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# Weak Concavity Properties of Indirect Utility Functions in Multisector Optimal Growth Models

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# Weak concavity properties of indirect utility functions in multisector optimal growth models\*

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**Abstract:** *Studies of optimal growth in a multisector framework are generally addressed in reduced form models. These are defined by an indirect utility function which summarizes the consumers' preferences and the technologies. Weak concavity assumptions of the indirect utility function allow one to prove differentiability of optimal solutions and stability of steady state. This paper shows that if the consumption good production function is concave- $\gamma$ , and the instantaneous utility function is concave- $\rho$ , then the indirect utility function is weakly concave, and its curvature coefficients are bounded from above by a function of  $\gamma$  and  $\rho$ .*

**Keywords:** *Indirect utility function, social production function, multisector optimal growth model, weak concavity.*

*Journal of Economic Literature* Classification Numbers: C62, E32, O41.

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# 1 Introduction

Studies of the qualitative properties of capital accumulation paths in continuous or discrete-time multisector optimal growth models are generally addressed in reduced form infinite-horizon problems ( $\mathcal{P}$ ) such that:

$$\begin{aligned}
 W(k_0) &= \max_{k(t)} \int_0^{+\infty} e^{-\delta t} V(k(t), \dot{k}(t)) dt, & W(k_0) &= \max_{\{k_t\}_{t \geq 0}} \sum_{t=0}^{+\infty} \beta^t V(k_t, k_{t+1}) \\
 \text{s.t. } &(k(t), \dot{k}(t)) \in D & \text{s.t. } &(k_t, k_{t+1}) \in D \\
 &k(0) = k_0 \text{ given} & &k_0 \text{ given}
 \end{aligned}$$

where  $D$  is a non-empty compact convex subset of  $\mathbb{R}^{2n}$ .  $V$  is the indirect utility function which summarizes the main characteristics of the consumers' preferences and the underlying technological structure,  $W$  the value function,  $\delta$  the discount rate and  $\beta$  the discount factor, which are respectively assumed to be positive and taken between 0 and 1. Differentiability and concavity of indirect utility functions are of the greatest importance to characterize the local and global dynamic properties of optimal capital accumulation paths. For example, McKenzie [12] shows that if the utility function is not differentiable at the long-run steady state, neither local nor global stability can be proved. In contrast, Yano [32] shows that if the slope of a marginal utility function is bounded from both below and above by quadratic approximations, the asymptotic stability can be proved even in the non-differentiable case.

Since the early work of McKenzie [13], Brock and Scheinkman [7], Cass and Shell [8], Magill [10] and Rockafellar [22],<sup>1</sup> it is also well-known that there exists a trade-off between the curvature of the indirect utility function, and the turnpike property. Strict concavity is indeed a central assumption to obtain stability results and to prove differentiability of the indirect utility function. Benhabib and Nishimura [2] have proved that if all goods are produced non-jointly under decreasing returns to scale, then  $V$  is strictly concave.

More recently, all these former contributions have been used to study the existence of optimal endogenous fluctuations. Montrucchio [16, 17, 18], Nishimura [20], Rockafellar [22] and Sorger [25] prove that the set of discount rate values for which the steady state is saddle-point stable depends

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<sup>1</sup>See also the more recent contribution of Marena and Montrucchio [11].

on some precise degrees of curvature of the indirect utility function. More precisely, periodic or even chaotic dynamics are shown to be compatible with low discounting provided the degree of concavity is adequately chosen (see Boldrin and Montrucchio [4], Montrucchio [15]). For instance, based on the example provided in Benhabib and Rustichini [3], Venditti [29] shows that the smaller are some degrees of concavity of the indirect utility function, the lower are the discount rate values compatible with the endogenous business cycles.

Concavity is also crucial to establish differentiability properties. Montrucchio [14, 15, 17], Gota and Montrucchio [9] and Sorger [26, 27] show that the policy function, i.e. the optimal capital accumulation path, is Lipschitz-continuous and that its Lipschitz constant depends on the degree of curvature of  $V$  (see also Montrucchio [19], Santos [23], Santos and Vila [24] in which the  $C^1$ -differentiability of the policy function is obtained). It is worth noting however that the way endogenous fluctuations and the stability of the long-run steady state relate to the curvature of the utility function is still not completely understood in the literature. Beside the conclusions of Yano [32], Nishimura and Yano [21] have established the existence of chaotic optimal paths for the case in which the indirect utility function is non-differentiable.

Whatever the conclusions, most of these results are proved using some precise concavity properties:

- the  $\alpha$ -concavity, or strong concavity, which provides a measure of the lower curvature of the function. The parameter  $\alpha$  is indeed related to the smaller eigenvalue in absolute value of the Hessian matrix. A function  $f$  is  $\alpha$ -concave if it is "at least as concave" as the quadratic form  $-(\alpha/2)\|x\|^2$ . A strongly concave function is necessarily strictly concave.

- the concavity- $\gamma$ , or weak concavity, which provides a measure of the upper curvature of the function.<sup>2</sup> Compared with strong concavity, weak concavity is the "Alice's mirror image". The parameter  $\gamma$  is indeed related to the greater eigenvalue in absolute value of the Hessian matrix. A function  $f$  is concave- $\gamma$  if it is "at most as concave" as the quadratic form  $-(\gamma/2)\|x\|^2$ . A weakly concave function may not be strictly concave.

Though the indirect utility function is a reduced form, which gives a summary of the representative consumer's utility function and the production functions, beside the contribution of Benhabib and Nishimura [2], the

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<sup>2</sup>Sorger [26, 27] uses the notation " $(-\gamma)$ -convexity" instead of "concavity- $\gamma$ ".

literature did not provide during many years any precise details on the link between these assumptions and the concavity properties of the fundamentals. As explicated in Boldrin and Woodford [5], it was simply stated that the curvature of the indirect utility function "depends (albeit in a very complicated way) on the curvature of the technology and the preferences". But as the literature on endogenous business cycles developed extensively in the 90's, a central problem was to provide some conditions on the fundamentals giving rise to a strongly and / or weakly concave indirect utility function.

In Venditti [30], we provide sufficient conditions for strong concavity mainly based on the  $\alpha$ -concavity of the consumption good's production function and on Lipschitz continuity the capital goods' technologies. The present paper focuses on weak concavity. We provide sufficient conditions based on the weak concavity of the consumption good's technology and of the utility function, and we give some upper bounds for the degree of concavity- $\gamma$  of the indirect utility function. Moreover, as this property is widely used in discrete-time optimal growth models,<sup>3</sup> we also study the weak concavity of the value function characterizing the standard Bellman equation.

In section 2, we present a step-by-step construction of the indirect utility function. We introduce in section 3 the definition of weak concavity and we establish one mathematical result which is used to prove our main result. Section 4 is devoted to the study of the indirect utility function concavity properties. We also provide some economic interpretations of our conclusions. In Section 5, we focus on discrete-time models and give conditions for the concavity- $\gamma$  of the value function of the Bellman equation. All the proofs are gathered in the Appendix.

## 2 The model

We consider a  $(n + 1)$ -sector competitive economy with one consumption good and  $n$  capital goods. Total labor is normalized to one, and the model is defined by the following equations:<sup>4</sup>

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<sup>3</sup>See Boldrin and Montrucchio [3], Montrucchio [15, 17], Sorger [26, 27].

<sup>4</sup>The lack of the time index  $t$  means that the model may be considered as either in discrete or continuous time.

$$\begin{aligned}
y_i &\leq f^i(k_{1i}, \dots, k_{ni}, l_i), & i = 0, 1, \dots, n \\
1 &= \sum_{i=0}^n l_i \text{ and } k_j = \sum_{i=0}^n k_{ji}, & j = 1, \dots, n
\end{aligned} \tag{1}$$

where  $y_0$  is the consumption good output,  $y_j$  the output of capital good  $j$ ,  $k_{ji}$  the amount of capital good  $j$  used in the production of good  $i$ ,  $l_i$  the amount of labour used in the production of good  $i$ ,  $k_j$  the stock of capital good  $j$ , and  $f^i$  the technology of good  $i$ .

**Assumption 1.** *The functions  $f^i : \mathbb{R}_+^{n+1} \rightarrow \mathbb{R}_+$ ,  $i = 0, \dots, n$ , are time-invariant,  $C^r$  with  $r \geq 2$ , strictly increasing in each argument and concave.*

The  $n$  stocks of capital goods  $k_j$  are such that  $k_j \in \mathbb{R}_+$ . Assuming a growth rate of labour force  $g > 0$ , a capital depreciation rate  $\mu \in [0, 1]$  which is constant and identical across sectors, we obtain the capital accumulation equations in continuous and discrete time for each good  $j = 1, \dots, n$ :

$$y_j(t) = \dot{k}_j(t) + (\mu + g)k_j(t), \quad y_{jt} = (1 + g)k_{jt+1} - (1 - \mu)k_{jt} \tag{2}$$

We assume that net investment vectors  $\dot{k}$  form a convex set  $\mathcal{I} \subseteq \mathbb{R}^n$ . Let us maximize the production of the consumption good  $y_0$  subject to the technological constraints, namely:

$$\begin{aligned}
&\max_{(k_{10}, \dots, k_{n0}, l_0)} f^0(k_{10}, \dots, k_{n0}, l_0) \\
&s.t. \quad y_j \leq f^j(k_{1j}, \dots, k_{nj}, l_j) \quad j = 1, \dots, n
\end{aligned} \tag{3}$$

$$1 = \sum_{i=0}^n l_i, \quad k_j = \sum_{i=0}^n k_{ji} \quad j = 1, \dots, n \tag{4}$$

$$k_{ji} \geq 0, l_i \geq 0 \quad i = 0, \dots, n \text{ and } j = 1, \dots, n \tag{5}$$

This program is denoted  $(\tilde{\mathcal{P}}_{k,y})$ . Let  $X_i = (k_{1i}, \dots, k_{ni}, l_i)$ ,  $i = 0, \dots, n$ , and  $k = (k_1, \dots, k_n)$  be the vector of capital goods stocks. Considering the  $n + 1$  linear constraints (4), we have

$$X_0 = \mathcal{A}(k, X_1, \dots, X_n) = a + A(k, X_1, \dots, X_n) \tag{6}$$

with  $a^T = (\overbrace{0, \dots, 0}^n, 1)$  and  $A : \mathbb{R}^{n(n+2)} \rightarrow \mathbb{R}^{n+1}$  (which is precisely defined in the Appendix). Let  $F^0(k, X_1, \dots, X_n) \equiv (f^0 \circ \mathcal{A})(k, X_1, \dots, X_n)$ . The optimization program  $(\tilde{\mathcal{P}}_{k,y})$  becomes

$$\begin{aligned} \max_{(X_j)_{j=1}^n} \quad & F^0(k, X_1, \dots, X_n) \\ \text{s.t.} \quad & y_j \leq f^j(X_j) \quad j = 1, \dots, n \end{aligned} \quad (7)$$

$$\mathcal{A}(k, X_1, \dots, X_n) \geq 0 \text{ and } X_j \geq 0 \quad j = 1, \dots, n \quad (8)$$

This new program is denoted  $(\mathcal{P}_{k,y})$ . For each given  $(k, y)$ , the set of admissible vectors for  $(\mathcal{P}_{k,y})$ , denoted

$$\mathcal{Q}_{k,y} = \left\{ (\mathcal{A}(k, X_1, \dots, X_n), X_1, \dots, X_n) \in (\mathbb{R}^{n+1})^{n+1} / X_j \geq 0, \right. \\ \left. \mathcal{A}(k, X_1, \dots, X_n) \geq 0, f^j(X_j) \geq y_j, j = 1, \dots, n \right\} \quad (9)$$

is convex. We thus obtain:

**Lemma 1.** *Consider the optimization program  $(\mathcal{P}_{k,y})$  and the set of admissible vectors  $\mathcal{Q}_{k,y}$ . Under Assumption 1, if for a given  $(k, y) \in \mathbb{R}_+^{2n}$ ,  $\mathcal{Q}_{k,y}$  is non empty, then  $\mathcal{Q}_{k,y}$  is a compact set and  $(\mathcal{P}_{k,y})$  has an optimal solution.*

The optimal solution gives the maximal level of consumption as a function of capital goods stocks  $k_j$  and output  $y_j$ , i.e.:

$$y_0^* = c = T(k_1, \dots, k_n, y_1, \dots, y_n) \quad (10)$$

The *social production function*  $T$  is defined over a convex set  $\mathcal{K} \subseteq \mathbb{R}_+^{2n}$ , and gives the frontier of the production possibility set. Consider equations (2). Using notation which is consistent with continuous and discrete time, we get

$$T(k, y) = (T_o B)(k, z) \quad (11)$$

where  $B : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$  is a linear map defined by the matrix

$$B = \begin{pmatrix} I_n & 0 \\ bI_n & dI_n \end{pmatrix} \quad (12)$$

with  $b, d \in \mathbb{R}$  and  $I_n$  the  $n \times n$  identity matrix. We thus have either  $z = \dot{k}$ ,  $b = (\mu + g)$ ,  $d = 1$ , or  $z = k_{t+1}$ ,  $b = (\mu - 1)$ ,  $d = (1 + g)$ .

Labor supply is inelastic and the preferences of the representative agent are described by some utility function  $u(c)$  such that:

**Assumption 2.**  *$u : \mathbb{R}_+ \rightarrow \mathbb{R}$  is time-invariant,  $C^r$ ,  $r \geq 1$ , increasing and concave.*

Let us introduce the set

$$\mathcal{D} = \{(k, z) \in \mathbb{R}^n \times \mathbb{R}^n / B(k, z) = (k, y) \in \mathcal{K}\} \quad (13)$$

The indirect utility function is finally defined as  $V : \mathcal{D} \rightarrow \mathbb{R}$  with

$$V(k, z) \equiv (u_o T_o B)(k, z) \quad (14)$$



### 3 On weak concavity

Concavity assumptions used in economics do not in general provide precise restrictions on the degree of curvature of a function. The concept of curvature is associated with the eigenvalues and the determinant of the Hessian matrix. In this paper we are concerned with the concept of weak concavity which relies on the greater eigenvalue in absolute value, and which provides an upper bound for the curvature.

**Definition 1.** Let  $\mathbb{R}^n$  be endowed with the Euclidean norm  $\|\cdot\|$ , and  $\mathcal{D} = X \times Y \subseteq \mathbb{R}^n \times \mathbb{R}^n$  be a non empty convex set. Let  $U : \mathcal{D} \rightarrow \mathbb{R}$  be a real-valued concave function. Let  $\gamma$  and  $\eta$  be the greatest lower bounds of the set of real numbers  $g$  and  $h$  such that the function  $U(x, y) + (1/2)g\|x\|^2 + (1/2)h\|y\|^2$  is convex over  $\mathcal{D}$ , i.e.

$$\begin{aligned} U(tx_1 + (1-t)x_2, ty_1 + (1-t)y_2) &\leq tU(x_1, y_1) + (1-t)U(x_2, y_2) \\ &\quad + (1/2)gt(1-t)\|x_1 - x_2\|^2 + (1/2)ht(1-t)\|y_1 - y_2\|^2 \end{aligned}$$

for all  $(x_1, y_1), (x_2, y_2) \in \mathcal{D}$  and all  $t \in [0, 1]$ . If  $\gamma > 0$  or  $\eta > 0$ ,  $U$  is called concave- $(\gamma, \eta)$  or equivalently weakly-concave.<sup>5</sup>

In Section 2 we have shown that the indirect utility function  $V$  is obtained at the end of a process which combines maximization and composition of several functions. To prove that  $V$  may be weakly concave, we need the following mathematical result:

**Proposition 1.** Let  $\mathcal{F} \subset \mathbb{R}^n \times \mathbb{R}^n$  be a non-empty, compact convex set, and  $f : \mathcal{F} \rightarrow \mathbb{R}$  be a differentiable concave- $(\alpha, \beta)$  function, with  $\alpha > 0$  or  $\beta > 0$ . Let  $g : \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable, monotone increasing, and concave- $\gamma$  function with  $\gamma > 0$ . Let  $\|Df(x)\| = \sup_{\nu \neq 0} \|Df(x)\nu\|/\|\nu\|$  and  $p = \sup_{x \in \mathcal{F}} \|Df(x)\|$ . If  $g'(x)$  is bounded from above by a finite number  $q$ , then the composite function  $g \circ f$  is concave- $(\varphi, \chi)$  over  $\mathcal{F}$  with  $\varphi \leq \alpha q + \gamma p^2$  and  $\chi \leq \beta q + \gamma p^2$ .

This proposition is an extension to the global case of a result proved by Bougeard and Penot [6] (see also Vial [31]). Moreover, assuming that  $g$  is weakly concave allows us to provide a more precise characterization of the concavity coefficients of  $g \circ f$ .

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<sup>5</sup>See Vial [31], Bougeard and Penot [6].

## 4 Weak concavity of indirect utility functions

Let us first consider the social production function defined by program  $(\mathcal{P}_{k,y})$ . We prove the robustness of weak concavity with respect to maximization.

**Proposition 2.** *Let  $f^0$  be concave- $\alpha$  with  $\alpha > 0$ , and  $f^j$  be concave,  $j = 1, \dots, n$ . Then under Assumption 1, the value function  $T(k, y)$  of program  $(\mathcal{P}_{k,y})$  is such that for any given  $y \geq 0$ ,  $T(\cdot, y)$  is concave- $\gamma$  with  $\gamma \leq \alpha(1+n)$ .*

Benhabib and Nishimura [2] also study the concavity properties of  $T(k, y)$ . With standard arguments of concave programming, it is easy to prove that under Assumption 1,  $T$  is a concave function. However strict concavity is more difficult to obtain. Benhabib and Nishimura [2] provide two results depending on the returns to scale of the consumption and capital goods technologies. Indeed, under the assumption of non-joint production:

- if each good is produced under decreasing returns to scale, then the Hessian matrix of  $T(k, y)$  has full rank, i.e.  $T$  is strictly concave;
  - on the contrary, if the consumption good and one capital good at least are produced under constant returns to scale, then the Hessian matrix of  $T(k, y)$  cannot have full rank. Therefore,  $T$  may not be strictly concave.
- Their results differ drastically from ours since weak concavity is fully compatible with non strict concavity, and with constant or decreasing returns to scale. Let us now consider the following assumptions on  $T$  and  $u$ :

**Assumption 3.**  $\tau = \sup_{(k,y) \in \mathcal{K}} \|DT(k, y)\| < +\infty$  with  $\|DT(k, y)\| = \sup_{\nu \neq 0} \|DT(k, y)\nu\|/\|\nu\|$ .

**Assumption 4.**  $\sup_{x \in \mathbb{R}_+} u'(x) = r < +\infty$ .

Before establishing our main result, we have to comment on Assumption 3. Let us consider program  $(\mathcal{P}_{k,y})$  which gives the social production function. It is well-known that the static optimization conditions imply:

$$DT(k, y) = (T_1(k, y), T_2(k, y)) = (\omega(k, y), -\pi(k, y))$$

with  $\omega$  and  $\pi$  the vectors of the rental rates and prices of the capital goods in terms of the price of the consumption good. Assumption 3 then implies that over the production possibility set, the competitive prices remain bounded. Note also that Assumption 4 rules out the Inada condition.

Denoting  $p = \tau\|B\|$  with  $\|\cdot\|$  the Euclidean norm, we obtain the main result of the paper:

**Theorem 1.** *Let Assumptions 1-4 hold. Assume that  $f^0$  is concave- $\alpha$  with  $\alpha > 0$ , and that  $u$  is concave- $\rho$  with  $\rho > 0$ . Let  $\mathcal{D}_k = \{\zeta \in \mathbb{R}^n / (k, \zeta) \in \mathcal{D}\}$ . Then the indirect utility function  $V(., z)$  is concave- $\varphi$  for every given  $z \in \mathcal{D}_k$  with  $\varphi \leq \alpha(1+n)r + \rho p^2$ .*

Note that if the utility function  $u$  is assumed to be linear, we have  $u(c) = rc$  and  $\rho = 0$ . The indirect utility function has thus a parameter of weak concavity which satisfies  $\varphi \leq \alpha(1+n)r$ . Moreover, if Assumption 3 does not hold, i.e. if  $\tau$  is not finite, then the linearity of  $u$  becomes a sufficient condition for the weak concavity of  $V(., z)$ .

Our goal is to understand how a modification of the curvature of both the production function  $f^0$  and the utility function  $u$  may lead to a modification of the indirect utility function's degree of weak concavity. In particular, is it possible to have an indirect utility function which is less and less concave? To answer this question we have to study the interdependence between the parameters  $\alpha$ ,  $r$ ,  $\rho$  and  $p$ . The parameter  $r$  may depend on  $\rho$  since the shape of the utility function  $u$  varies when its curvature is modified. But recall that since  $u$  is weakly concave,  $u$  is at most as concave as the quadratic form  $-(\rho/2)||x||^2$ . Then, if for instance  $\rho$  goes toward zero,  $u$  is closer and closer to a linear function and  $r$  is finite. Similarly, the parameter  $\tau$  may depend on  $\alpha$ . As in the previous case, it is easy to see that if  $\alpha$  goes toward zero, the social production function is closer and closer to a linear function and the norm of its gradient remains bounded. Then a weakening of the curvature of the consumption good technology  $f^0$  and the utility function  $u$  allows to decrease the indirect utility function's degree of weak concavity.

One may then wonder how to economically interpret these properties. The degree of weak concavity  $\alpha$  gives a measure of the transformations of the consumption good production function which are necessary to obtain some non-decreasing returns to scale. In other words, let us consider the set of all production functions divided in two subsets according as the returns to scale are non-increasing or non-decreasing.  $\alpha$  provides information on the distance between the given technology and the frontier of the subset containing the non-decreasing returns to scale technologies.  $\rho$  may be interpreted as a measure of the inverse of the intertemporal elasticity of substitution in consumption: if  $\rho$  tends to 0,  $u$  becomes linear and the elasticity tends to infinity. Note also that the greater  $\rho$  is, the lower the agent's level of consumption saturation is.

In continuous-time infinite-horizon models, the weak concavity of  $V(\cdot, z)$  for every given  $z \in \mathcal{D}_k$  has been used to establish the existence of endogenous business cycles with low discounting. Indeed, we know since Rockafellar [22] that the more concave is the indirect utility function, the higher is the value of the discount rate  $\delta$  below which the turnpike property holds. This result then suggests that if the degree of concavity of  $V$  is low enough, the steady state can become unstable with a low discount rate. Assuming that  $V(\cdot, z)$  is concave- $\varphi$  for every given  $z \in \mathcal{D}_k$ , Venditti [29] shows that endogenous persistent fluctuations occur through a Hopf bifurcation when the discount rate  $\delta$  is larger than a bound  $\delta^*$  which is bounded above by an increasing function of  $\varphi$ . As a consequence of Theorem 1, the smaller are the degrees  $\alpha$  and  $\rho$  of weak concavity of the pure consumption good's production function  $f^0$  and of the utility function  $u$ , the lower are the discount rate values compatible with the endogenous business cycles. This general result is illustrated using the example of a three-sector economy with Cobb-Douglas technologies and a linear utility function provided by Benhabib and Rustichini [3].

## 5 On the value function of discrete-time models

As we explained in the introduction, weak concavity has been extensively used in discrete-time models to prove Lipschitz-continuity of the policy function and turnpike results.<sup>6</sup> In such a framework, the infinite-horizon problem ( $\mathcal{P}$ ) is strictly related to the Bellman equation

$$W(k) = \max_{z \in \mathcal{D}_k} \{V(k, z) + \beta W(z)\} \quad (15)$$

More precisely, it is shown in Montrucchio [15, 17] and Sorger [26, 27] that if the value function  $W(k)$  is concave- $\gamma$ , then the policy function as defined by  $z = h(k) = \operatorname{argmax}_{z \in \mathcal{D}_k} \{V(k, z) + \beta W(z)\}$  is Lipschitz-continuous. Moreover, under the same restriction, Montrucchio [17] shows that the turnpike property holds for any  $\beta > \beta^*$  with  $\beta^*$  an increasing function of  $\gamma$ .

Based on Theorem 1, we can provide precise restrictions on the fundamentals that ensure the weak concavity of the value function  $W(k)$ .

**Corollary 1.** *Let Assumptions 1-4 hold. Assume that  $f^0$  is concave- $\alpha$  with*

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<sup>6</sup>See Boldrin and Montrucchio [3], Montrucchio [15, 17], Sorger [26, 27].

$\alpha > 0$ , and that  $u$  is concave- $\rho$  with  $\rho > 0$ . Then the value function  $W(k)$  as defined by (15) is concave- $\varphi$  with  $\varphi \leq \alpha(1+n)r + \rho p^2$ .

According to Theorem 4.1 in Montrucchio [17], this Corollary shows that in discrete-time infinite-horizon models, other things being equal, the smaller are the degrees  $\alpha$  and  $\rho$  of weak concavity of the pure consumption good's production function  $f^0$  and of the utility function  $u$ , the closer to 0 will be the lower bound  $\beta^*$  above which the turnpike property holds. This conclusion drastically differs from the one established by Benhabib and Rustichini [3] and Venditti [29] within continuous-time models where the turnpike property becomes less robust when  $\alpha$  and  $\rho$  are closer to 0.

## 6 Appendix

### 6.1 Proof of Lemma 1

Consider program  $(\tilde{\mathcal{P}}_{k,y})$ . Let  $X_i = (k_{1i}, \dots, k_{ni}, l_i)$ ,  $i = 0, \dots, n$ , and  $k = (k_1, \dots, k_n)$  be the vector of capital goods stocks.  $(\tilde{\mathcal{P}}_{k,y})$  may be more precisely defined from a mathematical point of view using the following procedure. Defining a new objective function  $\tilde{f}^0(X_0, X_1, \dots, X_n) \equiv f^0(X_0, (\tilde{\mathcal{P}}_{k,y})$  becomes: maximize  $\tilde{f}^0(X_0, X_1, \dots, X_n)$  with respect to  $(X_0, X_1, \dots, X_n)$ , subject to the constraints (3)-(5). For a given vector  $(k, y)$ , the set of admissible vectors for  $(\tilde{\mathcal{P}}_{k,y})$  is defined as follows:

$$\Theta_{k,y} = \left\{ (X_0, X_1, \dots, X_n) \in (\mathbb{R}^{n+1})^{n+1} / X_i \geq 0, i = 0, \dots, n, \right. \\ \left. f^j(X_j) \geq y_j, j = 1, \dots, n \right\}$$

Using the constraints (4) and (5), we obtain:

$$\|X_i\| \leq \sqrt{\sum_{j=1}^n k_j^2 + 1} < +\infty \text{ for } i = 0, 1, \dots, n$$

so that

$$\|(X_0, \dots, X_n)\| = \sqrt{\sum_{i=0}^n \|X_i\|^2} \leq \sqrt{n+1} \sqrt{\sum_{j=1}^n k_j^2 + 1} < +\infty$$

Then there exists  $C > 0$  such that  $(X_0, \dots, X_n) \in \Theta_{k,y}$  implies  $\|(X_0, \dots, X_n)\| \leq C$ , i.e.  $\Theta_{k,y}$  is a bounded subset of  $(\mathbb{R}^{n+1})^{n+1}$ . Therefore,

under Assumption 1,  $\Theta_{k,y}$  is a compact set. Let us now consider the optimization program  $(\mathcal{P}_{k,y})$  which gives the social production function  $T$ . For every given  $(k, y)$ , the set of admissible vectors is now

$$\mathcal{Q}_{k,y} = \left\{ (\mathcal{A}(k, X_1, \dots, X_n), X_1, \dots, X_n) \in (\mathbb{R}^{n+1})^{n+1} / X_j \geq 0, \right. \\ \left. \mathcal{A}(k, X_1, \dots, X_n) \geq 0, f^j(X_j) \geq y_j, j = 1, \dots, n \right\}$$

The problem is then to know the link between  $\Theta_{k,y}$  and  $\mathcal{Q}_{k,y}$ . Using  $\tilde{f}^0$ , we obtain  $F^0(k, X_1, \dots, X_n) \equiv \tilde{f}^0(\mathcal{A}(k, X_1, \dots, X_n), X_1, \dots, X_n)$ . Therefore, we have  $\Theta_{k,y} = \mathcal{Q}_{k,y}$  since the constraint  $\mathcal{A}(k, X_1, \dots, X_n) \leq 0$  of program  $(\mathcal{P}_{k,y})$  is equivalent to the constraints (4) and  $X_0 \geq 0$  of program  $(\tilde{\mathcal{P}}_{k,y})$ . It follows that  $\mathcal{Q}_{k,y}$  is a compact set, and under Assumption 1,  $(\mathcal{P}_{k,y})$  has an optimal solution.  $\square$

## 6.2 Proof of Proposition 1

Let  $x, y \in \mathbb{R}$  be such that  $x = f(a)$  and  $y = f(b)$  with  $a = (a_1, a_2)$ ,  $b = (b_1, b_2)$  in  $\mathcal{F}$ . For all  $t \in [0, 1]$  we have

$$g(tf(a) + (1-t)f(b)) \leq tg(f(a)) + (1-t)g(f(b)) + \frac{\gamma}{2}t(1-t)\|f(a) - f(b)\|^2$$

Moreover,  $f$  is a concave- $(\alpha, \beta)$  function such that

$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b) + \frac{\alpha}{2}t(1-t)\|a_1 - b_1\|^2 + \frac{\beta}{2}t(1-t)\|a_2 - b_2\|^2$$

Since  $g$  is a monotone increasing function we get

$$g\left(f(ta + (1-t)b) - \frac{\alpha}{2}t(1-t)\|a_1 - b_1\|^2 - \frac{\beta}{2}t(1-t)\|a_2 - b_2\|^2\right) \\ \leq t(g \circ f)(a) + (1-t)(g \circ f)(b) + \frac{\gamma}{2}t(1-t)\|f(a) - f(b)\|^2$$

Owing to the mean value theorem, there exist  $\theta, \eta \in (0, 1)$  such that

$$g\left(f(ta + (1-t)b) - \frac{\alpha}{2}t(1-t)\|a_1 - b_1\|^2 - \frac{\beta}{2}t(1-t)\|a_2 - b_2\|^2\right) - g(f(ta + (1-t)b))$$

$$= -Dg(\theta X + (1-\theta)Y) \frac{1}{2}t(1-t) (\alpha\|a_1 - b_1\|^2 + \beta\|a_2 - b_2\|^2)$$

$$\text{with } X = f(ta + (1-t)b) - \frac{\alpha}{2}t(1-t)\|a_1 - b_1\|^2 - \frac{\beta}{2}t(1-t)\|a_2 - b_2\|^2,$$

$$Y = f(ta + (1-t)b), \text{ and}$$

$$f(a) - f(b) = Df(\eta a + (1-\eta)b)(a_1 - b_1, a_2 - b_2)$$

Assume now that  $g'(x)$  is bounded from above by a finite number  $q$ . Since

$$\|(x, y)\|^2 = \|x\|^2 + \|y\|^2, \text{ we finally obtain}$$

$$(g \circ f)(ta + (1-t)b) \leq t(g \circ f)(a) + (1-t)(g \circ f)(b)$$

$$+ \frac{qt(1-t)}{2} (\alpha\|a_1 - b_1\|^2 + \beta\|a_2 - b_2\|^2) + \frac{\gamma p^2 t(1-t)}{2} (\|a_1 - b_1\|^2 + \|a_2 - b_2\|^2)$$

with  $p = \sup_{x \in \mathcal{F}} \|Df(x)\|$  and  $\|Df(x)\| = \sup_{\nu \neq 0} \|Df(x)\nu\| / \|\nu\|$ . Therefore,  $g \circ f$  is concave- $(\varphi, \chi)$  over  $\mathcal{F}$  with  $\varphi \leq \alpha q + \gamma p^2$  and  $\chi \leq \beta q + \gamma p^2$ .  $\square$

### 6.3 Proof of Proposition 2

Before proving Proposition 2, we need to establish two useful Lemma.

**Lemma 6.1.** *Let  $\mathcal{A}$  be the affine map as given by (6). Let  $f^0$  be concave- $\alpha$  with  $\alpha > 0$ . Then under Assumption 1-2,  $f^0 \circ \mathcal{A}$  is concave- $\gamma$  with  $\gamma \leq \alpha(n+1)$ .*

*Proof:* Let  $x, x' \in \mathbb{R}^{n(n+2)}$ . Since  $f^0$  be a concave- $\alpha$ , and  $\mathcal{A}$  is an affine map, we have

$$\begin{aligned} f^0(\mathcal{A}(tx + (1-t)x')) &\leq tf^0(\mathcal{A}(x)) + (1-t)f^0(\mathcal{A}(x')) + \frac{\alpha t(1-t)}{2} \|\mathcal{A}(x - x')\|^2 \\ &\leq tf^0(\mathcal{A}(x)) + (1-t)f^0(\mathcal{A}(x')) + \frac{\alpha \|\mathcal{A}\| t(1-t)}{2} \|x - x'\|^2 \end{aligned}$$

The linear part  $A$  of  $\mathcal{A}$  is defined as follows

$$\begin{pmatrix} I_{n \times n} & -I_{n \times (n+1)} & \cdots & -I_{n \times (n+1)} \\ \underbrace{0 \cdots 0}_n & \underbrace{0 \cdots 0}_n - 1 & \cdots & \underbrace{0 \cdots 0}_n - 1 \end{pmatrix}$$

with  $I_{n \times m}$  the  $n \times m$  identity matrix. The matrix product  $AA^T$  is equal to the diagonal matrix  $I_{(n+1) \times (n+1)} \cdot \overbrace{(n+1, \dots, n+1, n)}^n$ . The greater eigenvalue in absolute value is then equal to  $n+1$ , and the Euclidean norm of  $A$  is:  $\|A\| = \sqrt{n+1}$  (see Strang [28]). Then the composite function  $f^0 \circ \mathcal{A}$  is concave- $\gamma$  with  $\gamma \leq \alpha(n+1)$ .  $\square$

We have now to state a technical result which allows to study the constraints of program  $(\mathcal{P}_{k,y})$ , and which is used to prove Proposition 2. Let us consider the following program which is equivalent to  $(\mathcal{P}_{k,y})$ :

$$\max f(x_1, \dots, x_m)$$

$$s.t. (x_1, \dots, x_m) \in \mathcal{D} \text{ and } g_1(x_1) \geq z_1, \dots, g_m(x_m) \geq z_m$$

with  $f$  and  $g_i$  some real-valued concave functions, and  $\mathcal{D}$  a convex set. This new program is denoted  $(\mathcal{P}_z)$ . In the following, let  $\mathcal{MI} = \{f : \mathbb{R}^m \rightarrow \mathbb{R} / \partial f / \partial x_i > 0, i = 1, \dots, m\}$  be the set of real-valued strictly increasing functions, and  $\mathcal{MD} = \{f : \mathbb{R}^m \rightarrow \mathbb{R} / \partial f / \partial x_i < 0, i = 1, \dots, m\}$  be the set of real-valued strictly decreasing functions.

**Lemma 6.2.** *Let  $f$  and  $g_i, i = 1, \dots, m$ , be some real-valued concave functions. Let  $z = (z_1, \dots, z_m)^T$ ,  $\bar{x}(z) = (\bar{x}_1(z), \dots, \bar{x}_m(z))$  be the optimal solution of program  $(\mathcal{P}_z)$ , and  $\mathcal{J}(z) = \{i / g_i(\bar{x}_i(z)) = z_i\}$  be the set of binding constraints. If  $f \in \mathcal{MD}$  (resp.  $\mathcal{MI}$ ) and if  $g_i \in \mathcal{MI}$  (resp.  $\mathcal{MD}$ ) for  $i = 1, \dots, m$ , then  $\#\mathcal{J}(z) = m$ .*

*Proof:* Let  $\bar{x}(z) = (\bar{x}_1(z), \dots, \bar{x}_m(z))$  be the optimal solution of program  $(\mathcal{P}_z)$ . Assume that  $\bar{x}(z)$  is such that a constraint  $i = i_0$  is not binding, i.e.  $g_{i_0}(\bar{x}_{i_0}(z)) > z_{i_0}$ . Let us then consider  $\hat{x} \in \mathcal{D}$  such that  $\hat{x}_i = \bar{x}_i(z)$  for every  $i \neq i_0$ , and  $g_{i_0}(\hat{x}_{i_0}) > z_{i_0}$ . Assume first that  $f \in \mathcal{MD}$  and  $g_i \in \mathcal{MI}$  for all  $i = 1, \dots, m$ . It is then easy to verify that  $\hat{x}_{i_0} < \bar{x}_{i_0}(z)$  and  $f(\hat{x}) > f(\bar{x}(z))$ . Therefore, if  $\bar{x}(z)$  is the optimal solution, we cannot have  $g_{i_0}(\bar{x}_{i_0}(z)) > z_{i_0}$ . Since this result holds for any  $i$ ,  $\bar{x}(z)$  is necessarily such that all the constraints are binding. The same result holds if  $f \in \mathcal{MI}$  and  $g_i \in \mathcal{MD}$ . On the other hand, if  $f, g_i \in \mathcal{MD}$ , or  $f, g_i \in \mathcal{MI}$ , the same reasoning proves that no constraint is binding.  $\square$

*Proof of Proposition 2:* Under Assumption 1, and from the definition of  $F^0$ , Lemma 6.2 implies that every constraint  $y_j \leq f^j(X_j)$ ,  $j = 1, \dots, n$ , is binding. Let  $k, k', y \in \mathbb{R}^n$  be such that  $(k, y), (k', y) \in \mathcal{K}$  and  $\bar{k} = tk + (1 - t)k'$ . Let  $X^*(k, y)$ ,  $X^*(k', y)$  and  $X^*(\bar{k}, y)$  be the corresponding solutions of  $(\mathcal{P}_{k,y})$  with  $T(k, y) = F^0(k, X^*(k, y))$ ,  $T(k', y) = F^0(k', X^*(k', y))$  and  $T(\bar{k}, y) = F^0(\bar{k}, X^*(\bar{k}, y))$ . If  $f^0$  is concave- $\alpha$ , Lemma 6.1 implies that  $F^0$  is concave- $\gamma$  with  $\gamma \leq \alpha(n + 1)$ . Moreover, under Assumption 1, for each given  $y \geq 0$ , the function  $X^*(\cdot, y)$  is continuous (see Benhabib and Nishimura [1], pp. 438-441). Then we get

$$T(\bar{k}, y) \leq tF^0(k, X^*(\bar{k}, y)) + (1 - t)F^0(k', X^*(\bar{k}, y)) + \frac{\gamma t(1-t)}{2} \|k - k'\|^2$$

But since  $T(k, y) \geq F^0(k, X^*(\bar{k}, y))$  and  $T(k', y) \geq F^0(k', X^*(\bar{k}, y))$ , we conclude that  $T(\bar{k}, y) \leq tT(k, y) + (1 - t)T(k', y) + \frac{\gamma}{2} t(1 - t) \|k - k'\|^2$  and the value function  $T(\cdot, y)$  is concave- $\gamma$  with  $\gamma \leq \alpha(1 + n)$ .  $\square$

## 6.4 Proof of Theorem 1

Before proving Theorem 1, we need to establish a last useful Lemma.

**Lemma 6.3.** *Let  $T(\cdot, y)$  be concave- $\gamma$  for each given  $y \geq 0$ , and  $B$  be the linear map as given by (12). Then  $(T_o B)(\cdot, z)$  is concave- $\gamma$  for any given  $z \in \mathcal{D}_k = \{\zeta \in \mathbb{R}^n / (k, \zeta) \in \mathcal{D}\}$ .*

*Proof:* Let  $(k, z), (k', z) \in \mathcal{D}$ . Since  $T(\cdot, y)$  is concave- $\gamma$  for any given  $y \geq 0$ , and  $B$  is linear, we have

$$(T_o B)(t(k, z) + (1-t)(k', z)) \leq t(T_o B)(k, z) + (1-t)(T_o B)(k', z) + \frac{\gamma}{2} t(1-t) \|k - k'\|^2$$

which proves that  $(T_o B)(\cdot, z)$  is concave- $\gamma$  for any given  $z \in \mathcal{D}_k$ .



□

*Proof of Theorem 1:* Lemmas 6.1-6.3 and Proposition 2 imply that if  $f^0$  is concave- $\alpha$ , then  $(T_oB)(\cdot, z)$  is concave- $\gamma$  for any given  $z \in \mathcal{D}_k$  with  $\gamma \leq \alpha(n+1)$ . Moreover,  $(T_oB)$  satisfies

$$\|D(T_oB)(k, z)\| = \|D(T(B(k, z))B)\| \leq \|D(T(B(k, z)))\| \cdot \|B\|$$

Under Assumption 3, we have  $\sup_{(k,z) \in \mathcal{D}} \|D(T_oB)(k, z)\| \leq \tau \|B\| \equiv p$ . If  $u$  is concave- $\rho$  and satisfies Assumption 4, Proposition 1 implies that  $V(\cdot, z)$  is concave- $\varphi$  for every given  $z \in \mathcal{D}_k$  with  $\varphi \leq \alpha(1+n)r + \rho p^2$ . □

## 6.5 Proof of Corollary 1

Theorem 1 shows that under Assumptions 1-4, if  $f^0$  is concave- $\alpha$  and  $u$  is concave- $\rho$ ,  $V(\cdot, z)$  is concave- $\varphi$  for every given  $z \in \mathcal{D}_k$  with  $\varphi \leq \alpha(1+n)r + \rho p^2$ . Consider the Bellman equation  $W(k) = \max_{z \in \mathcal{D}_k} \{V(k, z) + \beta W(z)\}$ . Let  $(k_1, y_1), (k_2, y_2) \in \mathcal{D}$  such that  $W(k_1) = V(k_1, z_1) + \beta W(z_1)$ ,  $W(k_2) = V(k_2, z_2) + \beta W(z_2)$  and  $\bar{k} = tk_1 + (1-t)k_2$  with  $t \in [0, 1]$ . We get

$$W(\bar{k}) = \max_{z \in \mathcal{D}_{\bar{k}}} \{V(\bar{k}, z) + \beta W(z)\} = V(\bar{k}, \bar{z}) + \beta W(\bar{z})$$

with  $\bar{z} = z(\bar{k})$  the solution of the Bellman equation. Since  $V(\cdot, z)$  is concave- $\varphi$  for every given  $z \in \mathcal{D}_k$  we conclude

$$W(\bar{k}) \leq t[V(k_1, \bar{z}) + \beta W(\bar{z})] + (1-t)[V(k_2, \bar{z}) + \beta W(\bar{z})] + \frac{\gamma t(1-t)}{2} \|k_1 - k_2\|^2$$

But as by definition  $W(k_1) \geq V(k_1, \bar{z}) + \beta W(\bar{z})$  and  $W(k_2) \geq V(k_2, \bar{z}) + \beta W(\bar{z})$ , we derive  $W(\bar{k}) \leq tW(k_1) + (1-t)W(k_2) + \frac{\gamma t(1-t)}{2} \|k_1 - k_2\|^2$  and the value function  $W(k)$  is also concave- $\gamma$ . □

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