Jumps in financial modelling: pitting the Black-Scholes model refinement programme against the Mandelbrot programme

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Abstract

This paper gives an overview of the financial modelling of discontinuities in the behaviour of stock market prices. I adopt an epistemological perspective to present to the two main competitors for this stake: Mandelbrot's programme and the non-stable Lévy processes based approach. I explain this contest using the De Bruin's notions of refinement programme, overmathematisation and model-tinkering: I argue that the non-stable Lévy based approach of discontinuities can be viewed as a "Black-Scholes model refinement programme" (BSMRP) in the De Bruin's sense, launched against the radical view of Mandelbrot. I use Sato's classification to contrast the two competitors. Next I present the two strands of research from an historical perspective between 1960 and 2000. Mandelbrot's initial model based on alpha-stable motions initiated huge controversies in the finance field and failed to fully describe the observed behaviour of returns due to the stronger fractal hypothesis. The mixed jump-diffusion non fractal processes began in the 1970s, followed after two decades by infinite activity processes in the 1990s. At the end, the time-change representation of the 2000s seems to unify the two competitors.

Keywords

finance, financial modelling, epistemology, Mandelbrot, refinement program, overmathematisation

La modélisation des discontinuités boursières : le programme de Mandelbrot et le programme pragmatique

Résumé

Deux programmes de recherche se partagent les travaux de modélisation des discontinuités des variations boursières entre les années 1960 et les années 2000 : le programme de Mandelbrot et le programme « pragmatique ». On présente ici ces deux programmes de recherche en les situant l'un face à l'autre. On montre comment le programme pragmatique a privilégié les améliorations techniques des modèles qui permettaient de conserver les manières usuelles de gérer le risque financier, sans remettre en cause les représentations collectives sur l'incertitude financière. On situe ces deux programmes au moyen de la classification de Sato. On montre comment le programme de Mandelbrot s'est trouvé confronté à un rejet radical par les « pragmatiques » dans les années 1970 et a finalement été abandonné dans les années 1980 pour des raisons de difficultés à la fois mathématiques et statistiques, l'hypothèse d'invariance de morphologie du risque selon l'échelle d'analyse des marchés (ou hypothèse « fractale ») n'étant pas corroborée par les tests effectués sur les marchés. On retrace l'évolution du programme pragmatique dans les années 1990 pour faire apparaître comment les hypothèses épistémologiques de ce programme retrouvent celles du programme de Mandelbrot sans l'hypothèse fractale. On suggère que, à partir des années 2000, les deux programmes se rejoignent épistémologiquement dans une nouvelle manière de comprendre la temporalité des marchés financiers : une dynamique qui évoluerait au gré d'un temps boursier intrinsèque différent du temps calendaire.

Mots-clefs

modélisation financière, épistémologie, finance, Mandelbrot, processus de Lévy

Sommaire

What is this paper about?

This chapter gives an overview of the financial modelling of discontinuities in the behaviour of stock market prices. I adopt an epistemological perspective to present to the two main competitors for this stake: Mandelbrot's programme and the non-stable Lévy processes based approach. I explain this contest using De Bruin's notions of refinement programme, overmathematisation and model-tinkering: I argue that the non-stable Lévy based approach of discontinuities can be viewed as a "Black-Scholes model refinement program" (BSMRP) in De Bruin's (2009) sense, launched against the radical view of Mandelbrot. I use the Sato classification to contrast the two competitors. Next I present the two competitors from an historical perspective.

The outline of the chapter is as follows. In section 2, I present the background of the debates related to the issue of discontinuity. Section 3 recalls some fundamental notions from Lévy processes such as activity and variation, thus allowing Sato's classification to be applied to financial modelling. Section 4 introduces what I define as the Mandelbrot programme and discusses the related problems. Section 5 presents the BSMRP with the two stages of the programme. Section 5.1 begins with mixed jump-diffusion processes in the 1970s. Section 5.2 follows with infinite activity processes in the 1990s. Section 6 ends with the time change representation, both in Mandelbrot's programme in the 1970s and in the BSMRP with infinite activity processes in the 2000s.

Continuity and discontinuity in financial modelling

There are two fundamentally different ways of viewing uncertainty in finance. One assumes the principle of continuity, the other doesn't. According to the first view, following Bachelier's (1900) legacy, price movements are modelled by continuous diffusion processes, as for instance Brownian motion. According to the other view, following Mandelbrot's (1962) legacy, price movements are modelled by discontinuous processes, as for instance Lévy processes. I now elaborate this point, which is of a great importance for contemporary debates in finance and the issue of financial modelling I aim to address here.

In physics, the principle of continuity states that change is continuous rather than discrete. Leibniz and Newton, the inventors of differential calculus, said "*Natura non facit saltus*" (nature does not make jumps). This same principle underpinned the thoughts of Linné on the classification of species and later Charles Darwin's theory of evolution (1859). In 1890, Alfred Marshall's *Principles of Economics* assumed the principle of continuity, allowing the use of differential calculus in economics and the subsequent development of neoclassical economic theory. Modern financial theory grew out of neoclassical economics and naturally assumes the same principle of continuity. One of the great success stories of modern financial theory was the valuation of derivatives. Examples include the formulas of Fisher Black, Myron Scholes, and Robert Merton (1973) for valuing options, and the subsequent fundamental theorem of asset pricing that emerged from the work of Michael Harrison, Daniel Kreps, and Stanley Pliska between 1979 and 1981. These success stories rest on the principle of continuity.

In the 20th century, both physics and genetics abrogated the principle of continuity. Quantum mechanics postulated discrete energy levels while genetics took discontinuities into account. But economics – including modern financial theory – stood back from this intellectual revolution. An early attempt by Benoit Mandelbrot in 1962 to take explicit account of discontinuities on all scales in stock market prices led to huge controversies in the profession. But by the 1980s the academic consensus reaffirmed the principle of continuity, despite the repeated financial crises following the 1987 stock market crash. Many popular financial techniques, such as portfolio insurance or the calculation of capital requirements in the insurance industry assume that (financial) nature does not make jumps and therefore promote continuity. Most statistical descriptions of time series in finance assume continuity.

It follows that Brownian representation became the standard model, part and parcel of finance curricula across the globe. It is the point of reference of most top journals in the field of finance; it is the dominant view in the financial industry itself; and it underlies almost all prudential regulation worldwide: for instance, the so-called square-root-of-time-rule underlying the regulatory requirements (Basle III and Solvency II) for calculating minimum capital is a very narrow subset of time scaling rule of risk, and comes directly from the hypothesis that returns follow a Brownian motion. Brownian motion increments have the important property of being independent and identically distributed (hereafter IID). The processes with IID increments are called Lévy processes after the French mathematician Paul Lévy. Brownian motion is a specific Lévy process: it assumes continuity. Other Lévy processes don't.

The reasons for questioning the rationality of the continuity assumption that Brownian representation makes are now well known. The explanatory success of the Brownian framework is restricted to a fairly small set or relatively standard cases. In other words, the explanatory power of non-Brownian representations is, in general, significantly larger. One important reason to question continuity is a negative spillover of the assumed continuity: the truncation of financial time series into "normal" periods and periods of "insanity" where markets are deemed "irrational". This dichotomy leaves the profession unable to explain the transition from one period to another. For example, in an editorial in the Financial Times (16.3.08), Alan Greenspan commented on the financial crisis of 2007-2008, "We can never anticipate all discontinuities in financial markets." For Greenspan, (financial) nature does not make jumps. This demonstrates the limits of traditional risk management when using a Brownian based representation of risk. Despite obvious disadvantages, however, the Brownian model remains vastly more popular than its discontinuous competitors.

One source of the problem is the interpretation of jumps in statistical descriptions. In the classic case of the Brownian representation of fluctuations, trajectories are continuous. However, a stock price trajectory is by construction discontinuous; because it comprises jumps at all quote times. The classic Brownian representation views quotes as points sampled in a continuous trajectory. Pro-continuity activists argue that, in the case of a Brownian representation, quote jumps are proportional to the volatility of Brownian motion and hence it is not necessary to change representation. This statement correctly addresses the puzzle of jumps. In a given representation, are the points separated by distances that are consistent with the postulated model for paths? In the Brownian representation, are the observed jumps consistent with the diffusive nature of Brownian motion, or are they too large? Strange as it may seem, this issue had not been tackled in finance literature until very recently. While normality tests have been well known for many years, it was not until the contributions of Yacine Aït-Sahalia and Jean Jacod (2009) that appropriate tests for the detection of jumps were constructed, adding discontinuity tests to the classical toolbox of financial statistics.

For these processes, each trajectory is by definition discontinuous everywhere. How shall we name markets that are "continually discontinuous"? Mandelbrot felt that the name should reflect the fractured nature of the paths representing price changes. He coined the term "fractal" (from the Latin *fractus*, meaning fractured) to characterize discontinuities at all scales.

The Sato classification

This section presents in the simplest and most intuitive way possible the main characteristics of Lévy processes. Many books present a comprehensive view of these processes, as for example Bertoin (1996) and Sato (1999). Broadly speaking, at the most simple level a random walk is just a stochastic process whose increments are IID. In continuous time, it is a Lévy process.

Lévy processes

To specify a Lévy process, there are two alternative routes: either to describe the marginal probability distribution of the process, i.e. the shape of the probability density function of the law, which describes the morphology of market uncertainty, considered from a static standpoint. Or to describe the Lévy measure, a mathematical object that captures the structure of the dynamics of jumps. The marginal probability distribution corresponds to a representation of uncertainty in the real world (here, the reality of the chance of the market, the reality of the stock price behaviour, the reality of the financial phenomenon), which can be used for real-world calibrations with market data; whereas the Lévy measure appears only in the transformed space of characteristic functions: the inverse Fourier transform of the probability density function. The characteristic functions can also be used as part of procedures for fitting probability distributions to samples of data. These two representations are equivalent for the specification of a Lévy process in the sense that knowing one of the functions always makes it possible to find the other. Both provide different insights for understanding the morphology of uncertainty in the real world.

However, the two cannot be used indifferently. The probability density function does not always exist (closed form expression is not available); whereas the characteristic function of any infinitely divisible distribution always exists. Thus, for reasons of mathematical convenience, we use the characteristic function to define in a simple way an infinitely divisible distribution and the Lévy processes corresponding to it. The characteristic function of a Lévy process has an equivalent meaning to the density function: it describes the morphology of uncertainty of the observed phenomenon.

The explicit form of the characteristic exponent (exponential transform of characteristic function) of a random walk was obtained in the most general case by Paul Lévy in 1934 from his theory of processes with IID increments. This so-called Lévy-Khintchine formula may be analysed as follows. The first term represents the Brownian motion with mean and standard deviation (the diffusion component of the random process). Financially speaking the standard deviation represents market volatility. The second term is the pure jump process component of the random process, which exhibits the Lévy measure. This is an important component of a Lévy process, which completely defines the structure of jumps. Intuitively, the Lévy measure provides the average number of jumps per time unit as a function of their amplitude. It is thus the mathematical object that can quantify the occurrence and size of jumps and create discontinuities in the trajectories of the random processes representing stock market movements. The Lévy measure was explicitly used in the models of the 1990s, whereas it was only implicit in those of the 1970s, with the exception of Mandelbrot's model (1962, 1963), in which it appeared in an integrated form of the characteristic exponent.

A very important property of Lévy processes due to the IID property is that the characteristic exponent is proportional to the time duration. That is to say that the marginal distributions of these processes are infinitely divisible. In practice this means that a random variable can be understood as the sum of identical random variables, at any order. When modelling market uncertainty on whatever scale, one uses this property. The characteristic exponent at a given time t (uncertainty at time t) is easily obtained from the characteristic exponent at time 1 (uncertainty at time 1). This is one of the main attractions of Lévy processes, making them preferable to other types of model where the IID property does not hold.

Activity and variation of a Lévy process

Intuitively, the greater the number of jumps per time unit, the more the trajectory of the stochastic process will have a high degree of irregularity and the more erratic the random walk will be. Hence a random walk will be highly erratic if the average number of jumps occurring per unit of time is very large. The average number of jumps per unit of time defines the so-called "intensity" of a Lévy process – also known as "activity" by analogy with turbulence. The activity can be finite or infinite. Consider for example a very simple case: a Poisson process with parameter lambda (the average number of jumps per unit of time). The activity of the process is simply lambda. In this extremely rudimentary case, the average number of jumps per unit of time is finite and is given by the Poisson parameter.

Let us continue with this simple example to get an intuitive idea of what the Lévy measure is. Whenever a jump in the Poisson process occurs, the magnitude of the jump must be specified. Suppose that this magnitude is random and is pulled into a given probability distribution with known density. Hence the product activity-density captures both the occurrence rate of discontinuities and their magnitude. We see that this product fully characterizes the jump structure of the process. In the simple case of a compound Poisson process with any law of probability, this product is precisely what is termed the Lévy measure. If for example the distribution of jumps is normal with a mean being the average size of jumps and a standard deviation being the volatility of the size of jumps, it will in this case be a compound Poisson process with a normal distribution so-called compound Poisson-normal (CPN). The activity of this new jump process is the Poisson parameter and the density of the distribution of jumps is the normal distribution. In all cases in which one constructs a compound Poisson process with another distribution, the number of jumps per unit of time (the occurrence rate of discontinuities) is finite and the resulting Lévy process is of finite activity. In this situation one can clearly separate the activity from the density. When the activity is finite, the product activity-density is precisely what is called the Lévy measure.

It makes sense to generalize this approach for moving from finite to infinite activity. Indeed there is no reason why the average number of small jumps per unit of time should stay finite. The advantage of generalizing in this way is that the very many small market movements can be taken into account. In the case of infinite activity, it is no longer possible to separate the activity from the density. Both are "mixed" in the Lévy measure, which entirely shapes the morphology of the irregularity of the financial phenomenon. In this situation, it becomes less necessary to add a Brownian component. It is only when the average number of jumps is finite that it is necessary to add this Brownian component for market movements occurring between the jumps.

In the case of infinite activity, how can the activity be isolated? To understand how to achieve this objective, let us consider the simple case of the compound Poisson process. In this case, if we calculate the integral of the Lévy measure (the product activity-density), it follows directly (because the integral of a probability density is 1) that the integral is equal to the average number of jumps per unit of time. Hence the integral of the Lévy measure allows us to obtain the activity of the process. With this simple example, one sees that, in order to "isolate" the activity of a Lévy process, it is sufficient to calculate the integral of the Lévy measure. The activity of a Lévy process is no more than the integral of the Lévy measure. This integral may be either finite or infinite.

Consider now the average distance between two points of the process. The average distance can be finite or infinite (the mean may or may not exist). This idea of average distance corresponds to what is called the variation of a Lévy process. We see that the variation may be finite or infinite. The variation is another feature of the morphology of financial uncertainty.

In brief, two alternatives exist for shaping a Lévy process: either the activity is finite or infinite, or the variation is finite or infinite. The Sato (1999) classification defines a process by its pair (activity, variation) according to the double criterion finite or infinite. Table 1 below exhibits the Lévy processes in financial modelling following this double criterion. It appears that there are three types of process depending on whether their activity and their variation are finite or infinite. In general, the models of the late 1990s and early 2000s all used processes with infinite activity but finite variation. As an outlaw, the Mandelbrot model (1962) comprised both infinite activity and infinite variation.

Let us summarize what has been presented so far. A Lévy process is fully defined by the specification of three quantities: the mean of the diffusive component (the trend of the process), the diffusion coefficient (the scale of fluctuations) and the Lévy measure (the morphology of uncertainty). The role of the Lévy measure is decisive. It contains all the information needed to characterize the trajectory of a Lévy process, apart from its tendency and its diffusive fluctuation scale ('volatility'). It is the quantity that shapes the size of the tails of distribution, and the patterns of jumpy fluctuations. The significance of

Table 1. The Sato classification and financial modelling

Activity	Variation of the jump part	Example of models in financial modelling	Epistemological view
Finite	Finite	Press (1967), Merton (1976), Cox and Ross (1976)	BSMRP stage 1
Infinite	Finite	Madan and Seneta (1990), Madan and Milne (1991), Eberlein and Keller (1995), Barndorff-Nielsen (1997), Eberlein, Keller and Prause (1998), Madan, Carr and Chang (1998), Prause (1999), Carr, Geman, Madan and Yor (2002, 2003)	BSMRP stage 2
Infinite	Infinite	Mandelbrot (1962)	Heterodox

the new approach adopted in the 1990s came precisely from this possibility of defining any market dynamics with irregularities at all scales by direct specification of the Lévy measure. Thus, the dynamic of stock prices being any Lévy process, the representation of market fluctuations shifted, in the 1990s, from exponentials of Brownian motions to exponentials of Lévy processes.

Another consideration also favoured Lévy processes. Lévy processes are semi-martingales. The work of Ross, Harrison and Pliska between 1976 and 1981 on arbitrage showed that the arbitraged prices of securities ought to be capable of being modelled by semi-martingales. Thus for these reasons applying both to financial modelling and to the technique of stochastic calculus, the Lévy processes disinterred in the early 1990s, after a decade of growing maturity in financial thinking around the theory of arbitrage and the usefulness of intrinsic market temporality, appeared extraordinarily well adapted to the new way of conceiving the modelling of arbitraged markets, whether in calendar time or market time. The match between the most modern finance (absence of arbitrage) and the development of working techniques on Lévy processes (representation in market time) was pivotal for the introduction of these processes into financial research.

Mandelbrot's programme: a fractal approach

The initial idea of discontinuities at any scale of the observation of markets behaviour came from Mandelbrot (1962, 1963). Following these first attempts at financial modelling, he developed his main ideas in a series of significant epistemological papers published in French but not well known today. In this paper, I term "Mandelbrot's programme" the research programme described in these papers corresponding to Mandelbrot (1966, 1967, 1973a, 1973b, 1973c). Some of these texts are translated in English, modified and reprinted in his 1997 *Fractals and Scaling in Finance*. The three main concepts of this research programme for financial modelling are summarized in Walter (2015). A second strand of papers came after his 1997 book, corresponding to Mandelbrot (2001a, 2001b, 2001c), which build on the generalized multifractal model put forward in 1997, outside the IID framework. This model was the result of the Mandelbrot's coming back to finance after the 1987 crash. The three papers of 2001 echoed the three of 1973 and represent a second stage of the programme, moving from unifractals to multifractals. Ultimately, multifractal modelling allows a comprehensive view of discontinuities with bypassing the limitations of the first approach.

The origin of the idea of discontinuity in price variations at any scale is closely related to the scaling view of price fluctuations (named "fractal description of markets"), a current of thought which is initially entangled with the chartist approach to markets, before being adequately mathematicized with fractals. I now elaborare this point.

The fractal view of price behaviour

Stock market charts representing changes in the stock prices over a given period of time look like irregular patterns that seem to be reproduced and repeated in all scales of analysis. Rising periods follow periods of decline and the rises are punctuated with intermediate falling phases and falls are interspersed with partial rises, and this goes on until the quotation scale limit is reached.

This mixture of repetitive patterns of rising and falling waves at all scales was Ralph Elliott's (1938) intuition, to whom this idea occurred while observing the ebb and flow of tides on the sands of the seashore. From this, he coined a financial symbolization known as "stock market waves" or "Elliott's waves", which he subdivided into huge tides, normal waves and wavelets. In mathematical terms, the so-called Elliott wave principle presents a deterministic self-similar fractal description of stock markets with self-similar geometric patterns found on all scales of observations. Elliott designed a toolbox to analyse these pattern, based on chart recognition and known since as the technical analysis of markets: "technical analysis is the study of market action, primarily through the use of charts, for the purpose of forecasting future price trends" (Murphy, 1986).

From a chartist standpoint, it clearly appears that Bachelier's hypothesis-Brownian based representation of markets which carries on random walks for modelling the behaviour of stock market prices contradicts totally this desire for prediction. Hence, the academic view of markets with continuous random walks based on Brownian representation clashed head-on with the entire professional technical analyst community:

as Murphy (1986) wrote, "the idea that the markets are random is totally rejected by the technical community". One of the reasons was the prescientific numerologist Phytagorician approach of this social group in analysing market dynamics. Because of the lack of appropriate mathematical tools, this conceptualization of stock market variations was like alchemy before chemistry (Walter, 1996): a kind of dirty Bricolage while simultaneously the "real" science worked efficiently with the rise of modern financial theory, based on the Brownian representation of uncertainty.

The fractals of Mandelbrot, though developed in a radically different intellectual context, fit in this understanding of stock market variations. It presents, like the common view with Elliott's waves, a method for disentangling the inextricable interlacing of stock markets moves at all scales. Fractals represented an adequate conceptualization allowing the translation of intuitions of technical analysts into rigorous mathematical representation, because this mathematics deals with two financial stylized facts: discontinuity and scaling. The notion of "roughness" addresses these facts by creating a strange nexus between two seemingly disparate cases: discontinuity and scaling. Random fractal curves adequately mimic stock market charts. In the following section, I elaborate on this.

Fractal modelling: discontinuity and scaling

Despite the promising results opened up with this new way of thinking financial modelling, the adventure of fractal modelling in finance doesn't display a smooth (continuous) history. It is more an eventful (discontinuous) progression of Mandelbrot's assumptions through the evolution of finance theory over forty years, from 1960 until 2000. A review over forty years of searching for scaling laws in distributional properties of price variations (Walter, 2002a) exhibits a turbulent story with fierce controversies which stirred up the academic community with regard to the continuous/discontinuous debate. So strong was the opposition to Mandelbrot's hypothesis that any kind of alternative model was preferred to the idea of infinite variance as embedded in the alpha-stable motion proposed by Mandelbrot. For example, in a paper devoted to the statistical properties of exchange rates, Elie *et al.* (1993) said that "ARCH models allowed us to solve largely this problem of heavy tails of distributions while keeping a Gaussian framework which turns out more tractable than that of stable laws". In his admirable book about the development of financial economics, MacKenzie (2006) stressed this point by hypothesizing that Mandelbrot's model was viewed by the financial academic community as a probability 'monster'. In a paper summarizing the Mandelbrot's state of research programme in the 1980s, Mirowski (1995) observed that "the economics profession dropped the Mandelbrot hypothesis largely for reasons other than empirical adequacy and concise simplicity. […] The only purpose of the negative studies was to refute Mandelbrot". Let us have a closer look on this point.

The leptokurtic phenomenon

At the origin of the fractal modelling in finance is the so-called "leptokurtic phenomenon" (Walter, 2002b), i.e. the presence of fat tails on the empirical distributions of returns. The classical view of the extreme values of the distribution considered these data as irrelevant. For example, Granger and Orr (1972) asserted that "If the long tails of empirical distributions are of concern to the time-series analyst or econometrician, it is natural to consider reducing the importance of these tails. The most obvious approach is to truncate the data". On the contrary, Mandelbrot viewed these extreme values as something extremely important for the understanding of market behaviour. But at this time, the Gaussian distribution was the predominant tool used to describe the empirical distribution of returns. Hence it was not possible to take account of the tails with this distribution.

Mandelbrot's idea for suggesting the simplest generalization of Brownian motion that takes account of the fractal appearance of trajectories was to put forward the simplest process, which was, in this sense, stable by addition, the so-called alpha-stable motion. But the price to pay for accounting discontinuities and fractality was the abandonment of finite variance, because the variance of alpha-stable motion is infinite. This infiniteness of a crucial financial quantity which just arose in the new models for portfolio management (Markowitz, 1952, 1959) and option pricing (Black, Scholes and Merton, 1973) was seen as horrific by the academic mainstream of the 1970s. For example, one can find in a textbook that "many researchers find the conclusion of infinite variance unacceptable" (Taylor, 1986).

On the other hand, there was a lack of statistical tools to tackle the estimation of the parameters of stable distributions. For example, Fama (1965) said that:

> most of these difficulties (practical use of stable distributions) are due to the fact that economic models involving stable Paretian generating processes have developed more rapidly than the statistical theory of stable Paretian distributions. It is our hope that papers like this will arouse the interest of statisticians in exploring more fully the properties of these distributions.

In a reference textbook on one-dimensional stable distributions, Zolotarev (1986) echoes Fama by saying that “it can be said without exaggeration that the problem of constructing statistical estimators of stable laws entered into mathematical statistics due to the work of Mandelbrot”.

An important point to be understood in this controversy is the following (Walter, 2002a). It is possible to tackle the fat tails puzzle with models other than alpha-stable motions, indeed an unlimited number of models. But if one wants to keep the IID hypothesis and have non-Gaussian tails with scaling property (Brownian motion) the only alternative is the alpha-stable motion. The controversies resulting from the leptokurtic phenomenon and extreme values of the distributions became entangled in the intrication of the static approach (Gaussian or non-Gaussian) and the dynamic approach (Brownian or non-Brownian). In the 1970s, the debates ignored the stochastic process issues and concentrated on the extreme values of the distributions[1].

The rejection of fractals

Underpinned by the desire to reject fractality (“anything but Mandelbrot” or “ABM” statement – Mandelbrot was working as an IBM fellow –, was the watchword of the pro-continuity approach activists), the debates shifted to the testing of the alpha-stability-under-addition-property. They ended with the empirical rejection of fractal modelling for distributional properties returns in the 1970s because the scale invariance principle was found too strong for adequately modelling price variations. The alpha-stable Lévy processes were abandoned by the mainstream. But the issue of partial scaling invariance on a given frequency range – or the breakdown of scaling – had been dealt with by Mandelbrot (1963). Using scale invariance with cut-offs is less costly in parameter estimations than other types of models, which can be more accurate, but also more complex. In the 1990s, physicists began to propose such models combining truncated alpha-stable distributions with exponential tails (Mantegna and Stanley, 1994; Koponen, 1995; Bouchaud and Potters, 1997) and physicist research activity enters the financial modelling field. As Mantegna and Stanley (2000) noticed, “since 1990, a research community has begun to emerge”. This new community baptized itself with the name “econophysics”.

In the 1990s, research in financial modelling then split into two separate communities: that of financial academics – the mainstream – and that of physicists – the heterodox view known as “econophysics”. Physicists continued along the way paved by Mandelbrot’s model, working in particular with the scaling concept: as Mantegna and Stanley (2000) pointed out, financial academics were “trying to determine a characteristic scale for a problem that has no characteristic scale”. While physicists launched this new strand of research, mathematical financial academics then moved to the development of Lévy processes, following the first jump-diffusion type models of the 1970s that were developed to tackle the discontinuity issue in the framework of the finiteness of the second moment. Following De Bruin’s (2009) terminology, I term this mainstream strand of research the “Black-Scholes model refinement programme” (hereafter BSMRP).

The BSMRP opened the first period of model tinkering in financial modelling. According to De Bruin (2015), model tinkering characterizes a situation in which researchers, confronted with descriptive inadequacy, prefer to ‘repair’ existing models by introducing ad hoc mathematical ‘epicycles’ to them. They do not develop new models. The story of jump processes represents an illustration of the model tinkering analysis following de Bruin’s approach. Compare for example, the simplicity of the Black-Scholes option pricing model generalized by physicists (Bouchaud and

1. It is worth noting that this intrication is sometimes a source of confusion in the existing literature of historical financial thought, based on an analysis which doesn’t distinguish between so-called “Lévy distributions” (actually stable distribution with Pareto tail) and Lévy processes (actually stochastic processes with IID increments). Here the semantics is misleading.

Sornette, 1994) and the overmathematisation of the option pricing models of the BSMRP. I will now elaborate on the BSRP.

The Black-Scholes model refinement programme

I begin this section with this excerpt from a paper by Applebaum (2004):

> A sociologist investigating the behaviour of the probability community during the early 1990s would surely report an interesting phenomenon. Many of the best minds of this (or any other) generation began concentrating their research in the area of mathematical finance. The main reason for this can be summed up in two words: option pricing.

Paraphrasing the comment by De Bruin (2009) about the Nash equilibrium, I suggest that the Black-Scholes model is a paragon of mathematical elegance and simplicity. This model is based on the assumption that returns from the underlying assets follow a diffusion-type process, in particular a geometric Brownian motion. A large number of empirical studies showed that this model was inadequate, partly because of the continuity assumption. For example, Ball and Torous (1985) pointed out that "empirical evidence confirms the systematic mispricing of the Black-Scholes call option pricing model". Merton (1976) admitted that "there is a *prima facie* case for the existence of jumps" and Cox and Ross (1976) agreed that "exploring alternative forms is useful to construct them as jump processes". Again, Ball and Torous say that "the Merton model which explicitly admits jumps in the underlying security return process, may potentially eliminate theses biases". The goal of the BSMRP was precisely to overcome these inadequacies by tackling the issue of discontinuities without accepting Mandelbrot's programme. For example, Carr *et al.* (2002) said that they "seek to replace this process with one that enjoys all of the fundamental properties of Brownian motion, except for pathwise continuity and scaling, but that permits a richer array of variation in higher moment structure, especially at shorter horizons". This will be achieved with a "non-Gaussian Merton-Black-Scholes Theory" (Boyarchenko and Levendorskii, 2002), which gained official recognition.

This section gives a brief survey of the BRSMP by following the evolution of the modelling of jump processes, from the rediscovery of Poisson's law in finance by S. James Press in 1967 through to the Lévy infinite activity processes of the 2000s. It came in two major stages. First, with the rediscovery of Poisson's law in the late 1960s, a jump component was added to the diffusion process (Brownian motion): this superposition of jump and diffusion processes opened the period of hybrid models known as jump diffusion-processes (1970-1990), which state that prices undergo large jumps followed by small continuous movements. These models were initiated by Press (1967) and Merton (1976). It is a simple case of Lévy process with finite activity and finite variation in the jump component. This is the first stage of BSMRP (hereafter BSMRP1). Then, in the second period, the diffusive component was removed leaving only the jump component, moving to Lévy processes keeping finite variation in the jump component but with infinite activity. This is the second stage of BSMRP (hereafter BSMRP2).

In contrast to these BSMRPs, the model proposed by Mandelbrot in 1962 had both infinite activity and infinite variation in the jump component. It was – for this reason – an heterodox view. A convenient way to grasp the conceptual difference between the framework of Mandelbrot's first representation (1962) and that of Press's (1967) successors is to consider the intuition underlying the modelling of trajectory discontinuities by jump-diffusion processes: the invalidation of the stability-under-addition-property, one of the cornerstones of Mandelbrot's models, precisely the scaling view of markets (fractal nature) embedded in the stability-under-addition-property.

This first approach to jump-diffusion processes (BSMRP1), initially limited to Lévy processes with finite activity and finite variation, was generalized and fully developed only in the 1990s: the second life of Lévy processes (BSMRP2) belongs to the late twentieth century.

The Jump-diffusion models in the 1970s

I argue that the emergence of BSMRP1 was prepared for a long time by research around Poisson's law and process. Next I present the BSMRP1 and BSMRP2.

The rediscovery of Poisson's law in financial modelling

The issue of the explicit modelling of jumps (discontinuities) was well-known to insurance companies as early as 1903. In the context of managing their contracts, insurance companies had used the Poisson process to model the assessment of claims in non-life insurance. Lundberg's thesis of 1903 on insurance risk theory was the equivalent of Bachelier's theory of risk quantification in finance: Bachelier's (1900) Brownian model corresponded to Lundberg's (1903) Poisson model. Subsequently Harald Cramer and the Stockholm school introduced Lundberg's ideas into the theory of random processes, resulting in the so-called Cramer-Lundberg actuarial model.

If the Gaussian and Brownian motion constituted the mathematical basis of classical financial modelling, Poisson's law and process were their counterparts in traditional actuarial models. Brownian motion and the Poisson process are two examples of simple Lévy processes. When researchers tried to model the discontinuity of stock paths with a non-stable approach, this law and these processes emerge as the most "natural" candidates for the production of heavy tailed distributions, since Poisson's law precisely creates these tails. The Poisson framework appeared as the first response of financial economics mainstream to Mandelbrot's programme.

Thus in 1967 the Cramer-Lundberg actuarial model made its entry into finance. In that year, five years after Mandelbrot, to tackle the jumpy nature of the price process, Press's (1967) proposition provided, for the first time in financial modelling, a non-stable generalization of Bachelier's model, by complementing the Brownian continuous diffusive component with a discontinuous Poisson component. This innovation was able to produce a representation of the morphology of static uncertainty with a non-Gaussian distribution tail, a tail resulting from the introduction of the Poisson law.

Poisson's formula enables us to determine the probability of the occurrence of infrequent events (sometimes called rare events), provided that we know the constant average frequency at which these events occur. This frequency is described by the parameter of the Poisson distribution. Imagine, for example, that we consider a trajectory discontinuity as a jump. One easily sees to what extent the Poisson process is applicable in financial modelling: this process includes moments of jumps, the amplitude of which then simply has to be modelled. The combination of a Poisson process (for periods of jumps) and any law of distribution (for the size of jumps) produces what is called a compound Poisson process, that is to say, a process where the jumps occur at times coming from a simple Poisson process and have a determinate size. The choice of the probability law of the size fitting the possible values of this amplitude will then constitute the second stage of modelling.

If we choose a Gaussian distribution to model the size of jumps, we will obtain a structure combining a Poisson process and a Gaussian distribution, also called the normal compound Poisson process (or NCP). But it is possible to choose any probability law for the distribution of the size of jumps, such as a power law, a Gamma distribution, a Pareto law, and so on. Any distribution can be arbitrarily used for modelling the amplitude of discontinuities, coupled with the Poisson counting process. This linkage will then produce a compound Poisson process with these other laws (exponential Poisson, Gamma Poisson, etc.). It is this insight that underlies the representation of market discontinuities by jump processes.

The mixed jump-diffusion processes

However, the Poisson component is not sufficient to model all market changes since, with this pure Poisson representation, nothing happens between two jumps: the market remains inert, except when it jumps. It is therefore necessary to supplement it with another model. In the 1960s and 1970s, the only way to model this change in the market, perceived as "smoother", between two jumps was to opt for a Brownian motion. That's why we added a Brownian component to the Poisson component, and this linear combination of a compound Poisson process and Brownian motion corresponds precisely to Press's (1967) model. This model is thus presented as a simple juxtaposition of a process producing a very large number of small stock market fluctuations (Brownian motion) and a process of producing a small number of market discontinuities (the normal compound Poisson – NCP – process). These two basic building blocks processes are completely separate ("orthogonal"). Thus it is a mixed process involving diffusion and jumps, termed mixed jump-diffusion (MJD):

MJD = Brownian + NCP

As the increments of MJD processes are IID, MJD processes are Lévy processes. These are special cases of general non-stable Lévy process. The mixed Press model thus represents the first introduction of non-stable Lévy processes into finance. These processes had already been highlighted by Samuelson in 1965, echoing the work of Mandelbrot, but without giving rise to an explicit use, since Samuelson preferred returning to the usual Brownian motion model.

The values of the Poisson parameter (average number of jumps per unit of time) allow us to localize the MJD processes in relation to the Bachelier and Mandelbrot models. By characterising these two models by the number of jumps occurring during the evolution of market prices, i.e. by the Poisson parameter, the value of zero (no jumps) leads back to the Bachelier model, and the value of infinity (infinite number of jumps) leads to the Mandelbrot model. Between these two values (0 and infinity), any finite value of lambda results in a finite number of jumps between two given quotes. There are an infinite number of possible MJD processes, all filling the range between the Bachelier and Mandelbrot representations. The Press model thus represented an intermediate solution between Bachelier and Mandelbrot.

In the BSMRP1, the market dynamics of a given stock resulted simultaneously from frequent small movements, forming the continuous part of its trajectory and resulting from the Brownian diffusive component of the process, and from less frequent sudden movements forming the discontinuities of its trajectory, stemming from the Poisson component of the process. As Merton (1976) said, "the total change in the stock price is posited to be the composition of two types of changes: diffusion and jumps. The natural prototype process for the continuous component of the stock price change is a Wiener process, so the prototype for the jump component is a 'Poisson-driven' process". Again Cox and Ross (1976) stated that "in contrast to the diffusion process, the jump process [introduced] follows a deterministic movement upon which are superimposed discrete jumps".

From a financial standpoint, the MJD processes modelled the fluctuation risk of any asset in terms of two dimensions: (classic) volatility risk corresponding to the Brownian diffusive component and a (new) jump risk corresponding to the Poisson component. This innovation was important because it indicated to professionals that usual risk diversification on the basis solely of the volatility dimension was not sufficient to protect against adverse stock market fluctuations. The market risk of any asset was therefore at least two-dimensional. The second component of risk, or jump risk, was soon seen to be non-diversifiable, as became apparent from the work undertaken on the equity valuation models (Jarrow and Rosenfeld, 1984) and on the term structure of interest rates (Ahn and Thompson, 1988). This jump component creates a specific uncertainty as regards the risk usually measured by volatility.

The impossibility of perfect hedging for this type of risk was no doubt an obstacle to the widespread use of these MJD processes in financial engineering for some fifteen years. Note that the use of alpha-stable motions also implied the need to take into consideration a second dimension of risk, namely jump risk. Because this second dimension of risk was not taken up in financial circles, it can be assumed that the professional community was not sufficiently mature in the 1970s to manage financial products with two risk dimensions.

Pure jump models in the 1990s

The rebirth of the random walk model in finance is due to the rediscovery of two important characteristics of Lévy processes. First, in order to describe the jumping behaviour of various asset prices and interest rates, it became clear that the use of Lévy processes with infinite activity was sufficient. Hence, it was no longer necessary to build superpositions of jump and diffusion process (Brownian motion) in price dynamics equations, namely what was called jump-diffusion processes (special case of very simple Lévy processes) in the 1970s. Second, it was rediscovered that any Lévy process has an interesting relation to the Brownian motion, considering the morphology of uncertainty. Using a subordinator process for measuring time that increases with a randomly varying speed, any Lévy process in calendar time (physical time) can be written as Brownian motion measured in a time distorted by the pace of trading. The randomly increasing time has been interpreted as an operational time or a trading time reflecting the market activity. The fact that a Lévy process can capture the time change of the markets opened a new strand of research about the nature of intrinsic time in markets. At the end, the random walk model is released from the prison of the Brownian representation in calendar time in which it was trapped, and becomes a powerful tool for financial modelling using these two characteristics that foster the understanding of market price behaviour: infinite activity and the distortion of time. I now elaborate on this.

Financial modelling with infinite activity

The separation between the two sources of market movements – the Brownian source, forming the continuous part of the trajectories, and the Poisson source, creating discontinuities – was simple and convenient, but limited the possibilities for modelling. Moreover, as we have seen, even those changes perceived as continuous (between two jumps) could be represented differently, since share quotes are by definition discontinuous, with the tick defining the smallest time interval between two quotes. The notion of discontinuity is essential for modelling stock market variations. In other words, the intrinsic bumpiness of the financial phenomenon did not require the diffusive Brownian part of models to be retained. It was necessary only to be able to account variously for a very large number of very small jumps (ticks), a large number of larger jumps, and a very small number of very large jumps (market discontinuities), to obtain a relevant model of stock market functioning. The probabilistic representation of market fluctuations did not ultimately entail the use of the Brownian diffusive component.

This idea slowly made its way into the academic community, up to the early 1990s. The diffusive part of probabilistic representations had been needed for the modelling of the small movements only in the case of finite activity: the finite activity of the process required the addition of another component. But as soon as it was admitted that infinite activity was possible, the usefulness of the diffusive component disappeared and a pure jump process seemed to be sufficient to represent the entire stock market phenomenon, i.e. its bumpiness at all scales. The argument is well described in the paper by Peter Carr, Hélyette Geman, Dilip Madan and Marc Yor published in 2002:

> The rationale usually given for describing asset returns as jump-diffusions is that diffusions capture frequent small moves, while jumps capture rare large moves. Given the ability of infinite activity jump processes to capture both frequent small moves and rare large moves, the question arises as to whether it is necessary to employ a diffusion component when modelling asset returns.

These studies and those that followed mark the turning point in the modelling of jumps processes in finance, confirming their disembeddedness from Brownian representation, even if complemented by compound Poisson processes as in the case of the mixed jump diffusion processes of the 1970s.

Let us summarize. By adopting a representation of market fluctuations using an infinite activity Lévy process, it appeared possible in the 2000s to manage without any diffusive component. The structure of trajectory discontinuities (the morphology of the bumpiness of the stock market phenomenon) is fully characterized by Lévy's measure. Compared to the MJD processes that followed the path opened up by Press and Merton, this new representation of small market fluctuations was instead situated in the tradition of NCP-type pure jump processes, as proposed by Cox and Ross in 1976 for evaluating options in markets with trajectory discontinuities. In jump-diffusion processes, jumps are considered as rare events. In Lévy processes with infinite activity, jumps are always present at any scale of the fluctuations.

The generalized hyperbolic family

The first studies focussing on general non-stable Lévy processes had been explored in a completely different context in Denmark and Germany, namely studies of sandstorms. Geophysicists like Ole Barndorff-Nielsen and Ernst Eberlein worked on a family of distributions called hyperbolic distributions. One of the arguments given from the beginning in favour of applying these distributions to finance was they were not stable. In this vein, Eberlein and Keller (1995) write that "real stock-price paths change drastically if we look at them on different time scales". The hyperbolic distributions are infinitely divisible and can therefore be used to construct Lévy processes by specifying the underlying marginal distribution. But the hyperbolic distributions are not stable. In other words, if the underlying distribution is hyperbolic at one given scale, then this does not imply that it will remain this way at any other scale. Hence a numerical computation will be useful for going from one scale to any other scale. The paper by Eberlein and Keller (1995), which introduces the class of hyperbolic distributions

– and as a consequence hyperbolic Lévy motions as driving processes for financial modelling – was the first used for analysing and modelling financial data. The hyperbolic model was next intensively examined by Eberlein, Keller and Prause (1998). Unlike previous work, the papers on these distributions aimed to fit the data; in other words, these distributions represent an "application-driven" approach, like an inflexion point in the BSMRP: "these distributions seem to be tailor-made to describe the statistical behaviour of asset returns" (Eberlein and Prause, 1998).

An intuitive understanding of what motivated the term "hyperbolic" and its fruitfulness in finance is the following. Let us consider the graph of a Gaussian density in a semi-logarithmic graph, i.e. a graph where one axis is plotted on a logarithmic scale. We will find a parabola because of the square power of the variable. This parabola is characterized by a rapid fall of the distribution tails. But empirical semi-log graphs of empirical returns at any scale exhibit a hyperbola, contrary to the parabola of the Gaussian density. This is the reason why these distributions are called hyperbolic. An heuristically bottom-up building of an hyperbolic distribution is given in Le Courtois and Walter (2014b). The usefulness for the modelling of price changes stems from the slower decrease of their tails. The hyperbola fits the empirical data better. Like alpha-stable distributions, hyperbolic distributions are defined by four parameters: localisation, asymmetry, dispersion and kurtosis of the distribution. Like alpha-stable distributions, hyperbolic distribution can characterize the risk of any stochastic change with two dimensions: their size (the parameter of dispersion, or scale parameter), and their form (the fatness of the tails and asymmetry). But, unlike alpha-stable distributions, hyperbolic distributions have all their moments. Hence these distributions modelled both the skewness and leptokurtic features encountered in empirical distributions from the real financial world rather well, without running into the perceived inconvenience of alpha-stable distributions of Mandelbrot's programme. The capacity of these processes to model in an extremely powerful way all trajectory irregularities, while not retaining the property of stability by addition, the cornerstone of the first stage of Mandelbrot's programme, made the family of Lévy processes a serious candidate for the probabilistic representation of market fluctuations.

Another interesting feature of hyperbolic distributions is the limiting case, when the dispersion parameter takes values between 0 and infinity. The two limit cases correspond to the two Laplace laws: Gaussian (Laplace's second law, of 1778) and double exponential (Laplace's first law, of 1774). This shows Laplace's two laws as limit laws of hyperbolic distributions.

Hyperbolic distribution is a subclass of the generalized hyperbolic distribution introduced by Barndorff-Nielsen (1977) for the study of particle size in wind-blown sand deposits. The generalized version of the hyperbolic distributions allows other distributions to be obtained depending of the value of the generalization parameter "lambda". For example, the hyperbolic distribution corresponds to lambda = 1. For lambda = 0.5, we obtain the density of the normal inverse Gaussian distribution (NIG). The normal inverse Gaussian distribution is obtained by mixing normal and inverse Gaussian (IG) distributions. Barndorff-Nielsen moved into finance in 1995. In his papers, Barndorff-Nielsen (1995, 1997) used the normal inverse Gaussian (NIG) distributions. As for the hyperbolic distribution, the NIG is a subclass of the generalized hyperbolic (GH) distributions. Next, the generalized hyperbolic (GH) distribution, which generates the generalized hyperbolic Lévy processes, was systematically analysed by Eberlein and Prause (1998) and Prause (1999). The first applications to the valuation of derivatives appeared and the BSMRP2 succeeded to price options.

The tempered stable family

To remedy the inconvenience of not having any moments for the alpha-stable models of the Mandelbrot's programme, other models were developed with a truncation principle. In the alpha-stable models, the Lévy measure displays a power law which produces Paretian tails. This power law is precisely the origin of the non-existence of the moments when the Paretian exponent is less than 2. A simple way of avoiding this problem is to weight the Lévy measure by an exponential quantity in order to reduce large fluctuations and therefore recover the moments. This idea corresponds to a class of Lévy processes whose marginal distributions are truncated stable distributions, so-called "tempered stable" models. The stable distributions are truncated by exponential functions, hence the term tempered stable processes. The distribution tails of these models,

Table 2. Examples of Lévy processes in financial modelling

	Financial models	Type of Lévy process
1900	Bachelier	Brownian motion
1962	Mandelbrot	Alpha-stable motion
1976	Merton	Brownian motion and Poisson component
1995	Eberlein and Keller	Hyperbolic motion
1997	Barndorff-Nielsen	Generalized hyperbolic motion
1998	Madan, Carr, Chang	Variance Gamma process
2002	Carr, Geman, Madan, Yor	Generalized Variance Gamma process

tempered by the truncation, are semi-light. The Variance Gamma and CGMY models are well-known special cases of this model.

The symmetric Variance Gamma model was introduced by Madan and Seneta (1990) to generalize the Black-Scholes formula in the case of the evaluation of options. The main impetus for constructing this process concerned a practical market problem: finding a suitable model for the so-called volatility "smile" or "smirk" phenomenon. It was extended to incorporate skewness by Madan and Milne (1991) and Madan *et al.* (1998) to become the so-called Variance Gamma model (VG). The terminology is due to the fact that the variance follows a Gamma distribution. The CGMY process of Carr *et al.* (2002) generalizes the Variance Gamma process by adding a parameter permitting finite or infinite activity and finite or infinite variation. The Variance Gamma process and the CGMY process are special cases of the Koponen's (1995) model. Here there is an overlap with the physicist's approach.

The time change representation

The operation of measuring changes with any stochastic process which models a random clock in the time of market events is termed a time change. The resulting process is said to be time deformed. I now elaborate on this deformation of time.

Mandelbrot's approach in the 1970s

The idea of working in distorted time to analyse stock market fluctuations was introduced by Mandelbrot. As exposed in *Fractals and scaling in Finance* (Mandelbrot, 1997, p. 39):

> "The key step is to introduce an auxiliary quantity called trading time. The term is self-explanatory and embodies two observations. While price changes over fixed clock time intervals are long-tailed, price changes between successive transactions stay near-Gaussian over sometimes long time periods between discontinuities. Following variations in the trading volume, the time intervals between successive transactions vary greatly. This suggests that trading time is related to volume"

First I recall the beginning of the time distortion in financial modelling. Second I give a heuristic example of time change.

The beginning

The first studies in finance to introduce the idea of time change were those of Mandelbrot and Taylor (1967) and of Clark (1973). Mandelbrot and Taylor had used the process of the cumulative number of transactions for counting time. The assumption made was that this process was a non-decreasing alpha-stable motion such that the periods between two successive quotes followed a Pareto distribution: the transaction rate was Paretian. In 1973, Clark proposed that, rather than calculating time through transactions, a better measure of the speed of changing time would be the volume of shares traded.

One of the most interesting findings that emerged early in this research on time change was the relativity of the probability of the financial phenomenon: in transaction-based time, the distribution of variations in returns was Gaussian. In the conclusion of their paper, Mandelbrot and Taylor (1967) stated that there was formally no difference between a Paretian marginal distribution in physical time and a Gaussian marginal distribution in market time counted by successive Paretian rate transactions: alpha-stable motions in physical time and Brownian motion in Paretian time corresponded through a change in the time reference system. This idea – measuring a Brownian motion in market time to explain non-Gaussian distributions – was subsequently extensively explored, though not immediately.

To interpret this Paretian phenomenon in terms of information, Mandelbrot had used the work of the American linguist George Kingsley Zipf (1902-1950) on the distribution of the frequency of words in discourse pertaining to the interpretation of trade volumes. Zipf's law states that the words of a text are distributed in such a way that, if they are listed in descending order of frequency, the frequency of a word is inversely proportional to its position in the list: this implies that the less frequently a word occurs in a text, the more information it contains. Transposed to market movements, this relationship means that large price movements, which are infrequent, carry information essential for understanding the mechanism governing the market (whether or not this information is economically justified). Their rarity is the sign of their information value.

To name his financial fractal models, Mandelbrot had used biblical references: the story of Noah (the idea of a "flood of information") for the 1962 alpha-stable model and the story of Joseph (the idea of "fat cows and lean cows of economic activity") for the 1965 fractional model. To stay in this biblical register and engage in dialogue with Mandelbrot, to name the distorted time model from the standpoint of information, I propose the text of the psalmist: "a thousand years are as a passing day, and a day is as a thousand years" (Psalms 90: 4).

Heuristic approach of change of time

Recall that the first hyperbolic distribution (H) used by Barndorff-Nielsen (1997) was obtained by mixing a normal distribution (N) with an inverse Gaussian distribution (IG). I introduce the mnemonic below to summarize the construction of hyperbolic distributions as mixture distributions. The nesting of the inverse Gaussian distribution (IG) inside the normal distribution (N) is applied ? next to the resulting simple hyperbolic distribution (H):

H = N[IG]

In the same way, just as hyperbolic distributions (H), generalized hyperbolic distributions (GH) can be interpreted as the result of a mixture of a normal (N) and a generalized inverse Gaussian distribution (GIG):

GH = N[GIG]

In each case, fat tails are obtained by a mixture of simple probability distributions.

The mnemonic draws an interesting conclusion regarding the interpretation of price changes in the IID framework. As already noted, we are able to construct infinitely divisible leptokurtic distributions by mixing distributions. We now move to market price dynamics and think in terms of stochastic processes instead of static distributions. Recall that the variance represents the physical time (linearity in the relation between variance and time). Hence the random variance of a mixture of distributions in a static framework becomes a stochastic clock in a dynamic context.

In a dynamic context with stochastic processes and IID increments, the normal distribution becomes a Brownian motion, the inverse Gaussian distribution becomes an inverse Gaussian Lévy process, the hyperbolic distribution becomes a hyperbolic Lévy process and the generalised hyperbolic distribution becomes a generalised hyperbolic Lévy process. The above mnemonic scheme N[IG] becomes:

Hyperbolic Lévy process = Brownian motion [IG clock]

And the same for the second:

Generalized Hyperbolic Lévy process = Brownian motion [GIG clock]

Now we can write the previous relations in a more compact form:

Lévy process = Brownian motion [stochastic clock]

The clock that measures time is given by any stochastic process. This new representation of the same market price dynamics in terms of change of clock is extremely fruitful because it gives us a mechanism with which to deform time in stock markets and to understand in another manner the opposition between continuity and discontinuity and the Black-Scholes model refinement programme. I address this topic in the next section.

The BSMRP approach in the 2000s

After the early 1970s, the stream of research on time change seemed to have no immediate aftermath. But then works on the distortion of time and the relationship with the market activity resumed in the 1990s thanks to access to high-frequency data (Müller et al., 1990, 1995, 1997; Gouriéroux, Jasiak and Le Fol, 1999; Engle and Russel, 1994). For example, the exhaustive statistical analysis carried out on high frequency data on Elf Aquitaine share prices between March and August 1996 (amounting to 179,958 quotes) made it clear that the distribution of volumes assumed a decrease in a Pareto-type power law (Maillet and Michel, 1997), a law that has since been observed for the distribution of trade volumes.

Obtaining Gaussian distributions by correcting the time with a Brownian motion was demonstrated by Ané and Geman (2000). Since then, numerous studies have come up with similar results (Geman, 2008). Hence, the time-changed Lévy processes approach seems to unify the two competitors in the attempt to model jumps: the first strand following the Mandelbrot programme, and the second termed BSMRP. The option-pricing problem can be solved with the time-changed framework.

The time change representation of price changes

The stochastic clock is any non-decreasing stochastic process, called the driving process of the return process. The non-decreasing requirement refers to the impossibility of time going backward. In a way, the new return process is driven by the time change process, hence the terminology:

> Observed process (physical time) = driven process (market time)

When the time change is a non-decreasing process with IID increments, i.e. a non-decreasing Lévy process, the random clock is termed the *subordinator* of the return process. Every subordinator process is a driven process, but the converse is not true. For example, the Poisson process, the inverse Gaussian Lévy process, the hyperbolic Lévy process are subordinators. The stochastic clock chosen for deforming the time represents the metronome of the social market place. Hence the time change representation allows the rediscovery of the notion of "exchange time" in Fernand Braudel's terminology.

If the return is a Lévy process, the subordinated process remains a Lévy process. Conversely, we can interpret any Lévy process as another process measured in market time. The question then is the choice of market time. There are unlimited possibilities for the change of clock: one simply has to define a stochastic clock, which is in fact the operator of the change in time measurement, to go from calendar time (the time of natural events) to market time (the time of social events) and apply it to any Lévy process. The advantage of this representation in distorted time is being able to interpret any Lévy process in physical calendar time (or time without social events) as a Brownian motion in stock market time.

Rethinking discontinuity with time change

One consequence of the change of clock is the emergence of non-normality in physical time that occurs at the moments of the distribution. The marginal distributions of financial uncertainty will be non-Gaussian, but the distributions of uncertainty as regards market time will Gaussian. The Monroe (1978) theorem demonstrates that if a random process is a semi-martingale, then it can be written as Brownian motion measured with a different temporality. This means that it is possible to consider any Lévy non-Brownian process with long-tailed non-Gaussian distributions in calendar (physical) time as a Brownian motion in social time (intrinsic time), as shown in the crucial relationship below:

> Lévy process (physical time) = Brownian motion (social time)

For example, a Brownian motion measured in a social time with a social clock following a Gamma distribution (meaning that the inter-quote time lags are Gamma distributed) – in other words a Brownian motion in Gamma time – is the so-called Variance Gamma process of Madan and Seneta (1990). The name comes from the fact that the stochastic variance of the Brownian motion follows a Gamma distribution. In physical time, Le Courtois and Walter (2014a) demonstrate that the Variance Gamma process is a Laplace motion, a new Lévy process based on Laplace's first law of errors:

> Laplace motion (physical time) = Brownian motion (Gamma time)

We now understand that there is a close relationship between the choice of a discontinuous non-Brownian representation of risk in calendar time and the understanding of the "relativity" of time on the markets. In other words, the acceptance of jumps in the physical time representation leads to a new way of thinking time on the markets: a time in which human decisions can take place.

Example with the Variance Gamma model

The Gamma process is a Lévy process whose increments follow a strictly increasing Gamma distribution. It can therefore be viewed as a potential subordinator.

Let us now change the time frame of Brownian motion by measuring it in market time by means of a random clock whose density follows a Gamma distribution. With this clock, the times between successive quotes are distributed according to a Gamma law: trades are produced according occurrences defined by a Gamma process. Brownian motion measured by Gamma time is exactly the "Variance Gamma process" (VG) of Madan and Seneta (1990). The parameter of the Gamma process expresses the variance, which allows a direct intuitive interpretation: the greater the value of the parameter, the more quickly the stock market becomes agitated. Think of the variance as duration. We then see that the greater the variance of the Gamma process, the greater will be the impact on the trend of the Brownian motion. This will produce an asymmetric effect precisely because of this trend.

Going back to calendar time, Le Courtois and Walter (2010) demonstrated that the VG process is a Laplace process without drift. Writing it as a Laplace process allows us to consider the usual financial entries as special cases and to supplement them. If we add a non-zero drift to construct a VG with drift (VGD), we obtain the Laplace process of Le Courtois and Walter (2014b). The variance of the Laplace distribution is equal to the variance of Brownian motion multiplied by the variance of the Gamma process. The 'volatility' corresponds to the degree of agitation of Brownian motion. This parameter acts directly on the shape of the distribution and is used to bring out leptokurticity, acting in a way on opposite sides of the dispersion and allowing a second risk dimension to be controlled. A high degree of agitation for this time change process produces a strong leptokurticity effect on the resulting process.

To summarize; the way in which a random walk follows the pace of trading is illustrated by the example of Brownian motion measured in Gamma time. This change of time enables a Laplace process to be found in calendar time. With this example, the link between Laplace's first law (static morphology of uncertainty), Lévy processes in calendar time (dynamic view of uncertainty) and Brownian motion in Gamma time is established.

Conclusion

In this paper, I have defended the claim that the Mandelbrot programme contributed to a better understanding of the discontinuous nature of price change, even if the first Mandelbrot's models initially based on alpha-stable motions, a very specific subclass of Lévy processes, were not accepted by the mainstream financial academics community. I have argued that in the 1970s, the mainstream view of price changes made specific assumptions to defend the mathematical tractability of the financial modelling – based on continuous diffusion models – by using a model-tinkering approach, which gained highly recognition in the 1980s: the Black-Scholes model refinement programme (BSMRP). I have explained the successes of BSMRP in the 1990s by showing that an inflexion point seemed to appear with a reorientation of the BSMRP due to European researchers who put forward the fruitfulness of infinite activity of the Lévy processes in case of pure jumps models: this turning point was more application-driven than mathematical-driven. I indicated that this new view on price variations led to a reconciliation with one of the Mandelbrot's models, that of time change representation.

Table 3. From physical time to social time: some examples.

Lévy process (physical time)	Brownian motion (social time)
Alpha-stable motion (physical time)	Brownian motion (Paretian time)
Hyperbolic motion (physical time)	Brownian motion (inverse Gaussian time)
Laplace motion (physical time)	Brownian motion (Gamma time)

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