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Uniformity and games decomposition

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Abstract

We introduce the classes of uniform and non-interactive games. We study appropriate projection operators over the space of finite games in order to propose a novel canonical direct-sum decomposition of an arbitrary game into three components, which we refer to as the uniform with zero-constant, the non-interactive total-sum zero and the constant components. We prove orthogonality between the components with respect to a natural extension of the standard inner product and we further provide explicit expressions for the closest uniform and non-interactive games to a given game. Then, we characterize the set of its approximate equilibria in terms of the uniformly mixed and dominant strategies equilibria profiles of its closest uniform and non-interactive games respectively.

Keywords: decomposition of games, projection operator, uniformly mixed strategy.

JEL Classification: C70, C79 **AMS Classification**: 91A70.

1 Introduction

The class of finite games can be seen as a finite-dimensional vector space. Several approaches to decompose a game into simpler games which admit more tractable equilibrium analysis have been proposed so far in the literature. The study of components with distinct equilibrium properties allows to gain insights on the static and dynamic features of an arbitrary game. We next provide a brief description of two novel classes of games that appear as components in the decomposition result we contribute in this work.

The class of uniform games. The mixed strategy such that each action is selected with equal probability is called *uniform strategy* and the profile it induces, a *uniformly mixed*

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strategy profile. *Uniform games* are strategic-form finite games in which the uniformly mixed strategy profile appears as a Nash equilibrium. In other words, each player's sum of payoffs over the action profiles chosen by the rest of the players is constant. Well-known games such as matching pennies and rock-paper-scissors are uniform games. Since playing a uniform strategy is arguably the simplest way of mixing pure strategies, uniformly mixed strategy equilibria can be viewed as somewhat lying in between pure and mixed Nash equilibria. Uniform strategies are also easier to implement and may therefore be seen as a model of *bounded rationality* (see Rubinstein [1998]). *Group games*, introduced by Capraro and Scarsini [2013] belong to the class of uniform games. Likewise, *harmonic games* appeared in Candogan et al. [2011], are uniform.

Potential games and the non-interactive subclass. *Potential games* became a tractable research topic since the seminal paper of Monderer and Shapley [1996] due to their desirable equilibrium properties, i.e., existence of a pure Nash equilibrium. The structure of these games and the convergence of simple player dynamics to a Nash equilibrium (see Neyman [1997]) qualified them to play an important role in game-theoretic analysis. Consequently, such games found numerous applications in various control and resource allocation problems. They can be regarded as games in which the interests of players are aligned with a global potential function.
We introduce a particular subclass of potential games, the *non-interactive games*. In such games, each player's payoff depends only on his action. This class forms a subspace of potential games which admit a dominant strategy equilibrium with unique payoff that results when each player plays the action that maximizes his payoff. Therefore, players in such games do not interact with each other and every decision is taken ignoring the opponents' moves.

The idea of decomposing a game appears first in Von Neumann and Morgenstern [1945], where games with large number of players are decomposed into games with fewer players.
In the context of non-cooperative game theory, a decomposition of games in normal form appeared in Sandholm [2010], where for each subset of players, a component game is defined and the author achieves a decomposition of normal form games suggesting alternative ways to verify whether a game is potential or not. Given a two-player game, Kalai and Kalai [2010] propose a decomposition into zero-sum and common-interest components. Hence, the authors highlight cooperation-related issues that emerge in games where players strategically interact with each other. More precisely, in the cooperative component the payoff is defined at each action profile as the average of both players' payoffs while in the competitive one as the difference of players' payoffs divided by two. In general, the common-interest component is a potential game which can be used to approximate an arbitrary game. However, this approximation needs not yield the closest potential game to a given game. The better approximation, achieved in Candogan et al. [2011], lies in the fact that their decomposition result clearly separates the strategic and non-strategic components and further identifies components such as potential and harmonic games. To that end, the authors, associating to each finite game an indirected graph, propose a novel flow representation of the payoff structure in finite games, which enables to identify the fundamental characteristics in preferences that lead to potential games. Their main result relies on the *Helmholtz-Hodge decomposition*

theorem[1], a well-known mathematical tool of vector calculus that allows one to represent a vector field as the sum of a *divergence-free* and a *curl-free* vector fields. The authors provide a direct orthogonal sum decomposition of an arbitrary game into the *potential*, *harmonic* and *non-strategic* components, each with distinct equilibrium properties. The graph representation of games and the flows defined on this graph lead to a natural equivalence relation in the space of games such that games sharing identical payoff differences for any deviating player belong in the same class. Several notions of strategical equivalence[2] have been proposed in the game theory literature, to generalize the desirable static and dynamic properties of games to their equivalence classes. In Hwanga and Rey-Belletb [2014], following the definition of strategical equivalence introduced in Candogan et al. [2011], the authors provide decomposition results of an arbitrary finite game by identifying components such as potential games and games that are *strategically equivalent to zero-sum games*. The authors further prove analogous results in games with continuous strategy sets and establish an alternative proof of the charaterization for potential games given by Monderer and Shapley [1996]. Finally, Jessie and Saari [2013] present a *strategic* and *behavioral* decomposition of games with two actions. The authors highlight that certain solution concepts are either determined by a game's strategic part, or influenced by its behavioral component.

In this paper, we provide a direct-sum decomposition of an arbitrary game into three components, which we refer to as the *uniform with zero-constant*, the *constant* and the *non-interactive total-sum zero* components. The former component refers to games where each player's sum of payoffs over the action profiles chosen by the rest of the players yields zero. The second class describes games where each player's payoff is constant over all action profiles. The latter component refers to non-interactive games, in which each player's sum of payoffs over all action profiles is equal to zero. These components are orthogonal with respect to a natural extension of the standard inner product. Moreover, the sum of the two former and the sum of the two latter components constitute the space of uniform and non-interactive games respectively. As a consequence, using the induced distance in the space of games, given a finite game, we characterize its approximate equilibrium set in terms of the dominant strategy equilibria with unique payoff of its closest non-interactive game and the uniformly mixed strategy equilibrium of its closest uniform game. Namely, the dominant strategy equilibrium profile of the corresponding non-interactive game turns out to be an ε-equilibrium of the given game, for some ε whose upper bound is specified. Likewise, the uniformly mixed strategy equilibrium profile of the projection onto the uniform component is an $\tilde{\varepsilon}$-equilibrium of the original game for some $\tilde{\varepsilon}$, which is bounded by a constant as well.

Structure of the paper. The remainder of the paper is organized as follows: In Section 2, we present the relevant definitions and notations. In Section 3, we introduce the classes of uniform and non-interactive games and present the main results of this work. Section 4 deals with examples highlighting the value of the suggested decomposition. Section 5 concludes the paper and outlines future perspectives.

[1] An implementation of the Helmholtz decomposition in Statistical ranking is given by Jiang et al. [2011].

[2] The reader is referred to Moulin and Vial [1978], Hofbauer and Hopkins [2000], Mertens [2004], Morris and Ui [2004], Germano [2006], Candogan et al. [2011]

2 Basic definitions and notations

A *finite game* consists of:

- a finite set of players, denoted by $N = \{1, ..., n\}$.
- a finite set of actions S^i, for each $i \in N$. The joint strategy space is $S = \prod_{i \in N} S^i$.
- a payoff function of each player $i \in N$, denoted by $g^i : S \to \mathbb{R}$.

Accordingly, a finite game is given by the triplet $\big(N, (S^i)_{i \in N}, (g^i)_{i \in N}\big)$, which for notational convenience will often be abbreviated to $\mathbf{g}$, where $\mathbf{g} = (g^i)_{i \in N}$. We use the notation $s^i \in S^i$ for an action of player i and an action profile is given by $\mathbf{s} = (s^i)_{i \in N}$. A collection of actions for all players but the i-th one, is denoted by $s^{-i} \in S^{-i}$. We set $h_i = |S^i|$ for the cardinality of the action set of player i, and $|S| = \prod_{i \in N} h_i$ for the overall cardinality of the action space. Given N and S, every game is uniquely defined by its set of payoff functions. The payoff function of each player can be viewed as an element of $\mathbb{R}^{|S|}$, i.e., $g^i \in \mathbb{R}^{|S|}$ for any $i \in N$. Hence, the space of games with set of players N and joint action space S can be identified as $\mathcal{G} \cong \mathbb{R}^{n|S|}$. We denote the *player-specific space of games* by $\mathcal{G}^i$ and so $\mathcal{G} = \prod_{i \in N} \mathcal{G}^i$. For any subspace $X \subseteq \mathcal{G}$, we have $X = \prod_{i \in N} X^i$.

The basic solution concept of a game is the one of *Nash equilibrium*. An action profile $\mathbf{s} = (s^1, ..., s^n)$ is an ϵ-equilibrium if for all $i \in N$ and all $t^i \in S^i$,

$$g^i(s^i, s^{-i}) \geq g^i(t^i, s^{-i}) - \epsilon.$$

A pure Nash equilibrium is said to be an ϵ-equilibrium with $\epsilon = 0$.

Games in which each player's payoff depends only on the actions selected by the rest of the players are called *non-strategic games*.

Definition 2.1. A *non-strategic game* is a finite game in which for any $i \in N$, there exists a function $\ell^i : S^{-i} \to \mathbb{R}$, such that $\ell^i(s^{-i}) = g^i(s^i, s^{-i})$. The space of non-strategic games will be denoted by $\mathcal{NS}$.

Actual payoffs are not required for the identification of Nash equilibria, as long as the payoff differences are well-defined. Taking advantage of this fact and following Candogan et al. [2011] and Hwanga and Rey-Belletb [2014], we define an equivalence relation in the space of games, such that each class consists of games that share identical payoff differences for any deviating player and thus, also identical equilibrium sets.

Definition 2.2. The games $\mathbf{g}_1$ and $\mathbf{g}_2$ are *strategically equivalent* ($\mathbf{g}_1 \sim \mathbf{g}_2$), if for all $i \in N$ and all $s^{-i} \in S^{-i}$,

$$g_1^i(s^i, s^{-i}) - g_1^i(t^i, s^{-i}) = g_2^i(s^i, s^{-i}) - g_2^i(t^i, s^{-i}),$$

for any $s^i, t^i \in S^i$.

Proposition 2.3. *If $\boldsymbol{g}_1 \sim \boldsymbol{g}_2$ then the game defined as $\boldsymbol{g} = \boldsymbol{g}_1 - \boldsymbol{g}_2$ is a non-strategic game.*

Proof. By Definition 2.2, for all $i \in N$, all $s^{-i} \in S^{-i}$ and all $s^i, t^i \in S^i$, we have:

$$g_1^i(s^i, s^{-i}) - g_2^i(s^i, s^{-i}) = g_1^i(t^i, s^{-i}) - g_2^i(t^i, s^{-i}),$$

that proves the result.

□

A common way to fix a representative for strategically equivalent games is by means of normalization.

Definition 2.4. A finite game $\mathbf{g}$ is *normalized*, if for all $i \in N$ and all $s^{-i} \in S^{-i}$,

$$\sum_{s^i \in S^i} g^i(s^i, s^{-i}) = 0.$$

The space of normalized games will be denoted by $\mathcal{NO}$.

In normalized games, each player's expected payoff, when playing all of his actions with equal probability (uniform strategy), is equal to zero.

The standard inner product in $\mathbb{R}^{|S|}$ is given by:

$$\langle f_1, f_2 \rangle_0 = \sum_{\mathbf{s} \in S} f_1(\mathbf{s}) f_2(\mathbf{s}). \tag{2.1}$$

We will use a natural extension of the standard inner product on $\mathcal{G}$, which was also adopted by Candogan et al. [2011] and Hwanga and Rey-Belletb [2014]:

$$\langle \mathbf{g}_1, \mathbf{g}_2 \rangle_{\mathcal{G}} = \sum_{i \in N} h_i \langle g_1^i, g_2^i \rangle_0, \tag{2.2}$$

where the inner product in the right-hand side is the inner product on $\mathbb{R}^{|S|}$, defined in (2.1). The inner product in (2.2) induces a norm that will help us to quantify the distance between games. The norm on $\mathcal{G}$ is defined as follows:

$$\|\mathbf{g}\|_{\mathcal{G}}^2 = \langle \mathbf{g}, \mathbf{g} \rangle_{\mathcal{G}}. \tag{2.3}$$

For any $X, Y \subseteq \mathcal{G}$, the notation $X \ominus Y$ stands for the direct orthogonal sum of X and Y.

3 Decomposition of the games space

In this section, we proceed as follows: In Paragraph 3.1, we present a first orthogonal decomposition of the space of finite games with respect to the inner product in (2.2). In Paragraph 3.2, we define the classes of uniform and non-interactive games and we end up establishing two more orthogonal decompositions of finite games. Finally, in Paragraph 3.3 we provide our main results.

3.1 First Decomposition

Given a finite game $\mathbf{g}$, we introduce for each player i, the linear operator $\Pi^i : \mathbb{R}^{|S|} \to \mathbb{R}^{|S|}$, defined by:

$$\Pi^i g^i(s^i, s^{-i}) = g^i(s^i, s^{-i}) - \frac{1}{h_i} \sum_{s^i \in S^i} g^i(s^i, s^{-i}), \tag{3.1}$$

and we further define $\Pi : \mathbb{R}^{n|S|} \to \mathbb{R}^{n|S|}$, such that $\Pi\mathbf{g} = \left(\Pi^1 g^1, ..., \Pi^n g^n\right)$.
Moreover, we introduce for each player i the linear operator $\Lambda^i : \mathbb{R}^{|S|} \to \mathbb{R}^{|S|}$, defined as:

$$\Lambda^i g^i(s^i, s^{-i}) = \frac{1}{h_i} \sum_{s^i \in S^i} g^i(s^i, s^{-i}). \tag{3.2}$$

Similarly, we define $\Lambda : \mathbb{R}^{n|S|} \to \mathbb{R}^{n|S|}$, such that $\Lambda\mathbf{g} = \left(\Lambda^1 g^1, ..., \Lambda^n g^n\right)$. The notation I stands for the identity operator and we clearly have $I = \Pi + \Lambda$.

Proposition 3.1. *The space of games is the direct orthogonal sum, with respect to the inner product introduced in (2.2), of the normalized and non-strategic games, i.e.,*

$$\mathcal{G} = \mathcal{NO} \oplus \mathcal{NS}, \tag{3.3}$$

where $\mathcal{NO} = \text{Im}(\Pi) = \text{Ker}(\Lambda)$ *and* $\mathcal{NS} = \text{Ker}(\Pi) = \text{Im}(\Lambda)$.

Proof. We work on the operators Π^i, Λ^i and we equivalently obtain the result on Π and Λ using their definitions. It is easy to check that $\Pi^i \circ \Pi^i = \Pi^i$ and $\Lambda^i \circ \Lambda^i = \Lambda^i$. Therefore, the operators Π^i and Λ^i are projections. It follows that $\mathcal{G}^i = \text{Im}(\Pi^i) \oplus \text{Ker}(\Pi^i)$ and since Λ^i is the complementary projection operator of Π^i, we get $\mathcal{G}^i = \text{Im}(\Pi^i) \oplus \text{Im}(\Lambda^i)$.
Let $\mathbf{g}$ be a normalized game. By (3.1), for any $i \in N$, $\Pi^i g^i = g^i$ and thus, Π^i is the projection operator onto the normalized payoffs of player i. Conversely, let $\mathbf{g}$ be a game such that for any $i \in N$, $g^i \in \text{Im}(\Pi^i)$. Using (3.1), we get $\sum_{s^i \in S^i} g^i(s^i, s^{-i}) = 0$ and by definition of Π it follows $\text{Im}(\Pi) = \mathcal{NO}$.
For the rest of the proof, let $\mathbf{g}$ be a game such that for any $i \in N$, $g^i \in \text{Im}(\Lambda^i)$. By (3.2), for all $i \in N$ and all $\mathbf{s} \in S$, we have:

$$g^i(s^i, s^{-i}) = \frac{1}{h_i} \sum_{s^i \in S^i} g^i(s^i, s^{-i})$$

and by Definition 2.1, it is immediate that $\mathbf{g} \in \mathcal{NS}$. Now, let $\mathbf{g} \in \mathcal{NS}$. Likewise, Definition 2.1 indicates that for any $\mathbf{s} \in S$, $g^i(s^i, s^{-i}) = (1/h_i) \sum_{s^i \in S^i} g^i(s^i, s^{-i})$ and thus, $\Lambda^i g^i = g^i$. Hence, $\text{Im}(\Lambda) = \mathcal{NS}$. To show orthogonality between the components, let $\mathbf{g}_0 \in \mathcal{NO}$ and $\mathbf{g}_{\mathcal{NS}} \in \mathcal{NS}$. Then,

$$\begin{aligned}\langle \mathbf{g}_0, \mathbf{g}_{\mathcal{NS}} \rangle_{\mathcal{G}} = \sum_{i \in N} h_i \langle g_0^i, g_{\mathcal{NS}}^i \rangle_0 &= \sum_{i \in N} h_i \sum_{\mathbf{s} \in S} g_0^i(\mathbf{s}) g_{\mathcal{NS}}^i(\mathbf{s}) \\ &= \sum_{i \in N} h_i \sum_{s^{-i} \in S^{-i}} \sum_{s^i \in S^i} g_0^i(s^i, s^{-i}) g_{\mathcal{NS}}^i(s^i, s^{-i}),\end{aligned}$$

where the middle equality follows from (2.1). In view of Definition 2.1, for all $s^{-i} \in S^{-i}$, $g^i_{\mathcal{NS}}(s^i, s^{-i}) = \ell^i(s^{-i})$ for some $\ell^i : S^{-i} \to \mathbb{R}$. Hence,

$$\sum_{s^i \in S^i} g^i_0(s^i, s^{-i}) g^i_{\mathcal{NS}}(s^i, s^{-i}) = \ell^i(s^{-i}) \sum_{s^i \in S^i} g^i_0(s^i, s^{-i}) = 0,$$

where the latter equality follows from Definition 2.3. □

3.2 Uniform and non-interactive classes of games

In this section, we introduce the classes of uniform and non-interactive games and we then characterize their equilibrium sets.

3.2.1 Uniform games

We first define the uniformly mixed strategy profile and the class of uniform games.

Definition 3.2. The *uniform strategy* of player $i \in N$ is the mixed strategy that selects each of his actions $s^i \in S^i$ with equal probability $(1/h_i)$. The induced strategy profile is called *uniformly mixed strategy profile.*

Definition 3.3. A finite game is *uniform* if and only if the uniformly mixed strategy profile is a Nash equilibrium. The class of uniform games is denoted by $\mathcal{U}$.

Clearly, uniform games form a subspace of games. An aftereffect of Definition 3.2, is that in uniform games the sum of each player's payoffs, fixing any of his actions, over the action profiles made up of the rest of the players, is constant. Namely, for any $i \in N$, there exists $c^i \in \mathbb{R}$, such that for all $s^i \in S^i$,

$$\sum_{s^{-i} \in S^{-i}} g^i(s^i, s^{-i}) = c^i. \tag{3.4}$$

Definition 2.1 and (3.4) imply that the class of non-strategic games belongs in uniform games. Next proposition states that uniform games are stable by projection onto the subspace of normalized games.

Lemma 3.4. *Let us consider a game $\boldsymbol{g}$ and a uniform game $\boldsymbol{g}_{\mathcal{U}}$, such that $\boldsymbol{g} \sim \boldsymbol{g}_{\mathcal{U}}$. Then, $\boldsymbol{g}$ is a uniform game.*

Proof. By assumption, for all $i \in N$ and all $(\mathbf{s}, \mathbf{t}) \in S \times S$, such that $s^i \neq t^i$ and $s^{-i} = t^{-i}$, we have:

$$g^i(\mathbf{t}) - g^i(\mathbf{s}) = g^i_{\mathcal{U}}(\mathbf{t}) - g^i_{\mathcal{U}}(\mathbf{s}).$$

Since $\mathbf{g}_{\mathcal{U}}$ is uniform (see (3.4)), adding over $s^{-i} \in S^{-i}$ in both sides of the last equation, we get:

$$\sum_{s^{-i} \in S^{-i}} \left(g^i(t^i, s^{-i}) - g^i(s^i, s^{-i})\right) = c^i - c^i = 0.$$

Therefore, for all $i \in N$ and all $s^i, t^i \in S^i$,

$$\sum_{s^{-i} \in S^{-i}} g^i(s^i, s^{-i}) = \sum_{s^{-i} \in S^{-i}} g^i(t^i, s^{-i}),$$

that concludes the proof. □

3.2.2 Non-interactive games

In this paragraph, we introduce the non-interactive games and we show that they constitute a subspace of potential games. Then, we prove that non-interactive games admit a dominant strategy equilibrium with unique payoff.

Definition 3.5. A finite game $\mathbf{g}$ is *potential* if there exists a function $\varphi : S \to \mathbb{R}$ such that, for all $i \in N$, all $s^i, t^i \in S^i$ and all $s^{-i} \in S^{-i}$.

$$\varphi(s^i, s^{-i}) - \varphi(t^i, s^{-i}) = g^i(s^i, s^{-i}) - g^i(t^i, s^{-i}).$$

The function φ is referred to as a *potential function* of the game. The space of potential games will be denoted by $\mathcal{P}$.

In this class of games, all players are interested in maximizing the potential function; hence,

Theorem 3.6. *Potential games admit a pure Nash equilibrium.*

Definition 3.7. A *non-interactive game* is a finite game in which for all $i \in N$, there exists a function $\lambda^i : S^i \to \mathbb{R}$, such that $\lambda^i(s^i) = g^i(s^i, s^{-i})$. The class of such games is denoted by $\mathcal{NI}$.

Lemma 3.8. *The class of non-interactive games forms a subspace of potential games with a potential function to be given by $\varphi : S \to \mathbb{R}$, defined as $\varphi(\boldsymbol{s}) = \sum\limits_{i \in N} \lambda^i(s^i)$.*

Proof. By definition, it is clear that non-interactive games constitute a subspace of games. Now, let $\mathbf{g}_{\mathcal{NI}}$ be a non-interactive game. By Definition 3.7, for all $i \in N$ and all $\mathbf{s}, \mathbf{t} \in S$, such that $s^i \neq t^i$, we get:

$$g^i(s^i, s^{-i}) - g^i(t^i, s^{-i}) = \lambda^i(s^i) - \lambda^i(t^i).$$

Let $\varphi : S \to \mathbb{R}$, such that $\varphi(s^i, s^{-i}) = \sum_{i \in N} \lambda^i(s^i)$. It is easy to see that φ is a potential function for $\mathbf{g}_{\mathcal{NI}}$ and thus, by Definition 3.5, $\mathbf{g}_{\mathcal{NI}}$ is a potential game. □

An action profile $\mathbf{s} = (s^1, ..., s^n) \in S$ is a *dominant strategy equilibrium*, if for all $i \in N$ and all $\mathbf{t} = (t^i, t^{-i}) \in S$,

$$g^i(s^i, t^{-i}) \geq g^i(t^i, t^{-i}),$$

Proposition 3.9. *Every non-interactive game admits a dominant strategy equilibrium with unique payoff.*

Proof. Let $\mathbf{g} \in \mathcal{NI}$. In view of Lemma 3.8, $\mathbf{g}$ is a potential game and thus, from Theorem 3.6, it admits a pure Nash equilibrium, denoted by $\mathbf{s} = (s^1, ...s^i, ..., s^n)$. By contradiction, let us assume that $\mathbf{s}$ is not a dominant strategy equilibrium. Then, there exist $i \in N$ and $\mathbf{t} = (t^i, t^{-i}) \in S$, such that $g^i(t^i, t^{-i}) > g^i(s^i, t^{-i})$. However, since $\mathbf{g}$ is a non-interactive game, for each $i \in N$, there exists $\lambda^i : S^i \to \mathbb{R}$, such that

$$\lambda^i(s^i) = g^i(s^i, s^{-i}) = g^i(s^i, t^{-i}) < g^i(t^i, t^{-i}) = g^i(t^i, s^{-i}) \leq g^i(s^i, s^{-i}) = \lambda^i(s^i),$$

where the right inequality follows, since $\mathbf{s}$ is a pure Nash equilibrium and the proof is completed. $\square$

3.2.3 Second Decomposition

Given a finite game $\mathbf{g}$, we introduce for each player i, the linear operator $\Sigma^i : \mathbb{R}^{|S|} \to \mathbb{R}^{|S|}$, defined as follows:

$$\Sigma^i g^i(s^i, s^{-i}) = g^i(s^i, s^{-i}) - \frac{1}{\prod\limits_{j \neq i} h_j} \sum_{s^{-i} \in S^{-i}} g^i(s^i, s^{-i}) \tag{3.5}$$

We also define $\Sigma : \mathbb{R}^{n|S|} \to \mathbb{R}^{n|S|}$, such that $\Sigma\mathbf{g} = \left(\Sigma^1 g^1, ..., \Sigma^n g^n\right)$.
We further introduce for each player i, the linear operator $\Theta^i : \mathbb{R}^{|S|} \to \mathbb{R}^{|S|}$, defined by:

$$\Theta^i g^i(s^i, s^{-i}) = \frac{1}{\prod\limits_{j \neq i} h_j} \sum_{s^{-i} \in S^{-i}} g^i(s^i, s^{-i}) \tag{3.6}$$

In like manner, we have $\Theta : \mathbb{R}^{n|S|} \to \mathbb{R}^{n|S|}$, such that $\Theta\mathbf{g} = \left(\Theta^1 g^1, ..., \Theta^n g^n\right)$ and it clearly holds true: $I = \Sigma + \Theta$.

Notation. The subspace of *uniform games with zero-constant*, i.e., in (3.4) $c^i = 0$ for all $i \in N$, will be denoted by $\mathcal{U}_0$.

Proposition 3.10. *The space of games is the direct orthogonal sum, with respect to the inner product in (2.2), of the uniform games with zero-constant and non-interactive games, i.e.,*

$$\mathcal{G} = \mathcal{U}_0 \oplus \mathcal{NI},$$

where $\mathcal{U}_0 = \mathrm{Im}(\Sigma) = \mathrm{Ker}(\Theta)$ *and* $\mathcal{NI} = \mathrm{Ker}(\Sigma) = \mathrm{Im}(\Theta)$.

Proof. We work on the player-specific operators Σ^i, Θ^i and we equivalently conclude the result on Σ and Θ, using their definitions. It is easy to check that $\Sigma^i \circ \Sigma^i = \Sigma^i$ and $\Theta^i \circ \Theta^i = \Theta^i$. Therefore, Θ^i and Σ^i are projection operators. It clearly follows that $\mathcal{G}^i = \mathrm{Im}(\Sigma^i) \oplus \mathrm{Ker}(\Sigma^i)$ and since Θ^i is the complementary projection operator of Σ^i, we get that $\mathcal{G}^i = \mathrm{Im}(\Sigma^i) \oplus \mathrm{Im}(\Theta^i)$.
Given $\mathbf{g} \in \mathcal{U}_0$, in view of (3.5), it is immediate that for any $i \in N$ we have $\Sigma_i g^i = g^i$ and thus, Σ^i is the projection operator onto the uniform with zero-constant payoffs of player i. Conversely, let $\mathbf{g}$ be a game, such that for all i, $g^i \in \mathrm{Im}(\Sigma^i)$. Since by (3.5), for all $s^i \in S^i$, $\sum_{s^{-i}\in S^{-i}} g^i(s^i, s^{-i}) = 0$, using the definition of Σ, it follows $\mathrm{Im}(\Sigma) = \mathcal{U}_0$.
Now, let $\mathbf{g}$ be a game such that for any $i \in N$, $g^i \in \mathrm{Im}(\Theta^i)$. Then, by (3.6), for all $i \in N$ and all $\mathbf{s} \in S$,

$$g^i(s^i, s^{-i}) = \frac{1}{\prod\limits_{j\neq i} h_j} \sum_{s^{-i}\in S^{-i}} g^i(s^i, s^{-i}).$$

Definition 3.7 implies that $\mathbf{g} \in \mathcal{NI}$. Conversely let us assume that $\mathbf{g} \in \mathcal{NI}$. Likewise, from Definition 3.7 we have $g^i(s^i, s^{-i}) = \left(1/\prod_{j\neq i} h_j\right) \sum_{s^{-i}\in S^{-i}} g^i(s^i, s^{-i})$, which is equivalent to $\Theta^i g^i = g^i$ that concludes this part of the proof.
To show orthogonality between the components, let $\mathbf{g}_0 \in \mathcal{U}_0$ and $\mathbf{g}_{\mathcal{NI}} \in \mathcal{NI}$. Then,

$$\begin{aligned}\langle \mathbf{g}_0, \mathbf{g}_{\mathcal{NI}} \rangle_{\mathcal{G}} = \sum_{i\in N} h_i \langle g_0^i, g_{\mathcal{NI}}^i \rangle &= \sum_{i\in N} h_i \sum_{\mathbf{s}\in S} g_0^i(\mathbf{s}) g_{\mathcal{NI}}^i(\mathbf{s}) \\ &= \sum_{i\in N} h_i \sum_{s^i\in S^i} \sum_{s^{-i}\in S^{-i}} g_0^i(s^i, s^{-i}) g_{\mathcal{NI}}^i(s^i, s^{-i}),\end{aligned}$$

where the middle equality follows from (2.1). In view of Definition 3.7, for all $s^i \in S^i$ we have $g_{\mathcal{NI}}^i(s^i, s^{-i}) = \lambda^i(s^i)$. Hence,

$$\sum_{s^{-i}\in S^{-i}} g_0^i(s^i, s^{-i}) g_{\mathcal{NI}}^i(s^i, s^{-i}) = \lambda^i(s^i) \sum_{s^{-i}\in S^{-i}} g_0^i(s^i, s^{-i}) = 0,$$

where the right equality follows by definition of $\mathcal{U}_0$. □

3.2.4 Third decomposition

We first define two classes of games that appear as components in the decomposition result of this paragraph.

Definition 3.11. A finite game $\mathbf{g}$ is a *total-sum zero game*, if for all $i \in N$,

$$\sum_{\mathbf{s}\in S} g^i(\mathbf{s}) = 0.$$

The space of total-sum zero games is denoted by $\mathcal{G}_*$.

Definition 3.12. A finite game $\mathbf{g}$ is a *constant game*, if for all $i \in N$, there exists c^i, such that for all $\mathbf{s} \in S$,

$$g^i(\mathbf{s}) = c^i.$$

The space of constant games is denoted by $\mathcal{C}$.

Constant games is a subspace of non-interactive and of non-strategic games (and thus, of uniform games too). It is easy to see that $\mathcal{C} = \mathcal{NI} \cap \mathcal{NS}$.

Given a finite game $\mathbf{g}$, we define for each player i, the linear operator $\Pi^i_* : \mathbb{R}^{|S|} \to \mathbb{R}^{|S|}$, defined by:

$$\Pi^i_* g^i(\mathbf{s}) = g^i(\mathbf{s}) - \frac{1}{\prod\limits_{i\in N} h_i} \sum_{\mathbf{s}\in S} g^i(\mathbf{s}). \tag{3.7}$$

We also have $\Pi_* : \mathbb{R}^{n|S|} \to \mathbb{R}^{n|S|}$ defined as $\Pi^*\mathbf{g} = \left(\Pi^1_* g^1, ..., \Pi^n_* g^n\right)$.
For any $i \in N$, we further define $\Gamma^i : \mathbb{R}^{|S|} \to \mathbb{R}^{|S|}$ as follows:

$$\Gamma^i g^i(\mathbf{s}) = \frac{1}{\prod\limits_{i\in N} h_i} \sum_{\mathbf{s}\in S} g^i(\mathbf{s}). \tag{3.8}$$

In like manner, we have $\Gamma : \mathbb{R}^{n|S|} \to \mathbb{R}^{n|S|}$ defined as $\Gamma\mathbf{g} = \left(\Gamma^1 g^1, ..., \Gamma^n g^n\right)$ and it clearly holds $I = \Pi_* + \Gamma$.

Proposition 3.13. *The space of games is the direct orthogonal sum with respect to the inner product in (2.2), of the total-sum zero games and constant games, i.e.,*

$$\mathcal{G} = \mathcal{G}_* \bigoplus \mathcal{C},$$

where $\mathcal{G}_* = \mathrm{Im}(\Pi_*) = \mathrm{Ker}(\Gamma)$ *and* $\mathcal{C} = \mathrm{Ker}(\Pi_*) = \mathrm{Im}(\Gamma)$.

Proof. We work on the player-specific operators Π^i_*, Γ^i and we equivalently conclude the result on Π_* and Γ, using their definitions. It is easy to check that $\Pi^i_* \circ \Pi^i_* = \Pi_*$ and $\Gamma^i \circ \Gamma^i = \Gamma^i$. Therefore, Π^i_* and Γ^i are projection operators. It follows that $\mathcal{G}^i = \mathrm{Im}(\Pi^i_*) \oplus \mathrm{Im}(\Gamma^i)$ due to Γ^i is the complementary projection operator of Π^i_*.
Given $\mathbf{g} \in \mathcal{G}_*$, in view of (3.7), for any $i \in N$, $\Pi^i_* g^i = g^i$. Conversely, let $\mathbf{g} \in \mathrm{Im}(\Pi^*)$, which using (3.7) implies $\sum_{\mathbf{s}\in S} g^i(\mathbf{s}) = 0$.
Now, let $\mathbf{g} \in \mathrm{Im}(\Gamma)$. By (3.8) and Definition 3.12, it follows $\mathbf{g} \in \mathcal{C}$. To prove the inverse inclusion, let $\mathbf{g}$ be a game such that for all $i \in N$, $g^i \in \mathcal{C}^i$. Definition 3.12 implies that for all $i \in N$ and all $\mathbf{s} \in S$, we have $g^i(\mathbf{s}) = \left(1/\prod_{i\in N} h_i\right) \sum_{\mathbf{s}\in S} g^i(\mathbf{s})$ and we therefore get, $\Gamma^i g^i = g^i$.
To prove orthogonality between the components, let $\mathbf{g}_{\mathcal{G}_*} \in \mathcal{G}_*$ and $\mathbf{g}_{\mathcal{C}} \in \mathcal{C}$. Using the inner product in (2.2), we have:

$$\langle \mathbf{g}_{\mathcal{G}_*}, \mathbf{g}_{\mathcal{C}} \rangle_{\mathcal{G}} = \sum_{i\in N} h_i \sum_{\mathbf{s}\in S} g^i_{\mathcal{G}_*}(\mathbf{s}) g^i_{\mathcal{C}}(\mathbf{s}) = \sum_{i\in N} h_i c^i \sum_{\mathbf{s}\in S} g^i_{\mathcal{G}_*}(s) = 0,$$

where the middle and right equalities follow from Definition 3.12 and Definition 3.11 respectively. $\square$

Remark 3.14. The decomposition result of Proposition 3.13 holds true by projection in the subspaces of uniform and non-interactive games. Namely,

$$\mathcal{U} = \mathcal{U}_* \oplus \mathcal{C}, \qquad \mathcal{NI} = \mathcal{NI}_* \oplus \mathcal{C}.$$

3.3 The offspring decomposition

In this section, we provide our main result. To that end, we first prove next lemma:

Lemma 3.15. *The following assertions hold true:*

i. $\mathcal{U}_0 = \mathcal{U}_*$

ii. $\mathcal{NI} \cap \mathcal{NO} = \mathcal{NI}_*$

Proof. To show assertion (i), we clearly have: $\mathcal{U}_0 \subseteq \mathcal{U}_*$. Conversely, let $\mathbf{g} \in \mathcal{U}_*$. Then, for any $i \in N$, we have:

$$0 = \sum_{\mathbf{s} \in S} g^i(\mathbf{s}) = h_i c^i,$$

where left equality is due to $\mathbf{g}$ is total-sum zero and the right one follows since $\mathbf{g}$ is uniform. Hence, $c^i = 0$, for any $i \in N$ and thus, $\mathbf{g} \in \mathcal{U}_0$.
To prove assertion (ii), let $\mathbf{g} \in \mathcal{NI}_*$. Then, for any $i \in N$ we equivalently have:

$$0 = \sum_{\mathbf{s} \in S} g^i(\mathbf{s}) = \sum_{s^i \in S^i} \prod_{j \neq i} h_j \lambda^i(s^i) = \sum_{s^i \in S^i} \lambda^i(s^i) \prod_{j \neq i} h_j,$$

where the first equality is due to $\mathbf{g}$ is total-sum zero and the middle one follows since $\mathbf{g}$ is non-interactive. Therefore, we equivalently get for all $i \in N$, $\sum_{s^i \in S^i} \lambda^i(s^i) = 0$ that is $\mathbf{g} \in \mathcal{NI} \cap \mathcal{NO}$.

□

Corollary 3.16. *The space of games is the direct orthogonal sum with respect to the inner product in (2.2), of the uniform with zero-constant, the non-interactive normalized and the constant games, i.e.,*

$$\mathcal{G} = \mathcal{U}_0 \oplus \mathcal{C} \oplus (\mathcal{NI} \cap \mathcal{NO})$$

Proof. The result follows from Proposition 3.10, Proposition 3.13, Remark 3.14 and Lemma 3.15. □

Lemma 3.17. *The projection operators* Θ, Π_* *commute. Namely,*

$$\Theta \Pi_* = \Pi_* \Theta$$

Moreover, $\Theta\Gamma = \Gamma\Theta = \Gamma$.

Proof. Let $\mathbf{g} \in \mathcal{G}$ and $i \in N$. Using (3.6) and (3.7), for all $\mathbf{s} \in S$, we have:

$$\Theta^i \Pi_*^i g^i(\mathbf{s}) = \frac{\sum_{s^{-i} \in S^{-i}} g^i(\mathbf{s})}{\prod_{j \neq i} h_j} - \frac{\sum_{\mathbf{s} \in S} g^i(\mathbf{s})}{\prod_{i \in N} h_i} = \Pi_*^i \Theta^i g^i(\mathbf{s}).$$

Since constant games are a subspace of non-interactive games, it is immediate that $\Theta^i \Gamma^i = \Gamma^i \Theta^i = \Gamma^i$. We conclude the result on the operators Θ, Π_* and Γ using their definitions. □

Lemma 3.18. *The linear operator defined by* $\Theta\Pi_* : \mathbb{R}^{n|S|} \to \mathbb{R}^{n|S|}$ *is the projection onto* $\mathrm{Im}(\Theta) \cap \mathrm{Im}(\Pi_*)$ *and the linear operator defined by* $\Theta\Gamma : \mathbb{R}^{n|S|} \to \mathbb{R}^{n|S|}$ *is the projection onto* $\mathrm{Im}(\Gamma)$.

Proof. We present the proof only for the former operator since for the latter one, the result follows from Lemma 3.17, due to $\mathrm{im}(\Gamma) \subset \mathrm{im}(\Theta)$. Indeed, in view of Lemma 3.17,

$$\left(\Theta\Pi_*\right)^2 = \left(\Theta\Pi_*\right)\left(\Theta\Pi_*\right) = \left(\Pi_*\Theta\right)\left(\Theta\Pi_*\right) = \Pi_*\Theta^2\Pi_* = \left(\Pi_*\Theta\right)\Pi_* = \Theta\Pi_*^2 = \Theta\Pi_*$$

and thus, $\Theta\Pi_*$ is a projection.
Let $\mathbf{g} \in \mathbb{R}^{n|S|}$, such that $\Theta\Pi_*\mathbf{g} = \mathbf{g}$. Then, $\Theta\mathbf{g} = \Theta^2\Pi_*\mathbf{g} = \Theta\Pi_*\mathbf{g} = \mathbf{g}$ that is $\mathbf{g} \in \mathrm{Im}(\Theta)$. Moreover, $\Pi_*\Theta\mathbf{g} = \Theta\Pi_*\mathbf{g} = \mathbf{g}$ and thus, $\mathbf{g} \in \mathrm{Im}(\Pi_*)$. Hence, $\mathbf{g} \in \mathrm{Im}(\Theta) \cap \mathrm{Im}(\Pi_*)$.
Conversely, let $\mathbf{g} \in \mathrm{Im}(\Theta) \cap \mathrm{Im}(\Pi_*)$. Then, $\Theta\mathbf{g} = \mathbf{g}$ and $\Pi_*\mathbf{g} = \mathbf{g}$. It follows $\Theta\Pi_*\mathbf{g} = \mathbf{g}$ that concludes the proof. □

Notation. In the sequel, $\mathbf{P}(S^i)$ stands for the simplex of probabilities over the set of actions of player i and $x^i(s^i)$ denotes the probability whereby player i chooses his action s^i.

Corollary 3.19. *Let* $\boldsymbol{g}$ *be a finite game. With respect to the norm in (2.3), we have:*

- *the closest non-interactive game is* $\boldsymbol{g}_{\mathcal{NI}} = \left(\Theta\Pi_* + \Gamma\right)\boldsymbol{g}$,
- *the closest uniform game is given by* $\boldsymbol{g}_{\mathcal{U}} = \left(\Sigma + \Gamma\right)\boldsymbol{g}$,
- *the closest constant game is given by* $\boldsymbol{g}_{\mathcal{C}} = \Gamma\boldsymbol{g}$,

Furthermore, each dominant strategy equilibrium of $\boldsymbol{g}_{\mathcal{NI}}$*, is an* ϵ*-equilibrium of* $\boldsymbol{g}$*, for some* $\epsilon \leq \max_{i \in N}\left(2/\sqrt{h_i}\right)\left\|\boldsymbol{g}_{\mathcal{U}_0}\right\|_{\mathcal{G}}$ *and the uniformly mixed strategy profile is an* $\tilde{\epsilon}$*-equilibrium of* $\boldsymbol{g}$*, for some* $\tilde{\epsilon} \leq \max_{i,s^i}\{\lambda^i(s^i)\}$*, where* $\lambda^i(s^i)$ *is the payoff of player* i *for his action* s^i*, in* $\boldsymbol{g}_{\mathcal{NI}_*}$.

Proof. Using the decomposition result of Corollary 3.16, due to (3.8), it is immediate that the closest constant game to an arbitrary game $\mathbf{g}$, is given by $\mathbf{g}_{\mathcal{C}} = \Gamma\mathbf{g}$. In view of Lemma 3.15 and Lemma 3.18, the operator $\Theta\Pi_*$ is the projection onto $\mathcal{NI} \cap \mathcal{NO}$ and since constant games is a subspace of non-interactive games, it clearly follows that its closest non-interactive game is given by $\mathbf{g}_{\mathcal{NI}} = (\Theta\Pi_* + \Gamma)\mathbf{g}$. Since constant games are also a subspace of uniform games, Corollary 3.17 implies that the closest uniform game to $\mathbf{g}$, is given by $\mathbf{g}_{\mathcal{U}} = (I - \Theta\Pi_*)\mathbf{g}$. By Lemma 3.18, it clearly follows that $\Sigma + \Theta\Pi_* + \Gamma = \Sigma + \Theta\Pi_* + \Theta\Gamma = \Sigma + \Theta(\Pi_* + \Gamma) = \Sigma + \Theta = I$.

Hence, $I - \Theta\Pi_* = \Sigma + \Gamma$, which concludes this part of the proof.
From Corollary 3.16 and definition of the norm in (2.3), for all $i \in N$ and all $\mathbf{s} \in S$,

$$\left|g^i(\mathbf{s}) - g^i_{\mathcal{NI}}(\mathbf{s})\right| = \left|g^i_{\mathcal{U}_0}(\mathbf{s})\right| \le (1/\sqrt{h_i}) \left\|\mathbf{g}_{\mathcal{U}_0}\right\|_{\mathcal{G}} \le \max_{i\in N}(1/\sqrt{h_i}) \left\|\mathbf{g}_{\mathcal{U}_0}\right\|_{\mathcal{G}} \tag{3.9}$$

Let $\mathbf{s} = (s^1, ..., s^i, ..., s^n) \in S$ be a dominant strategy equilibrium in $\mathbf{g}_{\mathcal{NI}}$. For all $\mathbf{t} \in S$, such that $t^i \neq s^i$ and $t^{-i} = s^{-i}$, we have:

$$\begin{aligned} g^i(\mathbf{t}) - g^i(\mathbf{s}) &\le g^i(\mathbf{t}) - g^i(\mathbf{s}) - \left(g^i_{\mathcal{NI}}(\mathbf{t}) - g^i_{\mathcal{NI}}(\mathbf{s})\right) \\ &= \left(g^i_{\mathcal{NI}}(\mathbf{s}) - g^i(\mathbf{s})\right) + \left(g^i(\mathbf{t}) - g^i_{\mathcal{NI}}(\mathbf{t})\right) \\ &= g^i_{\mathcal{U}_0}(\mathbf{t}) - g^i_{\mathcal{U}_0}(\mathbf{s}), \end{aligned}$$

where the inequality follows since $\mathbf{s}$ is a pure equilibrium in $\mathbf{g}_{\mathcal{NI}}$. Thus, using (3.9) we get:

$$\begin{aligned} g^i(\mathbf{t}) - g^i(\mathbf{s}) &\le g^i_{\mathcal{U}_0}(\mathbf{t}) - g^i_{\mathcal{U}_0}(\mathbf{s}) \\ &\le \max_{i\in N} \left(2/\sqrt{h_i}\right) \left\|\mathbf{g}_{\mathcal{U}_0}\right\|_{\mathcal{G}}, \end{aligned}$$

that concludes this part of the proof.
Now, let $\mathbf{x} = (x^1, ..., x^i, ..., x^n) \in \prod_{i\in N} \mathbf{P}(S^i)$ be the uniformly mixed strategy profile. By Definition 3.2, $\mathbf{x}$ is an equilibrium in $\mathbf{g}_{\mathcal{U}}$. For all $\mathbf{y} \in \prod_{i\in N} \mathbf{P}(S^i)$ such that $y^i \neq x^i$ and $y^{-i} = x^{-i}$ for some $i \in N$, we have:

$$\begin{aligned} g^i(\mathbf{y}) - g^i(\mathbf{x}) &= \sum_{\mathbf{s}\in S} \mathbf{y}(\mathbf{s}) g^i(\mathbf{s}) - \sum_{\mathbf{s}\in S} \mathbf{x}(\mathbf{s}) g^i(\mathbf{s}) \\ &= \sum_{\mathbf{s}\in S} \left(\mathbf{y}(\mathbf{s}) - \mathbf{x}(\mathbf{s})\right) g^i(\mathbf{s}) \\ &\le \sum_{\mathbf{s}\in S} \left(\mathbf{y}(\mathbf{s}) - \mathbf{x}(\mathbf{s})\right) g^i(\mathbf{s}) - \sum_{\mathbf{s}\in S} \left(\mathbf{y}(\mathbf{s}) - \mathbf{x}(\mathbf{s})\right) g^i_{\mathcal{U}}(\mathbf{s}), \end{aligned}$$

where the inequality follows since $\mathbf{x}$ is an equilibrium in $\mathbf{g}_{\mathcal{U}}$. Hence, using Corollary 3.16,

$$g^i(\mathbf{y}) - g^i(\mathbf{x}) \le \sum_{\mathbf{s}\in S} \left(\mathbf{y}(\mathbf{s}) - \mathbf{x}(\mathbf{s})\right) g^i_{\mathcal{NI}_*}(\mathbf{s}).$$

Notice that $\mathbf{y} = (x^1, ..., x^{i-1}, y^i, x^{i+1}, ..., x^n)$. If $y^i(s^i)$ stands for the probability whereby player i chooses his action s^i, then we have:

$$\begin{aligned} g^i(\mathbf{y}) - g^i(\mathbf{x}) &\le \sum_{\mathbf{s}\in S} \left[\left(y^i(s^i) \frac{1}{\prod\limits_{j\neq i} h_j} - \frac{1}{\prod\limits_{i\in N} h_i} \right) g^i_{\mathcal{NI}_*}(\mathbf{s}) \right] \\ &\le \sum_{\mathbf{s}\in S} \left(\frac{y^i(s^i)}{\prod\limits_{j\neq i} h_j} g^i_{\mathcal{NI}_*}(\mathbf{s}) \right), \end{aligned}$$

where the last inequality follows since the game $\mathbf{g}_{\mathcal{NI}_*}$ is total-sum zero. Then, it follows:

$$\begin{aligned} g^i(\mathbf{y}) - g^i(\mathbf{x}) &\leq \frac{1}{\prod\limits_{j\neq i} h_j} \sum_{s^i\in S^i} \sum_{s^{-i}\in S^{-i}} y^i(s^i) g^i_{\mathcal{NI}_*}(s^i, s^{-i}) \\ &\leq \sum_{s^i\in S^i} y^i(s^i)\lambda^i(s^i) \\ &\leq \max_{i,s^i}\{\lambda^i(s^i)\}, \end{aligned}$$

where the second inequality follows by Definition 3.7 and the proof is completed. □

Corollary 3.20. *The dimensions of the uniform, non-interactive and constant games subspaces are given as follows:*

- $\dim(\mathcal{U}) = \sum\limits_{i\in N} \left(\prod\limits_{i\in N} h_i - h_i \right) + n,$
- $\dim(\mathcal{NI}) = \sum\limits_{i\in N} h_i,$
- $\dim(\mathcal{C}) = n.$

Proof. The result on the dimension of constant games is trivial. Definition 3.7 implies that $\dim\left(\mathcal{NI}\right) = \sum_{i\in N} \dim\left(\mathcal{NI}^i\right) = \sum_{i\in N} h_i$. Thus, Remark 3.14 ensures that $\dim(\mathcal{NI}_*) = \dim(\mathcal{NI}) - \dim\left(\mathcal{C}\right) = \sum_{i\in N} h_i - n$. Since $\dim(\mathcal{G}) = n\prod_{i\in N} h_i$, using the decomposition result of Corollary 3.16, it follows the dimension of the uniform games subspace, i.e.,

$$\dim\left(\mathcal{U}\right) = n\prod_{i\in N} h_i - \left(\sum_{i\in N} h_i - n\right)$$

that concludes the proof. □

4 Examples

In this section, we provide two examples: The first is a celebrated game in which our decomposition result coincides with the one proposed in Candogan et al. [2011], the decomposition presented in Kalai and Kalai [2010], and this one of Jessie and Saari [2013]. The second example concerns a decomposition of a game that is a perturbation in terms of payoffs, of a uniform with zero-constant game.

Example 4.1. ***Rock-Paper-Scissors game.***
Let us consider the two-player game $\mathbf{g}$ given by:

$$\mathbf{g} = \begin{pmatrix} 0,0 & -w,w & w,-w \\ w,-w & 0,0 & -w,w \\ -w,w & w,-w & 0,0 \end{pmatrix}$$

where $w \in \mathbb{R}$. The uniformly mixed strategy profile is an equilibrium of $\mathbf{g}$ and thus, $\mathbf{g}$ is a uniform game. It is also a harmonic game (see Candogan et al. [2011]).

Example 4.2. ***A perturbed uniform with zero-constant game.***
Let us consider the two-player game $\mathbf{g}$ given by:

$$\mathbf{g} = \begin{pmatrix} 100, 100 & 100, 105 & -190, 1 \\ 105, 100 & 95, 95 & -195, 200 \\ 1, -190 & 200, -195 & -190, -195 \end{pmatrix}$$

The uniform with zero-constant component-game corresponds to:

$$\mathbf{g}_{\mathcal{U}_0} = \begin{pmatrix} 290/3, 290/3 & 290/3, 310/3 & -580/3, -1 \\ 310/3, 290/3 & 280/3, 280/3 & -590/3, 594/3 \\ -8/3, -580/3 & 589/3, -590/3 & -581/3, -591/3 \end{pmatrix}$$

and the non-interactive one by:

$$\mathbf{g}_{\mathcal{NI}} = \begin{pmatrix} 10/3, 10/3 & 10/3, 5/3 & 10/3, 2 \\ 5/3, 10/3 & 5/3, 5/3 & 5/3, 2 \\ 11/3, 10/3 & 11/3, 5/3 & 11/3, 2 \end{pmatrix}$$

Moreover, the non-interactive normalized component-game is:

$$\mathbf{g}_{\mathcal{NI}\cap\mathcal{NO}} = \begin{pmatrix} 4/9, 1 & 4/9, -6/9 & 4/9, -3/9 \\ -11/9, 1 & -11/9, -6/9 & -11/9, -3/9 \\ 7/9, 1 & 7/9, -6/9 & 7/9, -3/9 \end{pmatrix} = \mathbf{g}_{\mathcal{NI}_*}$$

and the uniform one is given by:

$$\mathbf{g}_{\mathcal{U}} = \begin{pmatrix} 896/9, 99 & 896/9, 951/9 & -1714/9, 12/9 \\ 956/9, 99 & 866/9, 861/9 & -1744/9, 1803/9 \\ 2/9, -191 & 1793/9, -1749/9 & -1717/9, -1752/9 \end{pmatrix}$$

Finally, the constant component corresponds to:

$$\mathbf{g}_{\mathcal{C}} = \begin{pmatrix} 26/9, 7/3 & 26/9, 7/3 & 26/9, 7/3 \\ 26/9, 7/3 & 26/9, 7/3 & 26/9, 7/3 \\ 26/9, 7/3 & 26/9, 7/3 & 26/9, 7/3 \end{pmatrix}$$

The uniformly mixed strategy profile $\mathbf{x} = (x^1, x^2)$, where $x^1 = x^2 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, is an equilibrium of $\mathbf{g}_{\mathcal{U}}$. Notice that in $\mathbf{g}_{\mathcal{NI}_*}$, $\max_{i,s^i}\{\lambda^i(s^i)\} = 1$. Corollary 3.19 yields that $\mathbf{x}$ is an ϵ-equilibrium in $\mathbf{g}$, for some $\epsilon \leq 1$.

5 Conclusion and perspectives

In this work, we introduce the classes of uniform and non-interactive games. Using an appropriate projection operator, we classify games in terms of uniform with zero-constant games. More precisely, games that share equal payoff differences, in deviations realized by the rest of the players, are represented by the same uniform with zero-constant game. With respect to a natural extension of the standard inner product, the structure of the proposed decomposition enables the appearance of non-interactive games as the orthogonal complement in the space of games, of uniform with zero-constant games. The main feature of this debut class of games, is that each player has a best choice, regardless of the other players' choices and thus, players do not need to interact with each other in order to take their decisions. In this context, the uniform with zero-constant component forms the interactive part of an arbitrary game.

Following Candogan et al. [2011], we define a strategical equivalence in the space of finite games. Games that share identical payoff differences for any deviating player, are represented by the same normalized game. Using a similar projection operator, it follows an orthogonal decomposition with respect to the standard inner product, where the complement of normalized games consists of games in which unilateral deviations are identically zero. This decomposition distinguishes the strategic and non-strategic parts of a given game. Non-strategic games belong in the uniform class and thus, they can be decomposed in non-strategic with zero-constant and constant games. As a consequence, the *strategic-interactive* part of a given game lies in the intersection of uniform and normalized components in the corresponding decompositions.

We introduce the total-sum zero games and using an appropriate projection operator, a decomposition result follows, in which games are uniquely decomposed in two components: the total-sum zero and constant games. Hence, we establish a classification of games in terms of total-sum zero games. Precisely, any player, in between games that are represented by the same total-sum zero game, has identical payoff differences in any unilateral deviation of him and further in any coalitional deviation comprised by the rest of the players. Constant games belong in the intersection of non-interactive and uniform games, leading us to define an offspring canonical decomposition of games in three components, where the former is the uniform with zero-constant games, the second is the non-interactive normalized games and the latter one consists of constant games. In this context, we provide explicit expressions of the closest uniform and non-interactive games to an arbitrary finite game. Furthermore, it is given a characterization of the approximate equilibria of a given game in terms of the uniformly mixed strategy profile that appears an equilibrium in its closest uniform game. An additional characterization of its approximate equilibrium set can be given through the equilibria in dominant strategies of its closest non-interactive game. If the non-interactive component admits more than one equilibrium, then all of them are ϵ-equilibria in the original game with the same ϵ since equilibria in dominant strategies admit a unique payoff in non-interactive games.

Let us mention that in view of Candogan et al. [2011], Kalai and Kalai [2010], and Jessie and Saari [2013], one may show that their decomposition results coincide with the one we establish in this work over the subspace of two-player uniform games with equal number of actions. Finally, it might be interesting to study projections under different norms.

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