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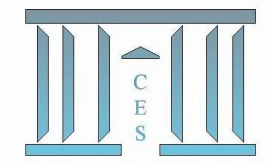
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Francis BLOCH, David CANTALA

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Dynamic Allocation of Objects to Queuing Agents: The Discrete Model *

Francis Bloch† David Cantala‡

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Abstract

This paper analyzes the optimal allocation of objects which arrive sequentially to agents organized in a waiting list. Applications include the assignment of social housing, deceased donor organs and daycare slots. A mechanism is a probability distribution over all priority orders which are consistent with the waiting list. We consider three efficiency criteria: first order stochastic dominance in the vector of agents' values, the probability of misallocation and the expected waste. We show that the strict seniority order dominates uniform random order according to the two first criteria, and the uniform random order dominates strict priority according to the third criterion. If agents values are perfectly correlated, strict priority dominates all other probabilistic mechanisms for all agents values.

JEL Classification Numbers: C78, D83, R31

Keywords: dynamic matching, queuing, queuing disciplines, social housing, organ transplant

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†Université Paris 1 and Paris School of Economics, 106-112 Boulevard de l'Hopital, 75647 Paris CEDEX 13, France. Email francis.bloch@univ-paris1.fr.

‡El Colegio de Mexico, Col. Pedregal de Sta. Teresa C.P. 10740, Mexico, D.F.Mexico. Email dcantala@colmex.mx.

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1 Introduction

This paper analyzes the allocation of objects which arrive sequentially to agents organized in a waiting queue. This situation arises when there is a huge imbalance between demand and supply and monetary transfers cannot be used to match the two sides of the market. Examples include the assignment of social or subsidized housing, of deceased donor organs for transplant, or of spots in daycare. In all these examples, objects are heterogeneous – apartments which become available have different sizes and locations, organs are harvested on deceased donors of different ages and health conditions, daycares have different staffs and amenities. Agents have preferences over the objects which arrive in sequence, and these preferences may or may not be correlated across agents.

In most allocation mechanisms, when a new object becomes available, it is proposed to agents according to a priority order. The priority order is based explicitly or implicitly on a point system, weighing different criteria and usually placing great importance on seniority in the queue. For example, the priority order in the assignment of deceased organs in the United States is strictly based on the seniority rank in the waiting queue[1], and the assignment of social housing in cities like Toronto or Paris put a large weight on the seniority rank in the queue.[2] In sharp contrast to these priority orders based on waiting time, the City of New York has decided to allocate some social housing units through lotteries.[3].

Because objects are heterogeneous, when an agent receives an offer, he faces an optimal stopping problem. Should he accept the current object or wait to receive a better object in the future? The answer to this question depends on the priority order for future objects. If the priority order puts a big weight on waiting time, agents who are the top of the seniority rank have a higher continuation value and hence are more likely to be selective and to reject current proposals. This may result in a sequence of rejections. When the object has a short lifetime (like organs for transplants), objects end up being wasted.[4] For objects with a longer life, like apartments, sequences of

[1]See the policies of the Organ Procurement and Transportation Network available at http://optn.transplant.hrsa.gov.

[2]For Toronto, this is the unique criterion to allocate housing (see http://www.housingconnections.ca.). The allocation of social housing in Paris is implicitly based on a point system but the weighing of the different criteria is not transparent even though in practice waiting time is one of the major criteria (see the report on social housing allocation in 2012 at http://www.apur.org).

[3]See http://www.nyc.gov.

[4]See the article in the New York Times dated September 20, 2012,

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rejections result in vacancies and delays.[5]. Inefficiencies due to waste and delay have led some social housing allocation boards to punish agents for multiple refusals.[6]

Confronted with the diversity of practices for allocation of social housing and deceased organs, and the perceived need for reform from the authorities in charge, our objective in this paper is to model the game played by agents in a waiting queue with fixed priority order, and characterize the effect of parameters of the allocation mechanism (priority order, punishment schemes) on efficiency. We define an allocation mechanism as a probability distribution over the set of priority orders. The mechanism must be consistent with the waiting list so that the probability that an agent who is ranked higher in the seniority queue is proposed the object is higher. There are thus two extreme mechanisms: the strict seniority order, which lets agents choose exactly in their sequence in the waiting list, as for deceased donor organs or social housing in Toronto, and the uniform random order where all priority orders are chosen with equal probability, so that in effect agents face a uniform lottery as in the New York City affordable housing allocation scheme.

In the model of dynamic allocation of objects to queuing agents there is no single obvious efficiency criterion. In order to rank allocation mechanisms, we consider three different criteria. We first consider the (dynamic) values of agents in the queue. As agents are differentiated by their rank in the seniority queue, values are measured by a *vector* of utilities for each agent according to his rank. We then consider the static efficiency of the allocation mechanism. Absent monetary transfers, an object can be allocated to an agent whereas another agent has a higher value for it. The second criterion we consider is then the probability that the object is misallocated to an agent when another agent with higher value would have accepted it. We finally consider the speed at which objects are allocated, and the turnover in the waiting queue. Our third efficiency criterion is the expected waste, measured by the probability that an object is wasted. We note that the first two criteria are related to the welfare of insiders – agents who are currently in the queue. The third criterion reflects the welfare of outsiders – agents who are not currently waiting for the object but will potentially join the queue.

http://www.nytimes.com/2012/09/20/health/transplant-experts-blame-allocation-system-for-discarding-kidneys.html

[5]See for example the recent report on social housing reform to the Department of Social Development in Northern Ireland , http://www.dsdni.gov.uk/social-housing-allocations-report1.pdf

[6]In Toronto, an applicant who rejects three offers is moved at the back of the waiting list.

More precisely, in this paper, we consider a *discrete model* where agents pick a value for the object in a two-point distribution. We suppose that all agents face a homogeneous, known, additive waiting cost. Each agent decides whether to accept the object when it is proposed to him. Except in the special case of the strict priority order, where agents recursively face individual optimal stopping problems, the actions of different agents in the seniority queue are interdependent. Hence, agents play a *game* with other agents in the seniority queue and we characterize behavior through the Markov perfect equilibria of this dynamic game. We analyze a simple framework where the size of the queue is fixed to n agents, and an agent only enters the queue when another agent leaves it.[7] In principle, each agent has the choice between two strategies: accepting both objects or only accepting the object of high value, so that there are 2^n possible equilibrium configurations. Our first result shows that, instead, there can only be n candidate equilibria. Agents who are ranked higher in the seniority queue are more selective so that an equilibrium is characterized by the rank k of the first agent who accepts both objects. Our second result shows that the game exhibits strategic complementarities: if an agent is more selective, the value of all other agents goes up, because these agents are more likely to be proposed the object, and hence other agents in turn become more selective. Because of strategic complementarities, the game may exhibit multiple equilibria, and we show that there always exist values of the waiting cost for which multiple equilibria (with different values) exist. Finally, we notice that, for a fixed equilibrium configuration, the value of all agents is highest in the strict seniority order and lowest in the uniform random order. The intuition underlying this observation can be understood as follows. The value of an agent when he accepts the object is independent of the priority order. When the agent does not accept the object, his objective is to move as fast as possible up the seniority ladder, as values are increasing according to the seniority rank. The strict seniority order is the fastest way to climb up the seniority ladder, as it maximizes the probability that agents ranked higher in the seniority order are proposed the object. By contrast, with a uniform random order, there is no advantage to move up the seniority ladder, and values are thus minimized.

Equipped with these preliminary results we show that *the values of all agents in the queue are higher in the strict seniority order than in the uniform random mechanism. In addition, if values are perfectly correlated, the strict seniority order dominates any other probabilistic priority mechanism*

[7]Considering a stochastic queue, where agents enter and leave the queue through a birth-and-death process would greatly complicate the analysis without providing additional insights. We prefer to leave this for future research.

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This result stems from the fact that, in the strict seniority order, all agents are more selective, and that for a fixed equilibrium configuration, all values are maximized when the allocation mechanism is the strict seniority order. Finally, we observe that waste is increasing in selectivity, so that *the expected waste is minimized in the least selective equilibrium of the uniform random order.* Our results thus vindicate the use of a strict seniority mechanism when the designer's objective is to maximize the value of agents in the queue. If the designer's objective is to maximize the turnover in the queue he should instead use the random order mechanism.

Our analysis also sheds light on the comparison between different existing allocation mechanisms. If agents apply to the object after they learn their values (as in London and New York), the continuation value increases, and agents become typically more selective than in the sequential acceptance mechanism. Hence, agents in the queue obtain a higher value with prior applications, but the expected waste is larger. role of punishment mechanisms. If agents who reject an object are evicted from the queue with a positive probability, the continuation value after rejection decreases, making agents less selective. This effect increases the turnover in the queue but lowers the values of agents inside the queue. Hence, punishment mechanisms increase the welfare of outsiders at the expense of insiders. We also consider an example where agents belong to two populations –one with high waiting costs (like families who apply to social housing in emergency after losing their homes) – and one with low waiting costs. There are then two sources of heterogeneity in the population – waiting time and waiting costs. The incentives of the first and second agent in the queue are no longer aligned. The first agent always wants to be the first to choose, but the second agent prefers to be the first to choose whenever he has a high waiting cost and to let the first agent choose when he has a low waiting cost.

Our paper belongs to the emerging literature on dynamic matching models, enriching assignment mechanisms with dynamic considerations. Other papers in this literature include Abdulkadiroglu and Loerscher (2007), Unver (2010), Bloch and Houy (2012), Bloch and Cantala (2013) , Kurino (2014) and Kennes, Monte, Tumennasan (2014) for models without transfers and Gerkshov and Moldovanu (2009a), (2009b), and Dizdar, Gerkshov and Moldovanu (2011) for models with transfers. Our analysis is also related to the extensive literature in operations research and computer science on queuing disciplines (see the classic book by Kleinrock (1976), a recent up-to-date account in Stidham (2009)). Recent applications of queuing theory to dynamical allocation of kidneys have been proposed by Zenios (1999), Zenios et al. (2000), Su and Zenios (2004) and Su and Zenios (2005), who

allow agents to reject objects. Two papers bear a strong relation to our analysis. Su and Zenios (2004) analyze a continuous model where agents in a stochastic queue choose whether to accept or reject an object. They compare two queueing disciplines: the First Come First Serve and Last Come First Serve rules. Like us, they find that the First Come First Serve rule makes agents at the top of the seniority rank more selective and may result in waste. In their work, agents' preferences are perfectly correlated (organs are vertically differentiated), making the equilibrium characterization easier than in a setting with imperfect correlation. They focus attention on two deterministic queueing disciplines: FCFS and LCFS. As the mechanisms are deterministic, the equilibrium can be computed as a recursive sequence of optimal stopping problems. Leshno (2012) studies dynamic allocation of objects to queueing agents when agent's preferences are unknown. He shows that, in the absence of transfers, a dynamic allocation mechanism where agents are placed in a priority buffer queue is incentive compatible and furthermore that it is optimal to choose uniformly among agents in the buffer queue. While his study is inspired by the same applications to social housing and organ transplant as ours, the two papers differ in many respects. First, in Leshno (2012), agent's preferences are perfectly correlated across time – some agents prefer to live in the North, other in the South, and the mechanism is designed to elicit this persistent type. In our model, there is no persistence of types and hence no information to be elicited from agents. Second, Leshno (2012) only considers two types of objects – there is no vertical quality differentiation – and assumes that there is always an agent who is assigned the object. Hence, there is no waste in his model, and the efficiency discussion is limited to one criterion: the probability of misallocation.

The remainder of the paper unfolds as follows. We present our model in Section 2. In Section 3, we characterize the equilibrium of the model with two agents in the queue. Section 4 contains our main results, comparing the efficiency of allocation mechanisms in general queues. In Section 5, we analyze extensions of the two-agent model to heterogeneous waiting costs, mechanisms with prior applications and with eviction probabilities. We conclude and give directions for future research in Section 6. All proofs are collected in the Appendix.

2 The Model

2.1 Queues, values and waiting costs

We consider a society with an infinite number of agents, organized in a queue of fixed size n. We let $i = 1, 2, ..., n$ denote the order of agents in the queue. Time is discrete, and at each period $t = 1, 2, ...,$ a new object becomes available. Agents in the queue draw a value for the object, $\theta \in \Re$. This value is observed privately by the agent, but not by the other agents nor by the mechanism designer. In the discrete model, θ can only take on two values, $\theta \in \underline{\theta}, \overline{\theta}$.

We assume that each object is different, and that there is no persistence in agents' valuations. Hence, the values $\theta_t, \theta_{t'}$ drawn by an agent for the objects available at periods t and $t' \neq t$ are uncorrelated. At any period t, the values drawn may be correlated across agents, and we will consider the two extreme cases of *private values* where the values are independent and *common values* where the values are perfectly correlated. When $n = 2$, we also allow for more general correlation in the discrete model. Each time an agent waits in the queue, she incurs an additive cost $c > 0$. Given the additive structure of the model, we can without loss of generality renormalize the value of the waiting cost c and pick values $\underline{\theta} = 0, \overline{\theta} = 1$. Assuming that the reservation utility of an agent outside the queue is a large negative number, we can guarantee that individual rationality constraints are satisfied, so that agents always have an incentive to enter the queue.

2.2 The dynamic allocation mechanism

The allocation mechanism specifies a probability distribution p over the finite set R of all possible orders on the n agents in the queue. We denote by $\rho : N \to N$ a typical order in R. For most of the analysis, we assume that the probability distribution is consistent with the waiting list according to the following definition.

Assumption 2.1 *For any two agents $i < j$, and any orders ρ, ρ' such that $\rho(k) = \rho'(k)$ for all $k \neq i, j$, $\rho(i) = \rho'(j) < \rho(j) = \rho'(i)$, $p(\rho) \geq p(\rho')$.*

Assumption **??** states that, whenever two agents i and j are permuted in the orders ρ and ρ', then the order in which the agent with the highest seniority rank is chosen first is picked with a probability at least as large as

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the order in which he is chosen last. [8] In the special case where there are only two agents in the queue, R only contains two orders $\rho_1 = 1, 2$ and $\rho_2 = 2, 1$ and assumption **??** reduces to: $p(\rho_1) \geq p(\rho_2)$. In general, we can define a partial order in the set of probability distributions satisfying assumption **??**, $\mathcal{P}$, by letting $p \succeq p'$ if and only if, for any two agents $i < j$, any any orders ρ, ρ' such that $\rho(k) = \rho'(k)$ for all $k \neq i, j$, $\rho(i) = \rho'(j) < \rho(j) = \rho'(i)$, $p(\rho) - p(\rho') \geq p'(\rho) - p'(\rho')$. It appears that the set of probability distributions satisfying assumption **??** $\mathcal{P}$ admits a minimal and maximal element with respect to the partial order $\succeq$. The maximal element is the *strict priority ranking* where all probability weight is placed on the order $\hat{\rho}$, where $\hat{\rho}(i) = i$ for all i, that we denote $\hat{p}$. The minimal element is the *uniform random order* where all probability distributions ρ in R are chosen with equal probability, $p(\rho) = \frac{1}{n!}$ that we denote $\tilde{p}$. This observation singles out two probability distributions which will play an important role in the analysis. In the strict priority rule, players are strictly ordered according to their seniority in the queue. In the uniform random order rule, agents are treated equally, and an agent's ordering is independent of his rank in the queue.

2.3 Agents' strategies and values

Given a probability distribution $p \in \mathcal{P}$, the mechanism designer picks an order ρ of the agents, and proposes the object to the agents in the sequence ρ. If agent i is proposed an object of value θ and accepts it, he collects the value θ and leaves the queue. All other agents in the queue incur the waiting cost c, a new agent enters the queue at position n and all agents whose rank in the queue is higher than i move up one position in the seniority queue. If no agent in the queue accepts the object, all agents in the queue incur the waiting cost c, keep their rank in the seniority queue, and no new agent is allowed to enter the queue.

In the discrete model, agent i always accepts the object with value $\overline{\theta} = 1$. Hence, the only choice of agent i is whether he accepts the object with value $\underline{\theta} = 0$ or not. A (Markovian) strategy for agent i is thus the probability $q(i) \in [0, 1]$ that it accepts the object with value $\underline{\theta} = 0$ when it is proposed. With this notation, we write the value of agent i as:

[8] The assumption is rather strong – it could actually be weakened by considering only the order ρ truncated to the rank of the second of the two players i and j and computing the expected probability over all orders of the subsequent players.

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$$
\begin{aligned}
V(i) \;=\; & \Pr[\text{object accepted by } j<i](V(i-1)-c) \\
& +Pr[\text{agent } i \text{ is proposed the object}](Pr[\theta_i=\overline{\theta}]\overline{\theta}+q(i)Pr[\theta_i=\underline{\theta}]\underline{\theta}) \\
& +(1-\Pr[\text{object accepted by } j\leq i])(V(i)-c).
\end{aligned}
$$

The above expression takes into account the fact that agent i's rank in the seniority queue will increase by one if an agent above i accepts the object. If agent i is proposed the object, he will always accept it and obtain value $\overline{\theta}$ if the object is of high quality, and will pick it with probability $q(i)$ – and receive value $\underline{\theta}$ – if the object is of low quality. If neither agent $j<i$ nor agent i accepts the object, agent i will wait one period, retain the same rank in the seniority queue, and obtain an expected continuation value $V(i)$ next period.

In order to characterize the optimal strategies of the agents, we distinguish between the unique *deterministic* order satisfying assumption **??** – the strict seniority order – and any other mechanism which assigns positive probability to more than one order in R. In the deterministic mechanism, the optimal strategy of player 1 is a classical optimal stopping problem, and results in a single optimal strategy both in the discrete and continuous cases. Given the optimal strategy of player 1, player 2's problem becomes an optimal stopping problem with a unique optimal strategy. Continuing down the seniority queue, the optimal strategy of agent i can be characterized recursively, once the optimal strategies of agents $1,2,...,i-1$ have been determined. Hence, there is a unique optimal strategy vector in the strict seniority order model, which can be characterized by a recursive system. On the other hand, when the mechanism is probabilistic, the value of all agents – including agent 1 – depend on the strategies of other agents. Agents' strategies are interdependent, and the optimal strategies of the players cannot be characterized as solutions to individual optimization problems. Agents play a game, and optimal strategies arise as Nash equilibria of the game played by all agents in the queue.

2.4 Efficiency criteria

We consider a society with a varying population – agents enter and leave the queue over time – so that there is no obvious efficiency criterion to apply to our allocation mechanism. We use as a first criterion the *vector of values of all agents in the queue*, $\mathbf{V}=(V_1,V_2,...,V_n)$. This criterion is dynamic, as it captures the history of agents as they enter the queue, move up in the

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seniority ranking, and finally exit the queue. For our main results, we will focus on *Pareto dominance* and characterize mechanisms which result in the highest value of the *entire* vector $\mathbf{V}$. Alternatively, we could focus on the value of an agent entering the queue, V_n, or consider the sum of values of all agents in the queue, $\mathcal{V} = \sum_i V(i)$. Second, given that monetary transfers are not allowed and that the designer can only choose the order in which agents get to take the object, the mechanism may result in a (static) misallocation of the objects. Namely, the object may be given to agent i whereas there exists another agent j who would have accepted the object and such that $\theta_j > \theta_i$. As a second efficiency criterion, we consider the *misallocation loss* measured by the expected probability that the object is given to an agent i when there exists another agent j who accepts the object and for whom $\theta_j > \theta_i$. Finally, notice that the allocation mechanism may result in waste, as objects may be rejected by all agents in the queue. While this waste does not affect agents currently in the queue, it slows down the turnover in the queue, and prevents outsiders from joining the queue of fixed size. Hence, the *expected waste*, measured by the probability that any object is rejected by all agents, captures the speed at which the queue is served, and the welfare of agents who are not yet in the queue.

3 The two-agent case

In order to illustrate the model, we analyze in this section the simple case of a two-agent discrete-value dynamic allocation mechanism. When the queue only consists of two agents, under Assumption **??**, the dynamic mechanism is characterized by a single parameter $p \in [\frac{1}{2}, 1]$ denoting the probability that the order $\rho_1 = 1, 2$ is chosen by the designer.

3.1 Private values

We first consider the *private values case* and assume that the valuations drawn by the two agents in the queue at any period are independent. The probability that any agent draws a high value $\overline{\theta} = 1$ is given by π and the probability that he draws $\underline{\theta} = 0$ is $1 - \pi$. We restrict attention to pure strategies and characterize the Markov perfect equilibria in pure strategies of the game played by the two agents. We first note that, whenever $p \geq \frac{1}{2}$, there cannot be an equilibrium where agent 1 is selective whereas agent 2 is not.

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Lemma 3.1 *In the two agent discrete value model, if p is not the uniform random order, there is no equilibrium where $q(1) = 1$ and $q(2) = 0$.*

We thus consider the three possible equilibria: one where both agents are selective, $q(1) = q(2) = 0$, one where the top agent in the queue is selective but not the second, $q(1) = 0, q(2) = 1$ and one where both agents accept the low quality object, $q(1) = q(2) = 1$. We index each equilibrium by the number of selective agents, $k = 0, 1, 2$.

Except for the two extreme mechanisms $p = \frac{1}{2}$ and $p = 1$, whenever agent 1 is selective, agent 2 when proposed the object, does not know whether the object has been proposed to agent 1 before or not. He can however use the mechanism p to update his belief about his rank in the proposal order. This information is valuable because agent 2 can only be promoted to a higher rank in the seniority queue if agent 1 has not yet been proposed the object. More precisely, when agent 2 is proposed the object, her expected continuation value if she refuses is given by

$$p(V(2) - c) + (1 - p)(\pi(V(1) - c) + (1 - \pi)(V(2) - c)).$$

Notice that, as $V(1) - c > V(2) - c$, for fixed values of $V(1)$ and $V(2)$ the expected continuation value is *decreasing in* p: when the probability that agent 1 is proposed the object increases, it becomes less likely that agent 2 will be promoted up the seniority ladder if she rejects the object, and hence agent 2 is less likely to be selective.

3.1.1 Both agents are selective

In this equilibrium , when agent 1 chooses first, with probability π, he picks the object and agent 2 moves up in the seniority queue; with probability $(1-\pi)\pi$, agent 2 picks the object, and with probability $(1-\pi)^2$, none of the agents picks the object which is wasted. If agent 2 chooses first, he picks the objects with probability π ; with probability $\pi(1-\pi)$ agent 1 picks the object and agent 2 moves up the seniority queue, and with probability $(1 - \pi)^2$ no agent picks the object. We compute the values of the two agents as

$$\begin{aligned} V^2(1) &= p[\pi + (1 - \pi)(V^2(1) - c)] + (1 - p)[\pi(1 - \pi) + (1 - \pi(1 - \pi))(V^2(1) - c)], \\ V^2(2) &= p[\pi(1 - \pi) + \pi(V^2(1) - c) + (1 - \pi)^2(V^2(2) - c)] \\ &\quad +(1 - p)[\pi + \pi(1 - \pi)(V^2(1) - c) + (1 - \pi)^2(V^2(2) - c)]. \end{aligned}$$

Simplifying, we obtain

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$$\begin{aligned} V^2(1) - c &= 1 - \frac{c}{\pi(1-\pi+p\pi)}, \\ V^2(2) - c &= 1 - \frac{2c}{\pi(2-\pi)} \end{aligned}$$

Notice that the values of both agents are decreasing in the waiting cost c and increasing in the probability π. Higher values of p result in higher values of $V^2(1)$ but do not affect the value of the second agent. We also note that the first agent in the queue receives a higher value than the second agent, $V^2(1) > V^2(2)$. To check for which parameters this equilibrium exists, we need to consider that both agents prefer to reject object $\underline{\theta}$. Agent 1 rejects the low value object if

$$0 \leq V^2(1) - c,$$

and agent 2 rejects the low value object if

$$0 \leq p(V^2(2) - c) + (1-p)(\pi(V^2(1) - c) + (1-\pi)(V^2(2) - c)).$$

As $V^2(2) < V^2(1)$, a 2 equilibrium exists if and only if

$$0 \leq p(V^2(2) - c) + (1-p)(\pi(V^2(1) - c) + (1-\pi)(V^2(2) - c)).$$

Finally, note that in that equilibrium, the misallocation loss is zero – the object will always be assigned to an agent with value $\overline{\theta}$ – and the expected waste is $W = (1-\pi)^2$.

3.1.2 Agent 1 is selective, agent 2 is not

In this equilibrium, when agent 1 chooses first, with probability π he picks the object and agent 2 moves up in the seniority queue while with probability $1-\pi$, agent 2 picks the object. When agent 2 chooses first, he immediately picks the object. The values of the two agents are

$$\begin{aligned} V^1(1) &= p[\pi + (1-\pi)(V^1(1) - c)] + (1-p)(V^1(1) - c), \\ V^1(2) &= p[\pi(V^1(1) - c) + \pi(1-\pi)] + (1-p)\pi, \end{aligned}$$

yielding

$$\begin{aligned} V^1(1) - c &= 1 - \frac{c}{p\pi}, \\ V^1(2) - c &= \pi(1 + p - p\pi) - 2c. \end{aligned}$$

We observe again that both values are increasing in π and decreasing in c. In addition, both values are increasing in the probability p. The first agent in the queue receives a higher value than the second agent, $V^1(1) > V^2(1)$. In addition, the values of both agents are lower than in the equilibrium where both agents are selective, $V^1(1) < V^2(1)$ and $V^1(2) < V^2(2)$. Finally, in this equilibrium, static misallocation arises when the second agent receives the object and values it at zero, while the first agent has a value of 1, an event which occurs with probability $\pi(1-\pi)(1-p)$. Hence misallocation is decreasing in p. As agent 2 always picks the object, the expected waste in equilibrium is equal to zero. This equilibrium exists if and only if

$$\begin{aligned} 0 &\leq V^1(1) - c, \\ 0 &\geq p(V^1(2) - c) + (1-p)(\pi(V^1(1) - c) + (1-\pi)(V^1(2) - c) \end{aligned}$$

3.1.3 No agent is selective

In this equilibrium, both agents immediately pick the object, and the values are given by

$$\begin{aligned} V^0(1) &= p\pi + (1-p)(V^0(1) - c), \\ V^0(2) &= p(V^0(1) - c) + (1-p)\pi, \end{aligned}$$

so that

$$\begin{aligned} V^0(1) - c &= \pi - \frac{c}{p}, \\ V^0(2) - c &= \pi - 2c. \end{aligned}$$

Both values are increasing in π and decreasing in c, and the value of the first agent is increasing in p. Clearly, $V^0(1) > V^0(2)$, and the values of both agents are lower than in the equilibrium where agent 1 is selective, $V^0(1) < V^1(1)$ and $V^0(2) < V^1(2)$. In this equilibrium, misallocation occurs

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with probability $\pi(1-\pi)$, when the object is allocated to the first agent in the order who has a low value when the second agent in the order has a high value. This equilibrium exists if and only if $V^0(1) \leq c$ and $p(V(2)-c) + (1-p)(\pi(V(1)-c)+(1-\pi)(V(2)-c)) \leq 0$. Given that $V^0(1) > V^0(2)$, the equilibrium exists if and only if

$$c \geq \pi p.$$

3.1.4 Equilibria of the two agent model

We now bring together the three pure strategy equilibria, and illustrate in Figure 1 the regions of parameters in which these equilibria exist. We construct Figure 1 assuming that the two values are equiprobable, $\pi = \frac{1}{2}$. An equilibrium where both players are selective exists if and only if

$$c \leq \frac{3(1+p)}{2(5+p+2p^2)},$$

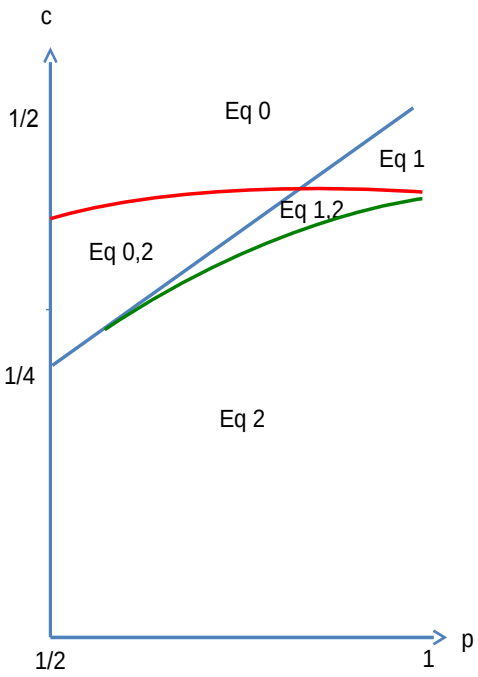
an equilibrium where player 1 is selective exists if and only if

$$\frac{p(6-p+p^2)}{8(1+p^2)} \leq c \leq \frac{p}{2},$$

and an equilibrium where both players accept both objects if

$$c \geq \frac{p}{2}.$$

We first note that there exist two parameter regions where multiple equilibria exist: one where equilibria 0 and 2 coexist – either both agents are selective or none is selective, and one where equilibria 1 and 2 coexist – either both agents are selective, or 1 is selective but not 2. The multiplicity of equilibria stems from the fact that agent's choices are *strategic complements*: if one agent is selective, the probability that the other agent gets to pick the object increases. This increases the value of the other agent, making him more selective. This strategic complementarity in the agent's selectivity levels results in multiple equilibria for some regions of the parameters. Notice however that for high and low values of the waiting cost, equilibrium is unique, with all agents being selective when c is sufficiently low, and no agent being selective for c sufficiently high. We also remark that the asymmetric equilibrium does not exist when p is small. When agents are treated symmetrically in the order, they do not adopt asymmetric selection strategies. Finally, we can analyze the comparative statics effect of an increase in c and p on the



equilibrium type. As c increases, agents become less selective, and equilibrium moves from a selective equilibrium to a less selective equilibrium. When p increases, two effects are at play. First, the values $V^2(1)$, $V^1(1)$ and $V^1(2)$ strictly increase – the other values remain constant – increasing the likelihood that agents are more selective. Second, the continuation value of the second agent after a rejection, $p(V(2)-c)+(1-p)(\pi(V(1)-c)+(1-\pi)(V(2)-c))$, decreases, decreasing the likelihood that the second agent is selective. The two effects result in a non-monotonic effect of p on the selectivity of equilibrium. In fact, the function $\frac{3(1+p)}{2(5+p+2p^2)}$ is first increasing, then decreasing in p, so that there exists a region of costs for which a 2 equilibrium exists for some value p but not for $p=1$. However, comparing the two extreme cases $p=\frac{1}{2}$ and $p=1$, we observe that the equilibrium are always more selective for the strict priority rule than for the uniform random order.

3.2 Correlated values

We suppose that the values of the two agents are correlated and denote by $\gamma \in [-1, 1]$ the correlation coefficient between the values. To simplify notations, assume that the two values are equiprobable, so that

$$\begin{aligned}
\Pr[\theta(1) = \overline{\theta}, \theta(2) = \overline{\theta}] = \Pr[\theta(1) = \underline{\theta}, \theta(2) = \underline{\theta}] &= \frac{1+\gamma}{4}, \\
\Pr[\theta(1) = \overline{\theta}, \theta(2) = \underline{\theta}] = \Pr[\theta(1) = \underline{\theta}, \theta(2) = \overline{\theta}] &= \frac{1-\gamma}{4}.
\end{aligned}$$

In an equilibrium where both agents are selective, we compute the values as

$$\begin{aligned}
V^2(1) - c &= 1 - \frac{4c}{1-\gamma+p(1+\gamma)}, \\
V^2(2) - c &= 1 - \frac{8c}{3-\gamma}.
\end{aligned}$$

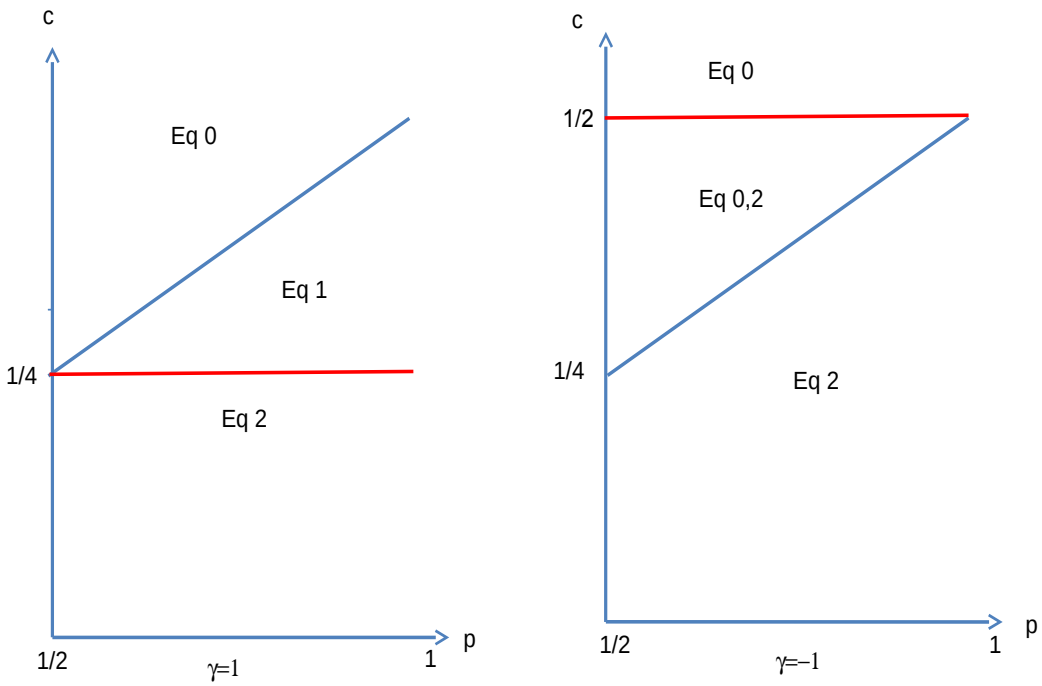
Notice that both values are decreasing in γ – agents benefit from less correlation in their values – and that the value of the first agent is increasing in p for all γ, while the value of the second agent is independent of p. In an equilibrium where only the top agent is selective,

$$\begin{aligned}
V^1(1) - c &= 1 - \frac{2c}{p}, \\
V^1(2) - c &= \frac{2 + p(1-\gamma) - 8c}{4}.
\end{aligned}$$

The values of the two agents are increasing in p – the value of agent 2 is strictly increasing in p if the values are not perfectly correlated and independent of p in the case of perfect correlation. The value of the first agent is independent of the correlation coefficient while the value of agent 2 is decreasing in the correlation coefficient γ. Finally, the value of the two agents in the zero equilibrium are independent of the correlation coefficient as the object is immediately picked up by the first agent to whom it is proposed.

Figure 3 illustrates the range of parameters for which the equilibria exist in the two polar cases of perfect correlation ($\gamma = 1$) and perfectly negative correlation ($\gamma = -1$). Notice that when $\gamma = 1$, for all parameters the equilibrium is unique and, as p increases, becomes more selective. When $\gamma = -1$,

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the two agents are in fact in a symmetric position – the order in which they are proposed the object does not matter as each agent wants the object exactly when the other one does not – and the equilibrium in which only the top agent is selective disappears. An increase in p again leads to equilibria where both agents are more selective. We conclude by observing that adding correlation in the values of the two agents does not change the main qualitative results of the baseline model: an increase in p always increases the values of the two agents, and results in an equilibrium where agents become more selective. Hence, the struct seniority order remains the optimal mechanism when correlation among values is allowed.

4 General queues

We now analyze the general model with discrete values, when the fixed size of the queue is an arbitrary number n. For any mechanism p, and vector of (pure) strategies $\mathbf{q}$, we can compute two vectors: $\gamma = \gamma(1), ..., \gamma(n)$ collects the expected probabilities that agent i receives the object at the end of the allocation process, where the expectation is taken over the realization of the values θ and of the order ρ. The vector $\omega = \omega(1), ..., \omega(n)$ collects the expected probabilities that agent i is proposed the object. Notice that, as at most one agent receives the object at the end of the allocation process, for any realization of θ and ρ, the sum of allocation probabilities is bounded by 1 and hence, $\sum_i \gamma(i) \leq 1$. On the other hand, as agents may reject the object when it is proposed to them, we may very well have $\sum_i \omega(i) > 1$. For example, if $n = 2$, both agents are selective, vales are private and the order 12 is chosen with probability p, we compute $\omega(1) = p + (1 - \pi)(1 - p)$ and $\omega(2) = p(1 - \pi) + 1 - p$, so that $\omega(1) + \omega(2) = 2 - \pi > 1$. Notice also that, if $q(i) = 1$, $\gamma(i) = \omega(i)$ as agent i accepts the object whenever it is proposed, whereas when $q(i) = 0$, $\gamma(i) = \pi\omega(i)$ as agent i only accepts the object when it is proposed and of high value. With these notations in hand, we prove the following Lemma.

Lemma 4.1 *Under Assumption* **??**, *if* $i < j$ *and* $q(i) = q(j)$, $\omega(i) \geq \omega(j)$ *and* $\gamma(i) \geq \gamma(j)$.

Lemma **??** establishes that, when the mechanism is consistent with the waiting list and two agents adopt the same strategy, the agent with higher seniority rank has a higher chance of being proposed the object (and accepting it) than the other agent. We use this observation next to show that the equilibrium values of agents are monotonic in the seniority rank.

Lemma 4.2 *Under Assumption* **??** *in any equilibrium of the game, if* $i < j$ *then* $V(i) \geq V(j)$.

The intuition underlying Lemma **??** is clear when one compares either two agents who are selective or two agents who are not selective. By Lemma **??**, our assumption on the probability mechanisms guarantees that agents with a higher rank in the seniority queue are proposed the object more often. Whenever agents adopt the same strategies in equilibrium, this implies that agents with a high rank get a higher expected utility. The only case where this reasoning may fail is when one compares the value of a selective agent with a value of an agent who is not selective. Then, by the definition of an

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equilibrium, the agent with the highest seniority rank chooses his optimal strategy which yields a higher value than the strategy of the other agent. This revealed preference argument completes the proof of the Lemma. Next we use Lemma **??** to show that there exists no equilibrium where agents with a higher seniority rank are *less* selective than agents with a lower seniority rank.

Lemma 4.3 *Under Assumption* **??**, *if p is not the uniform random order, there is no equilibrium where $q(i) = 1$ and $q(j) = 0$ for some $j > i$.*

Given Lemma **??**, we can focus attention on equilibria where the first k agents in the seniority ranking are selective and the last $(n-k)$ agents are not selective. We call these equilibria *k-equilibria*, with $k = 0, ..., n$. Given these strategies, we compute, for $0 < k < n$,

$$\begin{aligned}
V^k(1) &= \omega(1)\pi + (1-\gamma(1))(V^k(1) - c), \\
V^k(i) &= \sum_{t=1}^{i-1} \gamma(t)(V^k(i-1) - c) + \gamma(i) + (1 - \sum_{t=1}^{i})\gamma(t))(V^k(i) - c) \text{ for } i \leq k, \\
V^k(i) &= \sum_{t=1}^{i-1} \gamma(t)(V^k(i-1) - c) + \gamma(i)\pi + (1 - \sum_{t=1}^{i})\gamma(t))(V^k(i) - c) \text{ for } i > k,
\end{aligned}$$

Solving the recursive system, we obtain

$$\begin{aligned}
V^k(i) &= 1 - \frac{ic}{\sum_{t=1}^{i} \gamma(t)} \text{ for } i \leq k, \\
V^k(i) &= \frac{\sum_{t=1}^{k} \gamma(t) + \sum_{t=k+1}^{i} \gamma(t)\pi}{\sum_{t=1}^{i} \gamma(t)} - \frac{ic}{\sum_{t=1}^{i} \gamma(t)} \text{ for } i > k.
\end{aligned}$$

The next proposition establishes the following property of the equilibrium values $V^k(i)$.

Proposition 4.4 *The equilibrium values satisfy: For any k, i, $V^{k+1}(i) \geq V^k(i)$.*

Proposition **??** compares values of the same agent in two different equilibria – one where k agents are selective and one where $k+1$ agents are selective, and shows that an agent's value is higher in an equilibrium with more selective agents. The intuition underlying this result is again based on

the computation of the probability $\omega(i)$ that an agent is proposed the object. When agent $k+1$ switches from choosing $q(k+1)=1$ to $q(k+1)=0$, the probability that all other agents are proposed the object weakly increases whereas his probability remains constant. We show that an increase in the probability $\omega(i)$ for all $i \neq k+1$ together with a constant $\omega(k+1)$ weakly raises the value of all agents in equilibrium. The only case in which an agent's utility cannot increase is in the switch from the 0 equilibrium (where no agent is selective) to the 1 equilibrium, where the first agent in the seniority queue becomes selective. In that case, the probability that agent 1 is proposed the object remains the same, and as agent 1's value does not depend on the value of the other agents, his equilibrium value remains constant, $V^1(1) = V^0(1)$. When the mechanism puts all weight on the strict seniority order, for $i \leq k$, the probability of being proposed the object is independent of the strategy of agent $k+1$, so that the equilibrium vales are the same in the k and $k+1$ equilibria. Since in addition $\omega^k(k+1) = \omega^{k+1}(k+1)$, the equilibrium value of agent $k+1$ is also the same in both equilibria.[9] In any other equilibrium, as long as the probability p is not degenerate, all agents utilities increase when one agent switches from $q(k+1)=1$ to $q(k+1)=0$. This reflects the fact that agent's actions are *strategic complements*: when an agent becomes more selective, the value of other agents increases, increasing their incentives to become selective.

We noted in the case of two agents that strategic complementarities may induce multiplicity of equilibria for some values of the parameters. Clearly, multiplicity of equilibria will also arise in the general case. This multiplicity comes out as a simple corollary to Proposition **??**. For any agent i, let $\mu^k(i)$ denote the probability that an agent $j < i$ accepts the object after i's rejection. A k and $k+1$ equilibrium will coexist if

$$\begin{aligned}
\mu^k(k)V^k(k-1) + (1-\mu^k(k))V^k(k) &\geq c \\
\mu^k(k+1)V^k(k) + (1-\mu^k(k+1))V^k(k+1) &\leq c, \\
\mu^{k+1}(k+1)V^{k+1}(k) + (1-\mu^{k+1}(k+1))V^{k+1}(k+1) &\geq c \\
\mu^{k+1}(k+1)V^{k+1}(k+1) + (1-\mu^{k+1}(k+2))V^{k+1}(k+2) &\leq c.
\end{aligned}$$

Notice that whenever $V^{k+1}(k+1) > V^k(k+1)$, the four inequalities can simultaneously hold for some region of costs c. Hence multiplicity of

[9] Notice however that the equilibrium value of agent $k+2$ differs in the two equilibria, as agent $k+2$ has a positive probability of being proposed the object in the $k+1$ equilibrium but not in the k equilibrium. Recursively, the equilibrium values of agents $i > k+2$ are different in the two equilibria.

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equilibria arises naturally for intermediate values of the waiting costs. In the next Proposition, we compare the value of an agent at a k equilibrium in the strict seniority order, the uniform random order and in any other mechanism p. We denote by $\hat{V}^k(i)$ and $\tilde{V}^k(i)$ the value of agent i in a k equilibrium under the strict priority and uniform random rules.

Proposition 4.5 *For any k and any i, any mechanism $p \in \mathcal{P}$, $\hat{V}^k(i) \geq V^k(i)$ and for $k = 0, n$, $V^k(i) \geq \tilde{V}^k(i)$.*

Proposition **??** establishes that for a fixed k equilibrium *all agents* prefer the strict seniority order $\hat{p}$ to any other mechanism p and when all agents are selective or nonselective, they prefer any rule to the uniform random order $\tilde{p}$. While this result may at first glance seem counterintuitive – supposing that agents at the bottom of the seniority rank prefer a more egalitarian choice of the order ρ – it can be explained by considering the dynamics of the queue. Instead of the values $V(i)$, we can equivalently consider the ex post values $V(i) - c$. By Lemma **??**, given our assumption on the mechanism, all agents obtain a higher value by moving up in the queue. Because $V(i-1) - c \geq V(i) - c$, agent i always prefers a mechanism which increases the probability that the agent move up to obtain the ex post value $V(i-1) - c$. In the strict seniority order, the speed at which agents move up in the seniority queue is maximized. By contrast, in a uniform random order, there is no advantage to moving up in the queue. This explains why all agents prefer the strict seniority order to any other mechanism p and prefer mechanism p to the random order rule.

Proposition **??** compares the values of agents for different mechanisms for a fixed equilibrium structure. However, changes in the mechanism also affect the equilibrium structure: for fixed values of the parameters c and π, changes in p may induce a switch from a k equilibrium to an equilibrium with a different number of selective agents. Hence, in general the comparative statics effects of a change in the mechanism on the set of equilibria is difficult to ascertain. We focus attention on two cases: a general comparison between the strict priority order and the uniform random order for any correlation structure, and a general comparison of all mechanisms for common values.

4.1 Strict priority order vs. uniform random order

We first focus attention on the two extreme mechanisms $\hat{p}$ and $\tilde{p}$ and characterize all equilibrium structures under these mechanisms.

If $p = \hat{p}$, a k equilibrium exists if and only if

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$$V^k(k) - c = 1 - \frac{kc}{\pi[1 + ... + (1-\pi)^{k-1}]} \geq 0$$

$$\frac{\pi[1 + .. + (1-\pi)^k]}{\pi[1 + ... + (1-\pi)^{k-1} + (1-\pi)^k]} - \frac{(k+1)c}{\pi[1 + ... + (1-\pi)^{k-1} + (1-\pi)^k]} \leq 0.$$

or

$$\frac{1-(1-\pi)^{k+1}}{k+1} \leq c \leq \frac{1-(1-\pi)^k}{k}.$$

As the function $f(k) = \frac{1-(1-\pi)^k}{k}$ is decreasing in k, [10] we can partition the positive real line into intervals $A^k = [\frac{1-(1-\pi)^{k+1}}{k+1}, \frac{1-(1-\pi)^k}{k}]$ such that the unique equilibrium of the game is a k equilibrium when c belongs to A^k.

If $p = \tilde{p}$, the equilibrium must treat all agents symmetrically, and the only two candidate equilibria are the 0 and n equilibrium. The 0 equilibrium exists if and only if $c \geq \frac{\pi}{n}$ whereas the n equilibrium exists if and only if $c \leq \frac{1-(1-\pi)^n}{n}$. Notice that the n equilibrium only exists when the n equilibrium exists under the strict priority rule.

Proposition 4.6 *The equilibrium values of all agents are higher under the strict priority mechanism $\hat{p}$ than under the uniform random order $\tilde{p}$. The misallocation loss is lower under $\hat{p}$ than $\tilde{p}$ and the expected waste is lower under $\tilde{p}$ than $\hat{p}$.*

4.2 Common values

In the case of common values, if a player rejects the low value object, all agents with higher seniority rank reject the low value object as well. Hence, the continuation value of agent i after a rejection in a k equilibrium is simply $V^k(i) - c$. Ak equilibrium exists if and only if

$$V^k(i+1) \leq c \leq V^k(i).$$

A change in the mechanism p thus only affects the continuation value after a rejection through the values in the game. This simple observation allows us to show that the strict priority dominates all other mechanisms according to

[10] To see this consider k as a continuous variable and note that the sign of $f'(k)$ is the same as the sign of $(1-\pi)^k(1 - k\log(1-\pi)) - 1$. Differentiating $g(k) = (1-\pi)^k(1 - k\log(1-\pi))$, we obtain $g'(k) = -k(1-\pi)^k(\log(1-\pi))^2 < 0$, and finally observe that $g(0) = 1$.

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the two first efficiency criteria whereas the uniform random order dominates all other mechanisms for the minimization fo expected waste.

Theorem 4.7 *In the case of common values, the equilibrium values of all agents are always at least as large in the unique equilibrium of the strict priority mechanism $\hat{p}$ as in any equilibrium of any mechanism $p \in \mathcal{P}$. Similarly, the misallocation loss is at least as small in the unique equilibrium of the strict priority mechanism $\hat{p}$ as in any equilibrium of any mechanism $p \in \mathcal{P}$. The expected waste is at least as small in the least selective equilibrium of the uniform random mechanism as in any equilibrium of any mechanism $p \in \mathcal{P}$.*

5 Variants and extensions

In this Section, we discuss extensions based on the two agent model. We first discuss optimal mechanisms when agents have heterogeneous waiting costs. We then consider a model where agents simultaneously apply for the object after they learn their value and the object is assigned according to a probabilistic priority order. Finally, we discuss the impact of a punishment scheme where agents are evicted from the queue with positive probability if they reject the object.

5.1 Heterogeneous waiting costs

We suppose that agents are divided into two categories: a fraction λ of agents with low waiting costs $\underline{c}$, and a fraction $1-\lambda$ of agents with high waiting costs $\overline{c}$. For the sake of simplicity, suppose that the waiting cost is observable by the mechanism designer – for example, the designer can verify whether an agent currently lives in an apartment or not, or the health status of an agent waiting for a transplant. Agents are now characterized by two variables: their order in the seniority queue, and their waiting cost. In order to select an optimal mechanism, the designer faces a trade-off between these two characteristics, and must choose which weight to assign to seniority and waiting cost in the proposal order. More precisely, we assume that the designer, after observing the waiting costs of the two agents, (c,c) chooses the probability that the most senior agent is proposed the object first, $p(c,c)$. Because it may be optimal to let the second agent choose first when he has a high waiting cost, we do not put any restriction on $p(c,c)$.

Suppose that each agent knows the waiting cost of the other agent in the queue. The strategy of each agent assigns to each of the four possible vectors of waiting costs (c,c) a point in $\{0,1\}$. As each agent in the queue

chooses four actions, the total number of strategies makes it intractable to characterize admissible equilibrium configurations as a function of the parameters.[11] In order to understand the trade-off between waiting costs and seniority rank, we focus attention on one specific equilibrium configuration: one where the low waiting cost $\underline{c}$ is sufficiently low and the high waiting cost $\overline{c}$ sufficiently high so that *all agents with low waiting cost are selective, and all agents with high waiting costs accept both objects.*

In this equilibrium, we first compute the value of the first agent when his waiting cost is low.

$$\begin{aligned}
V_1(\underline{c},\underline{c}) - \underline{c} &= p(\underline{c},\underline{c})[\pi + (1-\pi)\pi[\lambda V_1(\underline{c},\underline{c}) + (1-\lambda)V_1(\underline{c},\overline{c})] \\
&+ (1 - p(\underline{c},\underline{c}))[\pi[\lambda V_1(\underline{c},\underline{c}) + (1-\lambda)V_1(\underline{c},\overline{c})] + (1-\pi)\pi + (1-\pi)^2 V_1(\underline{c},\underline{c})] - \underline{c}, \\
V_1(\underline{c},\overline{c}) - \underline{c} &= p(\underline{c},\overline{c})(\pi + (1-\pi)[\lambda V_1(\underline{c},\underline{c}) + (1-\lambda)V_1(\underline{c},\overline{c})]) \\
&+ (1 - p(\underline{c},\overline{c}))[\lambda V_1(\underline{c},\underline{c}) + (1-\lambda)V_1(\underline{c},\overline{c}) - \underline{c}.
\end{aligned}$$

resulting in

$$\begin{aligned}
V_1(\underline{c},\overline{c}) - \underline{c} &= \frac{p(\underline{c},\underline{c}) + (1 - p(\underline{c},\underline{c}))(1-\pi) + p(\underline{c},\underline{c})(1-\pi) + (1 - p(\underline{c},\underline{c}))A(p(\underline{c},\underline{c}), p(\underline{c},\overline{c}))}{2-\pi} \\
&- \frac{c}{\pi(2-\pi)}, \\
V_1(\underline{c},\overline{c}) - \overline{c} &= p(\underline{c},\overline{c})\pi + [p(\underline{c},\overline{c})(1-\pi) + 1 - p(\underline{c},\overline{c})]A(p(\underline{c},\underline{c}), p(\underline{c},\overline{c})) - \underline{c},
\end{aligned}$$

where

$$\begin{aligned}
A(p(\underline{c},\underline{c}), p(\underline{c},\overline{c})) &= \lambda V_1(\underline{c},\underline{c}) + (1-\lambda)V_1(\underline{c},\overline{c}) \\
&= 1 - \frac{(\lambda + (1-\lambda)\pi(2-\pi))\underline{c}}{\pi(\lambda[p(\underline{c},\underline{c}) + (1 - p(\underline{c},\underline{c}))(1-\pi))] + (1-\lambda)p(\underline{c},\overline{c})\pi^2(1-\pi))}.
\end{aligned}$$

Notice that the expected continuation value $A(p(\underline{c},\underline{c}), p(\underline{c},\overline{c}))$ is increasing in both probabilities $p(\underline{c},\underline{c})$ and $p(\underline{c},\overline{c})$. As $A < 1$, the values $V_1(\underline{c},\overline{c})$ and $V_1(\underline{c},\overline{c})$ are also increasing in the probabilities $p(\underline{c},\underline{c})$ and $p(\underline{c},\overline{c})$. We next compute the value of the first agent when his cost is high.

$$\begin{aligned}
V_1(\overline{c},\underline{c}) &= p(\overline{c},\underline{c})\pi + (1 - p(\overline{c},\underline{c}))[\pi(\lambda V_1(\overline{c},\underline{c}) + (1-\lambda)V_1(\overline{c},\overline{c})) + (1-\pi)\pi] - \overline{c}, \\
V_1(\overline{c},\overline{c}) &= p(\overline{c},\overline{c})\pi + (1 - p(\overline{c},\emptyset))(\lambda V_1(\overline{c},\underline{c}) + (1-\lambda)V_1(\overline{c},\overline{c})) - \overline{c},
\end{aligned}$$

[11] In principle, as each agent in the queue has $2^4 = 16$ choices, the total number of strategy vectors is $16 \times 16 = 256$. Characterizing equilibrium configurations with such a large strategy set becomes intractable.

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yielding

$$\begin{aligned} V_1(\overline{c},\underline{c}) &= p(\overline{c},\underline{c})\pi + (1-p(\overline{c},\underline{c})\pi(1-\pi) + \pi(1-p(\overline{c},\underline{c}))B(p(\overline{c},\underline{c}),p(\overline{c},\overline{c})) - \overline{c}, \\ V_1(\overline{c},\overline{c}) &= p(\overline{c},\overline{c})\pi + (1-p(\overline{c},\overline{c}))B(p(\overline{c},\underline{c}),p(\overline{c},\overline{c})) - \overline{c}, \end{aligned}$$

where

$$\begin{aligned} B(p(\overline{c},\underline{c}),p(\overline{c},\overline{c})) &= \lambda V_1(\overline{c},\underline{c}) + (1-\lambda)V_1(\overline{c},\overline{c}) \\ &= \pi\frac{\lambda(p(\overline{c},\underline{c}) + (1-p(\overline{c},\underline{c}))(1-\pi) + (1-\lambda)p(\overline{c},\overline{c}) - c}{\lambda(1-\pi(1-p(\overline{c},\underline{c})) + (1-\lambda)p(\overline{c},\overline{c}))}. \end{aligned}$$

Notice that the expected continuation value $B(p(\overline{c},\underline{c}),p(\emptyset,\overline{c}))$ is increasing in $p(\overline{c},\underline{c})$ and $p(\overline{c},\overline{c})$. As $B < \pi$, the values $V_1(\overline{c},\underline{c})$ and $V_1(\overline{c},\overline{c})$ are also increasing in the probabilities $p(\overline{c},\underline{c})$ and $p(\overline{c},\overline{c})$. Turning to the second agent we compute his value when his cost is low as

$$\begin{aligned} V_2(\underline{c},\underline{c}) &= p(\underline{c},\underline{c})[\pi A(p(\underline{c},\underline{c}),p(\underline{c},\overline{c})) + \pi(1-\pi) + (1-\pi)^2 V_2(\underline{c},\underline{c})] \\ &+ (1-p(\underline{c},\underline{c}))[\pi + (1-\pi)\pi A(p(\underline{c},\underline{c}),p(\underline{c},\overline{c})) + (1-\pi)^2 V_2(\underline{c},\underline{c})] - \underline{c}, \\ &= \frac{p(\underline{c},\underline{c})(1-\pi) + 1 - p(\underline{c},\underline{c}) + [p(\underline{c},\underline{c}) + (1-p(\underline{c},\underline{c}))(1-\pi)A(p(\underline{c},\underline{c}),p(\underline{c},\overline{c}))}{2-\pi} \\ &- \frac{\underline{c}}{\pi(2-\pi)}, \\ V_2(\overline{c},\underline{c}) &= p(\overline{c},\underline{c})A(p(\underline{c},\underline{c}),p(\underline{c},\overline{c})) + (1-p(\overline{c},\underline{c}))[\pi + (1-\pi)A(p(\underline{c},\underline{c}),p(\underline{c},\overline{c}))] - \underline{c} \end{aligned}$$

It is interesting to note that $V_2(\underline{c},\underline{c})$ is increasing in $p(\underline{c},\overline{c})$ but non monotonic in $p(\underline{c},\underline{c})$. The value $V(\overline{c},\underline{c})$ is increasing in $p(\underline{c},\underline{c})$ and $p(\underline{c},\overline{c})$ but decreasing in $p(\overline{c},\underline{c})$. For the second agent with high costs

$$\begin{aligned} V_2(\underline{c},\overline{c}) &= p(\underline{c},\overline{c})[\pi B(p(\overline{c},\underline{c}),p(\overline{c},\overline{c})) + (1-\pi)\pi] + (1-p(\underline{c},\overline{c}))\pi - \overline{c}, \\ V_2(\overline{c},\overline{c}) &= p(\overline{c},\overline{c})B(p(\overline{c},\underline{c}),p(\overline{c},\overline{c})) + (1-p(\overline{c},\overline{c}))\pi - \overline{c}. \end{aligned}$$

We observe that $V_2(\overline{c},\underline{c})$ is increasing in $p(\overline{c},\underline{c})$ and $p(\overline{c},\overline{c})$ but decreasing in $p(\underline{c},\overline{c})$ and $V_2(\overline{c},\overline{c})$ is increasing in $p(\overline{c},\underline{c})$ and non monotonic in $p(\overline{c},\overline{c})$.

Contrary to the case of homogenous waiting costs, the value of the second agent is not necessarily increasing in the probability that the first agent is proposed the object. In order to illustrate this fact, we consider the special

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case where $\lambda = \pi = \frac{1}{2}$ and compute the values of the probabilities which maximize the expected value of the second agent,

$$EV_2 = \frac{1}{4}V_2(\underline{c},\underline{c}) + V_2(\overline{c},\underline{c}) + V_2(\underline{c},\overline{c}) + V_2(\overline{c},\overline{c}),$$

In order to maximize EV_2 it is sufficient to maximize

$$\begin{aligned} E &= -\frac{7\underline{c}(5+2p(\underline{c},\underline{c})+p(\overline{c},\underline{c}))}{6(2+2p(\underline{c},\underline{c})+3p(\underline{c},\overline{c}))} - \frac{p(\underline{c},\overline{c})+2p(\overline{c},\overline{c})}{4} \\ &+ \frac{p(\underline{c},\overline{c})+2p(\overline{c},\overline{c})}{2}\frac{1+p(\overline{c},\underline{c})+2p(\overline{c},\overline{c})-8\overline{c}}{2+2p(\overline{c},\underline{c})+4p(\overline{c},\overline{c})}. \end{aligned}$$

We can check that $\frac{\partial E}{\partial p(\underline{c},\underline{c})} > 0$ and $\frac{\partial E}{\partial p(\overline{c},\overline{c})} < 0$. In addition, for sufficiently large values of $\overline{c}$ and sufficiently low values of $\underline{c}$, $\frac{\partial E}{\partial p(\underline{c},\overline{c})} < 0$ and $\frac{\partial E}{\partial p(\overline{c},\underline{c})} > 0$. Hence, in order to maximize the expected value of the second agent, the mechanism designer chooses $p(\underline{c},\underline{c}) = p(\overline{c},\underline{c}) = 1$ and $p(\underline{c},\overline{c}) = p(\overline{c},\overline{c}) = 0$. *The mechanism should always give the object to the second agent when he has a high waiting cost and to the first agent when he has a low waiting cost.*

5.2 Simultaneous allocation with prior application

In the baseline model, we suppose that agents are given the opportunity to accept the object in sequence. This sequential allocation rule is time consuming as some agents choose to reject the object which is proposed to them. As an alternative, we consider a simultaneous allocation rule, where agents, after they observe the value of the object, are asked to apply for the allocation. The mechanism then chooses which applicant is allocated the object using the priority mechanism $p \in \mathcal{P}$. This simultaneous allocation mechanism with prior application has been implemented in the lottery assignments in New York and the assignment of social housing in London and is currently considered to be used in Paris.

More precisely, we consider a mechanism where, after they learn their value, each agent announces $a(i) \in \{0,1\}$ where $a(i) = 1$ means that the agent applies to the object. A random order ρ is drawn by the mechanism designer and the first agent in the order ρ who chooses $a(i) = 1$ is allocated the object. Note that an agent with high value always applies to the allocation mechanism and accepts the object in the sequential mechanism. An agent with low value applies in the application mechanism if and only if the expected value of participating is higher than the value of not participating.

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The simultaneous allocation mechanism generates the same values as the sequential mechanism. There are three possible equilibria: (i) one where neither of the two agents participates when the value of the object is low, and agents obtain values $V^2(1)$ and $V^2(2)$, (ii) one where agent 1 only applies when the value of the object is low and agent 2 always applies, and the values are $V^1(1)$ and $V^1(2)$, and (iii) one where both agents always apply and the values are $V^0(1)$ and $V^0(2)$.

In the simultaneous model, the first agent with low value chooses to participate if and only if

$$(1-p)\pi(V(1)-c) \geq V(1)-c$$

if 2 is selective and

$$(1-p)(V(1)-c) \geq V(1)-c$$

if 2 is not selective. Both conditions amount to

$$V(1)-c \leq 0,$$

so that agent 1 participates when the value is low if and only if $V(1)-c \leq 0$, as in the sequential allocation model. The second agent with low value chooses to participate if and only if

$$p\pi(V(1)-c) \geq \pi[V(1)-c]+(1-\pi)[V(2)-c]$$

if 1 is selective and

$$p(V(1)-c) \geq \pi[V(1)-c]+(1-\pi)[V(2)-c]$$

if 1 is not selective. Hence, the equilibrium condition for agent 2 is not the same in the simultaneous and sequential models. The difference arises from differences in the computation of the continuation value in both cases. In the sequential case, agent 2 updates his belief about the order ρ taking into the account that he is offered the object. In the simultaneous case, agent 2, if he does not participate, computes an expected continuation value taking expectations over all possible orders ρ chosen by the mechanism. More precisely, an equilibrium where agent 2 is selective exists in the simultaneous allocation mechanism if and only if

$$\pi(1-p)[V(1)-c)]+(1-\pi)[V(2)-c] \geq 0.$$

Comparing with the condition guaranteeing equilibrium in the sequential model,

$$\pi(1-p)[V(1)-c)] + (1-\pi+p)[V(2)-c] \geq 0.$$

we observe that, as $V(2) < V(1)$, there is no parameter region where agent 2 is selective in the sequential model but not in the simultaneous model. Hence, *agents are more selective in the simultaneous model with prior application.* This observation suggests that the simultaneous allocation model with prior application dominates the sequential acceptance model in terms of the values of the agents inside the queue and the static probability of misallocation, but that the sequential model dominates the simultaneous model with respect to the probability of waste.

5.3 Eviction from the queue

We consider the effect of an eviction mechanism, where agents are taken away from the queue if they refuse an object with positive probability. Let $\beta(1)$ and $\beta(2)$ denote the probability that the first – respectively the second – agent remain in the queue if they refuse the object. In an equilibrium where both agents are selective, the equilibrium values are given by

$$\begin{aligned} V^2(1) - c &= \frac{p\pi + (1-p)(1-\pi)\pi - c}{p\pi + (1-p)(1-\pi)\pi + (1-\beta(1))[p(1-\pi) + (1-)(1-\pi)^2]}, \\ V^2(2) - c &= \frac{p\pi(1-\pi) + [p\pi + (1-pi)\pi\beta(2)][V^2(1) - c] - c}{1 - \beta(2)(1-\pi)^2}. \end{aligned}$$

Clearly, $V^1(1)$ is increasing in $\beta(1)$ and $V^2(2)$ is increasing in $\beta(1)$ and $\beta(2)$. Evicting the agents from the queue decreases their expected continuation values, making them less likely to be selective. In the equilibrium where only agent 1 is selective, the values become

$$\begin{aligned} V^1(1) - c &= \frac{p\pi - c}{p\pi + (1-\beta(1))p(1-\pi)}, \\ V^1(2) - c &= p\pi(V1(1) - c) + p\pi(1-\pi) + (1-p)\pi - c. \end{aligned}$$

The values $V^1(1)$ and $V^1(2)$ are both increasing in $\beta(1)$. Finally, the values in the equilibrium where both agents accept both objects are clearly unaffected by the eviction probabilities. We thus observe that introducing eviction probabilities *reduces the values of the agents in the queue* and makes them less lily to be selective. As agents are less selective, the misallocation probability increases and the expected waste decreases. Hence, introducing eviction

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probabilities can only reduce the welfare of agents currently in the queue. It also accelerates the turnover in the queue, improving the well being of agents who are waiting to be included in the queue.

6 Conclusion

This paper analyzes the optimal allocation of objects which arrive sequentially to agents organized in a waiting list. Applications include the assignment of social housing, deceased donor organs and daycare slots. A mechanism is a probability distribution over all priority orders which are consistent with the waiting list. We consider three efficiency criteria: first order stochastic dominance in the vector of agents' values, the probability of misallocation and the expected waste. We show that the strict seniority order dominates uniform random order according to the two first criteria, and the uniform random order dominates strict priority according to the third criterion.In addition, if values are perfectly correlated across agents, the strict priority mechanism dominates all other probabilistic mechanisms according to the first two criteria, and the uniform random order dominates all other mechanisms according to the third criterion.

Our analysis thus gives support to the use of waiting time as a primary criterion in the priority order in order to maximize the value of agents inside the queue, and of the use of lotteries in order to minimize waste. It also shows that punishment schemes like eviction from the queue harm agents currently inside the queue but accelerate turnover in the queue. There remain a number of aspects of dynamic allocation that we have not yet studied. Transplant agencies and social housing authorities maintain separate regional waiting lists. Would it be worthwhile to merge these waiting lists into a single national waiting list in order to improve matching? The current mechanisms elicit very little information about agent's characteristics and pay little attention to incentive issues. If agents have different privately known waiting costs, how can information about these costs be elicited without the use of transfers? We also assume that agents have perfect information about their order in the waiting list. Is it always useful to give agents information about their rank and the size of the waiting list? Finally we have abstracted away from agent's endogenous choice to participate int eh waiting list. How do agents choose to register on the waiting list? How is this decision affected by information about average waiting times? All these questions deserve further study.

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A Proofs

Proof of Lemma ??: This is a special case of Lemma **??**.

Proof of Lemma ??: To compute $\omega(i)$ and $\omega(j)$, consider a fixed realization of the order ρ and of the values θ and let $\omega(i, \rho, \theta)$ denote the probability that agent i is proposed the object. Fix one specific order ρ where $\rho(i) < \rho(j)$ and let ρ' be the order in which $\rho(k) = \rho'(k)$ for all $k \neq i, j$, $\rho(i) = \rho'(j)$ and $\rho(j) = \rho'(j)$. Clearly, the probability that the first of the two agents is proposed given ρ and θ is the same: $\omega(i, \rho, \theta) = 0 = \omega(j, \rho', \theta)$.
If $q(i) = q(j) = 1$, the probability that the second agent is proposed is equal to zero, so $\omega(i, \rho', \theta) = 0 = \omega(j, \rho, \theta)$.
If $q(i) = q(j) = 0$, if $\theta(i) = \theta(j) = 1$, the probability that the second agent is proposed is equal to zero ; if $\theta(i) = \theta(j) = 0$, the probability that the second agent is proposed is zero unless all agents are selective and pick low values, so that $\omega(i, \rho', \theta) = \omega(j, \rho, \theta) = 1$ if $\theta(k) = 1, q(k) = 0$ for all k such that $\rho(k) = \rho'(k) < \rho(j)$ and $\omega(i, \rho', \theta) = \omega(j, \rho, \theta) = 0$ otherwise . If $\theta(i) = 1, \theta(j) = 0$, $\omega(j, \rho, \theta) = 0$ and $\omega(i, \rho', \theta) = 1$ if $\theta(k) = 1, q(k) = 0$ for all k such that $\rho(k) = \rho'(k) < \rho(j)$ and $\omega(i, \rho', \theta) = 0$ otherwise. Similarly if $\theta(i) = 0, \theta(j) = 1$, $\omega(i, \rho', \theta) = 0$ and $\omega(j, \rho, \theta) = 1$ if $\theta(k) = 1, q(k) = 0$ for all k such that $\rho(k) = \rho'(k) < \rho(j)$ and $\omega(j, \rho, \theta) = 0$. Hence, if $q(k) = 1$ for some $k \neq i, j$ such that $\rho(k) < \rho(j)$, $\omega(i, \rho', \theta) = \omega(j, \rho, \theta) = 0$. If for all k, $\rho(k) < \rho(j)$, $q(k) = 0$, taking expectations over the realizations of θ,

$$\omega(i, \rho') = E_\theta \omega(i, \rho', \theta) \equiv \Pr[\theta(k) = 1 \forall k, \rho(k) < \rho(j)] = E_\theta \omega(j, \rho, \theta) \equiv \omega(j, \rho).$$

Finally, in order to compute $\omega(i)$, $\omega(j)$, we take expectations over all the orders. Given that the set of orders R can be decomposed into those orders for which $\rho(i) < \rho(j)$ and those for which $\rho(j) < \rho(i)$, we compute

$$\omega(i) = \sum_{\rho | \rho(i) < \rho(j)} p(\rho) \omega(i, \rho) + \sum_{\rho | \rho(j) < \rho(i)} p(\rho) \omega(i, \rho).$$

For any fixed order ρ such that $\rho(i) < \rho(j)$, consider the associated order ρ' such that $\rho'(j) = \rho(i), \rho'(i) = \rho(j)$.

$$\omega(i) = \sum_{\rho | \rho(i) < \rho(j)} (p(\rho) \omega(i, \rho) + p(\rho') \omega(i, \rho').$$

As noted above, $\omega(i, \rho) = \omega(j, \rho')$ and $\omega(i, \rho') = \omega(j, \rho)$. We also know that for a fixed order ρ, the expected probability of being proposed is higher

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for the first agent in the order. Hence, $\omega(i,\rho)-\omega(j,\rho) \geq \omega(j,\rho')-\omega(i,\rho') \geq 0$. Furthermore, by assumption **??**, $p(\rho) \geq p(\rho')$. We thus have:

$$p(\rho)\omega(i,\rho) + p(\rho')\omega(i,\rho') \geq p(\rho)\omega(j,\rho) + p(\rho')\omega(j,\rho'). \tag{1}$$

concluding the proof of the Lemma.

Proof of Lemma ??; The proof is by induction on the rank of agents. Consider first agents 1 and 2. Let $q(1)$ and $q(2)$ be the equilibrium strategies of the two agents and suppose first that $q(1) = q(2)$. We compute

$$\begin{aligned} V(1) - c &= [1 - (1-\pi)q(1)) - \frac{c}{\gamma(1)}], \\ V(2) - c &= \frac{\gamma(1)}{\gamma(1)+\gamma(2)}(V(1)-c) + \frac{\gamma(2)}{\gamma(1)+\gamma(2)}[1-(1-\pi)q(2)) - \frac{c}{\gamma(2)}]. \end{aligned}$$

By Lemma **??**, $\gamma(2) \leq \gamma(1)$ so that $[1-(1-\pi)q(1)) - \frac{c}{\gamma(2)}] \leq [1-(1-\pi)q(2)) - \frac{c}{\gamma(1)}]$, establishing that $V(1) - c \geq V(2) - c$. Next suppose that $q(1) \neq q(2)$ and let $\tilde{V}(1)$ denote the value of agent 1 if he plays strategy $q(2)$. By the preceding computation, $\tilde{V}(1) \geq V(2)$. But as agent 1 optimally chooses $q(1) \neq q(2)$, $V(1) \geq \tilde{V}(1) \geq V(2)$.

Suppose now that for all $j < i$, $V(j+1) \leq V(j)$. Assume first that $q(i) = q(i+1)$ and compute

$$\begin{aligned} V(i) - c &= \frac{\sum_{k=1}^{i-1}\gamma(k)}{\sum_{k=1}^{i}\gamma(k)}(V(i-1)-c) + \frac{\gamma(i)}{\sum_{k=1}^{i}\gamma(k)}[1-(1-\pi)q(i)) - \frac{c}{\gamma(i)}], \\ V(i+1) - c\frac{\sum_{k=1}^{i}\gamma(k)}{\sum_{k=1}^{i+1}\gamma(k)}(V(i)-c) &+ \frac{\gamma(i+1)}{\sum_{k=1}^{i+1}\gamma(k)}[1-(1-\pi)q(i+1)) - \frac{c}{\gamma(i+1)}]. \end{aligned}$$

By the induction hypothesis, $V(i) - c \leq V(i-1) - c$ and by Lemma **??**, $\gamma(i+1) \leq \gamma(i)$ so that $[1-(1-\pi)q(i)) - \frac{c}{\gamma(i+1)}] \leq [1-(1-\pi)q(i)) - \frac{c}{\gamma(i)}]$. Hence $V(i+1) \leq V(i)$. If $q(i+1) \neq q(i)$, let $\tilde{V}(i)$ denote the value fo agent i when he chooses $q = q(i+1)$. By the same revealed preference argument as above, $V(i) \geq \tilde{V}(i) \geq V(i+1)$, completing the proof of the Lemma.

Proof of Lemma ??: Let $\mu(i)$ denote the probability that an agent $k < i$ accepts the object after is refusal so that the continuation value of agent i after a rejection is

$$W(i) \equiv \mu(i)V(i-1) + (1-\mu(i))V(i) - c.$$

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Suppose by contradiction that there exists a (pure strategy) equilibrium where $q(i) = 1$ and $q(i+1) = 0$. This implies that $W(i) \leq 0 \leq W(i+1)$. By Lemma **??**, $V(i+1) \leq V(i) \leq V(i-1)$. Furthermore, if p is not the uniform random order, an inspection of the proof of Lemma **??** shows that $\gamma(i) > \gamma(i+1)$ if $q(i) = q(i+1)$ so that the proof of Lemma **??** can be adapted to show that these inequalities are strict. Hence, if $W(i) \leq 0$ necessarily $V(i) < 0$ and if $W(i+1) > 0$, we must have $V(i) > 0$, yielding a contradiction.

Proof of Proposition ??: Notice first that, along any realization of ρ where i precedes $k+1$, the probability that i is proposed the object remains the same under the k and the $k+1$ equilibrium, but along any realization of ρ where $k+1$ precedes i, the probability that i is proposed the object weakly increases. Hence, denoting by $\omega^k(i)$ the expected probability that i is proposed the object in the k equilibrium, $\omega^k(i) \leq \omega^{k+1}(i)$ for all $i \neq k+1$ and $\omega^k(k+1) = \omega^{k+1}(k+1)$. Now consider first $i \leq k$, as

$$V^{k+1}(i) - c1 - \frac{c}{\pi \sum_{t=1}^{i} \omega^{k+1}(t)},$$

and $\omega^{k+1}(t) \geq \omega^k(t)$ for all t, $V^{k+1}(i) - c \geq V^k(i) - c$. Next, consider $i \geq k+1$. We will prove the stronger statement:

$$[1 - \sum_{t=i+1}^{n} \omega^{k+1}(t)](V^{k+1}(i) - c) \geq [1 - \sum_{t=i+1}^{n} \omega^k(t)](V^k(i) - c).$$

Notice that for $i = k+1$ this statement is equivalent to:

$$\pi \sum_{t=1}^{k} \omega^{k+1}(t)(V^{k+1}(k) - c) + \pi\omega^{k+1}(k+1) - c \geq \pi \sum_{t=1}^{k} \omega^k(t)(V^k(k) - c) + \pi\omega^k(k+1) - c,$$

Using the fact that $\omega^{k+1}(t) \geq \omega^k(t)$ for all $t \neq k+1$, $\omega^{k+1}(k+1) = \omega^k(k+1)$ and $V^{k+1}(k) - c \geq V^k(k) - c$, , the inequality follows. Suppose now that the statement is true for all $t < i$ and consider i, we compute

$$[1 - \sum_{t=i+1}^{n} \omega^{k+1}(t)](V^{k+1}(i) - c) = \sum_{t=1}^{i-1} \gamma^{k+1}(t)(V^{k+1}(i-1) - c) + \pi\omega^{k+1}(i) - c.$$

Now, as $i - 1 > k$, $\sum_{t=1}^{i-1} \gamma^{k+1}(t) = 1 - \sum_{t=i}^{n} \omega^{k+1}(t)$, and using the induction hypothesis and the fact that $\omega^{k+1}(i) \geq \omega^k(i)$,

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$$
\begin{aligned}
[1-\sum_{t=i+1}^{n}\omega^{k+1}(t)](V^{k+1}(i)-c) &= [1-\sum_{t=i}^{n}\omega^{k+1}(t)](V^{k+1}(i-1)-c)+\pi\omega^{k+1}(i)-c \\
&\geq [1-\sum_{t=i}^{n}\omega^{k}(t)](V^{k}(i-1)-c)\pi\omega^{k}(i)-c \\
&= [1-\sum_{t=i+1}^{n}\omega^{k}(t)](V^{k}(i)-c),
\end{aligned}
$$

completing the proof of the Proposition .

Proof of Proposition ?? Consider first a selective agent $i \leq k$. Notice that, as all $t < i$ are selective in the k equilibrium, $\sum_{t=1}^{i}\omega(t) \leq (1+(1-\pi)+...+(1-\pi)^{k-1})$. The probability that the object is proposed to an agent $t = 1,..,i$ is maximized when p guarantees that the first agents who are proposed the object are agents $1,..,i$. In that case, irrespective of the order in which agents $1,..i$ are placed, the probabilities of being proposed are 1 for the first agent, $1-\pi$ for the second,...,$(1-\pi)^{-1}$ for the ith agent in the order ρ. Next notice that the strict seniority order is the only order which guarantees that for any $i \leq k$, $\sum_{t=1}^{i}\omega(t) \leq (1+(1-\pi)+...+(1-\pi)^{k-1})$. Now

$$
V^{k}(i) = 1-\frac{ic}{\pi\sum_{t=1}^{i}\omega(t)} \leq 1-\frac{ic}{\pi(1+(1-\pi)+...+(1-\pi)^{i})} = \hat{V}^{k}(i).
$$

Next consider an agent $i > k$. Since the $k+1$th player in the order ρ will necessarily be nonselective and pick up the object, the sum of probabilities that an agent $t = 1,...i$ is proposed the object is bounded by $(1+(1-\pi)+...+(1-\pi)^{k})$ for any $i > k$. Notice that the maximum is attained for any order ρ which places agents $1,2...,k$ at the top. Furthermore, the probability that an agent $t = 1,...i$ picks up the object is always bounded by 1. In the strict seniority order, this probability is exactly equal to 1, as agent $k+1$ will always pick up the object when none of the selective agents has accepted it. Next, notice that

$$
V^{k}(i) = \frac{\pi\sum_{t=1}^{i}\omega(t)}{\sum_{t=1}^{i}\gamma(t)} - \frac{ic}{\sum_{t=1}^{i}\gamma(t)}.
$$

Given that in equilibrium $\pi\sum_{t=1}^{i}\omega(t) \leq ic$,

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$$V^k(i) \leq \pi \sum_{t=1}^{i} \omega(t) - ic \leq \pi(1 + (1-\pi) + ... + (1-\pi)^k) - ic = \hat{V}^k(i).$$

Next suppose that $k = 0$, so that no agent is selective. Under the uniform random order $\omega(i) = \frac{1}{n}$ for all i (all agents have an equal probability to be chosen first) whereas, by Lemma **??** $\omega(i) \leq \omega(j)$ for $i < j$ in all mechanism $p \in \mathcal{P}$. Furthermore, as the first player picks the object, $\sum_{t=1}^{n} \omega(t) = 1$ and hence $\sum_{t=1}^{i} \omega(t) \geq \frac{i}{n} = \sum_{t=1}^{i} \tilde{\omega}(i)$. Now in a 0 equilibrium,

$$V^0(i) = \pi - \frac{ic}{\sum_{t=1}^{i} \omega(t)} \geq \pi - nc = \tilde{V}^0(i).$$

If $k = 1$, again under the uniform random order $\tilde{\omega}(i) = \tilde{\omega}(j) = \frac{1}{n} \sum_{t=0}^{n-1} (1-\pi)^t = \frac{1-(1-\pi)^n}{n\pi}$. For any other mechanism $p \in \mathcal{P}$, by Lemma **??**, $\omega(i) \geq \omega(j)$ if $i < j$. Furthermore, $\sum_{i=1}^{n} \omega(i) = \frac{1-(1-\pi)^n}{\pi}$. Hence,

$$V^n(i) = 1 - \frac{ic}{\pi \sum_{t=1}^{i} \omega(t)} \geq 1 - \frac{nc}{1-(1-\pi)^n} = \tilde{V}^n(i),$$

completing the proof of the Proposition.

Proof of Proposition ??: Notice that the equilibrium under $\hat{p}$ is always more selective than the equilibrium under $\tilde{p}$. If both mechanisms admit a n equilibrium, the expected values of all agents are higher under $\hat{p}$ by Proposition **??**. If the equilibrium under $\hat{p}$ is a k equilibrium and a 0 equilibrium under $\tilde{p}$, the expected values of all agents are higher under $\hat{p}$ by Propositions **??** and **??**. The results on misallocation and expected waste derive from the fact that the equilibrium under $\hat{p}$ is always more selective.

Proof of Theorem ??: Pick $c \in A^k$. We first show that there cannot be another mechanism p which admits an equilibrium with $k' > k$. Suppose by contradiction that this were the case. Then

$$V^{k'}k' - c \geq 0 \geq \hat{V}^k(k+1) - c.$$

By Proposition **??**, $\hat{V}^{k'}(k') \geq V^{k'}k'$, so that $\hat{V}^{k'}(k') \geq c$, implying that there is another equilibrium with strict seniority ordering and at least k' selective agents, a contradiction. This implies that for any other mechanism p, the number of selective agents is bounded by k. Hence, by Proposition **??** $\hat{V}^k(i) \geq V^k(i)$ for all other mechanism p, and by Proposition **??**, $V^k(i) \geq$

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$V^{k'}(i)$ for any $k' \leq k$, establishing that the equilibrium values of all agents are bounded above by $\hat{V}^k(i)$.

Misallocation arises when the object is given to a nonselective agent with value 0 rather than to a selective agent or a nonselective agent with value 1. By increasing the number of selective agents, the misallocation loss is reduced, as there exist realizations of the order ρ and of the values θ for which misallocation arises with fewer selective agents but not with more, but conversely, the switch of one agent from nonselective to selective cannot increase the number of realizations of ρ and θ for which misallocation occurs. Hence, misallocation is lower in an equilibrium with more selective agents, $k' > k$. Furthermore, n a k equilibrium, misallocation is lower when the order ρ places all k selective agents first. This shows that for $c \in A^k$, the misallocation loss is minimal when $p = \hat{p}$.

The expected waste is equal to zero except in an n equilibrium, where it is equal to $(1-\pi)^n$. For $c \geq \frac{\pi}{n}$, the least selective equilibrium of the random order mechanism produces no expected waste. We now show that for $c \leq \frac{\pi}{n}$, all mechanisms p induce an n equilibrium. Suppose by contradiction that a mechanism p admits a k equilibrium, so that $V^k(k+1) - c \leq 0$. By Proposition **??**, $V^k(k+1) - c \geq V^0(k+1) - c$, and by Proposition **??** $V^0(k+1) - c \geq \tilde{V}^0(k+1)$, so that

$$0 \geq V^k(k+1) - c \geq V^0(k+1) - c \geq \tilde{V}^0(k+1) - c > 0,$$

yielding a contradiction.