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ALFRED TARSKI: SEMANTIC SHIFT, HEURISTIC SHIFT IN METAMATHEMATICS

Metamathematics was created by Hilbert in a series of papers published between 1905 and 1931. Hilbert used the term 'metamathematics' for the first time in 1922 as synonymous with 'proof theory' (Beweistheorie). The *idea*, but not the term, of proof theory was already present in his 1905 paper and described in more detail in 1918. In the latter paper, Hilbert outlined a programme for opening a new branch of mathematical inquiry, namely the study of the concept of mathematical proof. He listed some problems in the "theory of [mathematical] knowledge" which, he believed, could be solved within the framework of this "new mathematics": first of all, the problem of the consistency of arithmetic, and also the questions of the solvability of every mathematical problem, of decidability by a finite number of steps, of checking results, of defining a criterion for testing the simplicity of mathematical proofs, and finally the question of determining the relations between content and formalism (*Inhaltlichkeit* und *Formalismus*)in mathematics and logic.[1] As Hilbert explicitly required in the case of the consistency of elementary arithmetic, the solution must be a mathematical one.[2] Thus, metamathematics consists in the use of mathematical tools to study logical or epistemological questions concerning mathematical structures or methods. Hilbert had such a definition in mind and this definition still suited the work of Hilbert's followers, namely, Gödel, Tarski, Church, Kleene or Kreisel (among others). One needs only to add that the meaning of 'mathematical tools' may be so wide as to include logical ingredients, since logic itself became in the meantime, and thanks precisely to the work of the above mentioned logicians and some others, much more involved in other mathematical branches.

With his proof theory, Hilbert created one of the three fundamental trends of metamathematical research which, along with recursion theory and model theory, nowadays constitute the scope of mathematical logic. I shall not look at recursion theory here. Rather, I will examine how Tarski developed Hilbertian metamathematics in order to create the basis of a new domain of mathematical logic, i.e., model theory.

1. TARSKIAN METAMATHEMATICS

Tarski knew very well the results and methods of what he himself named the "Göttingen School" of logic.[3] As we can see by reading his early writings, he used the expression "methodology of deductive sciences" to refer to his own work, borrowing this expression from K. Ajdukiewicz and stressing that it is synonymous with the Hilbertian term "metamathematics".[4] On his part, he defined the methodology of deductive sciences, or "methodology of mathematics", as the study of "formalized deductive theories" such as Peano's arithmetic or Hilbert's axiomatic geometry.[5] However, he soon came to dislike the term 'methodology', the meaning of which did not fit his current ideas:[6] Tarski wanted neither to develop "the science of method" nor to apply logic to philosophy. He emphasized indeed that "the analysis and the critical evaluation of the methods, which are applied in the construction of deductive sciences ceased to be the exclusive, or even the main task of methodology".[7] Eager to apply mathematics to logic and conversely logic to mathematics, he adopted the Hilbertian term from 1935 onwards.[8] He stressed however that the expression 'proof theory' seemed to him not to be very good and in any case less appropriate than 'metamathematics'.[9] Moreover, he strove to construct for the latter a meaning which no longer coincided with the Hilbertian one. If Hilbert *grosso modo* identified metamathematics with proof theory, Tarski needed to widen the scope of metamathematics in order to make it fit his own ideas and work. For him, the issue at stake was not only mathematical proof, but the whole field consisting of logic *and* mathematics – mathematics as generally practised by the working mathematician. This is one reason why Tarski more particularly rejected the restriction to "finitistic methods", i.e., methods for solving mathematical problems in a finite number of steps and with no appeal to the so-called "actual infinite".

In fact, after Gödel's results on the incompleteness of any first-order theory incorporating number theory (1931), metamathematical investigations could no longer be centered on Hilbert's main problem as formulated between 1904 and 1930, namely the consistency of elementary arithmetic, which was thought of as *the* foundational problem in mathematics. Among other Hilbertian metamathematical problems, the decidability of the first-order theory of quantification, viewed as *the* fundamental problem for symbolic logic, and the solvability of every mathematical question were the dominant concerns for logicians in the late 1920s. About both issues there were, however, strong misgivings.[10] As for the problem of the relations between a formal system and its content, it was Tarski who gave a radically new outlook, developing an original back-and-forth method between a formal language ant its interpretations and obtaining very important results by means of this method. One can say that developing this method, – which was going to be specific to model theory – was Tarski's own way of dealing with the Hilbertian problem of exploring the relations between content and formalism.

How did Tarski come to his views? What did he aim for in his own metamathematical investigations?

1.1. *A General Theory of Deductive Science*

As early as 1930, Tarski observed that "strictly speaking, metamathematics is not to be regarded as a single theory. For the purpose of investigating each deductive discipline a special metadiscipline should be constructed".[11] This observation means that there is no universal metatheory for the whole field of mathematics. In Tarski's view, it is possible, however, to construct the domain of "general metamathematics", whose task is to define the meaning of a series of metamathematical concepts which are common to the special metadisciplines, and to establish the basic properties of these concepts. Among these concepts we find consistency, completeness, decidability *and also* satisfiability and truth of a formula,[12] definability,[13] model, consequence,[14] theory,[15] etc. all of whose definitions the model theorists today mainly owe to Tarski. It was this precise definition of semantic concepts used informally by some great logicians who preceded him – such as Skolem or Gödel –, which constituted a crucial shift in metamathematics. Let me give one single example. Gödel used the term 'model' in the Introduction of his paper on 'The completeness of the calculus of logic' (1929).[16] As far as I know, this was the first occurrence of the term, but Gödel did not define it. In German the usual words were 'Realisierung', 'Erfüllbarkeit' or 'Erfüllung'. Hilbert, for instance, used to speak of the 'Erfüllbarkeit' or 'Erfüllung eines Axiomensystems'. Before 1935 he used the term 'model' only once, namely, in Hilbert and Bernays (1934, 18). On his part, Skolem generally used the periphrases "domain (*Bereich*), in which such proposition is satisfied (*erfüllt*)" or "set of elements in which such axioms are true". Skolem used the term 'modèle' (in French) with nearly the same meaning as the German "*Erfüllbarkeit*", in a single paper, read at the *Entretiens de Zürich*, in 1938.[17] At that time Tarski had already defined the concept of model in terms of the concept of satisfaction in his address at the International Congress of Scientific Philosophy in Paris (1936d).[18] He had stressed that the given definition coincides with the mathematical use of the word[19] and he had stated the definition of logical consequence which has become classic:

The sentence X *follows logically* from the sentences of the class K iff every model of the class K is also a model of the sentence X.

In chapter VI, paragraph 37 of his *Introduction to Logic and to the Methodology of Deductive Sciences* (1936b), Tarski had thoroughly explained the concept of model through the very simple example of congruent segments; he had specified that an axiomatic system can have several models and the way in which a theory can have a model in another theory.

In 1944 Tarski himself praised the merits of his semantical method: it allowed him to adequately define concepts which had been used until then only in an intuitive way.[20]

Thus, Tarski achieved a very specific semantic shift from a view which focused on formal systems, axioms and rules of proof, to a view focused: (1) on the *formal definition* and the *axiomatization* of semantic concepts, and (2) on the *interplay* between sets of elementary sentences and their mathematical models. In the model-theoretic tradition, which goes back to J. Lambert[21] in mathematics and to Ch. S. Peirce and E. Schröder in logic,[22] the relation of a sentence (or a set of sentences) to its models is the main concern. Tarski's specific contribution consisted in exploring the *mutual* relations between sentences and their *mathematical* models, in showing that semantic analysis is *furthered*, and not superseded, by syntactic analysis and in weaving metamathematical and mathematical threads in a much tighter fashion than anyone had done before him.

In a footnote to the last sentence of paragraph 42 of Tarski (1936b), Tarski dated the beginning of his general metamathematics to around 1920, and attributed it equally to the Göttingen School (with Hilbert and Bernays) and to the Warsaw School (with Leśniewski and Lukasiewicz among others). The originality and significance of his personal work, which may be rightly characterized as *axiomatic semantics*,[23] soon became evident to logicians. For instance, Rudolph Carnap stated in his *Introduction to Semantics* (1942) that it was Tarski who "first called my attention to the fact that the formal method of syntax must be supplemented by semantical concepts that can be defined by means no less exact than those of syntax".[24] About forty years later Jon Barwise declared that 'Tarski's view of logic has changed the way all of us think about the subject".[25]

1.2. *Examples of Tarski's Way of Defining a Concept and Solving a Problem*

Tarski fruitfully developed a specific kind of *conceptual analysis*. He combined the syntactic analysis inherited from Hilbert's School and the semantic methods of the algebraists of logic (Peirce, Schröder, Löwenheim, and Skolem). From Leśniewski – his dissertation adviser whose direct influence he acknowledged[26] – Tarski learned that any formalized theory consists of *meaningful* sentences.[27] Tarski's semantics did reorient syntactic analysis; it did not reject it. Thus, attention was focused on the interrelations between formal sentences and their possible meaning (i.e., their possible interpretations) in some mathematical domain, and the aim was to define semantic concepts as well as solve metamathematical problems mathematically. Let me give two examples of Tarski's conceptual analysis, which is at the same time synctactical and semantical.

1.2.1. In Tarski's view, no precise definition could be constructed in a completely abstract and general way. Tarski did not address such general questions as "What is a number?" – a major question for Frege – or "What is a set?" – a question which was much discussed since Dedekind's and Cantor's work on set theory –. In fact, Tarski did not expect a uniform and unambiguous answer to these kinds of questions. He thought, rather, that the answer to a question or, at least, to one of its aspects, depended on the presupposed framework and tools used to formulate precisely that question or its selected aspect. Even when he came to address the question 'What are logical notions?' in a lecture given in 1966 and published twenty years later, he stressed that he was not aiming to discuss "the general question 'What is logic?' " but wanted instead to restrict his comments to "one aspect of the problem, the problem of logical notions". Moreover, he pointed out how much his approach distances itself from attempts to catch "the proper, true meaning of a notion, something independent of actual *usage*, and independent of any *normative proposals*, something like the platonic idea behind the notion". On his part, he would "make a suggestion or proposal about a *possible use* of the term 'logical notion' ".[28] Thus, Tarski constructed an *algebraic definition* for this term: a notion is *logical* if it is invariant under all possible one-one transformations of the universe of discourse onto itself. Then, he concluded: "The suggestion I have made does not, by itself, imply any answer to the question of whether mathematical notions are logical". According to Tarski, the latter question has a positive answer if one takes a higher-order underlying logic (e.g., as in *Principia mathematica* of Russell and Whitehead), and a negative answer if one takes a first-order underlying logic. Thus, the logicist philosophy of Frege and Russell was in fact closely tied to the logical framework they had adopted. Another framework, another philosophy!

1.2.2. One finds another important example of Tarski's specific kind of conceptual analysis in his treatment of the concept of definability. In his seminal paper of 1931, he declared that "the meaning of this notion is not at all precise: a given object may or may not be definable with respect to the deductive system in which it is studied, the rules of definition which are adopted, and the terms that are taken as primitive".[29] Then, he determined the domain within which he would make precise the meaning of the notion, i.e., the domain of *interpretation* of the notion. This domain is one of those most familiar to mathematicians, namely, real numbers. Thus, Tarski did not address (in this paper) the general question of definability, but the restricted question of what are definable sets of real numbers. The construction of the meaning of the term 'definable set of real numbers' is carried out in two steps. First – in the metamathematical step –, a set of real numbers is said to be 'definable' if it is determined by a first-order sentential function (first-order formula),[30] i.e., if it is defined by what logicians usually call an "explicit definition". In a second step, Tarski showed how to reconstruct this definition, within mathematics itself, by using instead of the metamathematical notion of formulas (of a first-order language), its mathematical analogue, namely, the concept of sets of sequences, and by defining on these sets operations corresponding to the logical operations adopted as primitive (negation, conjunction, disjunction, universal and existential quantification). Going further, Tarski proposed to use the concept of a polynomial, defined as a function associating a certain real number $P(x_1, x_2, \ldots, x_n)$, to every sequence $x_1, x_2, \ldots, x_n$ of real numbers. Thus, a set E of real numbers is *definable* iff there is a polynomial P such that E is exactly the set of finite sequences $x_1, x_2, \ldots, x_n$ satisfying the equation $P(x_1, x_2, \ldots, x_n) = 0$ or the inequality $P(x_1, x_2, \ldots, x_n) > 0$. In other terms:

$$E = \{(x_1, x_2, \ldots, x_n) \in \mathbf{R}^n,\ P(x_1, x_2, \ldots, x_n) = 0 \vee P(x_1, x_2, \ldots, x_n) > 0\}.$$

Thus, Tarski stated for the ordered field of real numbers a notion analogous to that of algebraic set,[31] which was well-known for the field **C** of complex numbers and, more generally, for any commutative field. Tarski's statement is the very first definition of what modern mathematicians call real semi-algebraic sets, as I noted elsewhere.[32] Here, however, I only want to stress that the defining process is somewhat analogous to that which is used for the above-mentioned definition of the term 'logical notion'. Indeed, whenever it could be done, Tarski gave mathematical definitions, and preferably algebraic ones, for metamathematical concepts. He used likewise mathematical, and preferably algebraic tools, for solving metamathematical problems. For instance, he generalized Sturm's algorithm for counting the real roots of polynomials, by conceiving it as a decision procedure for the elementary theory of real numbers: not only the number of roots, but all questions formulated in the first-order language of the ordered field of real numbers are decided by applying Sturm's process in a generalized form. This result is a brilliant product of Tarski's conceptual analysis, which closely tied the study of syntactic properties of the axioms for the theory to the study of the structural properties of its model or models.[33] Tarski's ingenious trick was to understand Sturm's process as a mathematical instance of the logical method of effective elimination of quantifiers (EEQ). The older decision procedures by EEQ (Löwenheim, Skolem, Langford, Presburger, Herbrand) were also guided by the analogy with the successive elimination of unknowns in systems of polynomial equations, but they did not use any theorem borrowed from the theory of algebraic elimination. All the arguments of, say, Herbrand's proof for the decidability of the elementary theory of addition on integers, were of a purely logical nature. While Herbrand, among others, tried to mimic the algebraic elimination process, Tarski showed that the latter is a special case of effective elimination quantifier.[34] He noticed that from the point of view of its logical structure his proof was close to the older completeness proofs by quantifier elimination, but that it was singular in using "a much more powerful instrument".[35] Tarski's decision method for real algebra and geometry was indeed the first one being of a mathematical nature.

In a historical perspective, one can say that Tarski reversed the reduction of mathematics to logic, using mathematical methods for defining logical concepts and proving logical theorems. Developing this "reverse logic" was nothing else, to some extent, than carrying out Hilbert's aim to establish a "new mathematics". But Tarski went much further, willing not only to construct a new logic with the help of mathematics, but also to give an evidence for the mathematical efficiency of model-theoretical reasoning. He restored logic to power, but on a technical as well as – and perhaps more than – on a foundational level. He strove to extend our possibilities of *mathematical* reasoning.

2. A HEURISTIC SHIFT

Solomon Feferman, who studied with Tarski at Berkeley from 1948 to 1957, confirmed that Tarski did have "a very strong motivation" not only to make logic mathematical, but also "and at the same time to make it of interest to mathematicians".[36]

In fact, from the beginning of his career in Poland, Tarski expressed again and again his will to make metamathematics fully mathematical. In 1936 he argued that "*The methodology of the deductive sciences became a general theory of deductive sciences, in a sense analogous to that, in which* *arithmetic is the theory of numbers and geometry is the theory of geometrical figures*".[37] Metamathematics, which more particularly includes metatheories for arithmetic and geometry, became itself a science comparable in every respect to arithmetics or geometry, the oldest mathematical sciences. Indeed, since metamathematical concepts can be reconstructed by mathematical means, "they do not differ at all from other mathematical notions . . . their study remains entirely within the domain of normal mathematical reasoning".[38] This explains how metamathematics became able to borrow from the rest of mathematics results and methods, which it could then apply within its proper perspective, thus yielding results that might in turn be fruitful for what Tarski called "ordinary" or "normal" mathematics. This is indeed the case for the definition of definable sets of real numbers – which was a by-product of the decision method for real algebra and geometry –, even though "ordinary" mathematicians did not recognize the mathematical fruitfulness of this definition before the late 1970s and did not create real algebraic geometry starting from Tarski's work, but rather from Łojasiewicz's semi-analytic sets.[39] In one word, the logician Tarski opened a new mathematical branch, the mere possibility of which no mathematician could even conceive of in the thirties.

Hilbert made a strong separation between mathematics and metamathematics, although he proposed to use mathematical tools to deal with metamathematical problems. But solving the latter was, according to him, eliminating them once and for all. In such a perspective, Hilbert could not conceive of the fruitfulness of metamathematics for mathematics. In Hilbert's view, metamathematics was to yield a guarantee (*Sicherung*) for mathematics.[40] Tarski thought that this aim to provide for mathematicians "a feeling of absolute security" was "far beyond the reach of any normal human science"; it pertained to "a kind of theology".[41] Tarski wanted to transform theology into mathematics. In his view, metamathematics became similar to any mathematical discipline. Not only its concepts and results can be mathematized, but they actually can be *integrated into mathematics.* Hence they can help solve mathematical problems or throw new light on older mathematical fields or notions, such as real numbers, orderings, Sturm's algorithm, polynomials, etc. – and they do so independently of what philosophical thoughts they may involve or suggest.

Tarski destroyed the borderline between metamathematics and mathematics. He objected to restricting the role of metamathematics to the foundations of mathematics. He also strove to give metamathematics a *heuristic* role. He did not want to put logic to service as a tool of conceptual analysis; he wanted rather to develop conceptual analysis as a tool of mathematical research. In the 1930s this was the beginning of a watershed in modern mathematical logic, which would become conspicuous from the 1950s onward. While the semantic shift in logic has been and still is the subject matter of many papers and studies,[42] it seems to me that historians of logic more rarely paid attention to the heuristic shift, even though outstanding logicians, such as E. Beth, A. Robinson and G. Kreisel, pointed it out.[43] Now, it is Tarski who initiated this shift and, as noted by Kreisel (1985), "the passage *from* the foundational aims for which various branches of modern logic were originally developed *to* the discovery of areas and problems for which logical methods are effective tools . . . did not consist of successive refinements, a gradual evolution by adaptation . . . , but required radical changes of direction, to be compared to evolution by migration".

2.1. *Philosophical Attitude and Actual Doing*

Tarski thought that the foundational perspective was mostly philosophical, i.e., to some extent foreign to the technical work of logicians. Thus he was led to a pragmatic point of view, stressing that his methods and results were independent of philosophical assumptions concerning the foundations of mathematics.[44] Carrying Hilbert's guideline further, Tarski sharpened the distinction between logic and philosophy, or, more accurately, between mathematical logic and philosophical logic. But unlike Hilbert, he abolished the distinction between mathematics and metamathematics, for he thought it irrelevant from a practical point of view.[45] Now, the practical point of view was dominant in Tarski's *actual way* of *doing* logic.

Truly, Tarski oscillated between different stands, explicitly or implicitly taken, on the nature of mathematical and logical knowledge and between different *opinions on* his own activity. In his paper on 'The completeness of elementary algebra and geometry' (1939/1967), he noted that in order to determine whether or not a classical geometrical theorem belongs to his elementary system "it is only *the nature* of the concepts, not the character of the means of proof that matters".[46] What Tarski pointed out here is that a first-order theory may encompass concepts expressible and provable under non-elementary conditions which are satisfied in some particular model of the theory. That is to say that a first-order theory may grasp much more than first-order definable properties. From a logical (technical) point of view, this fact is by itself significant. Now, Tarski's insistence on a mathematical content independent of its possible formulation or proof might have led to a kind of realism. Although Tarski did not explicitly take such an attitude, it is compatible with that which is expressed in the final remarks of Tarski (1944). There, Tarski defended the *intrinsic* interest of metamathematical research,[47] arguing that it may be harmful to scientific progress to equate the importance of a work with its possible usefulness. He added, however, that this opinion did not affect the subject matter of his paper. The subject matter is also relatively independent of the formal language in the framework of which it is developed and totally unaffected by any informal philosophy associated with it.

Later on, Tarski changed his mind. Instead of stressing the independence of technical results from philosophical assumptions, he adopted the philosophical outlook fitting his actual practice. In particular, much as an "ordinary" mathematician, he accepted to raise the question of applicability and to show, for example, that his theory of arithmetical classes "has good chances to pass the test of applicability ... [and to] be of general interest to mathematicians".[48] At the same time, he substituted a more pragmatic philosophy to his former "intuitionistic formalism" (borrowed from Leśniewski).[49] In the above-mentioned lecture, delivered in 1966 and published twenty years later, Tarski related the way in which the Platonistic approach was "so foreign and strange to [him]".[50] In Tarski's words, the Platonistic approach consisted in trying to catch "the proper, true meaning of a notion, something independent of actual usage, and independent of any normative proposals". To some extent this view of Platonism was shared by Leśniewski.[51] But Tarski no longer agreed with Leśnewski's "intuitionistic formalism", proposing instead the attitude he described himself as follows:

> What I shall do is make a suggestion or proposal about a possible use of the term 'logical notion'. This suggestion seems to me to be in agreement, if not with all prevailing usage of the term 'logical notion', at least with one usage which actually is encountered in practice. I think the term is used in several different senses and that my suggestion gives an account of one of them.[52]

I think this declaration is very important for anyone who wishes to give a positive content to Tarski's negative statement that Lesniewski's philosophy did not "adequately reflect his present attitude" (footnote added in 1956 to Tarski (1930b).[53]) One may already observe the presence of such a pragmatic attitude in Tarski's lecture at the Princeton University Bicentennial Conference on the Problems of Mathematics, back in 1946.[54] There, Tarski said that he would "use the term 'logic' pragmatically to denote the work of people who regard themselves as logicians – or those who are considered logicians by mathematicians generally".

Now, a pragmatic attitude is, it seems to me, the most appropriate one to hold in defending the practical mathematical efficiency of metamathematics within mathematics itself.

2.2. *A Cross-Fertilization Between Mathematics and Metamathematics*

If methamathematics is itself a mathematical discipline in the fullest sense of the word 'mathematical', then, as Tarski stressed, "the results obtained in one deductive discipline can be automatically extended to any other discipline *in which the given one finds an interpretation*".[55] The process of interpretation – the formal definition of which would come in 1953[56] – yields a fruitful co-operation between metamathematics and other branches of mathematics. The back-and-forth method is to be applied not only to sentences and their models, but also to different theories. The interplay between different areas of mathematics, metamathematics included, brought new mathematical cross-concepts and problems, along with novel hybrid tools to tackle them. Gödel made a significant step towards this cross-fertilization with his coding of metamathematical sentences by natural numbers. Tarski, who often stressed that he independently discovered the possibility of interpreting metalogical sentences about a structure S in the structure S itself,[57] systematized the use of interpreting one theory in terms of another[58] and was very eager to construct tools for "ordinary" mathematicians from metamathematical results. Well-known successes of the process of interpretation may be found: (1) in undecidability proofs, and (2) in constructing algebraic translations of metamathematical notions and theories. As an example of (1) let us mention Julia Robinson's proof of the undecidability of the elementary theory of rational numbers, which was tantamount to showing the definability of integers within the field of rationals. As an example of (2) one should be reminded of Tarski's aim at specifying the significance of first-order formulas by describing the closure properties of classes of models defined by them and the subsequent use of ultrapowers – which are of algebraic nature – by J. H. Keisler (in 1964) to describe classes of models which can be defined by a first-order formula. Last, but not least, let us mention another important example: Tarski developed in collaboration with L. Henkin and J. D. Monk cylindric algebras, which are an algebraic version of predicate logic.

3. CONCLUSION

One cannot say that advances in metamathematics revolutionized the whole field of "pure" mathematics, as Gödel believed it should do.[59] However, Tarski's struggle to make metamathematics of interest to mathematicians succeeded, even though not immediately. Actually, new mathematical branches came out, which combine metamathematical and mathematical methods, such as model-theoretic algebra, model-theoretic analysis, real algebraic geometry, computer algebra, etc. These branches use many methods moulded by Tarski, and first of all tools deriving from Tarski's analysis of the notion of definability. Indeed, it may be the case that it is easier to state mathematical properties of some objects by analyzing the syntactical formula through which they can be defined rather than deducing these properties by purely mathematical means. This is the case, for instance, of proving that the projection of a semi-algebraic set is still semi-algebraic.[60] If metamathematics did not revolutionize the whole field of "pure" mathematics, there is nevertheless no doubt that much more of it has become an integral part of standard mathematics than "pure" mathematicians expected.

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NOTES

[1] Hilbert (1918, 153–5) References are made to volume III of the *Gesammelte Abhandlungen* (referred to as Hilbert (1935)), whenever a paper was reprinted in this volume.
[2] Hilbert (1922, 161).
[3] Tarski (1936a), in Tarski (1986a, Vol. II, 111).
[4] For example Tarski (1930a,b, 1935), in Tarski (1986a, Vol. I, 313, 347, 639). Ajdukiewicz's book on the methodology of deductive sciences (1921), which was the subject of discussions among Polish logicians, reported Hilbert's views: see Wolenski (1989, 162). Tarski called Hilbert "the father of metamathematics", "for he is the one who created metamathematics as an independent being" (Tarski 1995, 163).
[5] Tarski (1936b, Chap. VI, §36).
[6] Tarski (1936b, Chap. VI, §42).
[7] Ibid. p. 129 of the fourth English edition.
[8] Tarski (1935–1936), First part; in Tarski (1986a, Vol. II, 27).
[9] Tarski (1936b, Chap. VI, §42).
[10] See for example the Introductory Note to Gödel's 1930 paper, in Gödel (1986–1995), Vol. I, 49–50.
[11] Tarski (1930b), in Tarski (1986a, Vol. I, 347).
[12] Defined in Tarski (1936a); see Tarski (1986a, Vol. II, 101). Truth for formalized languages is defined in terms of satisfaction.
[13] See Tarski (1931), in Tarski (1986a, Vol. I, 529), Tarski (1935), Tarski (1986a, Vol. I, 643–4).
[14] 'Model' and 'consequence' are defined in Tarski (1936d) (see Tarski (1986a, Vol. II, 278–9). According to Etchemendy (1988), Tarski deeply influenced our global view of

logic by his emphasis on the concept of logical consequence over that of logical truth. In Tarski (1936d), indeed, logical truth is defined as the limiting case of consequence. Earlier, in Tarski (1930b), a deductive system was defined as a set of sentences which is identical to its set of consequences.

[15] Defined in Tarski (1930a): see Tarski (1986a, Vol. I, 314–6).

[16] Gödel (1986–1995, Vol. I, 60). For an early history of the *idea* – not the term – of model *in mathematics* see Webb (1995) and Sinaceur (1999).

[17] Skolem (1970, 464 sqq).

[18] See note 13 and Tarski (1956, 416–7).

[19] While mathematicians were familiar with the notion of a model for a long time, they did not use the term very early. For example, Beltrami, Klein and Poincaré used the term ‘interpretation’. Hilbert (1899), who systematically resorted to building (algebraic) models for testing compatibility, consistency, and independence of different sets of geometrical axioms, used the periphrase “system von Dingen, in dem sämtliche Axiome [der Geometrie] erfüllt sind” (e.g., Chap. 2, §9).

[20] Tarski (1944, paragraph 22), in Tarski (1986a, Vol. II, 692–3).

[21] See Webb (1995).

[22] See Vaught (1974) and Hintikka (1988).

[23] Wolenski (1989, 182).

[24] Carnap (1942, Preface, X).

[25] In Duren (1989, 396).

[26] Tarski (1930a), in Tarski (1986a, Vol. I, 313), Tarski (1930b), in Tarski (1986a, Vol. I, 349).

[27] Tarski stressed it, on his own part, in an early paper published in 1928: Tarski (1986a, Vol. IV, 55) (the underlining is mine).

[28] Tarski (1986b, 145) (the underlining is mine).

[29] In Tarski (1986a, Vol. I, 520).

[30] Tarski borrowed the expression ‘sentential function’ from the *Principia Mathematica* of Russell and Whitehead.

[31] *Algebraic* is a set whose elements are common solutions of a finite number of polynomial equations.

[32] Sinaceur (1991, 364–71).

[33] In the first version of his paper (Tarski 1939/1967), Tarski considered one single model, namely, real numbers. Later (Tarski 1948/1951, Footnote 9), he took into account Artin and Schreier’s theory and considered the whole class of real closed fields. For details see Sinaceur (1991).

[34] More details in Sinaceur (1991, 334–9).

[35] Tarski (1986a, Vol. III, 312).

[36] In Duren (1989, 402).

[37] Tarski (1936b), English fourth edition, p. 129 (the underlining is not mine).

[38] Tarski (1931), in Tarski (1986a, Vol. I, 520).

[39] Łojasiewicz (1964).

[40] Hilbert (1922, 174), Hilbert (1923, 179), Hilbert (1931, 192).

[41] Tarski (1995, 160).

[42] A recent study of the “Semantic revolution” and the history of the word ‘semantics’ may be found in Wolenski (1999)

[43] See, for example, Beth (1953), Robinson (1955), Introduction; Kreisel (1954, 1958, 156; 1985).
[44] Tarski (1930b), in Tarski (1986a, Vol. I, 349).
[45] Tarski (1944), in Tarski (1986a, Vol. II, 693).
[46] Tarski (1939/1967), in Tarski (1986a, Vol. IV, 305–6) (the underlining is mine). The remark is repeated in Tarski (1948/1951), in Tarski (1986a, Vol. III, 307).
[47] Tarski (1944, 693–4).
[48] Tarski (1952), in Tarski (1986a, Vol. III, 473).
[49] Tarski (1930b), in Tarski (1986a, Vol. I, 349).
[50] This is supported by another passage, in which Tarski described himself an an "extreme anti-Platonist", quoted by Feferman (1999, 61).
[51] See Wolenski (1995, Ad (4), 337–8).
[52] Tarski (1986b, 145).
[53] Tarski (1956, 62).
[54] Tarski (1946/2000). Jan Tarski found the draft of this talk in 1994 in his father's *Nachlass* in the Bancroft Library at Berkeley. This draft is to be published, with some editorial work, in a next issue of *The Bulletin of Symbolic Logic.*
[55] Tarski (1944), in Tarski (1986a, Vol. II, 693) (the underlining is mine).
[56] Tarski (1953). Finding an interpretation of a theory T_1 in a theory T_2 is tantamount to constructing in T_2 a model of T_1. In other words, T_1 has an interpretation in T_2 if, once one has chosen the interpretation in T_2 of the primitive terms of T_1, one can show that the statements obtained under this interpretation are axioms or theorems of T_2.
[57] Tarski (1939), in Tarski (1986a, Vol. II, 562, footnote).
[58] For a historical sketch of using this method *in mathematics*, first implicitely, then explicitely, see Sinaceur (1999).
[59] According to Wang (1988, 168).
[60] See Bochnak-Coste-Roy (1987, 23–4).

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