Human capital productivity and uncertainty

Mbaye Diene, Bity Diene, Théophile Azomahou

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Mbaye DIENE

Bity DIENE

Théophile T. AZOMAHOU

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CERDI
65 BD. F. MITTERRAND
63000 CLERMONT FERRAND – FRANCE
TEL. + 33 4 73 17 74 00
FAX + 33 4 73 17 74 28
www.cerdi.org

The authors

Mbaye Diene
Professor
University Cheikh-Anta-Diop and Consortium pour la Recherche Economique et Sociale (CRES) rue 10 Prolongée Cité Iba Ndiaye Diadji no 1-2. Dakar, Sénégal
Email: mdiene@cres-sn.org

Bity Diene
Associate Professor
Clermont Université, Université d'Auvergne, CNRS, UMR 6587, CERDI, F-63009 Clermont Fd
Email: bity.diene@udamail.fr

Théophile T. Azomahou
Professor
United Nations University (UNU-MERIT) and University of Maastricht, Keizer Karelplein 19, 6211 TC Maastricht, the Netherlands
Email: azomahou@merit.unu.edu

Corresponding author: Mbaye Diene

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Abstract

Several policies or interventions have been implemented in developing countries with the ultimate goal of improving educational outcomes and human capital. While lots of empirical studies have pointed to mixed results of these interventions, the role of uncertainty arising from the state of the nature about educational environment, household characteristics, along- side the efficiency of these interventions still lack economic mechanism. This paper aims at developing a theoretical framework that links policy interventions to educational outcomes. We characterize optimal policies and determine the conditions for enhancing social welfare. We also study the optimal growth of the economy under uncertainty and population heterogeneity when human capital is produced and used in the education sector. We show that the growth rate of the unskilled population has a direct impact on the growth of human and physical capitals.

Key words: Educational outcome, policy interventions, social welfare, skilled and unskilled labor, endogenous growth

JEL codes: I25, I31, J13, O15, C61

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1 Introduction

In many developing countries, the education sector has experienced several policy interventions, that is to say, the application of new resources or approaches that change practices to improve the accumulation of human capital and well-being, especially that of pupils. These interventions result in better achievements or performance of pupils at school, viz, improvement of academic outcomes, increased enrollment and attendance rates, and reduced drop-out and repetition rates. Hargreaves (2003) states that 'the opportunity to engage actively in innovations (or interventions) and the means of transferring successful interventions in some schools to other schools, are conditions that can support innovations in education'.[1] As outlined by Frank et al. (2004) and Groff and Mouza (2008), the success of these policies depends on the capacity and disposition of policy makers. The relationship between these two factors can be analyzed using two models: i) the 'distance and dependence model' that makes explicit how the specific context can affect a policy and help identifying its success by showing its difference from existing practice and resources, and ii) 'the layers of influence model' that distinguishes the influences that affect the conditions of a policy and the policy maker.

The first model sheds new light on the debate 'pedagogy before practice' by suggesting that the implementation of an intervention depends on close connection of practice and technological issues. The model was initially drawn from a study on technology-based innovation, but it can also be used for non-technology-based approaches in education. The model enables us to depict how an intervention can be assessed through its distance from current practice and its dependence on available resources. This model predicts that a policy has better chances of being accepted if it is close to existing practices. These practices can be related to educational environment. An intervention in education also needs resources that differ from existing ones.[2] For example, if an intervention requires a great change in home inputs practice and more financial resources, then it needs more support to succeed than an intervention that requires fewer resources or one that demands little change from the home's existing practice. As such, implementing a policy in educational environment can imply that one forgoes other policies so as to respond to resource need of implied by some initial intervention. The complexity of educational policies can follow from the level or mode of interventions: national, community based, school or individual levels. As a result, the success depends on the extent to which the change is understood and recharged. Hence, fostering educational policy requires an analysis of barriers and resistances to changes.

The second framework – model of layers of influence – analyzes the factors that affect policy makers ability to implement an intervention. It conceptualizes the layers of influence that affect both the intervention and the policy maker.[3] This model highlights the way layers interact and the role of environmental conditions surrounding the interventions. The environmental conditions that determine the failure or success of policies can be gathered into four categories:

Supportive informal social environment: This refers to the atmosphere and perception of agents that may help adopting new practices.[4]

Formal environment: This constitutes the organizational infrastructure of an educational system together with its formal policies and structures. This environment is crucial in providing resources for interventions, and allowing interventions to become accessible through technical

support policies.[5]

Risk aversion: Risk aversion is an important factor that inhibits the ability to innovate and it has implications on the extent to which any educational policy targets the appropriate conditions for interventions.

Shared visions: Refer to common perceptions of goals and requirements.[6]

Regardless of the approach adopted, the issue raised by policy interventions in education can be stated as follows: How does uncertainty affect the impact of policy interventions in the developing world? Indeed, in these countries, education is increasingly a crucial ingredient for development programmes. The role of uncertainty may come through different facets. In general, it is related to the state of nature, meaning exogenous factors unrelated to the policy that may affect the policy implemented. In developing countries, uncertainty is much more pronounced due in particular to the lack of resources and the level of development that ultimately can impact on the success of interventions. For example, it is common to observe that after the starting of implementation of an intervention at a given date, resource constraints are changing the object of the intervention, reduce its ambition, or sometimes even stop it.

This paper aims to develop theoretical frameworks to link every specific type of new intervention for each stakeholder to the global performance of education, taking into account the social welfare maximization problem. We consider the production function of the school as a black box where several factors combine (good school management, quality of school services and access to education) and whose outcome is the final performance of students. Our goal is to evaluate the evolution of this performance over time, when policy makers rely on the quality of the school production, e.g. educational outcome. Their interventions consist of changing the performance from an initial period to a final period, taking into account the constraints that may arise. To do this, we first link the vector of performance to the vector of constraints, assuming different types of relationships between these vectors.

In a first specification, we consider linear and nonlinear deterministic relationships and characterize the optimal interventions which give the best performance given the constraints and initial conditions. Then, recognizing that the lack of information on the socioeconomic characteristics of students and the educational environment in which interventions are implemented, among others, are uncertainty factors that may impede the achievement of performance objectives, we introduce uncertainty in the framework. Here again, we consider linear and nonlinear approaches. We find out the optimal conditions under which actions can be taken. Furthermore, we enlarge the analysis to the question of how the performance of the educational system can be integrated into a macroeconomic performance (in terms of well-being and economic growth).

Several results emerge from this study. Firstly, we consider the benchmark framework without uncertainty. In this set up, we consider both the linear and the nonlinear cases. For the linear model, we assume that the relationship between changes in performance and successive interventions are additive and separable. We study the growth rate of educational performance, their trend and the average change in performance due to a specific intervention. The main result is that interventions that allow to move from one level of initial performance to a final level are also additive and linear, and ultimately they may be constant in a regular time intervals. They also depend on the temporal growth rate of performances, that would have prevailed if there was no response. For the nonlinear model, we have shown that interventions are possible,

even in the case of resistance, meaning factors that preclude performances. These interventions can be coordinated, so common to all stakeholders.

Further, we illustrate these findings with some examples. In the first, interventions can fade over time, which means that students at a given date can be left at their free course, when they reached a sufficient level of performance which is high enough to be irreversible. In the second example, the intervention depends linearly on initial conditions in regular time intervals. This means that interventions are implemented gradually, until the desired level of performance is reached. In the last example, only one type of intervention is made to achieve the desired the performance, regardless of which decision maker applies it.

Secondly, the framework with uncertainty also considers linear and nonlinear probabilistic models. The occurrence of random events is integrated. Relying on normal distribution of random events, we express the optimality conditions of interventions, based on average probabilities. We propose a methodology to solve these conditions. The optimality is based on maximizing the probability of achieving the target performance from an initial period to the final period. We illustrate in an example how the construction of solutions relies on the correlation function of the random process and the initial conditions.

Thirdly, we link the performance levels to social welfare, on the assumption that the ultimate goal of policy makers is improving the well-being of all individuals. This can go by investing in education of students. As in the previous case, we use deterministic and probabilistic approaches. Taking the expected utility of consumption and investment in quality and access to education, we show analytically the optimality conditions of these variables.

Lastly, we deal with economic growth with heterogenous population. Skilled and unskilled groups have different demographic dynamics. The economy has two sectors: education and production of goods and services. We show that the demographic growth rates of the two populations have differentiated impacts on economic growth.

The remainder of the paper is organized as follows. Section 2 introduces a brief review of the educational production performance. Section 3 develops frameworks of interventions in education. Section 4 studies the optimality of interventions in terms of social welfare. Section 5 addresses the issue of global approach of interventions with heterogeneous populations and their dynamics on economic growth under uncertainty. Section 6 concludes the study.

2 Education performance: A brief review and empirical facts

The goal of achieving universal education in developing countries involves looking for ways to produce effective and efficient schools. Effective teaching methods, based on survey data acquired from schools, have shown their worth for almost fifteen years. In order to identify forms of effective schools, tools were developed primarily to measure whether countries can achieve the goal of enroling all the children of school age, and then to evaluate the effectiveness and quality of learning provided in schools.

Since the 1990s, PASEC has implemented in Francophone Africa, surveys to assess child learning, collecting information on their characteristics: origin of children, their living conditions (situation at home, medical, diet, economic well being of household, housing quality, parental care etc), characteristics of teachers and schools, etc. These elements are often used

as components of a production function of school (Bourdon, 2005). The problem is whether there is a form of this function that is appropriate to describe the effective provision of universal education and the performance of interventions in the education sector. A key challenge remains in describing the cost of education.

2.1 The production function of school

The identification of the determinants of quality educational service is not trivial. Hanushek (1986) shows that there is a bewildering range of issues including technical and esoteric conflicting results on the production process of the schools. He argues that there is still no clear answer as to what are the factors that influence pupils' performance. In this context, Pritchett (2001) finds that the choices that guide an educational allowance are not often based on academic performance.

Empirical facts contradict the hypothesis of an efficient allocation of resources that seeks to maximize the school performance. This contradiction is attributed to four reasons. First, the school is not a black box within which production technology follows market rules. Secondly, the impact of schooling on the attainment may be small compared to the role and importance of innate abilities of learners. Third, the demand for education is not facing a market, and the production function cannot be observed effectively from an economic standpoint. Finally, the education production function, if it is tested econometrically, cannot be generalized as already shown by Hanushek (1986).

2.2 Measurement of education effectiveness

The optimal timing of school programs has been studied by Farrell (1957) and Charnes et al. (1981). However, the difficulty lies in identifying stable parameters of the production function, most importantly those driven by the environment of school as well as households' characteristics. The school production is represented by the results of pupil assessment in language, calculation, the value of self-esteem reported by the pupils and also some more aggregate measures like enrollment, promotion, dropout, etc. For instance, Battese and Coelli (1995) show that the environmental variables can explain the remoteness of the border. Empirical studies are also interested in identifying the best performing schools. Relying on parametric and nonparametric approaches for envelope method, Cooper and Cohn (1997) have identified schools that are close to the efficiency frontier. The impact of intervention on the effectiveness of school have been examined as well by Stiefel et al. (1999) using randomized control trials. The authors show that there is a strong inertia between interventions and their effects on academic performance. Klein (2007) used a Becker-Stigler-Peltzman like model to determine the socially optimal level of intervention in education.

Other studies have tried to link school performance to the time of enrollment. For higher education, Dolton et al. (2003) described a production function where academic success, given by individual performance on the final exam, depends on the time spent at school. They show that the schooling time is four times less profitable than teaching in a working group. As a general form of intervention, they used public expenditure in education. Gupta and Verhoeven (2001) evaluated the effectiveness of public expenditures in 37 African countries over the period

1984-1995 and compared them with Asian and Western countries. They showed that on average, African countries are less efficient than Asian countries and countries in the Western hemisphere. Afonso et al. (2006) showed a clear distinction between countries according to indicators of absolute performance and cost effective type indicators. National structures for utility costs can play a crucial role and lead to situations where some systems offer public service and others do not. This may be due to allocation rules and routine border performance allowed by the technical frontier.

Kirjavainen and Loikkanen (1998) used a Tobit model powered by the levels of efficiency from Data Envelopment Analysis (hereafter DEA), to explain the determinants of efficiency of secondary schools in Finland. In their approach, the education of parents is a driving factor that determines the differences in school performance. Bradley et al. (2001) also used the DEA and Tobit model to evaluate the technical efficiency of English secondary schools. The average efficiency rate obtained were between 83% and 75%. The authors also found that competition between schools improves the efficiency level of schools. This finding is consistent the results of Waldo (2007) who studied the performance of Swedish secondary schools using DEA. In the case of Portugal, Oliveira and Santos (2005) examined institutional indicators. They were particularly interested in relaxing the convexity constraint. Simar (2003) found that the unemployment rate, access to health services, adult education and infrastructure endowments are determinants of academic performance. Rubenstein et al. (2007) used a sample of schools in the north-eastern USA and found that the effectiveness of policies is conditioned by structural elements including vocational training. This brief literature review outlines the ambiguity and difficulty of measuring the efficiency and performance of school. The school with superior performance can be the one that has better policy, but it can also be the one which is in a very favorable environment.

3 A Theory of interventions in education

In most countries, education is largely a national public service, whose organization and operation are provided by the government.[7] However, local administration can also be involved in the development of this public service. There are several stakeholders in the education sector, each with specific and complementary roles. At the national level, the *government* is competent in all aspects of pedagogy, curricula, national qualifications and management of teaching staff, etc. At the regional level, *local administrations* (counties, districts, municipalities, etc.) are in charge of the decentralized services of the ministry of education. The role of *communities* (e.g. association of parents) is also important. Indeed, parents are full members of the educational community. Through their representatives, they participate in school councils, class, and administration of the institutions which indirectly implies the application of education policy. It is worth to note that there often exist structures of consultation (which enable their opinion to guide decision-making or allow actors and partners of education to meet and take decisions) and sometimes technical committees dealing with issues of collective interest. Interventions by all these stakeholders in education have direct and indirect impact on pupils' performance.

However, these interventions are implemented in an environment with uncertainty which is related to the state of nature.[8] This environment may be favorable or unfavorable to the expected

result of the intervention. For example, unforeseen constraints (e.g. stochastic shocks) on resource availability can lead policy makers to modify or discontinue the intervention. Similarly, unobservable individual factors related to the environment can make the same intervention more efficient for some individuals and less for some others. Sometimes, the results can go in the opposite direction due to interaction with other factors. This raises the question as to how uncertainty affects the impact of interventions in education. In what follows, we develop simple models that account for these situations and help us to better understand the economic mechanisms through which these interventions operate as well as their effects on well-being.

3.1 The benchmark model without uncertainty

Let X_t denote a vector whose n components are the criteria measuring agents' (pupils') performance (e.g., achievements like score, repetition rate, etc). All interventions are captured by the vector U_t with r components. The aim is to start with an initial state X_0 and reach an optimal state X_T, where pupils' performance is better, T being the final time for the effects of interventions. The equation of variation of pupils' performance ia:

$$\dot{X}_t = F(t, X_t, U_t). \tag{1}$$

The optimality means that in the final state, the interventions lead to a state close to their objectives. At the level of an agent, it does not mean that all performance indicators' at period T are higher than those in the initial period. But the average level achieved with X_T is expected to be higher than the one with X_0. We will consider two cases for Equation (1): the linear and the nonlinear.

3.1.a The linear case

We assume that F can be written in the form

$$\dot{X}_t = P(t)X_t + Q(t)U_t + R_t \tag{2}$$

where $P(t)$ and $Q(t)$ are matrices of respective order $n \times r$ and $n \times n$. Equation 2 shows that variations in performance are additive with respect to successive separable interventions. All things being equal, $P(t)$ represents the rate of growth performance in the absence of policy interventions with trend $R(t)$. Similarly, $Q(t)$ denotes the average change in performance following an intervention. We can assume different frameworks: i) independence of interventions and ii) existence of a centralized public target (as a global education policy overseen by the government, in the form of recommendations to stakeholders) that guides the interventions. Let us consider each of these frameworks in turn.

Proposition 1 *The interventions leading X_0 to X_T can be written as:*

$$U_t = B(t)c + v(t) \tag{3}$$

where $c = A(T)[Y^{-1}(T)X_T - \int_0^T Y^{-1}(\theta)R(\theta)\,d\theta]$ is a vector of constants, v is a function of time, $B(T) = Y^t Q(t)$ and $Y(t)$ denotes the fundamental matrix of the system $\dot{X}_t = P(t)X_t$.

The intuition behind Proposition 1 is that the interventions are additive and linear and that they apply consistently in regular intervals of time. That is, these interventions are not highly variable and may be in the form of, for example, step functions so that each time a policy is taken, it is maintained for a while, before another intervention is undertaken. This opens the opportunity to regularly assess the effects of interventions and to modify them each time a new performance target is set. Indeed, if the interventions are continuous and regular, without the possibility of change, the costs could be unbearable and would lead to losses.

Proof. Given the fundamental matrix $Y(t)$, let us assume that the initial condition has the form $Y(0) = E$, with E being the unitary vector. Let us rewrite $X_t = Y(t)Z_t$ such that

$$\dot{Z}_t = B(t)U_t + G(t) \tag{4}$$

where $B(t) = Y^{-1}Q(t)$ and $G(t) = Y^{-1}R(t)$. By making the following transformation:

$$Z_t = \xi_t + \int_0^t G(\theta)\, d\theta \tag{5}$$

where ξ_t is a vectorial function that verifies the equation:

$$\dot{\xi}_t = B(t)U_t \tag{6}$$

It follows that

$$\begin{cases} \xi(0) = \xi_0 = X_0 \\ \xi_T = \xi_T = Y^{-1}(T)X_T - \int_0^T Y^{-1}(\theta)R(\theta)\, d\theta \end{cases} \tag{7}$$

For sake of simplicity, let us suppose that $r = 1$, meaning that we only have one intervention. The integration of Equation (6) gives the following system of integral equations:

$$\xi_T - \xi_0 = \int_0^T B(\theta)U(\theta)\, d\theta \tag{8}$$

the solution of which takes the form $U_t = B(t)c + v(t)$ with c being a constant vector and $v(t)$ is a function such that

$$\int_0^T b_s(\theta)v(\theta)\, d\theta = 0 \qquad s = 1, 2, \cdots, n \tag{9}$$

where the b's are the components of B. ■

Proposition 1 also states that at any point of time there exist actions guiding agents' performance. It can also be used when there are limit conditions. Indeed, if we state them as

$$\sum_{j=0}^{S} G_j X(t_j) := H \tag{10}$$

where H is a vector with m components and t_j are such that $0 \leq t_0 < t_1 \cdots < t_s \leq T$ and G_j are constant real matrices. Knowing $U(t) = U$, we can write

$$X_t = Y_t X_0 + \int_0^t Y_t Y^{-1}(\theta)Q(\theta)U\, d\theta + \int_0^t Y_t Y^{-1}(\theta)Q(\theta)\, d\theta. \tag{11}$$

Substituting (11) in (10), we have

$$\sum_{j=0}^{S}[G_jY(t_j)]X_0+\sum_{j=0}^{S}\int_0^{t_j} G_jY(t_j)Y^{-1}(\theta)Q(\theta)U(\theta)\,d\theta+\int_0^{t_j} G_jY(t_j)Y^{-1}(\theta)R(\theta)\,d\theta = H \quad (12)$$

By setting:

$$\varphi_j(\theta) = \begin{cases} 1 & \text{if } \theta \in [0, t_j[\\ 0 & \text{if } \theta \in [t_j, T] \end{cases}$$

and rewriting (12) accordingly, we have

$$A_1X_0 + \int_0^t B_1(\theta)U(\theta)\,d\theta = H_1 \quad (13)$$

with $H_1 = H - \sum_{j=0}^{S}\int_0^{t_j} G_jY(t_j)Y^{-1}(\theta)Q(\theta)R(\theta)\,d\theta$, $A_1 = \sum_{j=0}^{S}[G_jY(t_j)]$ and $B_1 = \sum_{j=0}^{S}\varphi_j(\theta)$ $G_jY(t_j)Y^{-1}(\theta)Q(\theta)$. Equation (13) is a system of integral equations. By setting $r = 1$, the solution is written as:

$$U = B_1^*c + v. \quad (14)$$

Substituting (14) into (13) yields

$$A_1X_0 + A_2c = H_1. \quad (15)$$

This means that when the ranks of the two matrices (A_1, A_2) and (A_1, A_2, H_1) are equal, then $X(t)$ is solution of Equation (2). The same expression of $U(t)$ holds when the limit conditions of (10) are continuous as:

$$\int_0^T dG(\theta)X(\theta) = H. \quad (16)$$

3.1.b The nonlinear case

We now consider a more general case in which the equation of variation of agents' performance is written as

$$\dot{X}_t = P(t)X_t + Q(t)U_t + R_t + \mu G(t, X_t, U_t, \mu). \quad (17)$$

The nonlinear case is more general than the linear one, with the additional term $\mu G(t, X_t, U_t, \mu)$. The nonlinear case takes into account two factors. First mixed interventions, then interventions where there are resistances, meaning for which people are reluctant. Indeed, it is possible that a government undertakes discrete interventions, which are supplemented by those undertaken by local authorities and households. In this case, it is difficult to isolate the specific effects of each intervention. For example, if a government undertakes health interventions, local authorities can specialize in nutrition interventions and households are involved in the monitoring of teaching of pupils, then these types of interventions are not exhaustive, but exclusive. Their specific effects on educational outcome are not controllable, and it can be difficult to identify the most effective intervention.

Similarly, community or individual resistance may occur, depending on the types of interventions and geographical locations. Often in developing countries, some ethnic backgrounds make people reluctant to nutrition and health policies (e.g. deworming) which are set up by government. Then sociocultural considerations may lead to distrust. If government wants to intervene, underperformance following from this kind of resistances may occur. The nonlinear specification integrates all these considerations, given that educational performances are not reached in a simple manner, and that there are complex factors that make policies difficult to implement or rarely effective.

In Equation (17), more specifically in the term $\mu G(t, X_t, U_t, \mu)$, μ is a parameter and G denotes a vector function. We shall assume impulse interventions meaning that interventions are constant. In other words, they are of the same type between t_0 (initial time) and the final time T. Specifically, the vector U_t is constant in time intervals $[t_j, t_{j+1}]$ with bounds, $t_0, t_1, t_2, \cdots, T$. The rationale of such assumption is that it is typically the kind of intervention encountered in the reality. For example: for school feeding programs, foods are provided at fixed hours and identically to all pupils; in the case of subsidies granted to pupils' parents, the scholarships are regularly and equally paid approximately at the same period; in the case of academic support consisting in helping students to catch up delays, this support usually occurs at fixed hours, etc. We have the following proposition.

Proposition 2 *Assume that we associate to every pair of bounded sets G_0 and G_T a scalar $\mu_0 = \mu_0(G_0, G_T)$ such that whatever $\mu_0 < \mu_T$. There exists an intervention U_t that leads X_0 from G_0 to X_T belonging to G_T.*

Proof. We set

$$A(T) = \int_0^T B(t)B^*(t)\,dt, \text{ where } B(t) = Y^{-1}(t)Q(t). \tag{18}$$

Suppose that we have $X(t)$ and $U(t)$ such that Equation (17) is verified. Then (17) can be written as:

$$X_t = Y(t)X_0 + Y(t)\int_0^t Y^{-1}(\theta)\left[Q(\theta)U(\theta) + R(\theta) + \mu G(\theta, X, U, \mu)\right]d\theta. \tag{19}$$

By setting $t = T$, we have

$$Y^{-1}(T)X_T - X_0 = \int_0^T B(\theta)U(\theta)d\theta + \int_0^T Y^{-1}(\theta)\left[Q(\theta)U(\theta) + R(\theta) + \mu G(\theta, X, U, \mu)\right]d\theta. \tag{20}$$

Writing $U = Bc + v$, it follows that

$$c = A(T)\left[Y^{-1}(T)X_T - X_0 - \int_0^T Y^{-1}(\theta)R(\theta)d\theta - \mu\int_0^T Y^{-1}(\theta)G(\theta, X, U, \mu)d\theta\right]. \tag{21}$$

Replacing for (21) in U, we have:

$$U(t) = U_0(t) + \mu B^* A^{-1}(T)\left[-\int_0^T Y^{-1}(\theta)G(\theta, X, U, \mu)d\theta\right] \tag{22}$$

$$U_0(t) = B^* A^{-1}(T)\left[Y^{-1}(T)X_T - X_0 - \int_0^T Y^{-1}(\theta)R(\theta)d\theta\right] + v(t). \tag{23}$$

Then, it is easy to find that:

$$X(t) = X_0(t) + \mu \left[\int_0^T Y(\theta) Y^{-1}(\theta) G(\theta, X, U, \mu) d\theta - Y(t) A(t) A^{-1}(T) \int_0^T Y^{-1}(\theta) G(\theta, X, U, \mu) d\theta \right] \tag{24}$$

with $X_0(t) = Y(t) \left[X_0 + \int_0^t B(\theta) U_0(\theta) d\theta + \int_0^t Y^{-1}(\theta) R(\theta) d\theta \right]$. Equations (23) and (24) show that for $\mu = 0$ and $U = U_0(t)$, Equation (17) has $X = X_0(t)$ as solution. ■

In what follows, we provide examples that illustrate different modes of interventions.

Example 1

Suppose that the intervention consists of providing school meals. We seek to measure the effects of such program on the concentration time of pupils in class. This concentration time is assumed to have a direct impact on pupils' academic performance. This problem can be described by the following equation:

$$\dot{X}_t = AX_t + GU_t \tag{25}$$

where X_t measures the difference between the maximum score and its current level:

$$X_t = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}; \qquad A = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \vdots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix}; \qquad G = \begin{pmatrix} G_{11} & \cdots & G_{1n} \\ \vdots & \vdots & \vdots \\ G_{n1} & \cdots & G_{nn} \end{pmatrix}; \qquad U_t = \begin{pmatrix} U_1 \\ \vdots \\ U_n \end{pmatrix}$$

Equation (25) can also be re-written as

$$\frac{dX_j}{dt} = \sum_{k=1}^{n} A_{jk} X_k + \sum_{l=1}^{r} A_{jl} U_l, \qquad j = 1, \cdots, n.$$

The solution of this given by

$$X_t = X_0 e^{At} + \int_0^t e^{A(t-\theta)} GU(\theta)\, d\theta. \tag{26}$$

It is straightforward to show that the interventions that can reduce the performance gap between $t = 0$ and $t = T$ take the form

$$\widetilde{U}_t = -G' e^{A't} M^{-1} X_0 \tag{27}$$

where $M = \int_0^t e^{-A\theta} GG' e^{-A'\theta} d\theta$, and A' and G' denote the transpose matrices. If one considers the much simpler form of education performance variation:

$$\dot{X}_t = a + U_t, \tag{28}$$

then, the intervention becomes

$$\widetilde{U}_t = -\frac{2a}{1 - e^{-2aT}} e^{-at} X_0. \tag{29}$$

This mode of intervention is typically the one undertaken by a government.

Example 2

Let us consider a discrete time problem where the change in performance over time is described by

$$X_t - X_{t-1} = aX_{t-1} + bU_t + r. \tag{30}$$

Equation (30) can be interpreted as follows: the difference between the performance before and after the access to the intervention depends on the initial performance and the intervention U_t. For the intervention to be optimal, in the sense that $X_t > X_{t-1}$, it must be of the form $U = Bc + v$. This implies that the intervention is constant in the time interval $[t-1, t]$. By induction we have:

$$X_t = (1+a)^t X_0 + \sum_{j=1}^{t} (1+a)^{j-1}(bU_{n-j+1} + r). \tag{31}$$

As long as the intervention is constant over time, we have:

$$\widetilde{U}_1 = \frac{X_t - (1+a)^t X_0}{b\sum_{j=1}^{t}(1+a)^{j-1}} - \frac{r}{b}. \tag{32}$$

This mode of intervention can be associated with community intervention.

Example 3

We consider a school located in an area where local government seeks to raise the level of academic achievement. The goal of the intervention is supporting a given number pupils to make sure that they will be enrolled. Let X denote the number of pupils who achieve the primary school and X_0 the number of pupils enrolled at the starting period. Let U be the proportion of pupils involved in a school achievement program. Assume that the decision is based on the mechanism

$$\dot{X}_t = aX + bU, \qquad 0 \leq U \leq 1. \tag{33}$$

The aim is to set the final value X_T as large as possible at the end of the process. It is easy to find that $\widetilde{U} = 1$ and $\max_{-1 \leq U \leq 1} X_T = X_0 e^{-aT} + \frac{b}{a}(1 - e^{-aT})$. This example illustrates an intervention that can be designed by local authorities.

3.2 Impacts of interventions under uncertainty

As discussed earlier, interventions depend on many uncontrolled factors that may impact their effectiveness. Informal and formal environments, risk aversion, shared common vision and other parameters have to be taken into account. In fact, uncertainty is the most realistic approach in modeling interventions and several factors support this approach. First, there is no deterministic causal relationship between the actions of authorities and educational achievement. Instead, authorities try to establish the best conditions (good home, school and community environments, etc.) for an improvement of pupils performance. Pupils are all different and their reactions to interventions can be highly variable from one individual to another. Indeed, their innate characteristics are not the same, some may be more effective in scientific topics, some in literary or artistic materials. Thus, the lack of information on certain socio-economic and personal

characteristics of pupils or students, is an uncertainty factor that weighs on the achievement of performance targets. Then the implementation of interventions is not always controllable. Communities, schools and households, may be vulnerable to unforeseen shocks that negatively affect pupils' performance (financing problems, floods, disease, etc.). Given these uncertain factors, a probabilistic approach suits better our purpose. The difference with the deterministic model is that the probability of occurrence of random events are embedded in the system of stochastic equations.

3.2.a The linear case

We assume that the factors that influence the intervention target lead to equations of motion:

$$\dot{X}_t = A(t)X_t + B(t)U_t + R(t) + C(t)Y_t, \tag{34}$$

where $A(t)$, $B(t)$ and $R(t)$ are vectors whose components depend on time. They are defined on the support $[0, T]$. The criteria measuring the performance is still the vector X_t and the intervention is captured by U_t. Y_t is a stochastic vector with components $y_1(t), y_2(t), \cdots, y_m(t)$. They are also defined onto $[0, T]$. We denote $Z(t)$ the fundamental matrix of the homogeneous system: $\dot{X}_t = A(t)X_t$ with $Z(0) = E$, where E is the unitary matrix. Suppose that D is a domain in the phase space and define:

$$J(U_t) = \mathbb{P}\left\{X(T) \in D\right\}. \tag{35}$$

The functional J is the probability that the end of the stochastic trajectory of the performances arrives at the region D. The issue is to find the better interventions U_t that lead to a maximum level of J. The mathematical expectation of X_t is:

$$\mathbb{E}(X_t) = Z(t)\mathbb{E}(X_0) + \int_0^t Z(\theta)Z^{-1}(\theta)\left[BU(\theta) + R(\theta) + C\mathbb{E}(Y)\right]d\theta. \tag{36}$$

This expression leads to a simple form of the variance of X_t,

$$\mathbb{V}(X_t) = \mathbb{E}[X_s(t) - a_s(t)]^2 \tag{37}$$

with

$$a_s = a_s^0(t) + \int_0^t \sum_{j=1}^{r} c_{sj}(t,\theta)U_j d\theta, \qquad s = 1, 2, \cdots, n.$$

Assume that the X_1 component follows a normal distribution with mean $a_1(t)$ and standard deviation $\sqrt{V_1(t)}$. The value $J_1(U) = \mathbb{P}\{X_1(t) \leq \delta\}$ is the probability of reaching the right end of the stochastic trajectory $X(T)$ in the overall performance delimited by $-\delta \leq X_1(T) \leq \delta$. Our goal is to find interventions that maximize $J_1(U)$ under the constraint that $\mid U_j \mid \leq j = 1, \cdots, r$. Let us consider the case of the functional $J_1(U)$ as normal by setting:

$$J_1(U) = \int_{-\delta}^{\delta} \frac{1}{\sqrt{2\pi V_1(T)}} \exp\left[-\frac{(X - a_1(T))^2}{2V_1(T)}\right] dX. \tag{38}$$

We have $\frac{\partial J_1(U)}{\partial a_1(T)} > 0$ for the following conditions: if $a_1(T) > 0$ and $a_1(T)$ is decreasing or if $a_1(T) < 0$ and $a_1(T)$ is increasing. As a result, $J_1(U)$ is minimum if $\mid a_1(T) \mid$ is the smallest

possible. Let us denote $(U_j^*)_{j=1,\cdots,r}$ the optimal intervention. Then, i) if $a_1^*(T) > 0$ and $a_1^*(T) + \int_0^T \sum_{j=1}^r \mid c_{1j}(T,\theta) \mid d\theta > 0$ then $U_j^* = -sgn\, c_{1j}(T,\theta)$; and ii) if $a_1^*(T) < 0$ and $a_1^*(T) + \int_0^T \sum_{j=1}^r \mid c_{1j}(T,\theta) \mid d\theta < 0$ then $U_j^* = sgn\, c_{1j}(T,\theta)$. As a result, whatever the case, we have $U_j^* = -sgn\, a_1^*(T) c_{1j}(T,\theta)$. With these optimal interventions, the value of the probability of the event 'reaching the ultimate goal of performance' is:

$$J_1^*(U) = \int_{-\delta}^{\delta} \frac{1}{\sqrt{2\pi V_1(t)}} \exp\left[-\frac{(X - a_1^*(T) + \int_0^T \sum_{j=1}^r \mid c_{1j}(T,\theta) \mid d\theta)^2}{2V_1(t)}\right] dX. \tag{39}$$

However, if $a_1^*(T) > 0$ and $a_1^*(T) + \int_0^T \sum_{j=1}^r \mid c_{1j}(T,\theta) \mid d\theta < 0$, then interventions become infinite. For a given interval $[0, t_0]$, we can take one of the actions, and for the complementary interval, we set that there is no intervention. Therefore, we have $U_j^* = -sgn\, a_1^*(T) c_{1j}(T,\theta)$ within $[0, t_0]$ and $U_j^* = 0$ within $(t_0, T]$. We chose t_0 as the smallest t where we have: $a_1^*(T) = \int_0^t \sum_{j=1}^r \mid c_{1j}(T,\theta) \mid d\theta < 0$. Similarly, if $a_1^*(T) < 0$, t_0 is the minimum time to have: $a_1^*(T) = -\int_0^t \sum_{j=1}^r \mid c_{1j}(T,\theta) \mid d\theta < 0$.

The vector of performance may be controlled not only on the last portion of its path, but also on one or more time intervals. However, these time intervals must comply with a condition. Let Ω be the union of these intervals and Ω^c its complement such that $\Omega \cap \Omega^c = \emptyset$ and $\Omega \cup \Omega^c = [0, T]$. If $U_j^* = sgn\, a_1^*(T) c_{1j}(T,\theta)$ when $t \in \Omega^c$ and $U_j^* = 0$ when $t \in \Omega$, this intervention is optimal for $a_1^*(T) = sgn\, a_1^*(T) \int_{\Omega^c} \sum_{j=1}^r \mid c_{1j}(T,\theta) \mid d\theta < 0$. The expectations of performances $X_1, \cdots, X_n$, are then $a_j = a_T^*(T) + \int_0^T \sum_{j=1}^r b_{ij} u_i \, d\theta$ for $j = 1, \cdots, n$ and $i = 1, \cdots, r$.

In the case where the variables $X_1(T), \cdots, X_k(T)$ are independent and follow a normal distribution, their cumulative distribution function is $F(\lambda_1, \cdots, \lambda_k) = F_1(\lambda_1) F_2(\lambda_2) \cdots F_k(\lambda_k)$ where $F_j(\lambda_j)$ denotes the cumulative distribution function of $(X_j(T))_{j=1,\cdots,k}$:

$$F_j(\lambda_j) = \int_{-\infty}^{\lambda_j} \frac{1}{\sqrt{2\pi V_j(T)}} \exp\left[-\frac{(X - a_1(T))^2}{2V_j(T)}\right] dX. \tag{40}$$

Let $J(U) = J_1(U) J_2(U) \cdots J_k(U) = \mathbb{P}\{X_1(T), \cdots, X_k(T) \in D\}$ denotes the probability that the final performance belong to domain D, with $-\delta_j \leq X_j(T) \leq \delta$, and $\delta > 0$. For $k = 1$, the intervention turns out to be following from Equation (38). What are the interventions when $k > 1$?

Suppose that $U^* = (U_1^*, \cdots, U_r^*)$ is optimal, and $U_j^* = -1, 0$ or 1. Consider $\theta \in [0, T]$ such that U_j^* does not change in the interval $[\theta - \varepsilon, \theta + \varepsilon]$ for ε sufficiently small. We can search $\bar{U}_j^*$ as follows: it is $U_j^*(\theta) = -\varepsilon$ for $t \in [\theta - \varepsilon, \theta + \varepsilon]$. Similarly, it is U_j^* for $\theta \in [0, T]$ and $t \in [\theta - \varepsilon, \theta + \varepsilon]$. Consider the intervention $\bar{U}^* = (U_1^*, \cdots, U_{j-1}^*, \bar{U}_j^*, U_{j+1}^*, U_r^*)$. We have $J(U^*) \geq J(\bar{U}^*)$ for all ε small. By substitution, we have $a_j = a_j(U^*) + \eta_j = a_j(\bar{U}^*)$ for $j = 1, \cdots, k$.
Observe that $J(\bar{U}^*)$ can be expanded in power series of η_i:

$$J(\bar{U}^*) = J(U^*) + \sum_{i=1}^{k} \frac{\partial J(\bar{U}^*)}{\partial \eta_i}\bigg|_{\eta_1 = \cdots = \eta_k = 0} \eta_i + \cdots \tag{41}$$

Let us set $\varpi_i = \frac{\partial J(\bar{U}^*)}{\partial \eta_i}\big|_{\eta_1 = \cdots = \eta_k = 0}$ for $i = 1, \cdots, k$. Then, $\varpi_i = \left[\frac{\partial J_i(\bar{U}^*)}{\partial \eta_i}\right] \frac{J(U^*)}{J_i(U^*)}\big|_{\eta_1 = \cdots = \eta_k = 0}$. We deduce that

$$\varpi_i = \frac{1}{\sqrt{2\pi V_i}} \left[\exp\left(-\frac{(-\delta - a_i(U^*))^2}{2V_i}\right) - \exp\left(-\frac{(\delta - a_i(U^*))^2}{2V_i}\right)\right] \frac{J(U^*)}{J_i(U^*)}. \tag{42}$$

Thus, $sgn\,\varpi_i = -sgn\,a_i(U^*)$, e.g., ϖ_i and $-a_i(U^*)$ are of the same sign. Suppose we have, whatever ε small enough $\sum_{i=1}^k \varpi_i\eta_i = 0$. Then, as U^* is optimal, $\sum_{i=1}^k \varpi_i\delta_i < 0$. Regarding η_i, we have:

$$\eta_i = a_i(\bar{U}^*) - a_i(U^*) = -2\int_{\theta-\varepsilon}^{\theta+\varepsilon} \varepsilon_j b_{ij}(t)dt = -2\int_{\theta-\varepsilon}^{\theta+\varepsilon} U^* b_{ij}(t)dt. \tag{43}$$

Similarly, we have $\sum_{i=1}^k \varpi_i\eta_i = -2\int_{\theta-\varepsilon}^{\theta+\varepsilon}\left(\sum_{i=1}^k \varpi_i b_{ij}U_j^* dt\right)$ and as the integrant is positive, $U_j^* = sgn\sum_{i=1}^k \varpi_i b_{ij}$.

As there is only one value ϖ_1 of the same sign as $-a_1(U^*)$ and therefore of the same sign as $-a^*$ if there is no part of the time when policy makers or other agents do not intervene, $U_j^* = -sgn(a^*b_j)$. If $k > 1$, we can use this method of calculation by approximation. Let U_1 be an intervention. We then use this framework to find out $\varpi_{11}, \cdots, \varpi_{k1}$ as we did for the computation of $\varpi_1, \cdots, \varpi_k$ thanks to U^*.

Then let $U_j^2 = sgn\sum_{i=1}^k \varpi_{i1}b_{ij}$ for $t \in [0, T]$. The intervention $U_j^2 = (U_1^2, \cdots, U_r^2)$ is the second approximation of the process, which results in the sequence $U_1, U_2, \cdots, U_r$ of successive approximations. The system $U_j^2 = sgn\sum_{i=1}^k \varpi_i b_{ij}$ is a system of integral equations that enables us to find optimal interventions. The following result holds:

Proposition 3 *Optimal interventions $U_j^* = sgn\sum_{i=1}^k \varpi_i b_{ij}$ for $j = 1, \cdots, r$ are solutions of the system of integral equations.*

We refer to appendix (see Appendix A) for the general resolution of this system.

3.2.b The nonlinear case

If the system is nonlinear, it can be written in the general form

$$\frac{dX_s}{dt} = f_s(x_1, \cdots, x_n; u_1, \cdots, u_r; y_1, \cdots, y_m), \quad s = 1, \cdots, n \tag{44}$$

and $J(u)$ is the functional defined on integral curves given by Equation (44). It defines the probability that the extreme performance $X(T)$ holds in a region of phase space. The interventions $u_1, \cdots, u_r$ are constrained and $u_1^*, \cdots, u_r^*$ are optimal interventions that maximize $J(U)$. Let $\xi_1, \cdots, \xi_n$ be a sequence of random variables such that $\mathbb{E}(\xi_i) = 0$, $\mathbb{E}(\xi_i\xi_j) = 0$, $\forall i \neq j$ and $\mathbb{E}(\xi_i^2) = 1$. We have $y_j \approx l_j(r)\sum_{i=1}^\infty l_{ji}(r)\xi_i$ for $j = 1, \cdots, m$. Moreover, we have $\mathbb{E}\left(y_i - l_j - \sum_{i=1}^k l_{ji}\xi_i\right)^2 \underset{k\longrightarrow\infty}{\longrightarrow} 0$ which is the convergence in mean towards the value y_i of the sequence. The system can be rewritten as:

$$\frac{dX_s}{dt} = f_s(t, x_1, \cdots, x_n; u_1, \cdots, u_r; \xi_1, \cdots, \xi_k). \tag{45}$$

We stop the sequence at ξ_k in amputating ξ_{k+1} to determine the solution of Equation (45). For $u_1, \cdots, u_r$ fixed, we set $\xi_1^j, \cdots, \xi_k^j$ for $j = 1, \cdots, l$ and determine the solution of Equation (45) associated with realizations ξ_j and initial conditions $x^0 = x^j$ for $t = 0$, $j = 1, \cdots, l$ (deterministic). We then form an interpolation polynomial

$$x = \pi(t, \xi_1, \cdots, \xi_k) \tag{46}$$

such that $x^j = \pi(t, \xi_1^j, \cdots, \xi_k^j)$ provides the solution to Equation (45) for any sequence $\xi_1^j, \cdots, \xi_k^j$. Then we determine $J(u)$ from Equation (46). For k and l large enough, we determine the optimal interventions as:

$$J = \int_0^T \mathbb{E}\left[(X(t) - Z(t))\right]^2 dt \tag{47}$$

for a given deterministic form. Then we substitute (46) into (47) to have:

$$\bar{J} = \int_0^T f_0\left(t_1, x^1(t) \cdots x^l(t)\right) \tag{48}$$

where X^j are deterministic solutions of Equation (44). Computations are then done by the Lagrangian method.

Example 4

We consider a problem of policy interventions in discrete time, on group of schools, by assuming that that pupils' performance evolves randomly and therefore require random interventions. These intervention requests are supposed to be Markovian, meaning that the decision makers face uncontrolled situations that do not depend on them. For example, school dropouts can lead to losses of performances, and can be supposed to follow a Markov process.[9] We assume that the performance is compared to a standard level, so that intervention requests depend on the difference between current and standard levels. Let us denote them v_t (assumed independent with the same probability distribution, $\mathbb{P}(v_t)$). The difference of performances is denoted X_t, and can be positive (the current performances are greater than the standard level) or negative (the current performances are less than the standard ones). For sake of simplicity, we still refer to X_t as performance, even if it represents the difference from the standard level. The performances are produced in each period, by interventions aiming to improve them. The interventions are noted $U_t \geq 0$.

There are two costs associated with the production of performance: i) the cost of performance itself, and the cost of lack of performance, denoted $f(X_t)$. Thus, a) when $X_t \geq 0$, $f(X_t)$ is interpreted as the cost to maintain the level of performance; b) when $X_t < 0$, $f(X_t)$ is the cost of weak performance. ii) The cost of implementation of the intervention U_t denoted $g(U_t)$.

The optimization consist in minimizing the costs of the intervention:

$$\min_{U_t} \mathbb{E} \sum_{t=0}^{T-1} \left[f(X_t) + g(U_t)\right] \tag{49}$$

under the dynamical constraints: $X_{t+1} = X_t - U_t + V_t$. The dynamic programming function is then:

$$V(t, x) = \min_{U_s} \mathbb{E}\left\{\sum_{t=0}^{T-1} \frac{\left[f(X_t) + g(U_t)\right]}{X_t} = x\right\}, \text{ with } s = t, \cdots, T-1. \tag{50}$$

The equation of the dynamic programming is stated as:

$$V(t, x) = \min_{U \in \mathbb{R}^+} \left\{\int_v V(t+1, X_t, U_t - v_t)\right\} \mathbb{P}(dv_t) + f(X_t) + g(U_t) \tag{51}$$

with $V(t, x) = 0$, where T is the final period.

Solving this equation gives the optimal cost V_t and a Markov decision strategy, as we suppose that intervention requests are Markovian. Over an infinite horizon, meaning that there are infinite renewal of the cohorts of pupils, if we discount the cost of interventions and school performances at the rate r, the dynamic programming problem is:

$$\min_{U_t} \mathbb{E}\left\{\sum_{t=0}^{\infty} \frac{[f(X_t)+g(U_t)]}{(1+r)^{t+1}}\right\} \tag{52}$$

under the dynamical constraints: $X_{t+1} = X_t - U_t + V_t$, and

$$V(t,x) = \frac{1}{1+r} \min_{U_t \geq 0} \left\{\int_v V(X_t + U_t - v_t)\mathbb{P}(dv_t) + g(U_t) + f(X_t)\right\}. \tag{53}$$

The period T can tend to infinity and the asymptotic behavior of the function of dynamic programming has solutions of the form: $e(X_t) = -mT + \rho(X_t)$ where m is the average cost per period. If we substitute $V(t, X_t)$ by $e(X_t)$, the dynamic equation becomes:

$$m + \rho(X_t) = \min_{U_t \geq 0} \left\{\int_v \rho(X_t + U_t - v_t)\mathbb{P}(dv_t) + g(U_t) + f(X_t)\right\}. \tag{54}$$

We check by recurrence that $e(X_t)$ is solution of the dynamic equation. Let us consider that a period of time has a length σ. We suppose the following particular case: i) $f(X_t) = \rho X_t^2$; ii) $\mathbb{P}(dv_t) = 1$ within the period of length σ and $\mathbb{P}(dv_t) = 0$ elsewhere; iii) $g(U_t) = 0$ if $U_t > 0$ and $g(U_t) = 1$ if $U_t = 0$; iv) We replace m by σm. The problem of optimality of the intervention is then:

$$\sigma m = \min\left\{\inf_{U_t > 0} [\rho(X_t + U_t - \sigma] - \rho(X_t) + 1, \rho(X_t - \sigma) - \rho(X_t)\right\} + \sigma X_t^2. \tag{55}$$

If σ tends to zero, we have:

$$\min\left\{\inf_{U_t > 0} [\rho(X_t + U_t - \sigma] - \rho(X_t) + 1, \rho'(X_t) + X_t^2 - m\right\} = 0 \tag{56}$$

The solution is

$$\rho(X_t) = \frac{1}{3}X_t^3 - mX_t. \tag{57}$$

Then the minimum is reached at $X_t = \sqrt{m}$ with $-\frac{2}{3}m^{\frac{3}{2}}$ as minimum value.

4 Optimality of interventions and social welfare

The issue regarding the role of education is wider than the impact of interventions on school performances. Education appears to be of great importance for development and economic growth (Lucas, 1988; Barro, 1991; Mankiw et al., 1992). All the stakeholders involved in the education system may have different targets in the short run. But the ultimate long run objectives are the same: promote economic development and social welfare. That is why governments in developing countries are inclined to undertake policies that enhance educational attainment and achievement. As shown by Glewwe (2002), many scholars and international organizations recognize that investment in education is a priority for development and welfare (Becker, 1995; Hanushek, 1987). In this section, we study the conditions of welfare improvement following from interventions.

4.1 The conditions of deterministic equilibrium

In our context, the relation between education and welfare can broadly be studied by taking into account the difference between the objectives of the stakeholders and the future labor productivity arising from skills acquired through education. The impact of different skills on future income and on other socioeconomic outcomes may have implications on the kinds of interventions in education. The previous two propositions define the existence of interventions. We can now extend this framework to social welfare that integrates utility functions of all stakeholders. As previous, the variation in performance is described by the equation $\dot{X}_t = F(t, X_t, U_t)$. Do interventions that maximize social welfare exist? We assume social welfare function to be of the form:

$$WF = \int_0^T L(S(t))w(t)dt \tag{58}$$

where S is the utility function which is defined on a set of variables including consumption and health of agents (for example pupils). The function L is weighted by $w(t)$ which represents the time allocation (either for education, or for work). The main issue we face here is that each stakeholder has a utility function. For each, we define an explicit form of the function and aggregate them in a specific social welfare function.

For the values to be optimal, we assume that the utility function $S(t) : \mathbb{R}_+ \to \mathbb{R}_+$ is C^∞ with $S'_C > 0$, $S''_C < 0$ and $\lim_{C(t)\to 0} S'_C \to \infty$ where $C(t)$ denotes any of the arguments of the function $S(t)$.

Observe that education provides human capital in the form of skills and ability and leads to higher probabilities to get into the labor market. The question is: how can the interventions in the education system lead to a better welfare for the whole population? A plausible approach is to run an optimization problem for each stakeholder taking into account his/her own objective and his/her budget constraints. However, it appears more tractable to start from a general framework where a social planner maximizes the social welfare function that integrates the parameters depending on the stakeholders' objectives.

The households have as objective to maximize a utility function with two arguments: consumption of goods and services and child cognitive skills. At the last period of schooling, T, pupils start working and they have earnings. Part of these earnings is given to parents and to the other stakeholders of the education system, such as the government and the local administrations, by the mean of taxes for example.

A utility function that embeds parents' consumption C_t from periods t_0 to T and child skills X_t as arguments is:

$$S_t = S(C_t, X_t, \sigma, \sigma') \tag{59}$$

where σ is a discount factor for future consumption and σ' is a value representing the households' incentive to have educated children. The higher σ' the more parents prefer educated children who will help them increase their future consumption. The production function of skills is given by:

$$X_t = I(q, \theta, e) \tag{60}$$

where q is the school quality, and θ denotes the years of schooling. The function I is increasing in q and θ. The parameter e is the child's learning efficiency (her/his own personal characteristics and those of parents: ability, motivation, etc.). It is an opportunity that enhances the possibility for children to better perform at school. Households' consumption is given by:

$$C_t = a(W_t - \pi\theta) + (1-a)(W_t + \delta W^c_{t,d}) \tag{61}$$

with

$$\begin{cases} a = 1 & \text{if } t \leq \theta \\ a = 0 & \text{if } t > \theta \end{cases} \tag{62}$$

The parameter π is the price of schooling, W_t is the households' income in period t, $W^c_{t,d}$ are respectively the child's gross and disposable income when she/he works, and d is the fraction of the disposable income spent in the household. The first term represents the income when the children are still attending school. The second term is the income when they do work. Let the income tax rates levied by the government be τ. We have:

$$C_t = W_t - a\pi\theta + \delta(1-a)(1-\tau)W^c_t. \tag{63}$$

The income of children increases with their skills by the equation:

$$W^c_t = z(X_t, U_t) \tag{64}$$

where z is a function that links skills and income in the labor market. Households' utility function can then be written as function of years of schooling θ and school quality q:

$$S_t = S(W_t - a\pi\theta + \delta(1-a)(1-\tau)z(X_t); X_t; \sigma; \sigma'). \tag{65}$$

Relying on Equation (58), the global welfare is given by

$$WF = \int_{t_0}^{T} L\left[S(W_t - a\pi\theta + \delta(1-a)(1-\tau)z(X_t); X_t; U_t; \sigma; \sigma')\right] w(t)dt \tag{66}$$

The welfare function is maximized under the constraint of motion equation of skills

$$\dot{X}_t = F(t, X_t, U_t). \tag{67}$$

The Hamiltonian of optimization problem for WF is

$$\mathbb{H} = L\left[S(W_t - a\pi\theta + \delta(1-a)(1-\tau)z(X_t); X_t; U_t; \sigma; \sigma')\right] w(t) + \mu_t F(t, X_t, U_t). \tag{68}$$

The optimality conditions are given by the state equation:

$$\frac{\partial L}{\partial S_t}\frac{\partial S_t}{\partial U_t}w(t) + \mu_t \frac{\partial F}{\partial U_t} = 0 \tag{69}$$

and by the co-state equation:

$$\dot{\mu}_t = -\delta(1-a)(1-\tau)\frac{\partial L}{\partial S_t}\frac{\partial S_t}{\partial z}\frac{\partial z}{\partial X_t}w(t) - \frac{\partial S_t}{\partial X_t} - \mu_t\frac{\partial F}{\partial X_t}. \tag{70}$$

The conditions of existence of optimal interventions depend on the expression of L and its characteristics (continuity, differentiability and concavity).

4.2 The case of stochastic welfare

Skills lead to higher future incomes and are an incentive for parents to get their children educated. Household's utility is supposed to depend on the current income, and on the expected income of children. The presence of skilled and unskilled labor in the production sector leads to unequal distribution of wages. Indeed, workers can be skilled or not and earn future income. The time spent in education differs between children, so do their innate abilities and their parents' education. The child's ability is a stochastic variable and does not dependent on parents' abilities. Parents pay their children's education and their involvement in the education is related to the probabilities of being skilled or unskilled. As a result, the utility function of households depends now on the fact that children may not acquire the skills they expected and this will have an impact on their expected incomes. The expected utility of the parents is:

$$\mathbb{E}(S_t) = S(\mathbb{E}(C_t), X_t, \sigma, \sigma'). \tag{71}$$

The expected income of the children from different types of households is:

$$\mathbb{E}(C_{s,t}) = W_{s,t} - a\pi\theta_s + \delta(1-a)(1-\tau_s)\mathbb{E}(W^c_{s,t}) \tag{72}$$

$$\mathbb{E}(C_{n,t}) = W_{n,t} - a\pi\theta_n + \delta(1-a)(1-\tau_n)\mathbb{E}(W^c_{n,t}). \tag{73}$$

Parents are supposed to educate their children when the expected utility with educated children is greater. Then we have:

$$\begin{aligned} S(W_{s,t} - a\pi\theta_s + \delta(1-a)(1-\tau_s)\mathbb{E}(W^c_{s,t}); X_t; U_t; \sigma, \sigma') &\geq \\ S(W_{s,t} + \delta(1-a)(1-\tau_s)\mathbb{E}(W^c_{s,t}); X_t; U_t; \sigma, \sigma') & \end{aligned} \tag{74}$$

$$\begin{aligned} S(W_{n,t} - a\pi\theta_n + \delta(1-a)(1-\tau_n)\mathbb{E}(W^c_{n,t}); X_t; U_t; \sigma, \sigma') &\geq \\ S(W_{n,t} + \delta(1-a)(1-\tau_n)\mathbb{E}(W^c_{n,t}); X_t; U_t; \sigma, \sigma'). & \end{aligned} \tag{75}$$

This means that, with the same household income, children from unskilled household drop earlier from education than the others, meaning that $\theta_n \leq \theta_s$. The global expected welfare is then:

$$WF = \int_{t_0}^{T} \left\{ L\left[S(\mathbb{E}(C_{s,t}); X_t; \sigma, \sigma')\right] + L\left[S(\mathbb{E}(C_{n,t}); X_t; \sigma, \sigma')\right] \right\} w(t)dt. \tag{76}$$

One of the objectives the authorities can have is to reduce the difference between skilled and unskilled wages, as the decision to invest in education depends on its cost and on the expected income from education, knowing that skilled labor is more profitable in terms of future income. The issue they face is to enhance the education of the children of unskilled parents.

5 Populations dynamics and growth under uncertainty

In this section, we now address the issue of global approach to the impacts of the different types of populations and their dynamics on economic growth. We have an economy with two sectors: the education sector and a commodity sector. We include leisure as argument in the utility functions. Physical capital is used in both sectors. There are also two types of consumers: skilled who use part of their leisure time for the education of their children and the unskilled.

5.1 The economy

Households live infinite periods, L_s and L_n represent respectively the skilled and non skilled labor. Skilled workers devote a fraction u_{st} of their time to the commodity production sector, while l_{st} is the fraction of leisure, which is divided into two elements: l_{st}^O and l_{st}^C. The first one is their own time of leisure and the second one is the part of l_{st} used to help children in their education.

5.1.a The firm problem

The firms are supposed to behave competitively and the production function of the commodity is:

$$\varphi_t = \varphi_t(\widetilde{K}_{1t}, \widetilde{H}_{1t}, \widetilde{L}_n) = \omega_t A (v_t \widetilde{K}_t)^{\alpha_1} (u_{st} \widetilde{H}_t)^{\alpha_2} \widetilde{L}_n^{\alpha_3} \tag{77}$$

where v_t is the part of the capital used in the production sector, $\widetilde{H}_t$ is the aggregate human capital stock, A is the technology and ω_t denotes a random perturbation in technology.[10] We assume that ω_t follows an autoregressive process (AR1):

$$\ln \omega_t = a_1 \ln \omega_{t-1} + \varepsilon_{1t}, \qquad |a_1| < 1 \tag{78}$$

and we assume that $\varepsilon_{1t} \sim N(0, \sigma_1^2)$. In the education sector, human capital is produced by using physical and human capital and the latter is supposed to depreciate at rate δ_h leading to:

$$\begin{aligned} \tilde{h}_{t+1} &= \Phi_t(\tilde{k}_{2t}, \tilde{h}_{2t}) + (1-\delta_h)\tilde{h}_t \\ &= \eta_t B \big[(1-v_t)\tilde{k}_t\big]^{\beta_1} \big[(1-u_{st}-l_{st})\tilde{h}_t\big]^{\beta_2} + (1-\delta_h)\tilde{h}_t \end{aligned} \tag{79}$$

where B is the level of technology, $\tilde{h}_t$ is the stock of human capital per worker, $(1-v_t)$ is the fraction of physical capital used for the production of human capital, $(1-u_{st}-l_{st})$ is the fraction of time used for the education, η_t is a random technology shock which is supposed to follow an AR(1) process:

$$\ln \eta_t = a_2 \ln \eta_{t-1} + \varepsilon_{2t}, \qquad |a_2| < 1 \tag{80}$$

and where we assume that $\varepsilon_{2t} \sim N(0, \sigma_2^2)$.

5.1.b The household problem

We have two types of households: the skilled and the unskilled. Each derives utility from consumption and leisure. The utility function for each of these households is assumed to be of a Constant Relative Risk Aversion (CRRA) type and respectively:

$$U_s = U_s(\tilde{c}_{st}, \tilde{l}_{st}) = \frac{\left(\tilde{c}_{st}^p \tilde{l}_{st}^{1-p}\right)^{1-\sigma} - 1}{1-\sigma} \tag{81}$$

$$U_n = U_n(\tilde{c}_{nt}, \tilde{l}_{nt}) = \frac{\left(\tilde{c}_{nt}^p \tilde{l}_{nt}^{1-p}\right)^{1-\sigma} - 1}{1-\sigma} \tag{82}$$

where s means skilled and n stands for unskilled and σ denotes the elasticity of substitution. The representative consumer of each type maximizes the following expected utility:

$$\max_{\{\tilde{c}_t, \tilde{l}_t, \tilde{i}_t \tilde{u}_t, \tilde{v}_t, \tilde{k}_{t+1}, \tilde{h}_{t+1}\}} \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t U\big(\tilde{c}_t, \tilde{l}_t\big)\right] \tag{83}$$

under the constraints:

$$
\begin{aligned}
(1-\tau_c)\tilde{c}_t + \tilde{i}_t &= (1-\tau_r)r_t v_t \tilde{k}_t + \tau_r \delta_k v_t \tilde{k}_t + (1-\tau_w) w_t u_t \tilde{h}_t && (84)\\
(1+n)\tilde{k}_{t+1} &= \tilde{i}_t + (1-\delta_k)\tilde{k}_t && (85)\\
\tilde{h}_{t+1} &= \Phi_t(\tilde{k}_{2t}, \tilde{h}_{2t}) + (1-\delta_h)\tilde{h}_t && (86)
\end{aligned}
$$

where $\tilde{i}_t$ is the investment in physical capital, r_t is the interest rate, w_t is the wage rate, τ_c, τ_r and τ_w stand for the tax rates on consumption, capital and labor income. β is the discount factor. This maximization program is specified for each type of household and its own parameters. For $\ell = (s, n)$, the Lagrangian of the problem is given by:

$$
\begin{aligned}
\mathbb{L}_\ell(\tilde{c}_{\ell t}, \tilde{l}_{\ell t}, \tilde{u}_{\ell t}, \tilde{v}_{\ell t}, \tilde{k}_{\ell,t+1}, \lambda_{\ell t}, \mu_{\ell t}) &= \mathbb{E}\Bigg\{ \sum_{t=0}^{\infty} \beta^t \frac{\left(\tilde{c}_{\ell t}^{p} \tilde{l}_{\ell t}^{1-p}\right)^{1-\sigma} - 1}{1-\sigma}\\
&+ \beta^t \lambda_{\ell t} \Big[-(1+\tau_c)\tilde{c}_{\ell t} - (1+n_\ell)\tilde{k}_{\ell,t+1} + (1-\delta_k)\tilde{k}_{\ell t}\\
&+ (1-\tau_{\ell r}) r_t v_t \tilde{k}_t + \tau_r \delta_k v_t \tilde{k}_t + (1-\tau_w) w_t u_t \tilde{h}_t \Big]\\
&+ \beta^t \mu_{\ell t} \Big[-\tilde{h}_{\ell,t+1} + \Phi\big((1-v_t)\tilde{k}_{\ell t}(1-u_{\ell t} - \tilde{l}_{\ell t})\tilde{h}_{\ell t}\big)\\
&+ (1-\delta_h)\tilde{h}_{\ell t} \Big] \Bigg\}. && (87)
\end{aligned}
$$

The first order conditions are:

$$
\begin{aligned}
p\tilde{c}_{\ell t}^{p(1-\sigma)-1} \tilde{l}_{\ell t}^{(1-p)(1-\sigma)} &= \lambda_{\ell t}(1+\tau_c) && (88)\\
(1-p)\tilde{c}_{\ell t}^{p(1-\sigma)} \tilde{l}_{\ell t}^{(1-p)(1-\sigma)-1} &= \mu_{\ell t}\eta_t B \beta_1 \big[(1-v_t)\tilde{k}_t\big]^{\beta_1} \big[(1-u_{\ell t} - l_{\ell t})\big]^{\beta_2 - 1} \tilde{h}_t^{\beta_2} && (89)\\
(1-\tau_w) w_{\ell t} \lambda_{\ell t} &= \mu_{\ell t}\eta_t B \beta_2 \big[(1-v_t)\tilde{k}_t\big]^{\beta_1} \big[(1-u_{\ell t} - l_{\ell t})\tilde{h}_t\big]^{\beta_2 - 1} && (90)\\
\big[\tau_{\ell r}\delta_k + (1-\tau_{\ell r}) r_t\big]\lambda_{\ell t} &= \mu_{\ell t}\eta_t B \beta_1 \big[(1-v_t)\tilde{k}_t\big]^{\beta_1 - 1} \big[(1-u_{\ell t} - l_{\ell t})\tilde{h}_t\big]^{\beta_2} && (91)\\
\lambda_{\ell t}(1+n_\ell) &= \beta\, \mathbb{E}_t \Big\{ \lambda_{\ell,t+1}\big((1-\tau_{\ell r}) r_{t+1} v_{t+1} + \tau_r \delta_k v_{t+1} + 1 - \delta_k\big) && (92)\\
&+ \mu_{\ell,t+1}\eta_{t+1}(1-v_{t+1}) B \beta_1 \big[(1-v_{t+1})\tilde{k}_{t+1}\big]^{\beta_1 - 1}\\
&\quad \big[(1-u_{\ell,t+1} - l_{\ell,t+1})\tilde{h}_{t+1}\big]^{\beta_2} \Big\}\\
\mu_{\ell t} &= \beta\, \mathbb{E}_t \Big\{ \lambda_{\ell,t+1}(1-\tau_w) w_{\ell,t+1} u_{t+1} + \mu_{\ell,t+1} \Big[\eta_{t+1} B \beta_2 \big[(1-v_{t+1})\tilde{k}_{t+1}\big]^{\beta_1}\\
&\quad \big[(1-u_{\ell,t+1} - l_{\ell,t+1})\big]^{\beta_2 - 1} \tilde{h}_{t+1}^{\beta_2} + 1 - \delta_h \Big] \Big\}. && (93)
\end{aligned}
$$

The limit conditions are:

$$
\begin{aligned}
\lim_{i\to\infty} \mathbb{E}_t \beta^t \big(\lambda_{\ell,t+i}\tilde{k}_{t+i+1}\big) &= 0 && (94)\\
\lim_{i\to\infty} \mathbb{E}_t \beta^t \big(\mu_{\ell,t+i}\tilde{h}_{t+i+1}\big) &= 0 \qquad \ell = s, n. && (95)
\end{aligned}
$$

The representative firm maximizes the profit of each period:

$$
\max_{\{\tilde{K}_{1t}, \tilde{H}_{1t}, \tilde{L}_n\}} \varphi_t(\tilde{K}_{1t}, \tilde{H}_{1t}, \tilde{L}_n) - r_k \tilde{K}_{1t} - w_{st}\tilde{H}_{1t} - w_{nt}\tilde{L}_n \qquad (96)
$$

with $\varphi_t(\tilde{K}_{1t}, \tilde{H}_{1t}, \tilde{L}_n) = \omega_t A(v_t\widetilde{K}_t)^{\alpha_1}(u_{st}\widetilde{H}_t)^{\alpha_2}\widetilde{L}_n^{\alpha_3}$. The first order conditions are then:

$$r_t = \omega_t\alpha_1 A(v_t\widetilde{K}_t)^{\alpha_1-1}(u_{st}\widetilde{H}_t)^{\alpha_2}\widetilde{L}_n^{\alpha_3} \tag{97}$$

$$w_{st} = \omega_t\alpha_2 A(v_t\widetilde{K}_t)^{\alpha_1}(u_{st}\widetilde{H}_t)^{\alpha_2-1}\widetilde{L}_n^{\alpha_3} \tag{98}$$

$$w_{nt} = \omega_t\alpha_3 A(v_t\widetilde{K}_t)^{\alpha_1}(u_{st}\widetilde{H}_t)^{\alpha_2}\widetilde{L}_n^{\alpha_3-1}. \tag{99}$$

5.1.c The government problem

The government collect taxes from consumption, capital and labour to finance its expenditures G_t. The global equality of resources and expenses holds:

$$\widetilde{C}_t + \widetilde{K}_{t+1} - (1-\delta_k)\widetilde{K}_t + G_t = \omega_t A(v_t\widetilde{K}_t)^{\alpha_1}(u_{st}\widetilde{H}_t)^{\alpha_2}\widetilde{L}_n^{\alpha_3}. \tag{100}$$

5.2 Competitive equilibrium: steady state analysis

We now characterize the competitive equilibrium, which is given by the values of the variables $\tilde{c}_t, \tilde{l}_t, \tilde{u}_t, \tilde{v}_t, \tilde{k}_{t+1}, \tilde{h}_{t+1}, r_t, w_t$ that maximize the profits and optimize the utilities under the constraints of each type of agent. We have the following results (see detail calculations in Appendix B):

$$\frac{(1-\tau_w)\omega_t\alpha_2 A(v_t\tilde{k}_t)^{\alpha_1}(u_{s,t}\tilde{h}_t)^{\alpha_2-1}\widetilde{L}_n^{\alpha_1+\alpha_2+\alpha_3-1}}{(1-\tau_{sr})\omega_t\alpha_1 A(v_t\tilde{k}_t)^{\alpha_1-1}(u_{s,t}\tilde{h}_t)^{\alpha_2}\widetilde{L}_n^{\alpha_1+\alpha_2+\alpha_3-1} + \tau_r\delta_k} = \frac{\beta_2(1-v_t)\tilde{k}_t}{\beta_1(1-u_{st}-\tilde{l}_{st})\tilde{h}_{st}}. \tag{101}$$

The marginal rates of transformation of capital and labor in the sectors of education and in the final good production are equal:

$$\frac{p\tilde{l}_{st}}{(1-p)(1-\tau_c)\tilde{c}_{st}} = \frac{1}{(1-\tau_w)\omega_t\alpha_2 A(v_t\tilde{k}_t)^{\alpha_1-1}(u_{s,t}\tilde{h}_t)^{\alpha_2}\widetilde{L}_n^{\alpha_1+\alpha_2+\alpha_3-1}}. \tag{102}$$

The marginal rate of substitution of consumption and leisure are equal to the marginal product of labor:

$$\begin{aligned}\tilde{c}_{\ell t}^{p(1-\sigma)-1}\tilde{l}_{\ell t}^{(1-p)(1-\sigma)} = \frac{1}{1+n_\ell}\beta\,\mathbb{E}_t\Big\{&\tilde{c}_{\ell,t+1}^{p(1-\sigma)-1}\tilde{l}_{\ell,t+1}^{(1-p)(1-\sigma)}\Big[(1-\tau_{\ell r})\omega_{t+1}\alpha_1 A(v_{t+1}\tilde{k}_{t+1})^{\alpha_1-1}\\ &(u_{\ell,t+1}\tilde{h}_{t+1})^{\alpha_2-1}\widetilde{L}_n^{\alpha_1+\alpha_2+\alpha_3-1} + 1 - (1-\tau_{\ell r})\delta_k\Big]\Big\} \quad \ell = s, n\end{aligned} \tag{103}$$

For each group of households, the marginal utility of giving up one unit of current consumption must be equal to the expected marginal utility of future consumption multiplied by the return from the investment of that unit of good for one period. This is equal to the marginal product of physical capital, net of taxes and depreciation rates. In what follows, we establish the steady state conditions.

A deterministic steady state occurs when the random shocks are all constant, for every value of t:

$$\ln\omega_t = \ln\eta_t = 0.$$

The variables $\tilde{l}_t, u_t, v_t$ are also constant at l, u, v and the growth rate of the variables $\tilde{c}_t, \tilde{k}_t, \tilde{h}_t, \varphi_t$ are constant at the respective values g_c, g_k, g_h, g_f. By making the log-linear approximation of the model, we have the following result:

Proposition 4 *If* $\frac{\beta_1}{1-\beta_2} > \frac{1-\alpha_1}{\alpha_2}$ *then the necessary conditions for endogenous growth of the economy are:*

$$\ln(1+g_{\tilde{k}}) = \frac{(1-\beta_2)(1-\alpha_1-\alpha_2-\alpha_3)}{\beta_1\alpha_2-(1-\alpha_1)(1-\beta_2)} n_n \left(1-\frac{n_n}{2}\right) \tag{104}$$

$$\ln(1+g_{\tilde{h}}) = \frac{\beta_1(1-\alpha_1-\alpha_2-\alpha_3)}{\beta_1\alpha_2-(1-\alpha_1)(1-\beta_2)} n_n \left(1-\frac{n_n}{2}\right). \tag{105}$$

Proposition 4 states that the growth rate of the unskilled population has a direct impact on the growth of human and physical capitals. The following figure gives a simulated relation between the growth rates of human and physical capital for different values of the parameters β_1 and β_2.

Include Figure 1

We observe that the two growth rates have the following equation:

$$\beta_1 \ln(1+g_k) + (\beta_2 - 1)\ln(1+g_h) = 0.$$

The two growth rates are equal in case of constant returns. Human capital grows faster than the physical capital when constant returns do not prevail in the economy.

6 Conclusion

In this study, we develop theoretical frameworks to relate interventions in education to educational performances. In that regard, we show that uncertainty plays a crucial role in shaping the optimality of interventions. Uncertainty may follow from lack of information on the socioeconomic characteristics of students and the educational environment in which interventions are implemented. We also enlarge the analysis to the issue of how the performance of the educational system can be integrated into a macroeconomic performance in terms of well-being and economic growth. We also link the performance levels to social welfare, on the assumption that the ultimate goal of policy makers is improving the well-being of individuals. Lastly, we argue that heterogenous populations where skilled and unskilled groups have different demographic dynamics have different implications for economic growth.

In this research, we focused on the study of equilibria by providing optimality conditions of interventions. Future research would be to examine the stability of optimal equilibria in both certain and uncertain cases. Indeed, there may be external shocks resulting in disturbances of the system. For example, natural events that may increase significantly the risk or more restrictive and unpredictable budgetary policies that policymakers may face. All these are likely to affect the stability of equilibria and the way they evolve over time.

Notes

[1]The term innovation is used here to refer to the implementation of new practices, the introduction of a new policy and mobilizing new resources to support the implementation of that policy.

[2]For instance, technical resources (equipment), human resources (extra staff to support activities and planning time), physical resources (classroom space), financial resources, etc.

[3]In the literature, four core layers are commonly identified and their influences analyzed: i) the intervention, ii) the micro level influences (factors that are relevant to the policy maker such as capacity to innovate), iii) the mid-level influences (local level influences such as educational environment), iv) the macro-level influences (national policy and programmes, government initiatives, etc.).

[4]For instance, allowing competition in schools can motivate and enhance efforts.

[5]For instance, food policies that promote pupils' access to enough calories. The formal environment has a key role in creating spaces for sharing existing or innovative practices. Furthermore, it facilitates the partnerships between all stakeholders in education.

[6]For example, teacher's perception on the effectiveness of new pedagogical practices implied by an intervention can influence its success. A shared vision provides clarity of purpose and direction for those who manage interventions.

[7]In developing countries, there is a growing contribution of the private sector to education (private school, universities, etc.)

[8]The empirical literature on impact assessment of interventions stresses the importance of taking uncertainty into account. But to date, no work has yet proposed a theoretical study.

[9]This means that students who leave school (dropout) are replaced by others who enter. This is a replacement process in which we do not master the reasons of dropout. Moreover, the performance of those who enter is independent of the performance of those who leave.

[10]In the sequel, the tilde notation denotes a contemporary variable, say x_t, from which we extract the trend at equilibrium, meaning when the growth rate is constant. That is: $\tilde{x}_t = \ln\left(\frac{x_t}{x_t^*}\right)$ and therefore $x_t = x_t^* e^{\tilde{x}_t}$.

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Appendix

Appendix A

To solve the system leading to the solution in the Proposition 3, one use the following approximation. For example, we search constants $c_1, \cdots, c_k$ using $c_i = \varphi(c_1, \cdots, c_k)$ for $i = 1, \cdots, k$. This turns out to solve an algebraic equation. The algebraic equation is obtained by substituting $U_j^* = -sgn(a^* b_j)$ in the expressions of c_i. Then we can solve by trial search approximation the minimum of the function $E = \sum_{i=1}^{k}(c_i - \varphi(c_1, \cdots, c_k))^2$. If $(c_1^n, \cdots, c_k^n)$ is the n-th approximation of the solution of the equation in c_i, the functions $\mu_j^n = sgn \sum_{i=1}^{k} c_i^n b_{ij}, j = 1, \cdots, k$ are the n-th approximation of the optimal intervention.

Let us move to the case of random functions $y_1(t), \cdots, y_m(t)$ which are independent of t and are random variables $\xi_1, \cdots, \xi_m$. Let $F_1, \cdots, F_m$ be respectively the associated response functions. Suppose (to simplify) that $\xi_1, \cdots, \xi_m$ are independent and the initial data are deterministic. The cumulative distribution function of $\xi_1, \cdots, \xi_m$ is $F(\lambda_1, \cdots, \lambda_m) = F_1(\lambda_1), \cdots, F_m(\lambda_m)$. We deduce the distribution function $G(\lambda)$ of the random variable $X_1(T)$, with:

$$X_1(T) = \sum_{i=1}^{m} \Gamma_i \varepsilon_i + \int_0^{+\infty} \sum_{j=1} b_j u_j dt + a_1^*. \tag{A-1}$$

Set in the space of m variables $\lambda_1, \cdots, \lambda_m$, the domain, say $\mathcal{S}(\lambda)$, defined by $X_1(T) \leq \lambda$. We have:

$$G(\lambda) = \int_{\mathcal{S}(\lambda)} dF(\lambda_1, \cdots, \lambda_m) = \int_{-\infty}^{+\infty} dF_1(\xi_1) \int_{-\infty}^{+\infty} dF_2(\xi_2) \cdots \int_{-\infty}^{\frac{\Gamma_m \xi_m + \lambda - X_1(T)}{\Gamma_m}} dF_m(\xi_m). \tag{A-2}$$

Hence the following representation:

$$G(\lambda) = \bar{G}\left[\frac{1}{\Gamma_m}\left(\lambda - a_1^* - \int_0^T \sum_{j=1}^{r} b_j u_j dt\right)\right]. \tag{A-3}$$

Let us seek the optimal intervention that maximizes $J(U) = \mathbb{P}\{| X_1 | \leq \delta\}$. Let α and β be respectively the minimum and maximum values of $\frac{1}{\Gamma_m}\left(a_1^* + \int_0^T \sum_{j=1}^{r} b_j u_j dt\right)$ and $\alpha^* \in [\alpha, \beta]$ the point where $\bar{G}\left(\frac{1}{\Gamma_m} - a\right) - \bar{G}\left(-\frac{1}{\Gamma_m} - a\right)$ takes its greatest known value $z \in [\alpha, \beta]$. If $\alpha^* = \alpha$ or $\alpha^* = \beta$, the optimal intervention is $\mu_j = sgn\,[(a_1^* - \alpha^*) b_j]$ for $j = 1, \cdots, r$. Otherwise, the intervention operates only on a portion of the time course. If the intervention is $\mu_j^* = 0$ in that case, the intervals or ranges of interventions are defined by:

$$\alpha^* = a_1^* + \int_0^T \sum_{j=1}^{r} b_j u_j^* dt. \tag{A-4}$$

The problem of determining the optimal intervention is fully resolved by any law of response of the random variables of the initial system.

Appendix B

The competitive equilibrium is characterized by the following conditions:

$$p\tilde{c}_{st}^{p(1-\sigma)-1}\tilde{l}_{st}^{(1-p)(1-\sigma)} = \lambda_{st}(1+\tau_c) \tag{B-1}$$

$$\begin{aligned}\lambda_{st}(1+n_s) &= \beta\,\mathbb{E}_t\Big\{\lambda_{s,t+1}((1-\tau_{sr})r_{t+1}v_{t+1}+\tau_r\delta_k v_{t+1}+1-\delta_k)\\ &+ \mu_{s,t+1}\eta_{t+1}(1-v_{t+1})B\beta_1\big[(1-v_{t+1})\tilde{k}_{t+1}\big]^{\beta_1-1}\\ &\quad \big[(1-u_{s,t+1}-l_{s,t+1})\tilde{h}_{t+1}\big]^{\beta_2}\Big\}\end{aligned} \tag{B-2}$$

Using $[\tau_{sr}\delta_k+(1-\tau_{sr})r_t]\lambda_{st} = \mu_{st}\eta_t B\beta_1[(1-v_t)\tilde{k}_t]^{\beta_1-1}[(1-u_{st}-l_{st})\tilde{h}_t]^{\beta_2}$, we can write:

$$\begin{aligned}\lambda_{st}(1+n_s) &= \beta\,\mathbb{E}_t\Big\{\lambda_{s,t+1}((1-\tau_{sr})r_{t+1}v_{t+1}+\tau_r\delta_k v_{t+1}+1-\delta_k) \qquad \text{(B-3)}\\ &+ \big[\tau_{sr}\delta_k+(1-\tau_{sr})r_{t+1}\big]\big(1-v_{t+1}\big)\lambda_{s,t+1}\\ &= \mathbb{E}_t\Big\{\lambda_{s,t+1}\big[(1-\tau_{sr})r_{t+1}+1-(1-\tau_{sr})\delta_k\big]\Big\} \qquad \text{(B-4)}\end{aligned}$$

Using the fact that $\frac{p\tilde{c}_{s,t+1}^{p(1-\sigma)-1}\tilde{l}_{s,t+1}^{(1-p)(1-\sigma)}}{(1+\tau_c)} = \lambda_{s,t+1}$, we have

$$\begin{aligned}\tilde{c}_{st}^{p(1-\sigma)-1}\tilde{l}_{st}^{(1-p)(1-\sigma)} &= \frac{1}{1+n_s}\beta\,\mathbb{E}_t\Big\{\tilde{c}_{s,t+1}^{p(1-\sigma)-1}\tilde{l}_{s,t+1}^{(1-p)(1-\sigma)}\Big[(1-\tau_{sr})r_{t+1}+1-(1-\tau_{sr})\delta_k\Big]\Big\} \qquad \text{(B-5)}\\ &= \frac{1}{1+n_s}\beta\,\mathbb{E}_t\Big\{\tilde{c}_{s,t+1}^{p(1-\sigma)-1}\tilde{l}_{s,t+1}^{(1-p)(1-\sigma)}\Big[(1-\tau_{sr}) \qquad \text{(B-6)}\\ &\quad \omega_{t+1}\alpha_1 A(v_{t+1}\widetilde{K}_{t+1})^{\alpha_1-1}(u_{s,t+1}\widetilde{H}_{t+1})^{\alpha_2}\widetilde{L}_n^{\alpha_3}+1-(1-\tau_{sr})\delta_k\Big]\Big\}\\ &= \frac{1}{1+n_s}\beta\,\mathbb{E}_t\Big\{\tilde{c}_{s,t+1}^{p(1-\sigma)-1}\tilde{l}_{s,t+1}^{(1-p)(1-\sigma)}\Big[(1-\tau_{sr}) \qquad \text{(B-7)}\\ &\quad \omega_{t+1}\alpha_1 A(v_{t+1}\tilde{k}_{t+1})^{\alpha_1-1}(u_{s,t+1}\tilde{h}_{t+1})^{\alpha_2}\widetilde{L}_n^{\alpha_1+\alpha_2+\alpha_3-1}+1-(1-\tau_{sr})\delta_k\Big]\Big\}\\ &= \frac{1}{1+n_n}\beta\,\mathbb{E}_t\Big\{\tilde{c}_{n,t+1}^{p(1-\sigma)-1}\tilde{l}_{n,t+1}^{(1-p)(1-\sigma)}\Big[(1-\tau_{nr}) \qquad \text{(B-8)}\\ &\quad \omega_{t+1}\alpha_1 A(v_{t+1}\tilde{k}_{t+1})^{\alpha_1-1}(u_{n,t+1}\tilde{h}_{t+1})^{\alpha_2}\widetilde{L}_n^{\alpha_1+\alpha_2+\alpha_3-1}+1-(1-\tau_{nr})\delta_k\Big]\Big\}\end{aligned}$$

Also,

$$\tilde{h}_{t+1} = \eta_t B\big[(1-v_t)\tilde{k}_t\big)^{\beta_1}\big]\big[(1-u_{st}-l_{st})\tilde{h}_t)\big]^{\beta_2}+(1-\delta_h)\tilde{h}_t \tag{B-9}$$

$$\widetilde{C}_t+\widetilde{K}_{t+1}-(1-\delta_k)\widetilde{K}_t+G_t = \omega_t A(v_t\widetilde{K}_t)^{\alpha_1}(u_{st}\widetilde{H}_t)^{\alpha_2}\widetilde{L}_n^{\alpha_3} \tag{B-10}$$

$$\tilde{c}_t+(1+n_n)\tilde{k}_{t+1}-(1-\delta_k)\tilde{k}_t = (1-\gamma)\omega_t A(v_t\tilde{k}_t)^{\alpha_1}(u_{st}\tilde{h}_t)^{\alpha_2}\widetilde{L}_n^{\alpha_1+\alpha_2+\alpha_3-1} \tag{B-11}$$

where $\gamma = \frac{\tilde{g}_t}{\tilde{y}_t}$ as we have:

$$\frac{\widetilde{C}_t+\widetilde{K}_{t+1}}{\widetilde{K}_t}-(1-\delta_k) = (1-\gamma)\frac{\widetilde{Y}_t}{\widetilde{K}_t} \tag{B-12}$$

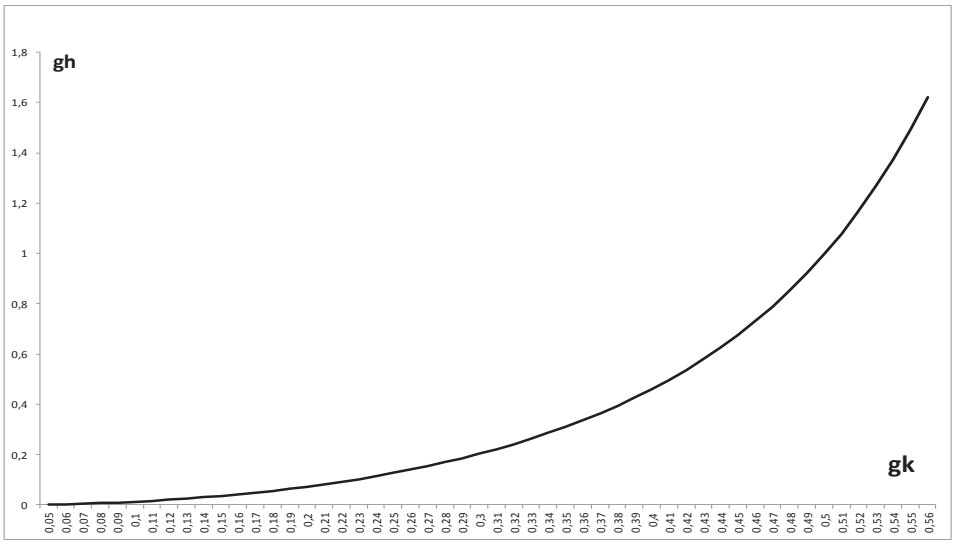

Figure 1: The relation between the growth of physical capital (g_k) and human capital (g_h)