functions analogous to Assumption 10 is not sufficient for establishing generic regularity in the space of households' endowments. [18] This is shown in the next section by means of an example.

3 Infinitely many competitive equilibria: An example

In the previous section, we have explained why standard arguments for the genericity of regular economies may not apply to economies with externalities. In this section, we provide an example of a private ownership economy with one household and production externalities between two firms where equilibria are indeterminate in an open set of the household's endowments. In the example, we get infinitely many equilibria because there are infinitely many mutually consistent supplies. Importantly, this allocations indeterminacy translates into a "price relevant indeterminacy". That is, one gets infinitely many equilibrium

[18] Further research is open to find some kind of auxiliary assumption on the first order effects of externalities on transformation functions. In Section 4, we choose another approach.

prices, and consequently the indeterminacy has an impact on the welfare of the economy.

Example. There are two commodities. There is one household, $x = (x^1, x^2)$ denotes the consumption of the household, $e = (e^1, e^2)$ is his initial endowment and the utility function is given by $u(x^1, x^2) = x^1 x^2$. Therefore, there are no externalities on the consumption side and then Assumption 10 is obviously satisfied.

There are two firms, the production plan of firm j is $y_j = (y_j^1, y_j^2)$. Without loss of generality, for simplicity of exposition, the subscript f denotes the subscript $-j$, so that the production plan of the firm other than j is $y_f = (y_f^1, y_f^2)$. Both firms use commodity 2 to produce commodity 1.

The production technology of firm j is affected by the production plan of the other firm in the following way. Given y_f, the production set of firm j is

$$Y_j(y_f) = \{y_j \in \mathbb{R}^2 : y_j^2 \leq 0 \text{ and } y_j^1 \leq f_j(y_j^2, y_f)\}$$

where the production function f_j is defined by $f_j(y_j^2, y_f) := 2\phi(y_f)\sqrt{-y_j^2}$ with

$$\phi(y_f) := \begin{cases} \dfrac{y_f^1}{2\sqrt{-y_f^2}} & \text{if} \quad y_f^2 < -\varepsilon^2 \\ \dfrac{y_f^1}{2\varepsilon} & \text{if} \;\; y_f^2 \in [-\varepsilon^2, 0] \end{cases}$$

where $0 < \varepsilon < 1$. The production function f_j is not completely smooth with respect to the externality. [19] But, it goes in the essence of the problem.

For every firm $j = 1, 2$, Assumption 1 is satisfied for every externality $y_f = (y_f^1, y_f^2)$ with $y_f^1 > 0$ and $y_f^2 < -\varepsilon^2$. Therefore, in what follows,

(1) we focus on equilibria where the amounts of output are strictly positive and the amounts of input are strictly lower than $-\varepsilon^2$,

(2) $\phi(y_f)$ is then given by $\dfrac{y_f^1}{2\sqrt{-y_f^2}}$ according to the definition above.

The price of commodity 2 is normalized to 1. By Definition 6, we have that $(x^*, y_1^*, y_2^*, (p^*, 1))$ is a competitive equilibrium if for every $j = 1, 2$, y_j^* solves

$$\begin{aligned} &\max_{y_j^1 > 0,\, y_j^2 < -\varepsilon^2} \; p^* y_j^1 + y_j^2 \\ &\text{subject to } \; y_j^1 - 2\phi(y_f^*)\sqrt{-y_j^2} \leq 0 \end{aligned}$$

[19] In order to get a smooth approximation, one might approximate the function ϕ around $-\varepsilon^2$ by a polynomial function.

x^* solves the following problem

$$\begin{aligned} &\max_{x \in \mathbb{R}^2_{++}} \; x^1 x^2 \\ &\text{subject to } \; p^* x^1 + x^2 \leq p^* e^1 + e^2 + \sum_{j=1}^{2} (p^* y_j^{*1} + y_j^{*2}) \end{aligned} \tag{7}$$

and markets clear.

For each firm $j = 1, 2$, let α_j be the Lagrange multiplier associated with firm j's technological constraint. The KKT conditions associated with firm j's maximization problem are given by

$$\begin{cases} p^* = \alpha_j, \; 1 = \dfrac{\alpha_j \phi(y_f^*)}{\sqrt{-y_j^2}} \\ y_j^1 = 2\phi(y_f^*)\sqrt{-y_j^2} \end{cases}$$

Consequently, one gets the following equilibrium equations for every $j = 1, 2$,

$$y_j^{*2} = -(p^*)^2 [\phi(y_f^*)]^2 \text{ and } y_j^{*1} = 2p^* [\phi(y_f^*)]^2$$

and one easily deduces that, at equilibrium,

$$y_1^{*1} = y_2^{*1} = -\frac{2y_2^{*2}}{p^*} \text{ and } y_1^{*2} = y_2^{*2} \text{ for any } y_2^{*2} < -\varepsilon^2 \tag{8}$$

Thus, at equilibrium, the aggregate profit is given by $\pi^* := -2y_2^{*2}$ and the optimal solution of the household is given by

$$x^{*1} = \frac{1}{2p^*}(p^* e^1 + e^2 + \pi^*) \text{ and } x^{*2} = p^* x^{*1} \tag{9}$$

Using the market clearing condition for commodity 2, the equilibrium price is

$$p^* = \frac{e^2 + 6y_2^{*2}}{e^1} \tag{10}$$

Notice that $p^* > 0$ if and only if $y_2^{*2} > -\dfrac{e^2}{6}$. Therefore, using (8), (9) and (10), any bundle

$$(x^*, y_1^*, y_2^*, (p^*, 1)) \text{ with } y_2^{*2} \in \left] -\frac{e^2}{6}, -\varepsilon^2 \right[$$

is a competitive equilibrium. Thus, for all initial endowments that belong to the open set $\{e = (e^1, e^2) \in \mathbb{R}^2_{++} \colon e^2 > 6\varepsilon^2\}$ we get infinitely many equilibria parametrized by y_2^{*2}.

We now show that the economy of the example exhibits well behaved second order external effects. For this purpose, we provide below the condition on the transformation functions analogous to Assumption 10.

Let $(x,y) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$ such that $t_j(y_j, y_{-j}, x) = 0$ for every $j \in \mathcal{J}$ and the gradients $(D_{y_j} t_j(y_j, y_{-j}, x))_{j\in\mathcal{J}}$ are positively collinear. Let $z \in \mathbb{R}^{CJ}$ such that $\sum_{j\in\mathcal{J}} z_j = 0$ and $z_j \in \operatorname{Ker} D_{y_j} t_j(y_j, y_{-j}, x)$ for every $j \in \mathcal{J}$. Then, $z_j \sum_{f\in\mathcal{J}} D^2_{y_f y_j} t_j(y_j, y_{-j}, x)(z_f) > 0$ whenever $z_j \neq 0$.

If there are no external effects, this condition is satisfied because t_j is differentiably strictly quasi-convex in y_j. Therefore, the sign of the inequality is strictly positive, whereas in Assumption 10 the analogous sign is strictly negative since u_h is differentiably strictly quasi-concave in x_h.

Consider the previous example. For every firm j, the transformation function is given by $t_j(y_j, y_f) = y^1_j - 2\phi(y_f)\sqrt{-y^2_j}$. As above, we focus on production plans for which $\phi(y_f)$ is given by $\dfrac{y^1_f}{2\sqrt{-y^2_f}}$.

Take $z = (z_1, z_2) \in \mathbb{R}^4$ such that

$$z_1 + z_2 = 0 \tag{11}$$

and $z_j \in \operatorname{Ker} D_{y_j} t_j(y_j, y_f)$. Then, one gets

$$z^1_j = -\frac{\phi(y_f)}{\sqrt{-y^2_j}} z^2_j \tag{12}$$

We compute below the two matrices involved in the condition above,

$$D^2_{y_j} t_j(y_j, y_f) = \begin{pmatrix} 0 & 0 \\ 0 & \dfrac{\phi(y_f)}{2(-y^2_j)\sqrt{-y^2_j}} \end{pmatrix}, \quad D^2_{y_f y_j} t_j(y_j, y_f) = \begin{pmatrix} 0 & 0 \\ \dfrac{\phi(y_f)}{y^1_f\sqrt{-y^2_j}} & \dfrac{\phi(y_f)}{2(-y^2_f)\sqrt{-y^2_j}} \end{pmatrix}$$

Using (11) and (12), it is an easy matter to compute $z_j D^2_{y_j} t_j(y_j, y_f)(z_j) + z_j D^2_{y_f y_j} t_j(y_j, y_f)(z_f)$ which is given by

$$\left[\frac{1}{(-y^2_j)} + \frac{1}{\sqrt{-y^2_j}\sqrt{-y^2_f}} - \frac{1}{(-y^2_f)}\right] \frac{\phi(y_f)}{2\sqrt{(-y^2_j)}} (z^2_j)^2 \tag{13}$$

Since $z_j \neq 0$, from (12) we have that $z^2_j \neq 0$. Since the gradients $(D_{y_j} t_j(y_j, y_f))_{j=1,2}$ are positively collinear, one gets $y^1_j = y^1_f$. Then $y^2_j = y^2_f$, because $t_j(y_j, y_f) = 0$ for every $j = 1, 2$. Therefore, the quantity in (13) becomes

$$\frac{y^1_j}{4(-y^2_j)^2} (z^2_j)^2$$

which is strictly positive. ∎

4 The general model with firms' endowments

In order to establish generic regularity, we incorporate firms' endowments in the model of Section 2. Every firm j is endowed with an *exogenously* given vector of commodities $\eta_j := (\eta_j^1, .., \eta_j^c, .., \eta_j^C)$, where η_j^c denotes the amount of commodity c held initially by firm j. Let $\eta := (\eta_j)_{j \in \mathcal{J}} \in \mathbb{R}^{CJ}$. Notice that in the revised version accepted for publication in *Economic Theory* (forthcoming 2017), we consider only positive firms' endowments. However, our results also apply for negative firms' endowments, under a suitable survival condition given in Assumption 12 below. As for the interpretation of negative endowments, we refer to Subsection 1.3.2 in Arrow and Debreu (1954). That is, we may extend η_j^c to include all debts payable in terms of commodity c. Debts owed to firm j are positive, debts owed by firm j are negative. [20]

For every j, the **total production plan y_j introduced in the market by firm** j is given by

$$y_j := y'_j + \eta_j \tag{14}$$

where y'_j is the **production decision** of firm j according to its technology and the externality (y_{-j}, x), that is, $y'_j \in Y_j(y_{-j}, x)$. Notice that in $Y_j(y_{-j}, x)$, the production externality is the total production plan $y_{-j} = y'_{-j} + \eta_{-j}$ introduced in the market by firms other than j, and not the production decision y'_{-j} of firms other than j. That is, in our model firm j's decision y'_j is constrained by its technology and its technology is affected by the total production plans introduced in the market by the other firms. This seems reasonable in many economic applications.

Remark 11 *One could consider a different model where the production externality is the production decision y'_{-j} of firms other than j, i.e., $y'_j \in Y_j(y'_{-j}, x)$. In this case, despite the presence of firms' endowments, one obtains the same Jacobian matrix $D_qG(q^*)$ as in (6) of Subsection 2.3. In other words, firms' endowments do not impact the technologies of the other firms. Therefore, perturbing the endowments η does not affect firms' supplies. Moreover, as regards market clearing conditions, perturbing the endowments η is completely redundant, i.e., it has exactly the same effect as perturbing the endowments e, because household h's budget constraint and market clearing conditions are given*

[20] If η_j is negative, then $p \cdot \eta_j$ represents a cost for firm j, i.e., a *penalty*. For instance, in the literature of game theory, adding penalties to the payoff functions of the agents has been often used to restore the generic regularity of Nash equlibria, see for instance van Damme (2002).

by $p \cdot (x_h - e_h - \sum_{j \in \mathcal{J}} s_{jh}(y'_j + \eta_j)) = 0$ *and* $\sum_{h \in \mathcal{H}} x_h^{\setminus} - \sum_{j \in \mathcal{J}} (y'_j + \eta_j)^{\setminus} - \sum_{h \in \mathcal{H}} e_h^{\setminus} = 0$. *Consequently, this way of introducing firms' endowments does not lead to the result of generic regularity.*

Given the price $p \in \mathbb{R}^C_{++}$ and the externality $(y_{-j}, x) \in \mathbb{R}^{C(J-1)} \times \mathbb{R}^{CH}_{++}$, the optimal production decision of firm j solves the following problem

$$\begin{aligned} &\max_{y'_j \in \mathbb{R}^C} \; p \cdot y'_j \\ &\text{subject to } \; t_j(y'_j, y_{-j}, x) \leq 0 \end{aligned}$$

We write this problem in terms of the total production plan y_j introduced in the market by firm j, that is,

$$\begin{aligned} &\max_{y_j \in \mathbb{R}^C} \; p \cdot y_j \\ &\text{subject to } \; t_j(y_j - \eta_j, y_{-j}, x) \leq 0 \end{aligned} \tag{15}$$

Problem (15) is equivalent to the previous maximization problem, because $y'_j = y_j - \eta_j$ by (14) and firm j takes as given its endowment η_j. Therefore, introducing firms' endowments entails a displacement of the argument of the transformation function t_j.

The return from firm j generated by the price p, the production decision y'_j and the endowment η_j is given by $p \cdot (y'_j + \eta_j)$. We write this return in terms of the total production plan introduced in the market by firm j, that is, $p \cdot y_j$.

According to the behavior of each firm j, we are led to introduce the following notation:

$$t_j(y_j, y_{-j}, x; \eta_j) := t_j(y_j - \eta_j, y_{-j}, x) \tag{16}$$

Notice that the function $t_j(y_j, y_{-j}, x; \eta_j)$ satisfies all the assumptions given in Assumption 1 except Point 2. However, $y_j = \eta_j$ acts for $t_j(y_j, y_{-j}, x; \eta_j)$ as $y_j = 0$ acts for $t_j(y_j, y_{-j}, x)$. Indeed, Point 2 of Assumption 1 implies that $t_j(\eta_j, y_{-j}, x; \eta_j) = 0$, and then using standard arguments for profit maximization, one gets $p \cdot y^*_j \geq p \cdot \eta_j$ whenever y^*_j solves problem (15). The individual wealth of household h associated with his endowment and his shares on the returns from the firms is given by $p \cdot (e_h + \sum_{j \in \mathcal{J}} s_{jh} y^*_j)$. Consequently, the individual wealth of household h is greater than $p \cdot (e_h + \sum_{j \in \mathcal{J}} s_{jh} \eta_j)$ which is strictly positive if $e_h + \sum_{j \in \mathcal{J}} s_{jh} \eta_j \gg 0$. Then, in the following assumption we focus on the set of endowments (e, η) for which $e_h + \sum_{j \in \mathcal{J}} s_{jh} \eta_j \gg 0$ for every $h \in \mathcal{H}$, from which one deduces the non-emptiness of the interior of all the individual budget constraints.

Assumption 12 (Survival Assumption) *(e, η) belongs to the following set.*

$$\mathcal{E} := \{(e, \eta) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ} : e_h + \sum_{j \in \mathcal{J}} s_{jh} \eta_j \gg 0, \ \forall \, h \in \mathcal{H}\}$$

We assume below that Assumption 3 (Uniform Boundedness) is satisfied in some open subset of endowments of households and firms. The following assumption is used to prove non-emptiness and properness properties of the equilibrium set, see the proof of Lemma 15 and Step 1 of the proof of Lemma 17 in Section 5.

Assumption 13 *There exists an open set $\Lambda \subseteq \mathcal{E}$ such that Assumption 3 is satisfied for every $r \geq \sum_{j \in \mathcal{H}} e_h + \sum_{j \in \mathcal{J}} \eta_j$ with $(e, \eta) \in \Lambda$.*

Remark 14 *In the setting of Remark 2, for every firm j the initial endowment η_j belongs to the set $\mathbb{R}^{C_j}$. In this case, in Assumptions 12 and 13, $\mathbb{R}^{CJ}$ is replaced by $\prod_{j \in \mathcal{J}} \mathbb{R}^{C_j}$, and for every commodity $c \in \mathcal{C}$ the sum over $j \in \mathcal{J}$ is replaced by the sum over $j \in \mathcal{J}(c)$.*

From now on,

(1) $t = (t_j)_{j \in \mathcal{J}}$ is fixed and satisfies Assumption 1 and Assumption 13,

(2) $u = (u_h)_{h \in \mathcal{H}}$ is fixed and satisfies Assumptions 5 and 10,

(3) a private ownership economy is completely parametrized by the endowments of households and firms (e, η) in the open set Λ given in Assumption 13,

(4) we simply denote $F_{e,\eta}$ the equilibrium function associated with an economy $(e, \eta) \in \Lambda$.

Competitive equilibrium and equilibrium function. The notions of competitive equilibrium and equilibrium function given in Subsection 2.2 are adapted to the economy (e, η) according to the behavior of firm j given in (15) and the notations introduced in (14) and (16). More precisely, in Definition 6, one replaces $t_j(y_j, y^*_{-j}, x^*)$ with $t_j(y_j, y^*_{-j}, x^*; \eta_j)$. Consequently, the equilibrium function $F_{e,\eta}$ coincides with the equilibrium function defined in (5), except for the first order conditions associated with problem (15) which are replaced by

$$F^{j.1}_{e,\eta}(\xi) := p - \alpha_j D_{y_j} t_j(y_j, y_{-j}, x; \eta_j) \text{ and } F^{j.2}_{e,\eta}(\xi) := -t_j(y_j, y_{-j}, x; \eta_j)$$

Under the previous assumptions one gets the result analogous to Theorem 8. The proof of the following lemma is given in Section 5.

Lemma 15 *For every* $(e, \eta) \in \Lambda$*, the equilibrium set* $F_{e,\eta}^{-1}(0)$ *is non-empty and compact.*

4.1 The regularity result

In this subsection, we provide and we prove our main theorem.

Theorem 16 *The set* Λ^* *of regular economies* (e, η) *is an open and full measure subset of* Λ*.*

In order to prove this theorem, we introduce the following notations and we provide two auxiliary lemmas, namely Lemmas 17 and 18. Both lemmas are proved in Section 5.

Point 1 of Assumption 1 and Point 1 of Assumption 5 imply that the equilibrium function $F_{e,\eta}$ is C^1 everywhere. By Definition 9, the economy (e, η) is regular if

$$\forall\, \xi^* \in F_{e,\eta}^{-1}(0),\ \operatorname{rank} D_\xi F_{e,\eta}(\xi^*) = \dim \Xi$$

Define the following set

$$G := \Big\{(\xi, e, \eta) \in F^{-1}(0) :\ \operatorname{rank} D_\xi F(\xi, e, \eta) < \dim \Xi\Big\}$$

where the function $F : \Xi \times \Lambda \to \mathbb{R}^{\dim \Xi}$ is defined by

$$F(\xi, e, \eta) := F_{e,\eta}(\xi)$$

and denote Π the restriction to $F^{-1}(0)$ of the projection of $\Xi \times \Lambda$ onto Λ, i.e.

$$\Pi : (\xi, e, \eta) \in F^{-1}(0) \to \Pi(\xi, e, \eta) := (e, \eta) \in \Lambda$$

We can now write the set Λ^* given in Theorem 16 as

$$\Lambda^* = \Lambda \setminus \Pi(G)$$

So, in order to prove Theorem 16, it is enough to show that $\Pi(G)$ is a closed set in Λ and $\Pi(G)$ is of measure zero.

We first claim that $\Pi(G)$ is a closed set in Λ. From Point 1 of Assumptions 1 and 5, F and $D_\xi F$ are continuous on $\Xi \times \Lambda$. The set G is characterized by the fact that the determinant of all the square submatrices of $D_\xi F(\xi, e, \eta)$ of dimension $\dim \Xi$ is equal to zero. Since the determinant is a continuous function and $D_\xi F$ is continuous on $F^{-1}(0)$, the set G is closed in $F^{-1}(0)$.

Thus, $\Pi(G)$ is closed since the projection Π is proper.[21] The properness of the projection Π is provided in the following lemma.

Lemma 17 *The projection $\Pi : F^{-1}(0) \to \Lambda$ is a proper function.*

To complete the proof of Theorem 16, we claim that $\Pi(G)$ has measure zero in Λ. The result follows by Lemma 18 given below and a consequence of Sard's Theorem (Theorem 23 in Subsection 6.1). Indeed, Lemma 18 and Theorem 23 imply that there exists a full measure subset Ω of Λ such that for each $(e, \eta) \in \Omega$ and for each ξ^* such that $F(\xi^*, e, \eta) = 0$, rank $D_\xi F(\xi^*, e, \eta) = \dim \Xi$. Now, let $(e, \eta) \in \Pi(G)$, then there exists $\xi \in \Xi$ such that $F(\xi, e, \eta) = 0$ and rank $D_\xi F(\xi, e, \eta) < \dim \Xi$. So, $(e, \eta) \notin \Omega$. This prove that $\Pi(G)$ is included in the complementary of Ω, that is in $\Omega^C := \Lambda \setminus \Omega$. Since Ω^C has measure zero, so too does $\Pi(G)$. Thus, the set of regular economies Λ^* is a full measure subset of Λ since $\Omega \subseteq \Lambda^*$, which completes the proof of Theorem 16.

Lemma 18 *0 is a regular value for F.*

Finally, one easily deduces the following proposition from Lemma 15 and Theorem 16, using a consequence of the Regular Value Theorem (Corollary 22 in Subsection 6.1) and the Implicit Function Theorem.

Proposition 19 (Properties of a regular economy) *For each $(e, \eta) \in \Lambda^*$,*

(1) the equilibrium set associated with the economy (e, η) is a non-empty finite set, i.e.

$$\exists\, r \in \mathbb{N} \setminus \{0\} : \ F_{e,\eta}^{-1}(0) = \{\xi^1, ..., \xi^r\}$$

(2) there exists an open neighborhood I of (e, η) in Λ^*, and for each $i = 1, \dots, r$ there exist an open neighborhood U_i of ξ^i in Ξ and a C^1 function $g_i : I \to U_i$ such that

(a) $U_i \cap U_k = \emptyset$ if $i \neq k$,

(b) $g_i(e, \eta) = \xi^i$ and $\xi' \in F_{e',\eta'}^{-1}(0)$ holds for $(\xi', e', \eta') \in U_i \times I$ if and only if $\xi' = g_i(e', \eta')$.

We conclude this section by remarking that most of the classical properties of the equilibrium manifold and the natural projection are established for the model studied in this section. 1) For any given economy, the equilibrium set is non-empty and compact (Lemma 15). 2) For any given economy, the topological degree modulo 2 of the equilibrium function is equal to 1 (Lemma 10 in Mercato and Platino, 2015).[22] Consequently, one gets that, at every regular economy, the number of equilibria is odd. 3) The properness of the natural

[21] See Definition 24 in Subsection 6.1.

[22] The proof of Lemma 10 of del Mercato and Platino (2015) have been adapted to our model in the proof of Lemma 15.

projection (Lemma 17). 4) 0 is a regular value of the equilibrium function (Lemma 18). Consequently, the equilibrium set is a sub-manifold of the same dimension as the space of the parameters describing the economy. 5) The local differentiability of the associated equilibrium selection map (Proposition 19).

5 Proofs

In this section, we prove all the lemmas stated in Section 4.

Proof of Lemma 15. The compactness of the equilibrium set is a direct consequence of Lemma 17. For the existence of an equilibrium it suffices to adapt the proof of Lemma 10 given in del Mercato and Platino (2015) by replacing the transformation functions $t_j(y_j, y_{-j}, x)$ with the functions $t_j(y_j, y_{-j}, x; \eta_j)$.

Notice that the functions $t_j(y_j, y_{-j}, x; \eta_j)$ satisfy all the assumptions given in Assumption 1 except Point 2. Whenever Point 2 of Assumption 1 is invoked in the proof of Lemma 10 of del Mercato and Platino (2015), the production plan $y_j = 0$ is replaced by $y_j = \eta_j$ which acts for $t_j(y_j, y_{-j}, x; \eta_j)$ as $y_j = 0$ acts for $t_j(y_j, y_{-j}, x)$. Furthermore, whenever Lemma 4 is invoked in the proof of Lemma 10 of del Mercato and Platino (2015), the bounded set $K(r)$ given by Lemma 4 is replaced by the set $K(\widetilde{r}) + \widetilde{\eta}$ where $\widetilde{r} := r + \sum_{j\in\mathcal{J}} \eta_j$ and $\widetilde{\eta} := (0, \eta)$ which is bounded by Assumption 13. Finally, in order to complete the adaptation of the proof of Lemma 10 of del Mercato and Platino (2015), one also adapts in the following way the proof of Proposition 15 in del Mercato and Platino (2015). Namely, in Step 1.2 of Proposition 15, one replaces the point $\widehat{e}_h(\tau^\nu)$ with the point $\tau^\nu(e_h + \sum_{j\in\mathcal{J}} s_{jh}\eta_j) + (1-\tau^\nu)\widetilde{x}_h$, and in Step 2.2, one replaces the point $\widehat{e}_h(\tau^\nu)$ with the point $e_h + \sum_{j\in\mathcal{J}} s_{jh}\eta_j$.

■

Proof of Lemma 17. We show that any sequence $(\xi^\nu, e^\nu, \eta^\nu)_{\nu\in\mathbb{N}} \subseteq F^{-1}(0)$, up to a subsequence, converges to an element of $F^{-1}(0)$, knowing that the sequence $\Pi(\xi^\nu, e^\nu, \eta^\nu)_{\nu\in\mathbb{N}} = (e^\nu, \eta^\nu)_{\nu\in\mathbb{N}} \subseteq \Lambda$ converges to some $(e^*, \eta^*) \in \Lambda$. We recall that $\xi^\nu = (x^\nu, \lambda^\nu, y^\nu, \alpha^\nu, p^{\nu\backslash})$.

Step 1. *Up to a subsequence, $(x^\nu, y^\nu)_{\nu\in\mathbb{N}}$ converges to $(x^*, y^*) \in \mathbb{R}^{CH}_+ \times \mathbb{R}^{CJ}$.*

For every $j \in \mathcal{J}$ and $\nu \in \mathbb{N}$, define the following production plan

$$y'^\nu_j := y^\nu_j - \eta^\nu_j \tag{17}$$

$F^{j.2}(\xi^\nu, e^\nu, \eta^\nu) = 0$ and the definition given in (16) imply $t_j(y'^\nu_j, y^\nu_{-j}, x^\nu) = 0$. Then, $y'^\nu = (y'^\nu_j)_{j\in\mathcal{J}}$ belongs to the set $Y(y^\nu, x^\nu)$ defined in (1). Summing

$F^{h.2}(\xi^\nu, e^\nu, \eta^\nu) = 0$ over h, from $F^M(\xi^\nu, e^\nu, \eta^\nu) = 0$ one gets $\sum_{h\in\mathcal{H}} x_h^\nu - \sum_{j\in\mathcal{J}} y_j^\nu = \sum_{h\in\mathcal{H}} e_h^\nu$ for all $\nu \in \mathbb{N}$. Consequently, using (17) we have that

$$\sum_{h\in\mathcal{H}} x_h^\nu - \sum_{j\in\mathcal{J}} y_j'^\nu = \sum_{h\in\mathcal{H}} e_h^\nu + \sum_{j\in\mathcal{J}} \eta_j^\nu$$

for every $\nu \in \mathbb{N}$. Now, for every commodity c consider the following compact set $Q^c := \{\sum_{h\in\mathcal{H}} e_h^{\nu c} + \sum_{j\in\mathcal{J}} \eta_j^{\nu c} \colon \nu \in \mathbb{N}\} \cup \{\sum_{h\in\mathcal{H}} e_h^{*c} + \sum_{j\in\mathcal{J}} \eta_j^{*c}\}$ and define

$$r^c := \max Q^c \quad \text{and} \quad r := (r^c)_{c\in\mathcal{C}}$$

From the equality above, $\sum_{h\in\mathcal{H}} x_h^\nu - \sum_{j\in\mathcal{J}} y_j'^\nu \leq r$ for all $\nu \in \mathbb{N}$, and $r \geq \sum_{h\in\mathcal{H}} e_h^* + \sum_{j\in\mathcal{J}} \eta_j^*$ by definition. Consequently, Assumption 13 implies that the sequence $(x^\nu, y'^\nu)_{\nu\in\mathbb{N}}$ is included in the bounded set $K(r)$ given by Lemma 4, because (x^ν, y'^ν) belongs to the set $A(x^\nu, y^\nu; r)$ for every $\nu \in \mathbb{N}$. Thus, the sequence $(x^\nu, y'^\nu)_{\nu\in\mathbb{N}}$ is included in the compact set $\operatorname{cl} K(r)$ and, up to a subsequence, it converges to some $(x^*, y'^*) \in \mathbb{R}_+^{CH} \times \mathbb{R}^{CJ}$. Finally, from (17) it follows that the sequence $(y^\nu)_{\nu\in\mathbb{N}}$ converges to $y^* := y'^* + \eta^*$ which completes the proof of the step.

Step 2. *The consumption allocation x^* is strictly positive, i.e. $x^* \gg 0$.* Define the bundle $\widehat{e}_h^* := e_h^* + \sum_{j\in\mathcal{J}} s_{jh}\eta_j^*$ which is strictly positive by Assumption 12. We show that for every $h \in \mathcal{H}$, x_h^* belongs to the closure of the following set

$$\{x_h \in \mathbb{R}_{++}^C : u_h(x_h, x_{-h}^*, y^*) \geq u_h(\widehat{e}_h^*, x_{-h}^*, y^*)\} \tag{18}$$

which is included in $\mathbb{R}_{++}^C$ by Point 4 of Assumption 5. Thus, $x_h^* \gg 0$.

By $F^{h.1}(\xi^\nu, e^\nu, \eta^\nu) = F^{h.2}(\xi^\nu, e^\nu, \eta^\nu) = 0$ and KKT sufficient conditions, x_h^ν solves the following problem for every $\nu \in \mathbb{N}$.

$$\max_{x_h\in\mathbb{R}_{++}^C} \; u_h(x_h, x_{-h}^\nu, y^\nu)$$
$$\text{subject to } p^\nu \cdot x_h \leq p^\nu \cdot e_h^\nu + p^\nu \cdot \sum_{j\in\mathcal{J}} s_{jh} y_j^\nu$$

We claim that the point $\widehat{e}_h^\nu := e_h^\nu + \sum_{j\in\mathcal{J}} s_{jh}\eta_j^\nu$ belongs to the budget constraint of the problem above. Notice that $\widehat{e}_h^\nu \gg 0$ by Assumption 12. By $F^{j.1}(\xi^\nu, e^\nu, \eta^\nu) = F^{j.2}(\xi^\nu, e^\nu, \eta^\nu) = 0$ and KKT sufficient conditions, y_j^ν solves the following problem for every $\nu \in \mathbb{N}$.

$$\max_{y_j \in \mathbb{R}^C} p^\nu \cdot y_j$$
$$\text{subject to } t_j(y_j, y^\nu_{-j}, x^\nu; \eta^\nu_j) \leq 0$$

Point 2 of Assumption 1 and the definition given in (16) imply $t_j(\eta^\nu_j, y^\nu_{-j}, x^\nu; \eta^\nu_j) = 0$, and so $p^\nu \cdot y^\nu_j \geq p^\nu \cdot \eta^\nu_j$ for every j. Thus, one gets $p^\nu \cdot (e^\nu_h + \sum_{j \in \mathcal{J}} s_{jh} y^\nu_j) \geq p^\nu \cdot \widehat{e}^\nu_h$ which completes the proof of the claim. Therefore, for every $\nu \in \mathbb{N}$

$$u_h(x^\nu_h, x^\nu_{-h}, y^\nu) \geq u_h(\widehat{e}^\nu_h, x^\nu_{-h}, y^\nu)$$

By Point 2 of Assumption 5, for every $\varepsilon > 0$ we have that $u_h(x^\nu_h + \varepsilon \mathbf{1}, x^\nu_{-h}, y^\nu) > u_h(\widehat{e}^\nu_h, x^\nu_{-h}, y^\nu)$ where $\mathbf{1} := (1, \ldots, 1) \in \mathbb{R}^C_{++}$. So, taking the limit for $\nu \to +\infty$ and using the continuity of u_h, one gets

$$u_h(x^*_h + \varepsilon \mathbf{1}, x^*_{-h}, y^*) \geq u_h(\widehat{e}^*_h, x^*_{-h}, y^*)$$

since $(\widehat{e}^\nu_h)_{\nu \in \mathbb{N}}$ converges to $\widehat{e}^*_h$. Thus, for every $\varepsilon > 0$ the point $(x^*_h + \varepsilon \mathbf{1})$ belongs to the set defined in (18), which implies that x^*_h belongs to the closure of this set.

Step 3. *Up to a subsequence,* $(\lambda^\nu, p^{\nu\,\backslash})_{\nu \in \mathbb{N}}$ *converges to some* $(\lambda^*, p^{*\,\backslash}) \in \mathbb{R}^H_{++} \times \mathbb{R}^{C-1}_{++}$. By $F^{h.1}(\xi^\nu, e^\nu, \eta^\nu) = 0$, fixing commodity C, for every $\nu \in \mathbb{N}$ we have $\lambda^\nu_h = D_{x^C_h} u_h(x^\nu_h, x^\nu_{-h}, y^\nu)$. Taking the limit over ν, by Points 1 and 2 of Assumption 5, we get $\lambda^*_h := D_{x^C_h} u_h(x^*_h, x^*_{-h}, y^*) > 0$.

By $F^{h.1}(\xi^\nu, e^\nu, \eta^\nu) = 0$, for all commodity $c \neq C$ and for all $\nu \in \mathbb{N}$ we have $p^{\nu\, c} = \dfrac{D_{x^c_h} u_h(x^\nu_h, x^\nu_{-h}, y^\nu)}{\lambda^\nu_h}$. Taking the limit over ν, by Points 1 and 2 of Assumption 5, we get $p^{*\, c} := \dfrac{D_{x^c_h} u_h(x^*_h, x^*_{-h}, y^*)}{\lambda^*_h} > 0$. Therefore, $p^{*\,\backslash} \gg 0$.

Step 4. *Up to a subsequence,* $(\alpha^\nu)_{\nu \in \mathbb{N}}$ *converges to some* $\alpha^* \in \mathbb{R}^J_{++}$. For every firm j, fix a commodity $c(j) \in \mathcal{C}$. By $F^{j.1}(\xi^\nu, e^\nu, \eta^\nu) = 0$, for every $\nu \in \mathbb{N}$ we have that $\alpha^\nu_j = \dfrac{p^{\nu\, c(j)}}{D_{y^{c(j)}_j} t_j(y^\nu_j, y^\nu_{-j}, x^\nu; \eta^\nu_j)}$ which is strictly positive by Point 4 of Assumption 1. Taking the limit, by Points 1 and 4 of Assumption 1, we get $\alpha^*_j := \dfrac{p^{*\, c(j)}}{D_{y^{c(j)}_j} t_j(y^*_j, y^*_{-j}, x^*; \eta^*_j)} > 0$.

■

Proof of Lemma 18. We show that for each $(\xi^*, e^*, \eta^*) \in F^{-1}(0)$, the Jacobian matrix $D_{\xi,e,\eta} F(\xi^*, e^*, \eta^*)$ has full row rank. It is enough to prove that

$\Delta D_{\xi,e,\eta}F(\xi^*, e^*, \eta^*) = 0$ implies $\Delta = 0$, where

$$\Delta := ((\Delta x_h, \Delta\lambda_h)_{h\in\mathcal{H}}, (\Delta y_j, \Delta\alpha_j)_{j\in\mathcal{J}}, \Delta p^{\setminus}) \in \mathbb{R}^{H(C+1)} \times \mathbb{R}^{J(C+1)} \times \mathbb{R}^{C-1}$$

The computation of $D_{\xi,e,\eta}F(\xi^*, e^*, \eta^*)$ is described in Subsection 6.2 and the system $\Delta D_{\xi,e,\eta}F(\xi^*, e^*, \eta^*) = 0$ is written in detail below.

We remind that

$$\begin{cases}
(1) \sum_{h\in\mathcal{H}} \Delta x_h D^2_{x_k x_h} u_h(x^*_h, x^*_{-h}, y^*) - \Delta\lambda_k p^* - \sum_{j\in\mathcal{J}} \alpha^*_j \Delta y_j D^2_{x_k y_j} t_j(y^*_j, y^*_{-j}, x^*; \eta^*_j) + \\
- \sum_{j\in\mathcal{J}} \Delta\alpha_j D_{x_k} t_j(y^*_j, y^*_{-j}, x^*; \eta^*_j) + \Delta p^{\setminus} [I_{C-1}|0] = 0, \ \forall\, k \in \mathcal{H} \\
(2) \ -\Delta x_h \cdot p^* = 0, \ \forall\, h \in \mathcal{H} \\
(3) \sum_{h\in\mathcal{H}} \Delta x_h D^2_{y_f x_h} u_h(x^*_h, x^*_{-h}, y^*) + \sum_{h\in\mathcal{H}} \Delta\lambda_h s_{fh} p^* - \sum_{j\in\mathcal{J}} \alpha^*_j \Delta y_j D^2_{y_f y_j} t_j(y^*_j, y^*_{-j}, x^*; \eta^*_j) + \\
- \sum_{j\in\mathcal{J}} \Delta\alpha_j D_{y_f} t_j(y^*_j, y^*_{-j}, x^*; \eta^*_j) - \Delta p^{\setminus} [I_{C-1}|0] = 0, \ \forall\, f \in \mathcal{J} \\
(4) \ -\Delta y_j \cdot D_{y_j} t_j(y^*_j, y^*_{-j}, x^*; \eta^*_j) = 0, \ \forall\, j \in \mathcal{J} \\
(5) \ \Delta\lambda_h p^* - \Delta p^{\setminus} [I_{C-1}|0] = 0, \ \forall\, h \in \mathcal{H} \\
(6) \ -\sum_{h\in\mathcal{H}} \lambda^*_h \Delta x^{\setminus}_h - \sum_{h\in\mathcal{H}} \Delta\lambda_h (x^{*\setminus}_h - e^{*\setminus}_h - \sum_{j\in\mathcal{J}} s_{jh} y^{*\setminus}_j) + \sum_{j\in\mathcal{J}} \Delta y^{\setminus}_j = 0 \\
(7) \ -\alpha^*_j \Delta y_j D^2_{\eta_j y_j} t_j(y^*_j, y^*_{-j}, x^*; \eta^*_j) - \Delta\alpha_j D_{\eta_j} t_j(y^*_j, y^*_{-j}, x^*; \eta^*_j) = 0, \ \forall\, j \in \mathcal{J}
\end{cases}$$

Using the definition given in (16), we have that

$D_{\eta_j} t_j(y^*_j, y^*_{-j}, x^*; \eta^*_j) = -D_{y_j} t_j(y^*_j, y^*_{-j}, x^*; \eta^*_j)$ and
$D^2_{\eta_j y_j} t_j(y^*_j, y^*_{-j}, x^*; \eta^*_j) = -D^2_{y_j} t_j(y^*_j, y^*_{-j}, x^*; \eta^*_j)$

Then, for every $j \in \mathcal{J}$ equation (7) becomes

$$\alpha^*_j \Delta y_j D^2_{y_j} t_j(y^*_j, y^*_{-j}, x^*; \eta^*_j) + \Delta\alpha_j D_{y_j} t_j(y^*_j, y^*_{-j}, x^*; \eta^*_j) = 0$$

Multiplying the equation above by Δy_j and using equation (4), since $\alpha^*_j > 0$ one gets $\Delta y_j D^2_{y_j} t_j(y^*_j, y^*_{-j}, x^*; \eta^*_j)(\Delta y_j) = 0$. So, equation (4) and Point 3 of Assumption 1 imply $\Delta y_j = 0$. Then, the equation above implies that $\Delta\alpha_j D_{y_j} t_j(y^*_j, y^*_{-j}, x^*; \eta^*_j) = 0$, and so $\Delta\alpha_j = 0$ by Point 4 of Assumption 1. Thus, $(\Delta y_j, \Delta\alpha_j) = 0$ for every $j \in \mathcal{J}$.

Since $p^{*C} = 1$, from equation (5) one gets $\Delta\lambda_h = 0$ for all $h \in \mathcal{H}$, and so

$\Delta p^{\setminus} = 0$. Thus, the above system becomes

$$\begin{cases} (1) \quad \sum_{h \in \mathcal{H}} \Delta x_h D^2_{x_k x_h} u_h(x^*_h, x^*_{-h}, y^*) = 0, \ \forall \ k \in \mathcal{H} \\ (2) \quad -\Delta x_h \cdot p^* = 0, \ \forall \ h \in \mathcal{H} \\ (3) \quad \sum_{h \in \mathcal{H}} \Delta x_h D^2_{y_f x_h} u_h(x^*_h, x^*_{-h}, y^*) = 0, \ \forall \ f \in \mathcal{J} \\ (6) \quad -\sum_{h \in \mathcal{H}} \lambda^*_h \Delta x^{\setminus}_h = 0 \end{cases} \tag{19}$$

$F^{h.1}(\xi^*, e^*, \eta^*) = 0$ and equation (2) in system (19) imply that $\Delta x_h \in \text{Ker } D_{x_h} u_h(x^*_h, x^*_{-h}, y^*)$ for every $h \in \mathcal{H}$. Now, for every $h \in \mathcal{H}$ define $v_h := \lambda^*_h \Delta x_h$. Thus, the vector $(x^*_h, v_h)_{h \in \mathcal{H}}$ satisfies the following conditions.

$$(v_h)_{h \in \mathcal{H}} \in \prod_{h \in \mathcal{H}} \text{Ker } D_{x_h} u_h(x^*_h, x^*_{-h}, y^*) \text{ and } \sum_{h \in \mathcal{H}} v_h = 0 \tag{20}$$

where the last equality comes from equation (6) in system (19). Multiplying by v_k both sides of equation (1) in system (19) and using the definition of v_h, one gets $\sum_{h \in \mathcal{H}} \frac{v_h}{\lambda^*_h} D^2_{x_k x_h} u_h(x^*_h, x^*_{-h}, y^*)(v_k) = 0$ for every $k \in \mathcal{H}$. Summing up $k \in \mathcal{H}$, we obtain $\sum_{h \in \mathcal{H}} \frac{v_h}{\lambda^*_h} \sum_{k \in \mathcal{H}} D^2_{x_k x_h} u_h(x^*_h, x^*_{-h}, y^*)(v_k) = 0$. $F^{h.1}(\xi^*, e^*, \eta^*) = 0$ implies that the gradients $(D_{x_h} u_h(x^*_h, x^*_{-h}, y^*))_{h \in \mathcal{H}}$ are positively proportional. So, by (20) all the conditions of Assumption 10 are satisfied, and then $v_h = 0$ for each $h \in \mathcal{H}$ since $\lambda^*_h > 0$. Thus, we get $\Delta x_h = 0$ for all $h \in \mathcal{H}$, and consequently $\Delta = 0$ which completes the proof.

■

6 Appendix

6.1 Regular values and transversality

The theory of general equilibrium from a differentiable view is based on results from differential topology. First, we remind the definition of a regular value. Second, we summarize the results used in our analysis. These results, as well as generalizations on these issues, can be found for instance in Guillemin and Pollack (1974), Mas-Colell (1985) and Villanacci et al. (2002).

Definition 20 *Let M, N be C^r manifolds of dimensions m and n, respectively. Let $f : M \to N$ be a C^r function with $r \geq 1$. An element $y \in N$ is a* *regular value for f if for every $x^* \in f^{-1}(y)$, the differential mapping $Df(x^*)$ is onto.*

Theorem 21 *(Regular Value Theorem) Let M, N be C^r manifolds of dimensions m and n, respectively. Let $f : M \to N$ be a C^r function with $r \geq 1$. If $y \in N$ is a regular value for f, then*

(1) if $m < n$, $f^{-1}(y) = \emptyset$,
(2) if $m \geq n$, either $f^{-1}(y) = \emptyset$, or $f^{-1}(y)$ is an $(m-n)$-dimensional submanifold of M.

Corollary 22 *Let M, N be C^r manifolds of the same dimension. Let $f : M \to N$ be a C^r function with $r \geq 1$. Let $y \in N$ a regular value for f such that $f^{-1}(y)$ is non-empty and compact. Then, $f^{-1}(y)$ is a finite subset of M.*

The following results is a consequence of Sard's Theorem for manifolds.

Theorem 23 *(Transversality Theorem) Let M, Ω and N be C^r manifolds of dimensions m, p and n, respectively. Let $f : M \times \Omega \to N$ be a C^r function, assume $r > \max\{m-n, 0\}$. If $y \in N$ is a regular value for f, then there exists a full measure subset Ω^* of Ω such that for any $\omega \in \Omega^*$, $y \in N$ is a regular value for f_ω, where*

$$f_\omega : \xi \in M \to f_\omega(\xi) := f(\xi, \omega) \in N$$

Definition 24 *Let (X, d) and (Y, d') be two metric spaces. A function $\pi : X \to Y$ is proper if it is continuous and one among the following conditions holds true.*

(1) π is closed and $\pi^{-1}(y)$ is compact for each $y \in Y$,
(2) if K is a compact subset of Y, then $\pi^{-1}(K)$ is a compact subset of X,
(3) if $(x^n)_{n\in\mathbb{N}}$ is a sequence in X such that $(\pi(x^n))_{n\in\mathbb{N}}$ converges in Y, then $(x^n)_{n\in\mathbb{N}}$ has a converging subsequence in X.

The conditions above are equivalent.

6.2 Jacobian matrix of the equilibrium function

The computation of $D_{\xi,e,\eta}F(\xi^*, e^*, \eta^*)$ is described below. Vectors are treated as row matrices. The symbol "T" means transpose. 0 denotes the zero vector. With innocuous abuse of notation, the dimension of 0 is C or $C-1$ depending on the dimension of the respective block of columns. $\mathbf{0}$ denotes the zero matrix. With innocuous abuse of notation, the size of $\mathbf{0}$ is $C \times C$ or $(C-1) \times (C-1)$ depending on the size of the respective block of rows and columns. $\widehat{I} := [I_{C-1}|0^T]_{(C-1)\times C}$ where I_{C-1} denotes the identity matrix of size $(C-1)$.

$$
\begin{array}{ccccccccccccccccccccccccccccc}
x_1 & \lambda_1 & \dots & x_h & \lambda_h & \dots & x_H & \lambda_H & y_1 & \alpha_1 & \dots & y_j & \alpha_j & \dots & y_J & \alpha_J & e_1 & \dots & e_h & \dots & e_H & p^{\backslash} & \eta_1 & \dots & \eta_j & \dots & \eta_J
\end{array}
$$

$$
\begin{pmatrix}
D^2_{x_1}u_1 & -p^{*T} & \dots & D^2_{x_hx_1}u_1 & 0^T & \dots & D^2_{x_Hx_1}u_1 & 0^T & D^2_{y_1x_1}u_1 & 0^T & \dots & D^2_{y_jx_1}u_1 & 0^T & \dots & D^2_{y_Jx_1}u_1 & 0^T & \mathbf{0} & \dots & \mathbf{0} & \dots & \mathbf{0} & -\lambda_1^*\widehat{I}^T & \mathbf{0} & \dots & \mathbf{0} & \dots & \mathbf{0} \\
-p^* & 0 & \dots & 0 & 0 & \dots & 0 & 0 & s_{11}p^* & 0 & \dots & s_{j1}p^* & 0 & \dots & s_{J1}p^* & 0 & p^* & \dots & 0 & \dots & 0 & -(x_1^{*\backslash}-e_1^{*\backslash}-\sum_{j\in\mathcal{J}} s_{j1}y_j^{*\backslash}) & 0 & \dots & 0 & \dots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
D^2_{x_1x_h}u_h & 0^T & \dots & D^2_{x_h}u_h & -p^{*T} & \dots & D^2_{x_Hx_h}u_h & 0^T & D^2_{y_1x_h}u_h & 0^T & \dots & D^2_{y_jx_h}u_h & 0^T & \dots & D^2_{y_Jx_h}u_h & 0^T & \mathbf{0} & \dots & \mathbf{0} & \dots & \mathbf{0} & -\lambda_h^*\widehat{I}^T & \mathbf{0} & \dots & \mathbf{0} & \dots & \mathbf{0} \\
0 & 0 & \dots & -p^* & 0 & \dots & 0 & 0 & s_{1h}p^* & 0 & \dots & s_{jh}p^* & 0 & \dots & s_{Jh}p^* & 0 & 0 & \dots & p^* & \dots & 0 & -(x_h^{*\backslash}-e_h^{*\backslash}-\sum_{j\in\mathcal{J}} s_{jh}y_j^{*\backslash}) & 0 & \dots & 0 & \dots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
D^2_{x_1x_H}u_H & 0^T & \dots & D^2_{x_hx_H}u_H & 0^T & \dots & D^2_{x_H}u_H & -p^{*T} & D^2_{y_1x_H}u_H & 0^T & \dots & D^2_{y_jx_H}u_H & 0^T & \dots & D^2_{y_Jx_H}u_H & 0^T & \mathbf{0} & \dots & \mathbf{0} & \dots & \mathbf{0} & -\lambda_H^*\widehat{I}^T & \mathbf{0} & \dots & \mathbf{0} & \dots & \mathbf{0} \\
0 & 0 & \dots & 0 & 0 & \dots & -p^* & 0 & s_{1H}p^* & 0 & \dots & s_{jH}p^* & 0 & \dots & s_{JH}p^* & 0 & 0 & \dots & 0 & \dots & p^* & -(x_h^{*\backslash}-e_h^{*\backslash}-\sum_{j\in\mathcal{J}} s_{jh}y_j^{*\backslash}) & 0 & \dots & 0 & \dots & 0 \\
-\alpha_1^*D^2_{x_1y_1}t_1 & 0^T & \dots & -\alpha_1^*D^2_{x_hy_1}t_1 & 0^T & \dots & -\alpha_1^*D^2_{x_Hy_1}t_1 & 0^T & -\alpha_1^*D^2_{y_1}t_1 & -D_{y_1}t_1^T & \dots & -\alpha_1^*D^2_{y_jy_1}t_1 & 0^T & \dots & -\alpha_1^*D^2_{y_Jy_1}t_1 & 0^T & \mathbf{0} & \dots & \mathbf{0} & \dots & \mathbf{0} & \widehat{I}^T & -\alpha_1^*D^2_{\eta_1y_1}t_1 & \dots & \mathbf{0} & \dots & \mathbf{0} \\
-D_{x_1}t_1 & 0 & \dots & -D_{x_h}t_1 & 0 & \dots & -D_{x_H}t_1 & 0 & -D_{y_1}t_1 & 0 & \dots & -D_{y_j}t_1 & 0 & \dots & -D_{y_J}t_1 & 0 & 0 & \dots & 0 & \dots & 0 & 0 & -D_{\eta_1}t_1 & \dots & 0 & \dots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
-\alpha_j^*D^2_{x_1y_j}t_j & 0^T & \dots & -\alpha_j^*D^2_{x_hy_j}t_j & 0^T & \dots & -\alpha_j^*D^2_{x_Hy_j}t_j & 0^T & -\alpha_j^*D^2_{y_1y_j}t_j & 0^T & \dots & -\alpha_j^*D^2_{y_j}t_j & -D_{y_j}t_j^T & \dots & -\alpha_j^*D^2_{y_Jy_j}t_j & 0^T & \mathbf{0} & \dots & \mathbf{0} & \dots & \mathbf{0} & \widehat{I}^T & \mathbf{0} & \dots & -\alpha_j^*D^2_{\eta_jy_j}t_j & \dots & \mathbf{0} \\
-D_{x_1}t_j & 0 & \dots & -D_{x_h}t_j & 0 & \dots & -D_{x_H}t_j & 0 & -D_{y_1}t_j & 0 & \dots & -D_{y_j}t_j & 0 & \dots & -D_{y_J}t_j & 0 & 0 & \dots & 0 & \dots & 0 & 0 & 0 & \dots & -D_{\eta_j}t_j & \dots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
-\alpha_J^*D^2_{x_1y_J}t_J & 0^T & \dots & -\alpha_J^*D^2_{x_hy_J}t_J & 0^T & \dots & -\alpha_J^*D^2_{x_Hy_J}t_J & 0^T & -\alpha_J^*D^2_{y_1y_J}t_J & 0^T & \dots & -\alpha_J^*D^2_{y_jy_J}t_J & 0^T & \dots & -\alpha_J^*D^2_{y_J}t_J & -D_{y_J}t_J^T & \mathbf{0} & \dots & \mathbf{0} & \dots & \mathbf{0} & \widehat{I}^T & \mathbf{0} & \dots & \mathbf{0} & \dots & -\alpha_J^*D^2_{\eta_Jy_J}t_J \\
-D_{x_1}t_J & 0 & \dots & -D_{x_h}t_J & 0 & \dots & -D_{x_H}t_J & 0 & -D_{y_1}t_J & 0 & \dots & -D_{y_j}t_J & 0 & \dots & -D_{y_J}t_J & 0 & 0 & \dots & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & -D_{\eta_J}t_J \\
\widehat{I} & 0^T & \dots & \widehat{I} & 0^T & \dots & \widehat{I} & 0^T & -\widehat{I} & 0^T & \dots & -\widehat{I} & 0^T & \dots & -\widehat{I} & 0^T & -\widehat{I} & \dots & -\widehat{I} & \dots & -\widehat{I} & \mathbf{0} & 0^T & \dots & 0^T & \dots & 0^T
\end{pmatrix}
$$

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