Ranking Distributions of an Ordinal Attribute

Nicolas Gravel, Brice Magdalou, Patrick Moyes

aix marseille school of economics

Working Papers / Documents de travail

Ranking Distributions of an Ordinal Attribute

Nicolas Gravel
Brice Magdalou
Patrick Moyes

WP 2014 - Nr 50

Ranking distributions of an ordinal attribute*

Nicolas Gravel[†], Brice Magdalou[‡] and Patrick Moyes[§]

Keywords: ordinal, qualitative, health, inequality, Hammond transfers, increments, dominance

JEL codes: C81, D63, I1, I3

November 14th 2014

Abstract

This paper establishes an equivalence between three incomplete rankings of distributions of an ordinally measurable attribute. The first ranking is that associated with the possibility of going from distribution to the other by a finite sequence of two elementary operations: increments of the attribute and the so-called Hammond transfer. The later transfer is like the Pigou-Dalton transfer, but without the requirement - that would be senseless in an ordinal setting - that the "amount" transferred from the "rich" to the "poor" is fixed. The second ranking is an easy-to-use statistical criterion associated to a specifically weighted recursion on the cumulative density of the distribution function. The third ranking is that resulting from the comparison of numerical values assigned to distributions by a large class of additively separable social evaluation functions. Illustrations of the criteria are also provided.

1 Introduction

When can we say that a distribution of *income* among a collection of individuals is more equal than another ? One of the greatest achievement of the modern theory of inequality measurement is the demonstration (made for the first time by Hardy, Littlewood, and Polya (1952) and popularized among economists by Kolm (1969) Atkinson (1970) Dasgupta, Sen, and Starrett (1973), Sen (1973) and Fields and Fei (1978)) that the following three answers to that question are equivalent:

1) When one distribution has been obtained from the other by a finite sequence of Pigou-Dalton transfers.

*We are indebted, with the usual disclaiming qualification, to Ramses Abdul Naga for his detailed comments on an earlier version of this paper.

†Aix-Marseille Univ. (Aix-Marseille School of Economics), CNRS & EHESS, Centre de la Vieille Charité, 2, rue de la Charité, 13002, Marseille France.

‡Université de Montpellier 1 & LAMETA, rue Raymond Dugrand, 34960 Montpellier, France.

§CNRS, GRETHA, Université Montesquieu (Bordeaux 4), Avenue Léon Duguit, 33 608 Pessac, France.

2) When one distribution would be considered better than the other by all utilitarian philosophers who assume that individuals convert income into utility by the same increasing and concave function.

3) When the Lorenz curve associated to one distribution lies nowhere below, and at least somewhere above, that of the other.

The equivalence of these three answers is an important result because it ties together three *a priori* distinct aspects of the process of inequality measurement.

The first one - contained in the famous Pigou-Dalton principle of transfers - is an "elementary transformation" of the distribution of income that captures in a crisp fashion the nature of the equalization process that is at stake. It provides a first immediately palatable answer to the question raised above. A distribution is more equal than another when it has been obtained from it by a finite sequence of such "clearly equalizing" Pigou-Dalton transfers.

The second aspect of the process of inequality measurement identified by the Hardy-Littlewood-Polya (HLP) theorem is the ethical principle underlying utilitarianism or, more generally, *additively separable* social evaluation. Attitude toward income inequality is clearly an ethical matter. It is therefore of importance to identify the ethical principles that rationalize the notion of inequality reduction underlying the Pigou-Dalton principle of transfers. While the HLP theorem points toward utilitarianism or additively separable social evaluation as a source of such rationalization, it can be shown (see e.g. Gravel and Moyes (2013)) that such a rationalization can also be obtained through a much more general class of social evaluation functions.

The third aspect of the process of inequality measurement captured by the HLP theorem is the empirically implementable criterion underlying Lorenz dominance. It is, indeed, immensely useful to have an implementable criterion like the Lorenz curve that enables one to check in an easy manner when one distribution dominates another. Comparisons of Lorenz curves have become a routine exercise that is performed every day by thousands of researchers all over the world. Moreover, compatibility with the Lorenz ranking of income distributions is now considered to be a minimal requirement that any numerical index of income inequality must satisfy. In this sense, the HLP theorem is, in the literal sense of the word, a *foundation* to income inequality measurement.

The current paper is concerned with the issue of establishing analogous foundations to the problem of comparing distributions of an *ordinal* or *qualitative* attribute among a collection of individuals. The last fifteen years have witnessed indeed an extensive use of data involving distributions of attributes such as access to basic services, educational achievements, health outcomes, and self-declared happiness to mention just some of the most popular of those. When performing normative comparisons of distributions of such attributes, it is not uncommon for researchers to disregard the ordinal measurability of the attribute and to treat it, just like income, as a variable that can be "summed", or "transferred" across individuals. Examples of those practices include Castelló-Clement and Doménech (2002) and Castelló-Clement and Doménech (2008) (who discusses inequality indices on human capital) and Pradhan, Sahn, and Younger (2003) (who decompose Theil indices applied to the heights of under 36 month children interpreted as a measure of health). Yet, following the influential contribution by Allison and Foster (2004), there has been a growing concern by researchers of duly accounting for the ordinal character of the numerical information conveyed by the indicators used in those studies. Examples of studies that have taken such a care, and have explicitly refused to use cardinal properties of the attribute when normatively evaluating its distribution, include Abul-Naga and Yalcin (2008) and Apouey (2007).

A difficulty with the normative evaluation of distribution of an ordinal attribute is that of defining an appropriate notion of inequality reduction in that context. What does it mean indeed for an ordinal attribute to be "more equally distributed" than another ? The usual notion of a Pigou-Dalton transfer used to answer this question in the case of a cardinally measurable attribute is of no clear use for that purpose. Recall indeed that a Pigou-Dalton transfer is the operation by which an individual transfers to someone with a lower quantity of the attribute a *certain quantity* of the attribute. Such a transfer is obviously meaningless if the "quantity" of the attribute is ordinal. While ordinal measurability of the attribute - provided that it is comparable across different individuals - enables one to identify which of two individuals has "more" of the attribute than the other, it does not enable one to quantify further the statement. It does not enable one to talk about a "certain quantity" of the attribute that can be transferred across individuals.

Some forty years ago, Peter J. Hammond (1976) has proposed, in the specific context of social choice theory, a so-called "minimal equity principle" that was explicitly concerned with distributions involving an ordinally measurable attribute. According to Hammond's principle, a change in the distribution that "reduces the gap" between two individuals endowed with different quantities of the ordinal attribute is a good thing, irrespective of whether or not the "gain" from the poor recipient is equal to the "loss" from the rich giver. A Pigou-Dalton transfer is clearly a particular case of a Hammond transfer (that imposes on the later the additional requirement that the amount given by the rich should be equal to the amount received by the poor). It seems to us that the purely ordinal nature of Hammond transfers qualifies them as a highly plausible instances of "ordinal inequality reduction". A distribution of an ordinal attribute is unquestionably more equal than another when it has been obtained from the later by a finite sequence of such Hammond transfers. This paper identifies a normative dominance criterion and a statistically implementable criterion that are each equivalent to the notion of equalization underlying Hammond transfers. It does so in the somewhat specific, but empirically important, case where the ordinal attribute can take only a finite number different values. This finite case is somewhat specific indeed for discussing Hammond equity principle. For, as is well-known in social choice theory (see e.g. D'Aspremont and Gevers (1977), D'Aspremont (1985), Sen (1977)), the Hammond equity principle is closely related to "Max-Min" or "Lexi-Min" types of criteria that rank distributions of an ordinal attribute on the basis of the smallest quantity of the attribute. Bosmans and Ooghe (2013) have even shown that any anonymous, continuous, and Pareto-inclusive transitive ranking of all vectors of $\mathbb{R}^n$ that are sensitive to Hammond transfer must be the Maxi-Min criterion that compares such vectors on the sole basis of the size of their smallest component. As shall be seen in this paper, this apparently tight connection between Maxi-Min or Lexi-Min principles and Hammond transfers becomes significantly looser when attention is restricted to distributions of an attribute that can take finitely many different values.

As for the normative principle, we stick to the tradition of comparing distributions of the attribute by means of an additively separable social evaluation function. Each attribute quantity is thus assigned a numerical value by some function, and alternative distributions of attributes are compared on the basis of the sum - taken over all individuals - of these values. While this normative approach can be considered utilitarian (if the function that assigns a value to the attribute is interpreted to be a "utility" function), it does not need to. One could also interpret the function more generally as an *advantage function* reflecting the value assigned to the attribute quantity by the social planner. It is also possible to justify axiomatically this additively separable normative evaluation in a non-welfarist setting (see e.g. Gravel, Marchant, and Sen (2011)). As the ordinal attribute (health indicator, education, access to basic services) is considered to be a good thing to the individual, it is assumed that the advantage function is increasing with respect to the attribute. We show in this paper that, in order for a ranking of distributions based on an additively separable social evaluation function to be sensitive to Hammond transfers, it is necessary and sufficient for the advantage function to satisfy a somewhat strong "decreasing increasingness" property. Specifically, any increase in the quantity of the attribute obtained from some initial level must increase the advantage more than would do any increase in the attribute taking place at some higher level of the attribute. Because of this result, we therefore consider the ranking of distributions of the ordinal attribute that coincides with the unanimity of all additively separable rankings who use an advantage function that satisfies this property.

The statistical criterion that we consider is, to the very best of our knowledge, a new one. Its construction is based on a curve that we call, provisionally, the H-curve, by reference to the Hammond principle of transfer to which it is, as it turns out, closely related. To draw such a curve, one first assigns to the lowest possible quantity of the ordinal attribute the fraction of the population that are endowed with this quantity. For this lowest category, this number is nothing else than the relative frequency of the category or, equivalently for the lowest category, its cumulative frequency. One then proceeds, for any quantity of the attribute that is strictly larger than this smallest quantity, by adding together the relative frequency of the population endowed with this quantity of the attribute *and* twice the fraction of the population endowed with a strictly lower quantity of the attribute. The innovation of this curve lies precisely in its addition, to any fraction of the population falling in some category, of *twice* the fraction of the population falling in a strictly lower category. This (doubly) larger weight given to the fraction of the population lying in a strictly worse category as compared to that of the population belonging to a given category reflects, of course, the somewhat strong redistributive flavour of the Hammond's equity principle. The criterion that we propose, and that we call H-dominance, is for the dominating distribution to have a $H-$curve nowhere above and somewhere below that of the dominated one. As we illustrate in the paper, the construction of these curves, and the resulting implementation of the criterion, is easy.

This paper provides some foundations for the use of such a H-curve. It does so by proving that the fact of having a distribution of an ordinal attribute that H-dominates another is equivalent to the possibility of going from the latter to the former by a finite sequence of Hammond transfers and/or increments of the attribute. The paper also shows that the H-dominance criterion coincides with the unanimity of all additively separable aggregation of advantage functions that use an advantage function that is strongly decreasingly increasing in the sense given above.

The plan of the rest of the paper is as follows. The next section introduces the notation and discusses the general problem of comparing distributions of an ordinal attribute. The third section introduces the normative dominance notions, the elementary transformations (Hammond transfers and increments) and the H-curve in the specific setting where the ordinal attribute can take finitely many different values. The main results are stated and proved in the fourth section and the firth section concludes.

2 Three perspectives on comparing distributions of an ordinal attribute

2.1 Normative evaluation

We consider societies made of a given number - n say - of individuals, indexed by i. Societies with varying number of individuals can be compared as usual by the Dalton principle of population replication. Every individual can fall into one out of k different categories, indexed by h. We denote by $\mathcal{C} = \{1, ..., k\}$ the set of these categories. These categories are assumed to be ordered from the worst (e.g. being gravely ill) to the best (e.g. being in perfect health). However the ordering of these categories is not assumed to be numerically represented by a cardinally meaningful function. If the integers $1, ..., k$ provide a numerical representation of the ordering of the categories, so do the numbers $f(1), ..., f(k)$ where f is any strictly increasing real valued function admitting C as its subdomain. A social situation or, more compactly, a society $s = (s_1, ..., s_n) \in \mathcal{C}^n$ is a particular assignment of these categories to the n individuals, where s_i is the category in which individual i falls in society s. For any society s, and every category j, one can define the number n_j^s of individuals who, in society s, falls in category j by:

$$n_j^s = \#\{i \in \{1, ..., n : s_i = j\}$$

We of course notice that $\sum_{i=1}^{k} n_i^s = n$ for every society s. If one adopts an anonymous perspective according to which "the names of the individuals do not matter", then the integers n_j^s (for $j = 1, ..., k$) summarize all the ethically relevant information about society s. The current paper adopts this anonymous perspective and examines more specifically the normative rankings $\succsim$ of societies in $\mathcal{C}^n$ that can be defined, for any two societies s and s' in $\mathcal{C}^n$, by:

$$s \succsim s' \Longleftrightarrow \sum_{j=1}^{k} n_j^s \alpha_j \geq \sum_{j=1}^{k} n_j^{s'} \alpha_j \tag{1}$$

for some list of k real numbers α_j (for $j = 1, ..., k$) satisfying $\alpha_1 < ... < \alpha_k$. These numbers can be seen as numerical evaluations of the corresponding categories. These valuations may reflect subjective utility (if a Utilitarian perspective is adopted) or a non-welfarist appraisal made by the social planner of the fact, for someone, to falling in the different categories to which these numbers are assigned. If such a non-welfarist perspective is adopted, the specific additive form of the numerical representation 1 of the social ordering can be axiomatically justified (see e.g. Gravel, Marchant, and Sen (2011) for such a justification in a variable population context).

The ordinal interpretation of the categories suggests that some care be taken in avoiding the normative evaluation exercise to be unduly sensitive to particular choices of the numbers α_j (for $j = 1, ..., k$). A standard way to exert such a care is to require the unanimity of evaluation over a wide class of such numbers. This underlies the following general definition of normative dominance.

Definition 1 *For any two societies s and s' in $\mathcal{C}^n$, we say that s normatively dominates s' for a family $\mathcal{A} \subset \mathbb{R}^k$ of evaluations of the k categories, denoted $s \succsim^{\mathcal{A}} s'$, if one has:*

$$\sum_{j=1}^{k} n_j^s \alpha_j \geq \sum_{j=1}^{k} n_j^{s'} \alpha_j$$

for all $(\alpha_1, ..., \alpha_k) \in \mathcal{A}$.

Given the assumed strict ordering of the categories, the largest set $\mathcal{A}$ over which a normative dominance could be looked for is the set $\mathcal{A}_1$ defined by:

$$\mathcal{A}_1 = \{(\alpha_1, ..., \alpha_k) \in \mathbb{R}^k : \alpha_1 < ... < \alpha_k\}$$

for any ranking ordinal ranking $\succsim$ represented by (1).

2.2 Elementary transformations

The definition of these transformations lies at the very heart of the problem of comparing alternative distributions of an ordinally measured attribute. One must indeed verify that the transformation only uses ordinal property of the attribute. In this paper, we consider two such transformations.

The first of these elementary transformations - the increment - is hardly new. It captures the intuitive idea that giving to someone - up to a permutation thanks to anonymity - additional quantities of the ordinal attribute without reducing - up to a permutation - the quantity of attribute of the others is a good thing. We actually formulate this principle in the following minimalist fashion.

Definition 2 (Increment) *We will say that society s has been obtained from society s' by means of an increment, if there exist $j \in \{1, \ldots, k-1\}$ such that:*

$$n_h^s = n_h^{s'}, \ \forall \ h \neq j, j+1\,; \tag{2}$$

$$n_j^s = n_j^{s'} - 1\ ;\ n_{j+1}^s = n_{j+1}^{s'} + 1\,. \tag{3}$$

In words, society s has been obtained from society s' by an increment if the move from s' to s is the sole result of the move of one individual from a category j to the immediately superior category $j + 1$.

The second elementary transformation considered in this paper is that underlying the principle of equity put forth by Peter J. Hammond (1976) some forty years ago. It reflects an appealing - if not strong - notion of *aversion to inequality* in contexts where the distributed attribute - taken to be utility in Hammond (1976) - is ordinal. This principle considers that a reduction in someone's endowment of the attribute that is compensated by an increase in the endowment of the other is a good thing if the looser remains, after his or her loss, better off than the winner. While a reduction in the endowment of someone compensated by an increase in that of someone else may be viewed as the result of a "transfer" of attribute between the two persons, it is worth noticing that, contrary to what is the case with standard Pigou-Dalton transfers, the "quantity" given by the donor needs not be equal to that received by the recipient. A Hammond transfer may involve taking a lot of attribute away from a "rich" person in exchange of giving just a little bit to a poorer one. It may, conversely, entail the transformation of a small "quantity" taken from the rich into a large quantity given to the poor. As the comparisons of gains and loss of an ordinal attribute is meaningless, the Hammond transfer may be viewed as the natural analogue, in the ordinal setting, of the Pigou-Dalton transfer used in the cardinal one. The precise definition of a Hammond transfer in our setting is as follows.

Definition 3 (Hammond's transfer) *We will say that society s is obtained from society s' by means of a Hammond's (progressive) transfer, if there exist four categories $1 \leq g < h \leq i < j \leq k$ such that:*

$$n_l^s = n_l^{s'}\,,\ \forall\, l \neq g, h, i, j\,; \tag{4a}$$

$$n_g^s = n_g^{s'} - 1\ ;\ n_h^s = n_h^{s'} + 1\,; \tag{4b}$$

$$n_i^s = n_i^{s'} + 1;\ n_j^s = n_j^{s'} - 1\,. \tag{4c}$$

2.3 Implementation criteria

Two implementation criteria are considered in this paper. The first one - first order stochastic dominance - is standard in the literature. Its formal definition in the current setting makes use of the cumulative distribution function associated to a society s, that is denoted, for every $i \in \mathcal{C}$, by $F(i;s)$ and that is defined by:

$$F(i;s) = \sum_{h=1}^{i} n_h^s/n\,. \tag{5}$$

Using this definition, one can define first order dominance as follows.

Definition 4 *We will say that society s first order dominates society s', which we write $s \succsim_1 s'$, if and only if:*

$$F(i;s) \leq F(i;s')\,, \quad \forall i = 1, 2, \ldots, k\,. \tag{6}$$

(remembering of course that $F(k;s) = \sum_{h=1}^{k} n_h^s/n = 1$ for any society s).

The second implementation criterion examined in this paper makes use of the following curve (defined for any society s and any $i \in \{1, ..., k\}$)

$$F_H(i;s) = \sum_{h=1}^{i} \left(2^{i-h}\right) n_h^s\,/n\ \ . \tag{7}$$

A few remarks can be made about this curve.

First, it verifies:

$$F_H(1; s) = F(1; s) \tag{8}$$

and:

$$F_H(i; s) = \sum_{h=1}^{i-1} \left(2^{i-h-1}\right) F(h; s) + F(i; s)\,,\ \forall i = 2, 3, \ldots, k. \tag{9}$$

The different values of $F_H(\cdot; s)$ are therefore nested. Moreover, for any $i = 2, 3, \ldots, k$ we have:

$$F_H(i; s) = 2\,F_H(i-1; s) + F(i; s) - F(i-1; s) = 2\,F_H(i-1; s) + n_i^s\,/n. \tag{10}$$

Hence, by successive decomposition, one obtains, for all $i = 2, 3, \ldots, k$:

$$F_H(i; s) = \left(2^j\right) F_H(i-j; s) + \sum_{h=0}^{j-1} \left(2^h\right) \frac{n_{i-h}^s}{n}\,,\ \forall j = 1, 2, \ldots, i-1\,. \tag{11}$$

This curve give rise to the following notion of dominance - called H dominance.

Definition 5 (H dominance) *We will say that society s H-dominates society s', which we write $s \succeq_H s'$, if and only if:*

$$F_H(i; s) \leq F_H(i; s')\,, \quad \forall i = 1, 2, \ldots, k\,. \tag{12}$$

As the curve associated to F_H is, to the best of our knowledge, new, it may be worthwhile to illustrate its construction by a simple example. For this sake, we consider the data used by Abul-Naga and Yalcin (2008) that describe the distribution of self-reported health status ("very bad" (1), "bad" (2),"so-so" (3), "good" (4) and "very good" (5)) in seven regions of Switzerland. The fractions n_i^s/n (for $i = 1, ..., 5$) of the population belonging to each of the five health categories in each of the seven regions are as follows:

	n_1^s/n	n_2^s/n	n_3^s/n	n_4^s/n	n_5^s/n
$s =$ Leman	0.01	0.04	0.11	0.56	0.28
$s =$ North-West	0.01	0.04	0.13	0.63	0.19
$s =$ Central	0	0.02	0.11	0.63	0.24
$s =$ Middle-Land	0.01	0.03	0.13	0.60	0.23
$s =$ East	0	0.03	0.11	0.64	0.22
$s =$ Ticino	0.01	0.05	0.11	0.70	0.13
$s =$ Zurich	0	0.03	0.10	0.65	0.22

Table 1

From this table, one can use definition (5) as well as equations (8) and (10) to obtain the values of $F(i; s)$ and $F_H(i; s)$ for $i = 1, ..., 5$ as in the two following tables:

	$F(1;s)$	$F(2;s)$	$F(3;s)$	$F(4;s)$	$F(5;s)$
Leman	0.01	0.05	0.16	0.72	1
North-West	0.01	0.05	0.18	0.81	1
Central	0	0.02	0.13	0.76	1
Middle-Land	0.01	0.04	0.17	0.77	1
East	0	0.03	0.14	0.78	1
Ticino	0.01	0.06	0.17	0.87	1
Zurich	0	0.03	0.13	0.78	1

Table 2

	$F_H(1;s)$	$F_H(2;s)$	$F_H(3;s)$	$F_H(4;s)$	$F_H(5;s)$
Leman	0.01	0.06	0.23	1.02	2.32
North-West	0.01	0.06	0.25	1.13	2.45
Central	0	0.02	0.15	0.93	2.10
Middle-Land	0.01	0.05	0.23	1.06	2.35
East	0	0.03	0.17	0.98	2.18
Ticino	0.01	0.07	0.25	1.20	2.53
Zurich	0	0.03	0.16	0.97	2.16

Table 3

A few remarks are in order on the F_H curves. In plain English, $F_H(i;s)$ is a (specifically) weighted sum of the fractions (if the number of individuals is variable) of the population in s that are in worse categories than i. The weight assigned to the fraction of the population in category h (for $h < i$) in that sum is 2^{i-h}. Hence the weights are (somewhat strongly) decreasing with respect to the categories. A nice feature of the F_H curve - that appears strikingly in formula (10) - is its recursive construction, that is quite similar to that underlying the cumulative distribution curve. The cumulative distribution F can indeed be defined recursively by:

$$nF(1;s) = n_1^s \tag{13}$$

and, for $i = 2, ..., k$, by:

$$nF(i;s) = nF(i-1;s) + n_i^s \tag{14}$$

The recursion that defines F_H starts just in the same way than as in (13) with:

$$nF_H(1;s) = n_1^s$$

but has the iteration formula (14) replaced by:

$$nF_H(i;s) = 2nF_H(i-1;s) + n_i^s$$

The H-curves associated to the Léman and the Central region in Switzerland are depicted on figure 1.

We can see that the Leman region H-dominates the central one. We can also see from table 3 that many (but not all) of the seven regions of Switzerland can be ranked by H dominance. The Hasse diagram corresponding to the ranking of the seven regions of Switzerland by the $H-$dominance criterion is shown in figure 2. By contrast, figure 3 shows the Hasse diagram associated to the ranking of these same regions by standard first order stochastic dominance.

As can be seen, the ranking of the regions by H-dominance, while obviously consistent with that of first order stochastic dominance, is significantly more discriminatory than the latter. The H-dominance ranking of these regions can be contrasted to the ranking of the same regions by the Allison and Foster (2004) criterion (provided by Abul-Naga and Yalcin (2008)) as well as to the complete rankings of those same regions provided by the comparisons of the value taken by some of the ordinal inequality indices proposed by Abul-Naga and Yalcin (2008).

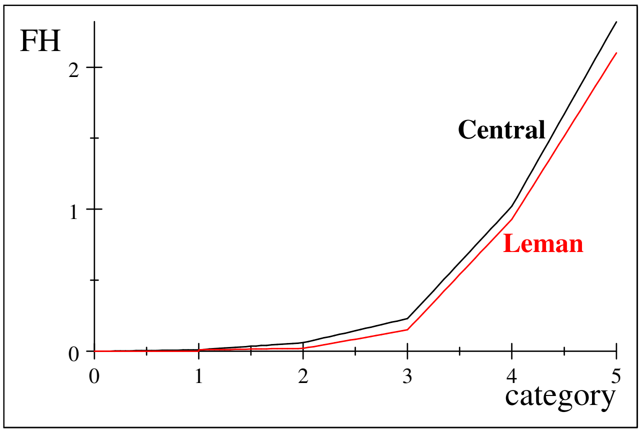

Figure 1: Two H-curves

In the next section, we provide some justification for using H-dominance as a criterion to compare distributions of an ordinal attribute.

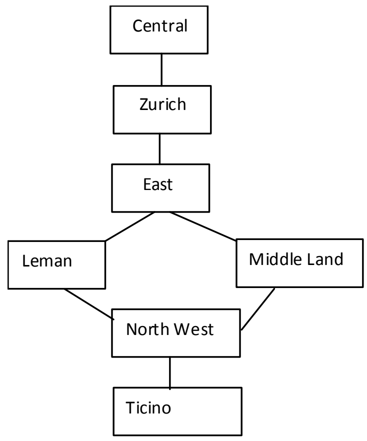

Figure 2: The ranking of the seven Swiss regions by H-dominance.

3 Results

We focus first on the notions of increments, normative dominance, and first order stochastic dominance. A preliminary question that can be asked is: what conditions on the set $\mathcal{A}$ on which normative dominance is defined are necessary and sufficient for normative dominance - as per definition 1 - to be sensitive to increments. The following proposition provides the obvious answer that the necessary and sufficient condition for this is that $\mathcal{A}$ be precisely equal to $\mathcal{A}_1$.

Proposition 1 *Let s be a society that has been obtained from s' by an increment as per definition 2. Then $s \succsim^{\mathcal{A}} s'$ if and only if $\mathcal{A} = \mathcal{A}_1$.*

Proof. *Let s be a society obtained from s' by an increment. By definition 2, there exists some $j \in \{1, ..., k-1\}$ such that:*

$$n_h^s = n_h^{s'}$$

for all $h \in \{1, ...k\}$ such that $h \neq j, j+1$,

$$n_j^s = n_j^{s'} - 1$$

and,

$$n_{j+1}^s = n_j^{s'} + 1.$$

Then $s \succsim^{\mathcal{A}} s'$ if and only if: (using definition 1):

$$\sum_{j=1}^{k} n_j^s \alpha_j \geq \sum_{j=1}^{k} n_j^{s\prime} \alpha_j$$

$$\Longleftrightarrow$$

$$\alpha_{j+1} - \alpha_j \quad \geq \quad 0$$

by definition of an increment. As this inequality must hold for any $j \in \{1, ..., k-1\}$, this completes the proof.

■

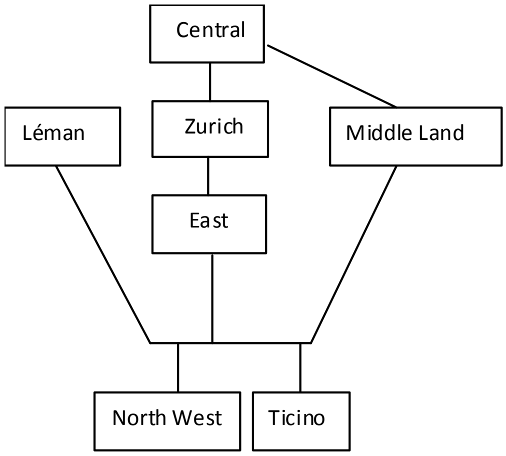

Figure 3: The 1st order stochastic dominance ranking of the seven Swiss regions

We now use this proposition to establish the following theorem. We provide the proof of this theorem for the sake of completeness (and for further use in the proof of our main theorem 2 below) even though the equivalences that it establishes have been known for a long time (see e.g. Lehmann (1955) or Quirk and R.Saposnik (1962)).

Theorem 1 *For any two societies s and $s' \in \mathcal{C}^n$, the following three statements are equivalent:*
(a) *s is obtained from s' by means of a finite sequence of increments,*
(b) *$s \succsim^{\mathcal{A}_1} s'$,*
(c) *$s \succsim_1 s'$.*

Proof. *The equivalence between (a) and (c) is well-known in the literature. We prove the equivalence between statements (b) and (c). We first notice that, for any society s, one has*

$$\sum_{j=1}^{k} n_j^s \alpha_j = \begin{cases} & [n_1^s\, \alpha_1 \\ + & n_2^s\, \alpha_2 \\ + & \cdots \\ + & n_k^s\, \alpha_k] \end{cases}$$

$$= \begin{cases} & n_1^s\, \alpha_1 \\ + & n_2^s\, \alpha_1 + n_2^s\, [\alpha_2 - \alpha_1] \\ + & n_3^s\, \alpha_1 + n_3^s\, [\alpha_2 - \alpha_1] + n_3^s\, [\alpha_3 - \alpha_2] \\ + & \cdots \\ + & n_k^s\, \alpha_1 + n_k^s\, [\alpha_2 - \alpha_1] + n_k^s\, [\alpha_3 - \alpha_2] + \ldots n_k^s\, [\alpha_k - \alpha_{k-1}]\,, \end{cases}$$

$$= \begin{cases} & n\, \alpha_1 \\ + & (n - n_1^s)\, [\alpha_2 - \alpha_1] \\ + & [n - (n_1^s + n_2^s)]\, [\alpha_3 - \alpha_2] \\ + & \cdots \\ + & \left[n - \sum_{h=1}^{k-1} n_h\right] [\alpha_k - \alpha_{k-1}]\,, \end{cases}$$

$$= n[\alpha_k - \sum_{h=1}^{k-1} F(h;s)(\alpha_{h+1} - \alpha_h)]$$

Given this, one can write statement (b) as:

$$\sum_{j=1}^{k} n_j^s \alpha_j \geq \sum_{j=1}^{k} n_j^{s\prime} \alpha_j \tag{15}$$

$$\Longleftrightarrow$$

$$\alpha_k - \sum_{h=1}^{k-1} F(h;s)\,[\alpha_{h+1} - \alpha_h] \geq \alpha_k - \sum_{h=1}^{k-1} F(h;s')\,[\alpha_{h+1} - \alpha_h]$$

$$\Longleftrightarrow$$

$$\sum_{h=1}^{k-1} [F(h;s) - F(h;s')]\,[\alpha_{h+1} - \alpha_h] \leq 0 \tag{16}$$

Hence having $F(h;s) \leq F(h;s')$ for every $h \in \{1,...,k-1\}$ is sufficient for inequality (15) to hold for all $(\alpha_1,...,\alpha_k) \in \mathcal{A}_1$. Conversely, suppose that one has $F(j;s) > F(j;s')$ for some category j. Notice that $j < k$ since $F(k;s) = F(k;s') = 1$. Consider then the list of real numbers $(\alpha_1,..,\alpha_k) \in \mathcal{A}_1$ such that $\alpha_h = 0$ for all $h = 1,\ldots,j$ and $\alpha_h = 1$ for all $h = j+1,\ldots,k$. This particular list of numbers combined to the statement $F(j;s) > F(j;s')$ leads immediately to the violation of inequality (16), as required. ■

We now turn to the notion of Hammond transfers. In a parallel fashion to what has been established in proposition 1 for increments, we first ask under what conditions on the set $\mathcal{A}$ is normative dominance - as per definition 1 - sensitive to both increments and Hammond transfers (as per definition 3). It turns out that the answer to that question involves the following subset $\mathcal{A}_2$ of $\mathcal{A}_1$:

$$\mathcal{A}_2 = \left\{(\alpha_1,...,\alpha_k) \in \mathbb{R}^k \,|\, (\alpha_h - \alpha_g) \geq (\alpha_j - \alpha_i) \text{ for } 1 \leq g < h \leq i < j \leq k\right\} \tag{17}$$

In words, $\mathcal{A}_2$ contains all lists of categories' evaluations that are "strongly concave" with respect to these categories in the sense that the "utility" or "advantage gain of moving from one category to a better one is always better when done from categories in the bottom part of the categories scale than when done in the upper part of it. This strong concavity is, perhaps, better seen if one restrict attention to those elements of $\mathcal{A}_2$ that are also in $\mathcal{A}_1$, as shown in the following lemma.

Lemma 1 *The list of k real numbers $(\alpha_1,...,\alpha_k)$ belongs to $\mathcal{A}_1$ and verifies $\alpha_{i+1} - \alpha_i \geq \alpha_k - \alpha_{i+1}$ for all $i \in \{1,...,k-1\}$ if and only if it belongs to $\mathcal{A}_1 \cap \mathcal{A}_2$.*

Proof. *Suppose that $(\alpha_1,...,\alpha_k) \in \mathcal{A}_1 \cap \mathcal{A}_2$ so that, among other things, the inequality $(\alpha_h - \alpha_g) \geq (\alpha_j - \alpha_i)$ for all $1 \leq g < h \leq i < j \leq k$. This implies that, for any $i \in \{1,...,k-1\}$, $\alpha_{i+1} - \alpha_i \geq \alpha_k - \alpha_{i+1}$. Conversely, consider any list of k real numbers $(\alpha_1,...,\alpha_k)$ in $\mathcal{A}_1$ that verifies $\alpha_{i+1} - \alpha_i \geq \alpha_k - \alpha_{i+1}$ for all $i \in \{1,...,k-1\}$. Since $(\alpha_1,...,\alpha_k) \in \mathcal{A}_1$, one has $\alpha_{i+1} - \alpha_g > \alpha_{i+1} - \alpha_i \geq \alpha_k - \alpha_{i+1} \geq \alpha_l - \alpha_j$ for all $g < i+1$ and l, j satisfying $k \geq l \geq j \geq i+1$. Replacing $i+1$ by h completes the proof that $(\alpha_1,...,\alpha_k) \in \mathcal{A}_2$.* ■

Hence, when examined in the light of lemma 1, the evaluations of categories allowed by the set $\mathcal{A}_2$ consider that the gain of moving upward between two adjacent categories is better than any gain that could result from an upward move initiated from an upper position in the categorical scale, no matter how important this latter move can be. It turns out that $\mathcal{A}_2$ is the largest set of evaluations of the categories over which normative dominance - as per definition 1 - is sensitive to Hammond transfers.

Proposition 2 *Suppose s is a society in $\mathcal{C}^n$ that has been obtained from another society s' in $\mathcal{C}^n$ by a Hammond transfer as per definition 3. Then $s \succsim^{\mathcal{A}} s'$ if and only if $\mathcal{A} = \mathcal{A}_2$.*

Proof. *Let s be a society that has been obtained from s' by a Hammond transfer as per definition 3. For $s \succsim^{\mathcal{A}} s'$ to hold, one must have (using definition 1):*

$$\sum_{j=1}^{k} n_j^s \alpha_j \geq \sum_{j=1}^{k} n_j^{s'} \alpha_j$$

for all $(\alpha_1, ..., \alpha_k) \in \mathcal{A}$. Using definition 3, this is equivalent to require that, for any $(\alpha_1, ..., \alpha_k) \in \mathcal{A}$, the inequality:

$$(\alpha_h - \alpha_g) - (\alpha_j - \alpha_i) \geq 0 \tag{18}$$

hold for any $1 \leq g < h \leq i < j \leq k$, as required by the definition of $\mathcal{A}_2$. In the other direction, it is clear that $s \succsim^{\mathcal{A}_2} s'$ if s has been obtained from s' by a Hammond transfer. ■

It is worth noticing that the set $\mathcal{A}_2$ does not need to belong to $\mathcal{A}_1$ for normative dominance to be sensitive to Hammond transfers. A normative criterion that is represented by an additively separable social evaluation function as per expression (1) may value favorably a Hammond transfer involving the categories $1 \leq g < h \leq i < j \leq k$ of definition 3 *even if* the evaluation of the categories used by this criterion is not increasing with respect to those categories.

We now turn to the main result of this paper.

Theorem 2 *For any societies s and $s' \in \mathcal{C}^n$, the following three statements are equivalent:*
(a) *s is obtained from s' by means of a finite sequence of Hammond's transfers and/or increments,*
(b) *$s \succsim^{\mathcal{A}_1 \cap \mathcal{A}_2} s'$,*
(c) *$s \succsim_H s'$.*

Proof. (c) $\Longrightarrow$ (b)
Assume that $s \succsim_H s'$ and, therefore, that the inequality:

$$F_H(i; s) \leq F_H(i; s')$$

holds for all $i = 2, 3, \ldots, k$. It suffices to show that this implies that, for any *vector of k numbers $(\alpha_1, ..., \alpha_k)$ in the set $\mathcal{A}_1 \cap \mathcal{A}_2$, one has:*

$$
\begin{aligned}
\sum_{j=1}^{k} n_j^s \alpha_j &\geq \sum_{j=1}^{k} n_j^{s\prime} \alpha_j \\
&\Longleftrightarrow \\
\sum_{j=1}^{k} n_j^s \alpha_j - \sum_{j=1}^{k} n_j^{s\prime} \alpha_j &\geq 0 \qquad (19)
\end{aligned}
$$

Using (16) in the proof of theorem 1, this is equivalent to:

$$
\sum_{h=1}^{k-1} [F(h; s') - F(h; s)] \, [\alpha_{h+1} - \alpha_h] \geq 0 \qquad (20)
$$

or:

$$
\sum_{h=1}^{k-1} \Delta F_h \theta_h \geq 0 \qquad (21)
$$

after defining θ_h by:

$$
\theta_h = \alpha_{h+1} - \alpha_h
$$

and ΔF_h by:

$$
\Delta F_h = F(h; s') - F(h; s)
$$

for $h = 1, ..., k-1$. Consider now the following decomposition of the right-hand side of expression (21) defined recursively by:
<u>For $j = 1$:</u>

$$
\begin{aligned}
\Delta F_1 \theta_1 &= \Delta F_1 [\theta_1 - \sum_{h=2}^{k-1} \theta_h] + \Delta F_1 \theta_2 + ... + \Delta F_1 \theta_{k-1} \\
&= \Delta F_1 [\theta_1 - \sum_{h=2}^{k-1} \theta_h] + \Delta F_1 [\theta_2 - \sum_{h=3}^{k-1} \theta_h] + 2\Delta F_1 \theta_3 + ... + 2\Delta F_1 \theta_{k-1} \\
&= \Delta F_1 [\theta_1 - \sum_{h=2}^{k-1} \theta_h] + \Delta F_1 [\theta_2 - \sum_{h=3}^{k-1} \theta_h] + 2\Delta F_1 [\theta_3 - \sum_{h=4}^{k-1} \theta_h] + 4\Delta F_1 \theta_4 + ... + 4\Delta F_1 \theta_{k-1} \\
&= .... \\
&= \Delta F_1 [\theta_1 - \sum_{h=2}^{k-1} \theta_h] + \Delta F_1 [\sum_{i=2}^{k-1} 2^{i-2} [\theta_i - \sum_{h=i+1}^{k-1} \theta_h]]
\end{aligned}
$$

<u>For $j = 2$:</u>

$$
\begin{aligned}
\Delta F_2 \theta_2 &= \Delta F_2 [\theta_2 - \sum_{h=3}^{k-1} \theta_h] + \Delta F_2 \theta_3 + ... + \Delta F_2 \theta_{k-1} \\
&= \Delta F_2 [\theta_2 - \sum_{h=3}^{k-1} \theta_h] + \Delta F_2 [\theta_3 - \sum_{h=4}^{k-1} \theta_h] + 2\Delta F_2 \theta_4 + ... + 2\Delta F_2 \theta_{k-1} \\
&= ... \\
&= \Delta F_2 [\theta_2 - \sum_{h=3}^{k-1} \theta_h] + \Delta F_2 [\sum_{i=3}^{k-1} 2^{i-3} [\theta_i - \sum_{h=i+1}^{k-1} \theta_h]]
\end{aligned}
$$

For any $j = 1, ...k-1$:

$$\Delta F_j \theta_j = \Delta F_j[\theta_j - \sum_{h=j+1}^{k-1} \theta_h] + \Delta F_j[\sum_{i=j+1}^{k-1} 2^{i-j-1}[\theta_i - \sum_{h=i+1}^{k-1} \theta_h]] \quad (22)$$

(under the convention that $\sum_{h=k}^{k-1} \theta_h = \sum_{i=k}^{k-1} 2^{i-k}[\theta_i - \sum_{h=k}^{k-1} \theta_h] = 0$*). Substituting the decomposition terms (22) into inequality (21) yields (up to a change of the summation index):*

$$\sum_{j=1}^{k-1} \Delta F_j \theta_j = \sum_{j=1}^{k-1} [\Delta F_j[\theta_j - \sum_{h=j+1}^{k-1} \theta_h] + \Delta F_j[\sum_{i=j+1}^{k-1} 2^{i-j-1}[\theta_i - \sum_{h=i+1}^{k-1} \theta_h]]] \geq 0$$

or (using the definition of F_H *given by (9):*

$$\sum_{j=1}^{k-1} [F_H(j; s') - F_H(j; s)] \left[\theta_j - \sum_{h=j+1}^{k-1} \theta_h \right] \geq 0 \quad (23)$$

Since:

$$\left[\theta_j - \sum_{h=j+1}^{k-1} \theta_h \right] = \alpha_{j+1} - \alpha_j - [\alpha_k - \alpha_{j+1}] \geq 0$$

for all $(\alpha_1, ..., \alpha_k) \in \mathcal{A}_1 \cap \mathcal{A}_2$ *thanks to lemma 1, we conclude therefore that having* $F_H(j; s) \leq F_H(j; s')$ *for all* $j = 1, ..., k-1$ *is sufficient for inequality (23) to hold for all* $(\alpha_1, ..., \alpha_k) \in \mathcal{A}_1 \cap \mathcal{A}_2$.

(b) $\Longrightarrow$ ***(c)***

Assume that inequality (19) holds for all $(\alpha_1, ..., \alpha_k) \in \mathcal{A}_1 \cap \mathcal{A}_2$. *Consider two real numbers* a *and* b *satisfying* $0 < a < b$ *and, for any* $i = 2, ..., k-1$, *define the vector* $\boldsymbol{\alpha}^i = (\alpha_1^i, ..., \alpha_k^i) \in \mathbb{R}_+^k$ *by:*

$$\begin{aligned}
\alpha_1^i &= a \\
\alpha_k^i &= b \\
\alpha_h^i &= (\alpha_{h-1}^i + b)/2 \text{ if } h = 2, ..., k-1 \text{ and } h \neq i \\
\alpha_i^i &= (\alpha_{i-1}^i + b + \varepsilon)/2
\end{aligned}$$

for some number ε *satisfying* $0 < \varepsilon < b - \alpha_{i-1}^i$. *The reader can check that the vector* $\boldsymbol{\alpha}^i$ *so constructed belongs to* $\mathcal{A}_1 \cap \mathcal{A}_2$. *Writing inequality (19) in the form of (23) with the vector* $\boldsymbol{\alpha}^i$ *yields:*

$$\sum_{j=1}^{k-1} [F_H(j; s') - F_H(j; s)] \left[\theta_j^i - \sum_{h=j+1}^{k-1} \theta_h^i \right] \geq 0 \quad (24)$$

where, for any $j = 1, ..., k-1$, θ_j^i *is defined by:*

$$\theta_j^i = \alpha_{j+1}^i - \alpha_j^i$$

Notice that, for any $j \notin \{1, i-1, k\}$, one has:

$$\begin{aligned}
\theta^i_j - \sum_{h=j+1}^{k-1} \theta^i_h &= \alpha^i_{j+1} - \alpha^i_j - \sum_{h=j+1}^{k-1} [\alpha^i_{h+1} - \alpha^i_h] \\
&= (b - \alpha^i_j)/2 - [b - \alpha^i_{j+1}] \\
&= (b - \alpha^i_j)/2 - [b - (\alpha^i_j + b)/2] \\
&= 0
\end{aligned}$$

while:

$$\begin{aligned}
\theta^i_{i-1} - \sum_{h=i}^{k-1} \theta^i_h &= \alpha^i_i - \alpha^i_{i-1} - \sum_{h=i}^{k-1} [\alpha^i_{h+1} - \alpha^i_h] \\
&= (b - \alpha^i_{i-1} + \varepsilon)/2 - [b - \alpha^i_i] \\
&= (b - \alpha^i_{i-1} + \varepsilon)/2 - [b - (\alpha^i_{i-1} + b + \varepsilon)/2] \\
&= \varepsilon
\end{aligned}$$

Hence, requiring inequality to hold for any such vector $\boldsymbol{\alpha}^i$ for any $i = 2, ..., k-1$ implies requiring inequality:

$$F_H(i; s') - F_H(i; s) \geq 0$$

to hold for any such i, as required by H dominance.
(a) $\Rightarrow$ (c).
Assume that s is obtained from s' by means of an increment. We know from theorem 1 that $s \succsim_1 s'$, which implies that $s \succeq_H s'$. Assume now that s is obtained from s' by means of a Hammond transfer. This implies that there are g, h, i and j satisfying $1 \leq g < h \leq i < j \leq k$ such that:

$$n^s_l = n^{s'}_l \;\; \forall \; l \neq g, h, i, j\,;$$

$$n^s_g = n^{s'}_g - 1 \; ; \; n^s_h = n^{s'}_h + 1\,;$$

$$n^s_i = n^{s'}_i + 1 \; ; \; n^s_j = n^{s'}_j - 1\,.$$

By using equation (7), one has:

$$F_H(l; s) - F_H(l; s') = \begin{cases} 0 & \text{for } l = 1, 2, \ldots, g-1\,, \\ -\left(2^{l-g}\right)/n & \text{for } l = g, \ldots, h-1\,, \\ -\left(2^{l-g} - 2^{l-h}\right)/n & \text{for } l = h, \ldots, i-1\,, \\ -\left(2^{l-g} - 2^{l-h} - 2^{l-i}\right)/n & \text{for } l = i, \ldots, j-1\,, \\ -\left(2^{l-g} - 2^{l-h} - 2^{l-i} + 2^{l-j}\right)/n & \text{for } l = j, \ldots, k. \end{cases}$$

from which we conclude that $F_H(l; s) - F_H(l; s') \leq 0$ for all $l = 1, 2, \ldots, k-1$ and, therefore, that $s \succeq_H s'$.
(c) $\Rightarrow$ (a).
Assume that $s \succsim_H s'$ so that $F_H(g; s) \leq F_H(g; s')$ for all $g = 1, 2, \ldots, k-1$ (avoiding the degenerate case where s is equal to s'). We know that $s \succsim_1 s'$ implies $s \succsim_H s'$. Hence, if $s \succsim_1 s'$, we conclude from theorem 1 that s can be obtained from s' by means of a finite sequence of increments and the proof is

complete. In the following, we shall therefore assume that $s \succsim_H s'$ holds but that $s \succsim_1 s'$ does not hold so that there exists some $g \in \{1, 2, \ldots, k-1\}$ for which one has $F(g; s) - F(g; s') > 0$. Define then the index h by:

$$h = \min \{g \mid F(g; s) - F(g; s') > 0\} \tag{26}$$

Given that index h, one can also define l by :

$$l = \min \{g > h \mid F(j; s) - F(j; s') \leq 0 \,, \forall j \in [g, k]\} \,. \tag{27}$$

Such a l exists because $F(k; s) - F(k; s') = 0$. Notice that, by definition of such l, one has:

$$F(l-1; s) - F(l-1; s') > 0 \textit{ and } F(l; s) - F(l; s') \leq 0 \,, \tag{28}$$

Hence, one has (using the definition of F provided by (5)), that $n_l^s < n_l^{s'}$. We now establish that there exists some $i \in \{1, 2, \ldots, h-1\}$ such that:

$$F(i; s) - F(i; s') < 0 \textit{ and } F(g; s) - F(g, s') = 0 \,, \forall g < i \,. \tag{29}$$

Indeed, since $F_H(g; s) \leq F_H(g; s')$ for all $g = 1, 2, \ldots, k-1$, one has either:

$$\begin{gathered} F_H(1; s) < F_H(1; s') \\ \iff \textit{(thanks to expression (8))} \\ F(1; s) < F(1; s') \end{gathered} \tag{30}$$

or:

$$F(1; s) = F(1; s') \tag{31}$$

If case (30) holds, then the existence of some $i \in \{1, 2, \ldots, h-1\}$ for which expression (29) holds is established (with $i = 1$). If, on the other hand, we are in case (31), then, since $F_H(2; s) \leq F_H(2; s')$ holds, we must have either:

$$\begin{gathered} F_H(2; s) < F_H(2; s') \\ \iff \textit{(thanks to expression (9))} \\ 2F(1; s) + F(2; s) < 2F(1; s') + F(2; s') \end{gathered} \tag{32}$$

or:

$$2F(1; s) + F(2; s) = 2F(1; s') + F(2; s') \tag{33}$$

Again, if we are in case (32), we can conclude (since $F(1; s) = F(1; s')$) that $F(2; s) < F(2; s')$, which establishes the existence of some $i \in \{1, 2, \ldots, h-1\}$ for which expression (29) holds (with $i = 2$ in that case). If we are in case (33), we iterate in the same fashion using the definition of F_H provided by (9). We notice that the index i for which (29) holds must be strictly smaller than h because assuming otherwise will contradict, given the definition of h and the (iterated as above) definition of F_H, the fact that $F_H(g; s) \leq F_H(g; s')$ holds for all $g = 1, 2, \ldots, k-1$. We finally note that, because of the definition of F provided by (5), the definition of the index i just provided entails that:

$$n_i^s < n_i^{s'} \tag{34}$$

and:

$$n_g^s = n_g^{s'}$$

for all $g = 1, ..., i-1$. *We now proceed by defining a new society -* s^1 *say - that belongs to* $\mathcal{C}^n$*, that has been obtained from* s' *by means of a Hammond's transfers and that is such that* $s \succsim_H s^1 \succ_H s'$. *For this sake, we define the numbers* δ_1 *and* δ_2 *and* δ *by:*

$$\delta_1 = n_i^{s'} - n_i^s \ ; \ \delta_2 = n[F(l-1; s) - F(l-1; s')] \quad \text{and} \quad \delta = \min(\delta_1, \delta_2) \quad (35)$$

We note that, by the very definition of the index i*, one has* $\delta_1 > 0$. *We notice also that, using (28) and the definition of the index* l*, one has* $0 < \delta_2 \leq n_l^{s'} - n_l^s$. *Define then the society* s^1 *by:*

$$n_g^{s^1} = n_g^{s'}, \ \forall \ g \neq i, i+1, l\,;$$

$$n_i^{s^1} = n_i^{s'} - \delta \ ; \ n_{i+1}^{s^1} = n_{i+1}^{s'} + 2\delta \ ; \ n_l^{s^1} = n_l^{s'} - \delta\,;$$

It is clear that such a society belongs to $\mathcal{C}^n$. *Moreover,* s^1 *has been obtained from* s' *by* δ *Hammond transfers as per definition 3 where the indices* g, h, i *and* j *of this definition are, here,* i*,* $i+1$*,* $i+1$ *and* l *(respectively)). By virtue of what has been established above, this implies that* $s^1 \succ_H s'$. *We further notice that:*

$$\begin{aligned} F(g, s^1) &= \sum_{e=1}^{g} n_e^{s^1}/n \\ &= \sum_{e=1}^{g} n_e^{s'}/n \\ &= F(g, s') \end{aligned} \quad (37)$$

for all $g = 1, ..., i-1$. *Also one has:*

$$\begin{aligned} F(i, s^1) &= \sum_{e=1}^{i} n_e^{s^1}/n \\ &= F(i-1, s') + n_i^{s^1}/n \\ &= F(i-1, s') + (n_i^{s'} - \delta)/n \\ &= F(i, s') - \delta/n \end{aligned} \quad (38)$$

$$\begin{aligned} F(i+1, s^1) &= F(i, s^1) + n_{i+1}^{s^1}/n \\ &= F(i, s') - \delta/n + n_{i+1}^{s^1}/n \\ &= F(i, s') - \delta/n + n_{i+1}^{s'}/n + 2\delta/n \\ &= F(i+1, s') + \delta/n \end{aligned} \quad (39)$$

Furthermore, for $g = i+2, ..., l-1$, one has:

$$\begin{aligned} F(g, s^1) &= F(i+1, s^1) + \sum_{e=i+2}^{g} n_e^{s^1}/n \\ &= F(i+1, s') + \delta/n + \sum_{e=i+2}^{g} n_e^{s^1}/n \\ &= F(i+1, s') + \delta/n + \sum_{e=i+2}^{g} n_e^{s'}/n \\ &= F(g, s') + \delta/n \end{aligned} \tag{40}$$

While finally, for $g = l, ..., k$:

$$\begin{aligned} F(g, s^1) &= F(l-1, s^1) + \sum_{e=l}^{g} n_e^{s^1}/n \\ &= F(l-1, s') + \delta/n + \sum_{e=l}^{g} n_e^{s^1}/n \\ &= F(l-1, s') + \delta/n + n_l^{s'}/n - \delta/n + \sum_{e=l+1}^{g} n_e^{s'}/n \\ &= F(g, s') \end{aligned} \tag{41}$$

We must now verify that $s \succsim_H s^1$ and, therefore, that $F_H(g; s) - F_H(g; s^1) \leq 0$ for all $g = 1, 2, \ldots, k-1$. Since $s \succsim_H s'$, we know already that $F_H(g; s) - F_H(g, s') \leq 0$ for all $h = 1, 2, \ldots, k-1$. We first observe that, by the definition just given of s^1 one has:

$$F_H(g, s^1) = \begin{cases} F_H(g, s') \text{ for } g = 1, .., i-1, \\ F_H(g, s') - \delta/n \text{ for } g = i, \\ F_H(g, s') \text{ for } g = i+1, ..., l-1, \\ F_H(g, s') - 2^{g-l}\delta/n \text{ for } g = l, ..., k. \end{cases} \tag{42}$$

The first line of (42) is indeed clear given expression (37) and the definition of F_H provided by (7).The second line of (42) results from (38) and the definition of F_H provided by (9). Consider now $g = i+1$. One has (using (9) again):

$$\begin{aligned} F_H(i+1, s^1) &= \sum_{g=1}^{i} \left(2^{i-g}\right) F(g; s^1) + F(i+1; s^1) \\ &= \sum_{g=1}^{i-1} \left(2^{i-g}\right) F(g; s') + F(i; s') - \delta/n + F(i+1; s^1) \text{ (by (38))} \\ &= \sum_{g=1}^{i-1} \left(2^{i-g}\right) F(g; s') + F(i; s') - \delta/n + F(i+1; s') + \delta/n \text{ (by (39))} \\ &= \sum_{g=1}^{i-1} \left(2^{i-g}\right) F(g; s') + F(i; s') + F(i+1; s') \\ &= \sum_{g=1}^{i} \left(2^{i-g}\right) F(g; s') + F(i+1; s') = F_H(i+1, s') \end{aligned} \tag{43}$$

Combined with (10) and the fact that $n_g^{s^1} = n_g^{s'}$ *for all* $g = i+2, ..., l-1$*, equality (43) establishes the third line of expression (42). As for the last line of (42), we start with* $g = l$ *and we use (10) to write:*

$$\begin{aligned} F_H(l, s^1) &= 2F_H(l-1; s^1) + n_l^{s^1}/n \\ &= 2F_H(l-1; s') + (n_l^{s'} - \delta)/n \\ &= F_H(l, s') - \delta/n \end{aligned} \tag{44}$$

Iterating on this expression using (10) yields:

$$\begin{aligned} F_H(l+1, s^1) &= 2F_H(l; s^1) + n_{l+1}^{s^1} \\ &= 2(F_H(l; s') - \delta/n) + n_{l+1}^{s'} \\ &= F_H(l+1, s') - 2\delta/n \end{aligned} \tag{45}$$

and therefore, for any $g \in \{l, ..., k\}$*:*

$$F_H(g, s^1) = F_H(g, s') - 2^{g-l}\delta/n$$

as required by the last line of expression (42). We now notice that expression (42) entails that:

$$F_H(g, s) - F_H(g, s^1) = \begin{cases} F_H(g, s) - F_H(g, s') \text{ for } g = 1, .., i-1, \\ F_H(g, s) - F_H(g, s') + \delta/n \text{ for } g = i, \\ F_H(g, s) - F_H(g, s') \text{ for } g = i+1, ..., l-1, \\ F_H(g, s) - F_H(g, s') + 2^{g-l}\delta/n \text{ for } g = l, ..., k. \end{cases} \tag{46}$$

Since by assumption $s \succsim_H s'$*, this establishes that* $F_H(g; s) - F_H(g; s^1) \leq 0$ *for all* $g \in \{1, 2, \ldots, i-1\} \cup \{i+1, ..., l-1\}$*. Consider now the case* $g = i$*. Using (10), we know that:*

$$F_H(i; s) - F_H(i, s') = 2\left(F_H(i-1; s) - F_H(i-1; s')\right) + (n_i^s - n_i^{s'})/n \tag{47}$$

By definition of i*, one has* $F(h; s) - F(h; s') = 0$ *for all* $h < i$*, so that the first term in the right hand side of equation (47) is 0. Recalling then from (35) that* $\delta_1 = n_i^{s'} - n_i^s > 0$ *and that* $\delta = \min(\delta_1, \delta_2)$*, it follows that:*

$$n_i^s - n_i^{s'} + \delta \leq 0$$

By combining equations (46) and (47), we conclude that:

$$F_H(i, s) - F_H(i, s^1) = F_H(i, s) - F_H(i, s') + \delta/n = n_i^s - n_i^{s'} + \delta \leq 0 \tag{48}$$

Consider finally the case where $g = l, \ldots, k-1$*. By using equation (10) (and recalling that* $\delta_2 = n[F(l-1; s) - F(l-1; s')]$*), one has:*

$$F_H(l; s) - F_H(l; s') = 2(F_H(l-1; s) - F_H(l-1; s')) + F(l; s) - F(l; s') - \delta_2/n. \tag{49}$$

Combining (49) with the last line of (46), and remembering that $\delta \leq \delta_2$*, one obtains:*

$$F_H(l, s) - F_H(l, s^1) = 2[(F_H(l-1; s) - F_H(l-1; s')] + F(l; s) - F(l; s') + (\delta - \delta_2)/n \leq 0\,. \tag{50}$$

Finally, using successive applications of equation (10), one obtains, for any $g = l+1, \dots, k-1$*:*

$$\begin{aligned} F_H(g;s) - F_H(g;s') &= 2^{g-l+1}[F_H(l-1;s) - F_H(l-1;s')] + \sum_{e=l}^{g-1} 2^{g-1-e}[F(e;s) - F(e;s')] \\ &\quad +F(g;s) - F(g;s') - 2^{g-l}\delta_2/n \\ &\leq 0 \end{aligned}$$

by assumption. Combined with the last line of (46) and the fact that $\delta \leq \delta_2$*, this completes the proof that* $s \succeq_H s^1$*. Hence, we have found a society* s^1 *obtained from society* s' *by means of a Hammond's transfers that is such that* $s \succsim_H s^1 \succ_H s'$*. We now show that, in moving from* s' *to* s*, one has "brought to naught" at least one of the differences* $|F(h;s) - F(h;s')|$ *that distinguishes* s *from* s'*. That is to say, we establish the existence of some* $h \in \{1, ..., k-1\}$ *for which one has:*

$$\left|F(h;s) - F_H(h;s^1)\right| = 0$$

and:

$$|F(h;s) - F(h;s')| < 0$$

This is easily seen from the fact that, in the construction of s^1*, one has either:*

$$\delta = \delta_1 = n_i^{s'} - n_i^s \tag{51}$$

or:

$$\delta = \delta_2 = n[F(l-1;s) - F(l-1;s')] \tag{52}$$

If we are in the case (51), one has by definition of the index i *and the function* F*:*

$$F(i;s) - F_H(i;s^1) = 0$$

and:

$$F(i;s) - F(i;s') < 0$$

If on the other hand we are in case (52), then, we have (using (39)):

$$\begin{aligned} F(l-1;s) - F(l-1;s^1) &= F(l-1,s) - [F(l-1,s') + \delta_2/n] \\ &= 0 \end{aligned}$$

while, by definition of the index l*, one has:*

$$F(l-1,s) - F(l-1,s') > 0$$

Now, if $s = s^1$*, then the proof is completed. If* $\neg(s = s^1)$ *but* $s \succsim_1 s^1$*, then we conclude that society* s *can be obtained from society* s' *by means of a finite sequence of Hammond's transfers and increments. If* $\neg(s = s^1)$ *and* $\neg\left\{s \succsim_1 s^1\right\}$*, then we can find three categories* i, h *and* l *just as in the preceding step and construct a new distribution - say* s^2 *- that can be obtained from distribution* s^1 *by means of an (integer number of) Hammond' transfers, satisfying* $s \succsim_H s^2 \succ_H s^1$ *and so on. Generically, after a finite number -* t *say - of iterations, we will find a distribution* s^t *such that* $s \succsim_H s^t \succ_H s^{t-1}$ *. In that case, we will have either* $s = s^t$ *or* $s \succsim_1 s^t$*. As* t *is finite, since there are only finitely many differences of the kind* $|F(h;s) - F(h;s')|$ *to bring to naught, this completes the proof.* ■

4 Conclusion

The paper has identified a statistically implementable criterion, called $H-$dominance, that can be viewed as the analogue, for comparing distributions of an *ordinally measurable* attribute, of the generalized Lorenz curve used for comparing distributions of a *cardinally measurable* attribute. It is well-known (see e.g. Shorrocks (1983)) that a distribution of a cardinally measurable attribute dominates another for the generalized Lorenz domination criterion if and only if it is possible to go from the dominated distribution to the dominating one by a finite sequence of increments of the attribute and/or Pigou-Dalton transfers. The main result of this paper - theorem 2 - has established an analogous result for the H-dominance criterion by showing that the latter criterion ranks two distributions of an attribute in the same way than would the fact of going from the dominated distribution to the dominating one by a finite sequence of increments and/or Hammond transfers of the attribute.

We believe the H-dominance criterion, and the Hammond principle of transfers that justifies it, to be a useful tool for comparing distributions of an ordinally measurable attribute that can not be meaningfully transferred *à la* Pigou-Dalton. Beside the fact of being justified by clear and meaningful elementary transformations, the H-dominance criterion has the advantage of being applicable to a much wider class of situations than, for instance, the widely discussed criterion proposed by Allison and Foster (2004) who is limited to distributions that have the same median category. The illustration of the criterion that we have done with the data provided in Abul-Naga and Yalcin (2008) suggests that its discriminatory power is significant, and that it could be much useful in practice to perform various kinds of normative evaluation exercises involving allocations of qualitative or ordinal attribute among individuals. The criterion could also be useful for comparing distributions of a cardinally measurable attribute provided that one is willing to accept the rather strong egalitarian ethics underlying the principle of Hammond transfers in such a setting.

A somewhat specific feature of the results established in this paper, as compared to the standard dominance results involving Pigou-Dalton transfers, is that we have not identified a statistically implementable criterion that coincides with Hammond transfers *only*. The result provided by theorem 2 shows that H-dominance coincides with the possibility of going from the dominated distribution to the dominating one by means of *either* Hammond transfers *or* increments. Those who are interested in obtaining a "pure" notion of inequality reduction in an ordinal context may feel a bit disappointed by this presence of increments, that are often seen as reflecting "efficiency", rather than "equity", considerations. Could we find a criterion that would be associated to Hammond transfers only? In the standard income-inequality framework, the statistically implementable criterion that coincides with the possibility of going from a dominated distribution to a dominating one by a finite sequence of Pigou-Dalton transfers only is generalized Lorenz dominance applied to distributions with the same mean. Unfortunately, we have not obtained such a "pure" inequality reduction criterion that underlies the notion of inequality reduction captured by Hammond transfers. We clearly believe that obtaining such a criterion to be a worthwhile objective for future research.

References

Abul-Naga, R., and T. Yalcin (2008): "Inequality Measurement for ordered response health data," *Journal of Health Economics*, 27, 1614–1625.

Allison, R. A., and J. E. Foster (2004): "Measuring health Inequality Using Qualitative Data," *Journal of Health Economics*, 23, 505–524.

Apouey, B. (2007): "Measuring Health Polarization with Self-Assessed Health Data," *Health Economics*, 16, 875–894.

Atkinson, A. (1970): "On the Measurement of Inequality," *Journal of Economic Theory*, 2, 244–263.

Bosmans, K., and E. Ooghe (2013): "A Characterization of Maximin," *Economic Theory Bulletin*, 1, 151–156.

Castelló-Clement, A., and R. Doménech (2002): "Human Capital Inequality and Economic Growth: some New Evidence," *The Economic Journal*, 112, C187–C200.

——— (2008): "Human Capital Inequality, Life Expectancy and Economic Growth," *The Economic Journal*, 118, 653–677.

Dasgupta, P., A. K. Sen, and D. Starrett (1973): "Notes on the Measurement of Inequality," *Journal of Economic Theory*, 6, 180–187.

D'Aspremont, C. (1985): "Axioms for Social Welfare Orderings," in *Social Goals and Social Organization*, ed. by L. Hurwicz, D. Schmeidler, and H. Sonnenschein, pp. 19–76. Cambridge University Press, Cambridge, UK.

D'Aspremont, C., and L. Gevers (1977): "Equity and the Informational Basis of Social Choice," *Review of Economic Studies*, 46, p.199–210.

Fields, G., and J. Fei (1978): "On inequality Comparisons," *Econometrica*, 46, 305–316.

Gravel, N., T. Marchant, and A. Sen (2011): "Comparing Societies with Different Numbers of Indiviuals on the Basis of their Average Advantage," in *Social Ethics and Normative Econonomics: Essays in Honour of Serge-Christophe Kolm*, ed. by M. Fleurbaey, M. Salles, and J. A. Weymark, pp. 261–277. Springer Verlag.

Gravel, N., and P. Moyes (2013): "Utilitarianism or Welfarism: does it Make a Difference ?," *Social Choice and Welfare*, 40, 529–551.

Hammond, P. J. (1976): "Equity, Arrow's Conditions and Rawls's Difference Principle," *Econometrica*, 44, 793–803.

Hardy, G. H., J. E. Littlewood, and G. Polya (1952): *Inequalities, 2nd edition*. Cambridge University Press, Cambridge, UK.

Kolm, S. C. (1969): "The Optimal Production of Social Justice," in *Public Economics*, ed. by H. Guitton, and J. Margolis. Macmillan, London.

LEHMANN, E. L. (1955): "Ordered Family of Distributions," *The Annals of Mathematics and Statistics*, 26, 399–419.

PRADHAN, M., D. A. SAHN, AND S. D. YOUNGER (2003): "Decomposing World Health Inequality," *Journal of Health Economics*, 22, 271–293.

QUIRK, J. D., AND R.SAPOSNIK (1962): "Admissibility and Measurable Utility Functions," *Review of Economic Studies*, 29, 140–146.

SEN, A. K. (1973): *On Economic Inequality.* Oxford, Clarendon.

——— (1977): "On Weights and Measures: Informational Constraints in Social Welfare Analysis," *Econometrica*, 45, 1539–1572.

SHORROCKS, A. F. (1983): "Ranking Income Distributions," *Economica*, 50, 3–17.