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Abstract

We investigate the effects of collaterals and monetary policy on growth rate dynamics in a Ramsey economy where agents have heterogeneous discount factors. We focus on the existence of business-cycle fluctuations based on self-fulfilling prophecies and on the occurrence of deterministic cycles through bifurcations. We introduce liquidity constraints in segmented markets where impatient (poor) agents without collaterals have limited access to credit. We find that an expansionary monetary policy may promote economic growth while making endogenous fluctuations more likely. Conversely, a regulation reinforcing the role of collaterals and reducing the financial market imperfections may enhance the economic growth and stabilize the economy.

Key words: Collaterals, heterogeneous agents, balanced growth, endogenous fluctuations, stabilization policies.

1 Introduction

The financial crisis of 2007-2008, known as subprime mortgage crisis, led at the very end to tightening credit and slowing economic growth in the U.S. and Europe. The drop in the value of collaterals, the titrization of bad credits and the spread of toxic assets promoted a liquidity crisis together with financial and real market instability. Two main ingredients are at the origin of such a global crisis: a strong development of subprime loans distributed to households with insufficient collaterals and new banking practices with respect to risk based on insufficient banks' own funds. The U.S. and European governments and central banks have then decided to increase the collateral constraints to both households and banks to avoid bad credits, and to increase the supply of money to fight against the lack of liquidity. However, besides the problems related

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to sovereign debts, the effects of these policies on the growth rate have been contrasted. While the U.S. seem to slightly recover its pre-2007 growth rate, most European countries suffer from stagnation or even recession.

In this paper, we address the stability issue of growth rate dynamics considering the role of collaterals in a monetary economy with credit market imperfections. In the spirit of Barbar and Bosi (2010), we assume that agents pay a part of their consumption purchases in cash, while the remainder is financed on credit whose extent depends on the amount of collaterals. One of our goal is to understand the role of collaterals in the occurrence of economic crises viewed as self-fulfilling prophecies.

We consider a credit constraint, formulated as an extended cash-in-advance constraint, which is similar in spirit to the credit constraint considered in Kiyotaki and Moore (1997). In our framework, consumers can borrow to consume more than the cash they hold but they are constrained by the amount of collateral they own. Similarly to Kiyotaki and Moore, we show that the credit constraint crucially affects the existence of endogenous fluctuations. As explained below, depending on whether this constraint is binding for all agents or only a part of them, we can get two different regimes, one with expectation-driven fluctuations based on the existence of sunspot equilibria, and one without transitional dynamics in which the equilibrium immediately jumps on the long run balanced growth path (BGP). Although we do not consider a micro-founded formulation based on monetary search models *à la* Lagos and Wright (2005), our results are also similar to the main conclusions of Ferraris and Watanabe (2011).[1] Embedding a model of credit *à la* Kiyotaki and Moore (1997) into a money search model *à la* Lagos and Wright (2005), these authors show that larger fluctuations arise when the credit constraint is binding and agents are at their borrowing limit.[2]

In this paper, we consider an economy where agents have different discount rates, cannot borrow against future labor income, but are allowed to pay a part of current consumption on credit if collaterals are positive. We know that heterogeneous discounting promotes the accumulation of capital in the hands of the most patient agents (the capitalists). If individuals are allowed to borrow against future income, the impatient agents borrow from the patient ones and spend the rest of their life to work in order to refund the debt. In the spirit of Becker (1980), Becker and Foias (1987, 1994), but in contrast to Le Van and Vailakis (2003), we also introduce a borrowing constraint: we rule out the possibility of negative capital (of debt). Through the credit constraint, a positive capital works as collateral to reduce the amount of balances needed to finance the current consumption. Thus markets are incomplete and collaterals weaken the cash-in-advance.

We consider an endogenous growth model where technology contains a Romer-type (1986) learning-by-doing externality leading to constant returns to scale

[1] See also Ferraris and Watanabe (2008).

[2] See also Fostel and Geanakoplos (2008) and Geanakoplos and Zame (2007) in which the existence of collateralized assets affects market prices and allocations and can generate fluctuations.

in capital accumulation. Our aim is indeed to study the impact of borrowing constraints and collaterals on the fluctuations of the growth rate. We will then focus on the existence of business-cycle fluctuations based on self-fulfilling prophecies and on the occurrence of deterministic cycles through bifurcations.

Under heterogeneous discounting, the population splits into two classes: patient capitalists and impatient workers.[3] We prove that there is room for two regimes: the one where the capitalist holds capital and no balances, the other where money and capital enter her/his portfolio. In the first regime, the credit constraint is not binding and the liquidity constraint does not affect patient agents' capital accumulation. The economy jumps on the balanced growth path. This first conclusion is similar to the configuration with agents below their borrowing limits considered by Ferraris and Watanabe (2011). In this case, they show that fluctuations never occur in capital. In the second regime, the credit constraint of patient agents binds and equilibrium transitions take place: endogenous fluctuations (two-period cycles and indeterminacy) may arise around the balanced growth path. This conclusion is now similar to the configuration with agents at their borrowing limits considered by Ferraris and Watanabe (2011) where fluctuations in capital occur. We also prove that an expansionary monetary policy may be growth-enhancing and affects welfare but may at the same time have a destabilizing effect.

More precisely, in the first regime (capitalists hold no money), money growth improves the patient agents' welfare, while worsening that of impatient agents, who bear the opportunity cost of holding balances (nominal interest rate). It affects the initial consumption, but has no effect on the growth rate. Focussing on the welfare differential between the two classes of agents along the BGP, we find an increase in social inequalities: the patient agents are richer because they own the entire stock of capital and experience a higher welfare level through an increase of consumption; the impatient ones are poorer and experience a welfare deterioration.

In the second regime (capitalists hold money), money growth has a positive impact on the growth rate but an ambiguous effect on initial consumptions and, therefore, on agents' welfare. This regime may be very different from the first one: to highlight the possibility of the opposite effects, we provide sufficient conditions on structural parameters and a numerical exercice with plausible parameter values showing that money growth worsens the welfare of capitalists (who now also bear the opportunity cost of holding balances) and improve the welfare of workers. Mimicking the previous interpretations, we conclude that an increase of the money growth rate may reduce the inequalities measured as welfare differential between the agents along the BGP.

In addition, money growth promotes the occurrence of endogenous fluctuations through self-fulfilling prophecies and crises. Indeed, it raises the inflation rate and the nominal interest rate, that is the opportunity cost of holding money.

[3]Woodford (1986) also studies an economy with patient capitalists and impatient workers financially constrained. Even if these agents are infinite-lived, his model looks like an overlapping-generations model because of the form of the borrowing constraint. Our model is closer to more traditional Ramsey models with cash-in-advance.

The imperfection (liquidity constraint) has a larger impact on the consumption smoothing of patient agents and makes indeterminacy more likely. Conversely, the possibility of collateralization moderates the effect of the credit market imperfection and makes the endogenous fluctuations less likely. Then, we conclude that, if the goal of the government is to avoid growth cycles, any increase of money supply should be followed by an increase of the sensitivity of credit grants to collaterals. In the light of the recent crisis, our results could be interpreted as follows: since central banks increase the money supply through quantitative easing policies, raising the weight of collaterals for all agents could also represent a mean to shelter the economy from the destabilizing effects of quantitative easing.

This paper is organized as follows. In Sections 2 and 3, we present the model and we derive the first-order conditions. In Section 4, we study the equilibrium properties when capitalists hold no money. Section 5 considers the case where they are also liquidity-constrained. Section 6 concludes. Mathematical details are gathered in the Appendix.

2 Fundamentals

In this model, firms produce a single good, households work, consume and save through capital and balances provided by a monetary authority. We specify the fundamentals (technology, preferences, endowments) as follows.

2.1 Homogeneous firms

Firms are represented by a technology with private constant returns and increasing social returns. There is a large number (J) of identical competitive firms. The representative firm rents capital and labor to produce the good under constant (private) returns to scale. External effects of capital intensity spill over the other firms. Technology is rationalized by a Romer-type (1986) production function that allows the existence of endogenous growth.

Assumption 1. *The production function is given by* $F(K_j, L_j) \equiv A\bar{k}^{1-s}K_j^s L_j^{1-s}$ *with* $s \in (0,1)$*, and where* K_j *and* L_j *are the* j*th firm's inputs and* $\bar{k}$ *is the average capital intensity in the economy.*

In the following, we denote by $K \equiv \sum_{j=1}^{J} K_j$ and $L \equiv \sum_{j=1}^{J} L_j$ the aggregate capital and labor force, respectively. The TFP ($A\bar{k}^{1-s}$) is affected by productive externalities of average capital intensity $\bar{k} \equiv K/L$, such as knowledge spillovers. Since the firms share the same technology, we obtain at equilibrium $k = \bar{k} = K_j/L_j$ for every j. Each producer is price-taker and maximizes the profit: $A\bar{k}^{1-s}K_j^s L_j^{1-s} - rK_j - wL_j$. Under Assumption 1, profit maximization requires productivities to be equal to factor prices:

$$r_t = sA \equiv r \quad (1)$$

$$w_t = (1-s)Ak_t \quad (2)$$

where r_t and w_t are the real interest rate and the wage bill at period t. The equilibrium interest rate is constant over time as in standard AK endogenous growth models à la Romer (1986).

2.2 Heterogenous households

There are n individuals in the economy who are differently endowed with capital. Each individual has indeed an initial endowment of capital $k_{i0} \geq 0$. His/her utility function has a constant elasticity of intertemporal substitution σ.

Assumption 2. *The utility function is given by* $u(c_i) \equiv c_i^{1-1/\sigma}/(1-1/\sigma)$ *for* $\sigma \neq 1$, *with* $\sigma > 0$, *and* $u(c_i) \equiv \ln c_i$ *for* $\sigma = 1$.

Individuals are also characterized by different time preferences. Without loss of generality, we consider from now on two types of heterogeneous infinite-lived agents: $i = 0, 1$, with heterogeneous discount factors β_i. We call 0 the patient type and 1 the impatient one.

Assumption 3.1. $\beta_1 < \beta_0$.

There are n_i individuals of type i with $n_0 + n_1 = n$. Each consumer i maximizes the following intertemporal additive utility function

$$\sum_{t=0}^{\infty} \beta_i^t u(c_{it}) \tag{3}$$

with respect to a sequence $(k_{it+1}, c_{it})_{t=0}^{\infty}$ under a sequence of budget constraints:

$$M_{it+1} - M_{it} + p_t (k_{it+1} - \Delta k_{it}) + p_t c_{it} \leq p_t r_t k_{it} + p_t w_t l_{it} + p_t \tau_{it} \tag{4}$$

and borrowing constraints

$$p_t c_{it} - \gamma p_t k_{it} \leq M_{it} \tag{5}$$

with $k_{it} \geq 0$ and $\delta = 1 - \Delta \in (0, 1)$, the depreciation rate of capital. Consumption (c_i) and capital (k_i) are the same good: they are produced with the same technology and have the same price (p).

The RHS of constraint (4) is the monetary income at the individual disposal. In real terms, it is the sum of capital income ($r_t k_{it}$), labor income ($w_t l_{it}$) and transfers from the monetary authority (τ_{it}). This disposable income is partly saved in form of money ($M_{it+1} - M_{it}$) and capital ($k_{it+1} - \Delta k_{it}$), and partly consumed. In addition, the representative household faces the partial cash-in-advance constraint (5): in the spirit of Grandmont and Younès (1972), we assume that a share of purchases is paid cash, while the rest on credit. We are also close to Lucas and Stokey (1987) with a cash and a credit good.

The individual's wealth matters in order to pay current consumption on credit. We consider indeed a cash-in-advance constraint which is extended to introduce a credit constraint in the spirit of the constraint considered by Kiyotaki and Moore (1997). The amount of credit depends on household's collaterals. We assume for simplicity that the credit grant is proportional to nominal collaterals ($\gamma p_t k_{it}$). γ represents either the common credit sensitivity to collaterals or the market rules. More precisely, γ captures the lenders' or sellers' prudential attitude towards borrowers in presence of asymmetric informations, but also credit market regulation policies, that is a legal constraints to credit grants in order to ensure borrowers' solvability. To buy a constant amount of consumption on credit $d = \gamma k_{it}$, the required amount of collateral is given by $k_{it} = d/\gamma$: so, the higher the parameter γ, the lower the needs of collateral to buy the amount d on credit. If γ is sufficiently high, the cash-in-advance is no longer binding and the capitalist no longer needs balances to finance the consumption purchases. Put differently, increasing γ can then be interpreted as a policy generating higher consumption based on a larger agents' solvability.

Note also that the introduction of collaterals in a model of capital accumulation implies that the velocity of money with respect to consumption becomes endogenous. Our formulation then takes into account one of the criticisms addressed to the cash-in-advance models: the implausibility of a constant velocity of money.[4]

In order to simplify the model, we finally assume that patient agents supply no labor, while impatient agents supply one unit of labor.

Assumption 4. $l_{0t} = 0$ *and* $l_{1t} = 1$.

It is worth noting that Assumption 4 is not restrictive. Indeed, similar models with endogenous labor supply may easily exhibit a steady state where patient agents supply no labor (see Bosi and Seegmuller (2010)). Since our main results are robust to the consideration of endogenous labor, we will focus on the case with inelastic labor for purpose of tractability.

2.3 Monetary authority

Let M_t denotes the aggregate supply of money at time t. Each period, an amount $M_{t+1} - M_t$ of money is "helicoptered" to agents. More precisely, an exogenous share θ_i is given to agents of type i. For simplicity, we assume also the same shares as initial conditions: $n_i M_{i0} = \theta_i M_0$ for $i = 0, 1$. In addition, money supply grows at a constant factor $\eta \equiv M_{t+1}/M_t$. The monetary policy consists in a pair of instruments (η, θ_0) because the other share is determined: $\theta_1 = 1 - \theta_0$. Normally, real economies experience nonnegative money growth rates.

Assumption 5. $\eta \geq 1$.

[4]Money-In-the-Utility (MIU) models are immunized against this criticism: the functional equivalence highlighted by Feenstra (1986) between CIA and MIU, no longer holds with a Cash-When-I'm-Done (CWID) timing or a CIA timing and strictly positive elasticity of substitution between consumption and real balances (see Carlstrom and Fuerst, 2003).

3 First-order conditions

The individual i maximizes the intertemporal utility (3) under the constraints (4) and (5). Under Assumption 2, her/his demand for money and for goods rests on the first-order conditions:

$$\beta_i^t c_{it}^{-1/\sigma} = p_t \left(\lambda_{it} + \mu_{it}\right) \tag{6}$$

$$p_t \lambda_{it} \geq p_{t+1} \left[\lambda_{it+1} \left(r_{t+1} + \Delta\right) + \gamma \mu_{it+1}\right] \tag{7}$$

$$\lambda_{it} \geq \lambda_{it+1} + \mu_{it+1} \tag{8}$$

where λ_{it} and μ_{it} are the nonnegative multipliers associated to constraints (4) and (5) respectively.

Let $\pi_{t+1} \equiv p_{t+1}/p_t$ denote the inflation factor. Equation (7) and (8) hold with equality if $k_{it+1} > 0$ and $M_{t+1} > 0$ respectively. The additional Karush-Kuhn-Tucker conditions hold:

$$\begin{aligned} p_t \lambda_{it} \left(r_t k_{it} + w_t + \tau_{it} - \pi_{t+1} m_{it+1} + m_{it} - k_{it+1} + \Delta k_{it} - c_{it}\right) &= 0 \\ p_t \mu_{it} \left(m_{it} - c_{it} + \gamma k_{it}\right) &= 0 \end{aligned}$$

where $m_{it} = M_{it}/p_t$ is the individual demand for real balances. If $\mu_{it} > 0$, the cash-in-advance constraint is binding, while, if $\mu_{it} = 0$, the constraint becomes superfluous. In such a case, under the standard assumption of positive nominal interest rate, the agent holds no money because of the opportunity cost of money holding ($M_{it} = 0$).[5]

The initial and the final conditions are given by the initial endowments: $M_{i0}, k_{i0} \geq 0$ and the transversality conditions:

$$\lim_{t \to +\infty} \beta_i^t c_{it}^{-1/\sigma} \left(k_{it+1} + \pi_{t+1} m_{it+1}\right) = 0 \tag{9}$$

for each individual $i = 0, 1$.

In the following, we consider two regimes.

(1) The patient agent's liquidity constraint is always nonbinding.

(2) The patient agent's liquidity constraint is always binding.

For the sake of simplicity, we do not consider the case in which the equilibrium switches from a regime to another.

4 Patient agents buy on credit

When the patient agent is sufficiently wealthy, s/he does not need money: s/he holds enough collaterals to buy on credit (the liquidity constraint (5) is ineffective). In this section, we study the regime where money is demanded only by impatient agents and we characterize the equilibrium in the long run. The economy will appear to move along the balanced growth path without transition.

[5] Indeed, when $\mu_{it+1} = 0$, the inequality (7) becomes $\lambda_{it} \geq \lambda_{it+1} \pi_{t+1} \left(\Delta + r_{t+1}\right)$. Then, if, at the same time, $M_{it+1} > 0$, we get $\lambda_{it} = \lambda_{it+1}$ and $1 \geq \pi_{t+1} \left(\Delta + r_{t+1}\right)$ which implies a negative nominal interest rate.

4.1 Patient agent

In the first regime, the patient agent's liquidity constraint becomes superfluous. Thus, in the case of positive interest rate, s/he holds no money ($M_{0t} = 0$) and $c_{0t} < \gamma k_{0t}$. From (1) and first-order conditions (6)-(8) with $\mu_{0t} = 0$, we derive the Euler inequality

$$\left(\frac{c_{0t+1}}{c_{0t}}\right)^{1/\sigma} \geq \beta_0 \left(\Delta + sA\right)$$

which holds with equality if $k_{0t} > 0$, and the budget constraint:

$$k_{0t+1} - \Delta k_{0t} + c_{0t} = sAk_{0t} + \tau_{0t} \tag{10}$$

The transversality condition simplifies to:

$$\lim_{t \to +\infty} \beta^t c_{0t}^{-1/\sigma} k_{0t+1} = 0 \tag{11}$$

4.2 Impatient agent

Focus now on the impatient agent who needs money to finance future consumption. From (1) and first-order conditions (6)-(8) with $\mu_{it} > 0$ and $\lambda_{it} = \lambda_{it+1} + \mu_{it+1}$ (the liquidity constraint is binding), we derive the Euler inequality:

$$\left(\frac{c_{1t+1}}{c_{1t}}\right)^{1/\sigma} \geq \beta_1 \frac{\pi_t}{\pi_{t+1}} \frac{\Delta + sA - \gamma}{1 - \gamma \pi_t}$$

which holds with equality if $k_{1t+1} > 0$, and the constraints:

$$m_{1t+1}\pi_{t+1} - m_{1t} + k_{1t+1} - \Delta k_{1t} + c_{1t} = sAk_{1t} + w_t + \tau_{1t} \tag{12}$$

$$c_{1t} - \gamma k_{1t} = m_{1t} \tag{13}$$

The transversality condition (9) holds.

4.3 Balanced growth path

As usual in endogenous growth models with AK technology, the long-run equilibrium is characterized by a stationary growth rate. On this basis, a Balanced Growth Path (BGP) is a sequence where the real variables grow at the same rate, namely $k_{it+1} = g^* k_{it}$ and $c_{it+1} = g^* c_{it}$, where g^* is the stationary balanced growth factor and $g^* - 1$ the corresponding rate.

Within standard exogenous growth models with infinitely lived agents, the heterogenous discounting implies the Ramsey conjecture (1928) later proved by Becker (1981): impatient agents hold no capital in the long run. We can show now that this property also holds in our model along a BGP.

Focus first on the money market. Demand for balances depend on the cash-in-advance constraint. On the supply side, each agent of type i receives in period t an amount $\theta_i (M_{t+1} - M_t) / n_i$ of nominal balances from the monetary authority:

$$M_{t+1} - M_t = \theta_0 (M_{t+1} - M_t) + \theta_1 (M_{t+1} - M_t) = n_0 p_t \tau_{0t} + n_1 p_t \tau_{1t} \tag{14}$$

where

$$\tau_{it} = (m_{t+1}\pi_{t+1} - m_t)\, \theta_i / n_i \tag{15}$$

Money market clears. The equality between the aggregate supply and demand writes:

$$m_t = n_1 m_{1t} \tag{16}$$

As usual, in dynamic monetary models, money growth has both inflationary and real effects (growth of real balances):

$$\eta = \pi_{t+1} m_{t+1}/m_t \tag{17}$$

Replacing (16) in (17) gives:

$$\pi_{t+1} = \eta m_{1t}/m_{1t+1} \tag{18}$$

Along the BGP, the inflation factor $\pi = \eta/g^*$ is constant and the decomposition $\eta = g^*\pi$ holds. Aggregating side by side the budget constraints (4) now binding across individuals, we obtain

$$\begin{aligned} & n_0 \left(k_{0t+1} - \Delta k_{0t} + c_{0t}\right) + n_1 \left(m_{1t+1}\pi_{t+1} - m_{1t} + c_{1t}\right) \\ = \; & n_0 \left(r_t k_{0t} + \tau_{0t}\right) + n_1 \left(w_t + \tau_{1t}\right) \end{aligned}$$

Replacing (16) in the LHS and (14) in the RHS, we obtain

$$n_0 \left(k_{0t+1} - \Delta k_{0t}\right) + n_0 c_{0t} + n_1 c_{1t} = n_0 r_t k_{0t} + n_1 w_t$$

Replacing the firms' equilibrium prices (1) and (4.2), and noticing that $k_t \equiv K_t/L_t = n_0 k_{0t}/n_1$, that is $n_0 k_{0t} = n_1 k_t$, we have the good market clearing:

$$n_1 \left(k_{t+1} - \Delta k_t\right) + n_0 c_{0t} + n_1 c_{1t} = n_1 A k_t \tag{19}$$

that is, the aggregate investment plus the aggregate consumption is equal to the aggregate production.

Usually, the heterogeneity of discounting ($\beta_1 < \beta_0$) implies that patient agents hold the entire stock of capital in the long run. In our monetary economy where patient agents do not hold money, $\beta_1 < \beta_0$ (Assumption 3.1) no longer ensures this degenerate distribution of capital. Assumption 3.1 has indeed to be replaced by a more restrictive assumption.

Assumption 3.2. $\frac{\beta_0}{\beta_1} > \frac{(\Delta+sA-\gamma)[\beta_0(\Delta+sA)]^\sigma}{(\Delta+sA)([\beta_0(\Delta+sA)]^\sigma-\gamma\eta)}$

Obviously, when $\gamma = 0$, Assumption 3.2 reduces to Assumption 3.1. As we have shown that along a BGP, the inflation factor π is constant and given by $\eta/g^* = \pi$, Assumption 3.2 is actually equivalent to

$$\beta_0 \left(\Delta + sA\right) > \beta_1 \tfrac{\Delta+sA-\gamma}{1-\gamma\pi}$$

Let us then explain why Assumption 3.1 no longer implies a degenerate distribution of capital. In a monetary economy, the nominal interest rate has to be positive: $\pi\left(\Delta + sA\right) > 1$ (Fisher decomposition and zero lower bound). This inequality is equivalent to

$$\tfrac{(\Delta+sA-\gamma)[\beta_0(\Delta+sA)]^\sigma}{(\Delta+sA)([\beta_0(\Delta+sA)]^\sigma-\gamma\eta)} > 1$$

If $\beta_1 < \beta_0$ is chosen too close to β_0, we have also

$$\frac{(\Delta+sA-\gamma)[\beta_0(\Delta+sA)]^\sigma}{(\Delta+sA)([\beta_0(\Delta+sA)]^\sigma-\gamma\eta)} > \frac{\beta_0}{\beta_1}$$

or, equivalently,

$$\beta_0\left(\Delta+sA\right) < \beta_1 \frac{\Delta+sA-\gamma}{1-\gamma\pi}$$

However, along a BGP,

$$\frac{c_{0t+1}}{c_{0t}} = \frac{c_{1t+1}}{c_{1t}} = g^*$$

and, because of inequalities (4.1) and (4.2),

$$\beta_0\left(\Delta+sA\right) < g^{*1/\sigma} = \beta_1 \frac{\Delta+sA-\gamma}{1-\gamma\pi}$$

In this case, patient agents hold no capital. Since they hold no money (first regime) and do not work (Assumption 4), they have no mean to consume, that is a contradiction.

Conversely, under Assumption 3.2, patient agents hold capital while the impatient ones do not:

$$\frac{c_{0t+1}}{c_{0t}} = \frac{c_{1t+1}}{c_{1t}} = g^* = \left[\beta_0\left(\Delta+sA\right)\right]^\sigma > \left(\beta_1 \frac{\Delta+sA-\gamma}{1-\gamma\pi}\right)^\sigma$$

The impatient agent indeed wants to consume more today (c_{1t}) instead of tomorrow (c_{1t+1}) in order to lower the term c_{1t+1}/c_{1t} in the LHS of inequality and restore the equality. To this purpose, s/he dissaves over time. In order to finance the current consumption at any period, s/he ends up to hold no capital. The assumption of borrowing constraints requires $k_{it} \geq 0$ (Becker, 1981). Thus, in the long run $k_{1t} = 0$ for every t (even during transition). In the long run, only the patient agents hold capital.

The positivity of interest rate and the nonnegativity of the growth rate along the BGP means respectively:

$$\pi\left(\Delta+sA\right) = \frac{\eta(\Delta+sA)}{[\beta_0(\Delta+sA)]^\sigma} > 1 \text{ and } g^* = \left[\beta_0\left(\Delta+sA\right)\right]^\sigma \geq 1$$

or, equivalently, $1/\left(\Delta+sA\right) \leq \beta_0 < \eta^{1/\sigma}\left(\Delta+sA\right)^{1/\sigma-1}$. Under Assumption 5, a more restrictive condition is $1/\left(\Delta+sA\right) \leq \beta_0 < \left(\Delta+sA\right)^{1/\sigma-1}$. Clearly, the left inequality makes sense only if $\Delta+sA > 1$**.**

Assumption 6. $\Delta+sA > 1$ *and* $1/\left(\Delta+sA\right) \leq \beta_0 < \left(\Delta+sA\right)^{1/\sigma-1}$.

Around a BGP, equations (12) and (13) become

$$m_{1t+1}\pi_{t+1} - m_{1t} + c_{1t} = w_t + \tau_{1t} \quad (20)$$

$$c_{1t} = m_{1t} \quad (21)$$

and the transversality condition (9) simplifies:

$$\lim_{t\to+\infty} \beta_1^t c_{1t}^{-1/\sigma} \pi_{t+1} m_{1t+1} = 0 \quad (22)$$

To simplify the analysis, let us set

$$\left(g_{t+1}, x_{0t}, x_{1t}\right) \equiv \left(\frac{k_{t+1}}{k_t}, \frac{m_{0t}}{k_t}, \frac{m_{1t}}{k_t}\right)$$

In the following proposition, we prove that, as in the initial Romer (1986) model, the BGP is the only possible equilibrium and there is no room for transitional dynamics.

Proposition 1. *Let Assumptions 1, 2, 3.2, 4, 5 and 6 hold. Any equilibrium where patient agents only hold capital and impatient agents only hold money is a BGP with* $g_t = g^* \equiv [\beta_0 (\Delta + sA)]^\sigma \geq 1$ *and*

$$\begin{array}{rcl} (x_{0t}, x_{1t}) & = & \left(0, \frac{(1-s)A}{1+(\eta-1)\theta_0}\right) \\ c_{00} & = & k_0 \frac{n_1}{n_0} \left[\Delta + sA + \frac{(1-s)A(\eta-1)\theta_0}{1+(\eta-1)\theta_0} - g^*\right] > 0 \\ c_{10} & = & \frac{(1-s)Ak_0}{1+(\eta-1)\theta_0} > 0 \end{array} \tag{23}$$

Proof. See Appendix 7.1. ■

Since capitalists do not hold money, the parameter γ does not have any impact on the growth rate g^* and the initial consumptions c_{i0}. The fact that the BGP is the only equilibrium means that, under rational expectations, the forward-looking variable (consumption) adjusts to ensure that the economy jumps from the very beginning on the BGP, and there is no transition. This conclusion is similar to one of the main results of Ferraris and Watanabe (2011). They show indeed that when agents are below their borrowing limits, fluctuations never occur in capital.

Recall that the regime we consider is $c_{0t}/k_{0t} < \gamma$ with $k_{0t} = k_t$ since patient agents hold no money and impatient agents hold no capital. Using Proposition 1, we find that this inequality is equivalent to

$$\gamma > (\Delta + sA)\left[1 - \beta_0^\sigma (\Delta + sA)^{\sigma-1}\right] + \frac{(1-s)A(\eta-1)\theta_0}{1+(\eta-1)\theta_0} > 0$$

This inequality gives a lower bounds for the sensitivity to collaterals. Indeed, if γ is lower, the capitalists have not enough collaterals to pay all the consumption on credit and they partially needs balances, while if γ satisfies (4.3), they can pay all the consumption on credit.

4.4 Monetary policy

Along the BGP, the monetary policy does not affect the growth factor g^* but it has an impact on the consumption and the welfare of both types of agents. The impact on $c_{it} = g^{*t} c_{i0}$ and

$$U_i \equiv \sum_{t=0}^{\infty} \beta_i^t u(c_{it}) = \sum_{t=0}^{\infty} \beta_i^t \frac{(c_{i0} g^{*t})^{1-1/\sigma}}{1-1/\sigma} = \frac{\sigma}{\sigma-1} \frac{c_{i0}^{1-1/\sigma}}{1-\beta_i g^{*1-1/\sigma}} \tag{24}$$

is qualitatively the same because, under the transversality condition $\beta_i g^{*1-1/\sigma} < 1$, $\partial U_i / \partial c_{i0} > 0$. However, the impact on the initial consumption of each agent is different as it is shown in the following proposition.

Proposition 2. *Under Assumptions 1, 2, 3.2, 4, 5 and 6, along the BGP, we have*

$$\frac{\partial U_0}{\partial \eta} > 0 \text{ and } \frac{\partial U_1}{\partial \eta} < 0$$

and, if $\eta > 1$,

$$\frac{\partial U_0}{\partial \theta_0} > 0 \text{ and } \frac{\partial U_1}{\partial \theta_0} < 0$$

Proof. See Appendix 7.2. ■

The impact of the money growth factor η can be interpreted as follows: the patient agent receives more money today and bears no opportunity cost of holding money, because s/he spends these balances today (s/he does need money to finance consumption tomorrow). The impatient agents bears an opportunity cost in order to finance future consumption with money. This opportunity cost is given by the nominal interest rate and increases with the monetary growth rate because of the inflation. Her/his cost is greater than the benefit of additional monetary transfers to impatient agents from the monetary authority. Since patient agents are richer than impatient ones as they own the whole stock of capital, we can interpret social inequalities as a difference between the utility levels along the BGP of the two classes of agents. We show here that inequalities increase with η and that the impact of the share θ_0 of money "helicoptered" to patient agents is similar.

5 Patient agents pay cash

Consider now the second regime where the patient agent's liquidity constraint is also always binding. The household is aware of the credit function, which is a sort of institutional constraint (endogenous structure of the credit market or legal constraints). Both types of agents have a binding cash-in-advance so that both hold money.

5.1 Patient agent

Patient agent's consumption smoothing over time is affected by the liquidity constraint. The Euler equation captures the intertemporal arbitrage and includes now the monetary effects through the inflation rate. Indeed, the capitalist also bears the opportunity cost of money holding. In order to understand the effects of inflation on the nominal interest rate and this opportunity cost, we compute the patient agent's Euler equation. From (1) and the first-order conditions (6)-(8) with $\mu_{0t} > 0$, we derive the Euler inequality:

$$\left(\frac{c_{0t+1}}{c_{0t}}\right)^{1/\sigma} \geq \beta_0 \frac{\pi_t}{\pi_{t+1}} \frac{\Delta + sA - \gamma}{1 - \gamma \pi_t}$$

which holds with equality if $k_{0t} > 0$, and the constraints:

$$\begin{aligned} m_{0t+1}\pi_{t+1} - m_{0t} + k_{0t+1} - \Delta k_{0t} + c_{0t} &= r_t k_{0t} + \tau_{0t} && (25) \\ c_{0t} - \gamma k_{0t} &= m_{0t} && (26) \end{aligned}$$

Comparing (5.1) with (4.1), we observe that now inflation matters. Since patient agent's wealth accumulation coincides with the aggregate capital accumulation, inflation has a direct effect on economic growth.

5.2 Impatient agent

As in the first regime, equations (4.2), (12), (13) and the transversality condition (9) still apply.

5.3 Intertemporal equilibrium and balanced growth path

Equations (14) and (15) still hold. However, now the capitalists hold money and the market clearing becomes:

$$m_t = n_0 m_{0t} + n_1 m_{1t} \tag{27}$$

Replacing (27) in (17) gives the inflation dynamics:

$$\pi_{t+1} = \eta \tfrac{m_t}{m_{t+1}} = \eta \tfrac{n_0 m_{0t} + n_1 m_{1t}}{n_0 m_{0t+1} + n_1 m_{1t+1}}$$

Equation (19) (good market clearing) still holds.

Under Assumption 3.1, considering simultaneously equations (4.2) and (5.1), we immediately conclude that now there will be some transitional dynamics and that along the transition equation (5.1) must be satisfied with equality while equation (4.2) must be satisfied with a strict inequality. Put differently, we conclude as in the first regime that the Ramsey conjecture still holds. Impatient agents hold no capital and their consumption depends only on money holding, i.e. equations (20) and (21) hold.

Consider again the ratios (4.3). We may then compute the dynamical system that drives the transitional dynamics.

Proposition 3. *Let Assumptions 1, 2, 3.1, 4 and 5 hold. Then, when the patient agent cash-in-advance constraint is binding, the intertemporal equilibrium satisfies the following system*

$$\eta n_1 x_{1t+1} + (1-\eta)\,\theta_1 \left(n_0 x_{0t+1} + n_1 x_{1t+1}\right) = n_1 (1-s)\, A \tfrac{n_0 x_{0t+1} + n_1 x_{1t+1}}{n_0 x_{0t} + n_1 x_{1t}}$$

$$\tfrac{g_t}{g_{t+1}} \tfrac{\eta n_1 x_{1t} + (1-\eta)\theta_1 (n_0 x_{0t} + n_1 x_{1t})}{\eta n_1 x_{1t+1} + (1-\eta)\theta_1 (n_0 x_{0t+1} + n_1 x_{1t+1})} \left[1 - \gamma \tfrac{\eta}{g_t} \tfrac{n_1 (1-s) A}{\eta n_1 x_{1t} + (1-\eta)\theta_1 (n_0 x_{0t} + n_1 x_{1t})}\right]$$

$$= \beta_0 \left(sA + \Delta - \gamma\right) \left(\frac{n_0 x_{0t} + \gamma n_1}{n_0 x_{0t+1} + \gamma n_1} \frac{1}{g_{t+1}}\right)^{1/\sigma} \tag{28}$$

where g_t is given by

$$g_t = g\left(x_{0t}, x_{1t}\right) \equiv \Delta + sA - \gamma - (1-s)\, A \tfrac{\eta n_0 x_{0t} + (1-\eta)\theta_0 (n_0 x_{0t} + n_1 x_{1t})}{\eta n_1 x_{1t} + (1-\eta)\theta_1 (n_0 x_{0t} + n_1 x_{1t})}$$

Proof. See Appendix 7.3. ■

Equations (28) and (28) form a two-dimensional system in (x_{0t}, x_{1t}). As we will show, the steady state of this system is a balanced growth factor which represents a BGP. This system then describes not only the long run (the BGP) but also the short run (transition to or around the BGP) that can be characterized by endogenous fluctuations.[6]

Along the BGP, the Euler equations (4.2), (5.1) and Assumption 3.1 imply

$$\tfrac{c_{0t+1}}{c_{0t}} = \tfrac{c_{1t+1}}{c_{1t}} = g^* = \left(\beta_0 \tfrac{\Delta + sA - \gamma}{1 - \gamma\pi}\right)^{\sigma} > \left(\beta_1 \tfrac{\Delta + sA - \gamma}{1 - \gamma\pi}\right)^{\sigma}$$

[6]See also Boucekkine *et al.* (2005) for the existence of oscillatory dynamics in an AK model with vintage capital, or Boucekkine and Ruiz-Tamarit (2008) for the analysis of transitional dynamics in the Uzawa-Lucas model.

with $\pi = \eta/g^*$. We provide simple conditions for the existence and uniqueness of g^* along which all the variables (balances, capital and consumption) grow at a constant rate. To simplify the characterization, we introduce the following additional assumptions that provide sufficient conditions for the existence of endogenous growth, i.e. the existence of a balanced growth factor larger than 1.

Assumption 7.1. $\Delta + sA - \gamma > 1$.

Assumption 8. $\eta\gamma + \beta_0 (\Delta + sA - \gamma) \geq 1$.

Assumption 9. $\gamma < (\Delta + sA - \gamma) \left[1 - \beta_0 (\Delta + sA - \gamma)^{1-1/\sigma}\right]$.

We then derive the following result.

Proposition 4. *Let Assumptions 1, 2, 3.1, 4, 5, 7.1, 8 and 9 hold. There exists $\bar{\eta} > 1$ such that when $\eta \in [1, \bar{\eta})$, there is a unique BGP $g^* \geq 1$ which is solution of*

$$g = \gamma\eta + \beta_0 (\Delta + sA - \gamma) g^{\frac{\sigma-1}{\sigma}} \equiv \varphi(g) \tag{29}$$

and which satisfies $k_t = k_0 g^{*t}$, $c_{0t} = c_{00} g^{*t}$ and $c_{1t} = c_{10} g^{*t}$ with

$$x_0 = \frac{[1 + (\eta - 1)\theta_0](\Delta + sA - \gamma - g^*) + (1 - s) A(\eta - 1)\theta_0}{\eta} \frac{n_1}{n_0} > 0 \tag{30}$$

$$x_1 = \frac{(1 - s) A [1 + (\eta - 1)\theta_1] + \theta_1 (\eta - 1)(\Delta + sA - \gamma - g^*)}{\eta} > 0 \tag{31}$$

$$c_{00} = k_0 \left(\frac{n_1}{n_0}\gamma + x_0\right) > 0 \tag{32}$$

$$c_{10} = k_0 x_1 > 0 \tag{33}$$

Proof. See Appendix 7.4. ■

It is important to note here that the constraint $c_{0t}/k_{0t} > \gamma$ always hold along the BGP. Indeed, we derive from equations (30) and (32) in Proposition 4 that $c_{0t}/k_{0t} = n_0 c_{00}/(n_1 k_0)$ which is larger than γ.

5.4 Monetary policy

One may wonder whether either a monetary expansion or a financial markets regulation can have a positive impact on economic growth and citizens' welfare.

5.4.1 Effects on growth

First, let us focus on the effects on the growth rate. We know that the monetary growth factor is the product of the inflation and the growth factor (see (5.3)). However, monetary policy affects the extent of inflation and growth in turn. For simplicity, we consider the policy issue along the BGP.

Proposition 5. *Let Assumptions 1, 2, 3.1, 4, 5, 7.1, 8 and 9 hold. The impact of monetary growth is positive:*

$$\frac{\partial g^*}{\partial \eta} = \frac{\sigma\gamma}{1-\gamma\pi(1-\sigma)} > 0$$

with $\eta = \pi g^*$*. The impact of sensitivity to collaterals is also positive:*

$$\frac{\partial g^*}{\partial \gamma} = \frac{\sigma g^*}{1-\gamma\pi(1-\sigma)}\frac{\pi(\Delta+sA)-1}{\Delta+sA-\gamma} > 0$$

Proof. See Appendix 7.5. ■

Focus first on (5). Issuing money increases the inflation and the nominal interest rate. The opportunity cost of holding money increases and agents hold less money, consume less and accumulate more capital. In our model, money growth is growth-enhancing.

Let us now interpret (5). Under a higher γ, increasing capital lowers more money demand and the opportunity cost of holding money (which is positive because we assume a positive nominal interest rate). Then households want to hold more capital and this enhances growth.

5.4.2 Effects on welfare

The growth rate does not capture the welfare of society: because there is a trade off between the initial consumption and the growth rate (consuming less at the beginning promotes higher saving and investment) and because agents are heterogenous. Consider again the expression (24) where c_{00} and c_{10} are given by (32) and (33). We study both the effects of η and of γ.

Proposition 6. *Let Assumptions 1, 2, 3.1, 4, 5, 7.1, 8 and 9 hold. The impact of* η *(monetary growth) and* γ *(credit sensitivity to collaterals) on the welfare* U_i *of a social class (patient capitalists or impatient workers) is ambiguous. This impact can be disentangled in a positive effect on the growth rate and an ambiguous effect on the initial consumption* c_{0i}*. However, we get the following clear-cut conclusions:*

i) There exist $\bar{\gamma} > 0$ *and* $\bar{\theta}_0 \in (0,1)$ *such that when* $\gamma \in (0,\bar{\gamma})$ *and* $\theta_0 \in (0,\bar{\theta}_0)$*, then* $\partial U_0/\partial\eta < 0$ *and* $\partial U_1/\partial\eta > 0$*.*

ii) There exist $\tilde{\gamma} > 0$ *and* $\tilde{\theta}_1 \in (0,1)$ *such that when* $\gamma \in (0,\tilde{\gamma})$ *and* $\theta_1 \in (0,\tilde{\theta}_1)$*, then* $\partial U_0/\partial\gamma > 0$ *and* $\partial U_1/\partial\gamma > 0$*.*

Proof. See Appendix 7.6. ■

The positive effect on the growth rate has been highlighted in Proposition 5. The ambiguous effects on consumptions entail at the very end a global ambiguity for welfare which can however be clarified under some parameters' restrictions. It is worthwhile noting that, contrary to the first regime, money growth may now worsen the welfare of capitalists, who now also bear the opportunity cost of holding balances, and improve the welfare of workers. As patient agents are richer than impatient ones because they own the total stock of capital, this decrease in the utility differential along the BGP between the two groups can be interpreted as a reduction of social inequalities. However, the welfare of both agents is positively impacted by the credit sensitivity to collaterals. The plausibility of these effects is supported by a calibrated example in Section 4.7 where an interpretation is also provided.

5.5 Sunspot fluctuations and stabilization policy

Agents care about their own welfare and a benevolent monetary authority takes into account individual preferences to implement a citizens-oriented monetary policy. However individuals are risk averse and dislike in general inefficient fluctuations. In particular, they dislike expectation-driven fluctuations that may occur when multiple equilibria (indeterminacy and sunspots) arise from market imperfections. In this respect, there is a second scope for monetary policy and regulation: the stabilization issue.

Local indeterminacy occurs when the number of eigenvalues inside the unit circle exceeds the number of predetermined variables. Since there are no predetermined variables in our dynamical system (**??**)-(28), local indeterminacy occurs if at least one eigenvalue is inside the unit circle. In the case of a two-dimensional system, in the spirit of Grandmont *et al.* (1998), we do not need to study the eigenvalues but simply the trace and determinant (T, D) of the Jacobian matrix evaluated at the steady state. Let us draw in a space (T, D), the lines corresponding to $D = 1$, $D = T - 1$, $D = -T - 1$.

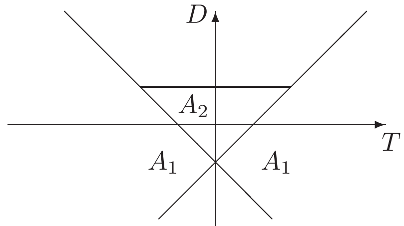

Figure 1: Areas of local indeterminacy.

Local indeterminacy then arises when the pair (T, D) lies inside the areas $\mathcal{A}_1$ with one eigenvalue within the unit circle, or $\mathcal{A}_2$ with two eigenvalues within the unit circle.

Let us introduce the reduced variables:

$$(y_0, z_0, a_0) \equiv \left(\frac{n_0 x_0}{n_0 x_0 + \gamma n_1}, \frac{n_0 x_0}{n_0 x_0 + n_1 x_1}, \frac{(1-\eta)\theta_1 z_0}{\eta(1-z_0)+(1-\eta)\theta_1} \right)$$

and the elasticities of the growth rate $g(x_0, x_1)$ as given by (3):

$$(\varepsilon_0, \varepsilon_1) \equiv \left(\frac{\partial g}{\partial x_0} \frac{x_0}{g}, \frac{\partial g}{\partial x_1} \frac{x_1}{g} \right)$$

To simplify notations, let us denote

$$h = 1/(1 - \gamma\pi) > 1 \tag{34}$$

with $\eta = \pi g^*$.

Lemma 7. *Let Assumptions 1, 2, 3.1, 4, 5, 7.1, 8 and 9 hold. The linearization of the two-dimensional system (**??**)-(28) around the steady state (30)-(31) yields the Jacobian matrix:*

$$J = \begin{bmatrix} \frac{\sigma(1-a_0-\varepsilon_0)+\varepsilon_0}{(z_0-a_0)(\sigma-y_0)} & \frac{\sigma}{\sigma-y_0} \\ \frac{y_0+\varepsilon_0-\sigma(a_0+\varepsilon_0)}{(z_0-a_0)(\sigma-y_0)} & \frac{\sigma}{\sigma-y_0} \end{bmatrix} \begin{bmatrix} z_0 & 1-z_0 \\ h(a_0+\varepsilon_0)-\frac{y_0}{\sigma} & h(1-a_0-\varepsilon_0) \end{bmatrix} \tag{35}$$

Proof. See Appendix 6.7. ■

The determinant and the trace of this Jacobian matrix are, respectively:

$$D = \frac{y_0(1-z_0)+h\sigma(z_0-a_0-\varepsilon_0)}{(\sigma-y_0)(z_0-a_0)} \text{ and } T = 1 + D + \frac{1+(h-1)\sigma}{(\sigma-y_0)(z_0-a_0)}\varepsilon_0$$

To apply the geometrical method developed by Grandmont *et al.* (1998), we choose the elasticity of intertemporal substitution σ as the bifurcation parameter. It does not enter directly the variables $(x_0, y_0, z_0, a_0, x_1, h)$ appearing in the Jacobian (35) (see the expressions (30), (31), (5.5), (34)), but only indirectly through g^*. Thus, the crucial question becomes whether we can keep g^* as fixed while moving σ. If we can normalize g^* such that moving σ no longer affects g^*, then the Jacobian matrix will depend only directly on σ and no longer indirectly through g^*. Focussing on the equation of steady state (29), we observe that the only possible way to normalize g^* is to scale another parameters which affects only g^* with no impact on the other variables $(x_0, y_0, z_0, a_0, x_1, h)$ in the Jacobian. It is easy to verify that there is only one candidate to play with: β_0. Indeed, β_0 does not appear directly in expressions (30), (31), (5.5), (34). In addition, we can set β_0 to normalize the endogenous growth factor g^* to one. Such particular normalization simplifies the checking of all the restrictions that ensure the existence of a BGP. Of course, this simplification implies that we focus on a configuration without growth. However, all our results are also compatible with a positive growth in the long run since they will hold by continuity for any $g^* \in [1, 1+\epsilon)$ with $\epsilon > 0$.

Setting $g^* = 1$ and solving equation (29), we find

$$\beta_0^* \equiv \frac{1-\gamma\eta}{\Delta+sA-\gamma}$$

that is a positive discount factor under Assumption 4, which is also less than 1 under Assumption 7.1. Of course, when g^* is slightly larger than 1 with $g^* \in (1, 1+\epsilon)$, the associated value of β_0^* remains close to expression (5.5).[7]

Consider then the line $\Sigma \equiv \{(T(\sigma), D(\sigma)) : \sigma > 0 \text{ given } (h, \varepsilon_0, a_0, y_0, z_0)\}$ in the (T, D)-plane. Let $S = D'(\sigma)/T'(\sigma)$ be the slope of Σ and $(T_0, D_0) \equiv (T(0), D(0))$ and $(T_\infty, D_\infty) \equiv \lim_{\sigma\to+\infty}(T(\sigma), D(\sigma))$ be its limit values. The following lemmas characterize the behavior of alternative economies obtained by varying the elasticity of intertemporal substitution.

First, as there exists a unique BGP, we can derive a useful property.

[7]The corresponding value of discounting is $\beta_0^* = \frac{g^*-\gamma\eta}{\Delta+sA-\gamma} g^{*\frac{1-\sigma}{\sigma}}$.

Lemma 8. *Let Assumptions 1, 2, 3.1, 4, 5, 7.1, 8 and 9 hold, and* $\eta \in [1, \bar{\eta})$ *with* $\bar{\eta}$ *as given by Proposition 4. For any* $\sigma \geq 0$, $T \neq 1+D$ *so that no eigenvalue is equal to* 1. *Moreover, there exist* $\epsilon > 0$ *and* $\beta_0^* (\epsilon) \in (0,1)$ *such that for any* $g^* \in [1, 1+\epsilon)$ *and* $\beta_0 = \beta_0^* (\epsilon)$, *there is a* $\bar{\sigma} > 0$ *such that* $D = \pm\infty$ *and* $T = \pm\infty$ *when* $\sigma = \bar{\sigma}$.

Proof. See Appendix 7.8. ■

Second, to characterize the location of the line Σ, we introduce an inequality alternative to Assumption 7.1 which is mildly stronger.

Assumption 7.2. $\Delta + sA - \gamma > 1 + (1-s) A$.

Note that this inequality can be also formulated as $\Delta + (2s-1) A - \gamma > 1$. For a given γ, it will be satisfied if $s > 1/2$ and A is large enough.

Lemma 9. *Let Assumptions 1, 2, 3.1, 4, 5, 7.2, 8 and 9 hold, and* $\eta \in [1, \bar{\eta})$ *with* $\bar{\eta}$ *as given by Proposition 4. There exist* $\epsilon > 0$ *and* $\beta_0^* (\epsilon) \in (0,1)$ *such that for any* $g^* \in [1, 1+\epsilon)$ *and* $\beta_0 = \beta_0^* (\epsilon)$, $-1 < D_0 < \min\{0, T_0 - 1\}$, $D_\infty > \max\{1, T_\infty - 1\}$ *and* $T_\infty > 2$. *Moreover,* $D'(\sigma) < 0$, $S \in (-\infty, 0) \cup (1, +\infty)$.

Proof. See Appendix 7.9. ■

Let us introduce the bifurcation value:

$$\sigma^* \equiv \frac{1}{1+h} \frac{1+2y_0\left(\frac{1-z_0}{\varepsilon_0}+\frac{a_0-z_0}{\varepsilon_0}\right)}{1+2\frac{a_0-z_0}{\varepsilon_0}} = \frac{1-\gamma\eta}{2-\gamma\eta} \frac{1+2y_0\left(\frac{1-z_0}{x_1}+\frac{1-z_0}{\varepsilon_0}\right)}{1+2\frac{1-z_0}{x_1}}$$

since $\pi = \eta$ when $g^* = 1$. Assumption 7.2 implies $D_0 > -1$ but is not crucial for the occurrence of indeterminacy. However, it is sufficient to ensure the existence of a positive value for σ^* generating an intersection between the Σ line and the locus corresponding to $D + T + 1 = 0$. Applying Lemmas 7, 8 and 9, we obtain the following geometrical configurations depending on the value of the slope S.

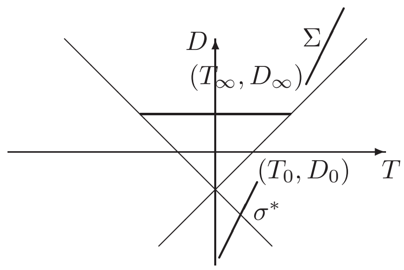

Figure 2: Local indeterminacy with $S > 1$.

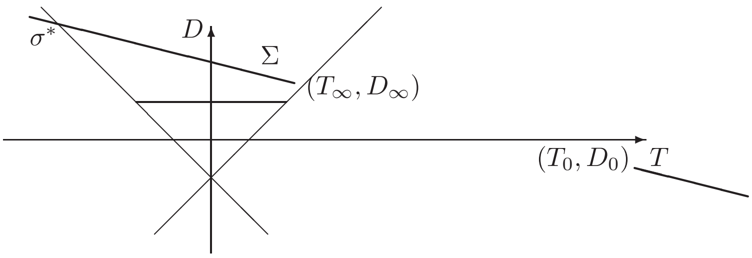

Figure 3: Local indeterminacy with $S \in (-1, 0)$.

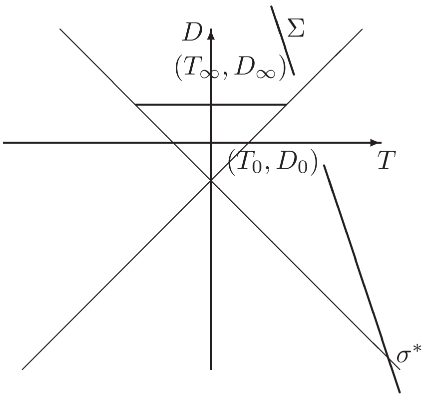

Figure 4: Local indeterminacy with $S < -1$.

We then derive the main proposition.

Proposition 10. *Let Assumptions 1, 2, 3.1, 4, 5, 7.2, 8 and 9 hold, and* $\eta \in [1, \bar{\eta})$ *with* $\bar{\eta}$ *as given by Proposition 4. There exist* $\epsilon > 0$ *and* $\beta_0^* (\epsilon) \in (0, 1)$ *such that for any* $g^* \in [1, 1 + \epsilon)$ *and* $\beta_0 = \beta_0^* (\epsilon)$*, the steady state is a saddle point for* $\sigma \in (0, \sigma^*)$ *(equilibrium indeterminacy) and a source for* $\sigma > \sigma^*$ *(determinacy). Moreover, at* $\sigma = \sigma^*$*, the system generically undergoes a flip bifurcation and two-period cycles arise.*

Proof. Simply observe that σ^* solves $D(\sigma) = -T(\sigma) - 1$ and apply Lemmas 7, 8 and 9 together with Figures 2, 3 and 4. Note that generically, when $S < 0$, $S \neq -1$. ■

The very interpretation of this proposition rests on the comparison between Ramsey models with or without cash-in-advance. In our model, capital accumulation is driven by the Euler equation of the patient agent. When the patient agent is constrained, our model behaves as a cash-in-advance Ramsey model with representative agent (see Stockman (1981), Svensson (1985) and Cooley and Hansen (1989)). In this case, cycles and indeterminacy may arise.

The intuition for two-period cycles and indeterminacy in the Ramsey models à la Stockman (1981) is simple. Consider the limit case $\gamma = n_1 = 0$ (the patient agent becomes the representative agent and collaterals no longer matter). The intuition for cycles is the following. Assume that k_t increases from its steady-state value. Then, the income Ak_t increases as well. If the intertemporal substitution σ is weak, the income effect prevails and raises current consumption c_t. If the intertemporal substitution is sufficiently weak, the response in terms of c_t exceeds the increase of $(\Delta + A)\, k_t$ and, according to the budget constraint, $k_{t+1} = (\Delta + A)\, k_t - c_t$ decreases. Thus, an increase of k_t is followed by a decrease of k_{t+1}; eventually, two-period cycles arise.

The intuition for self-fulfilling prophecies under equilibrium indeterminacy is the following. Assume that households anticipate an increase in the future price p_{t+1}, that is, in the inflation factor π_{t+1}. We want to show that this expectation can be self-fulfilling. Reconsider the Euler equation (5.1) with equality and the cash-in-advance constraint (26), set $\gamma = 0$ and omit the individual index 0:

$$\frac{c_{t+1}}{c_t} = \left[\beta\,(\Delta + sA)\,\frac{\pi_t}{\pi_{t+1}}\right]^\sigma \tag{36}$$

$$c_{t+1} = m_{t+1} \tag{37}$$

where $\Delta + sA$ is the gross real interest rate. From (36), we obtain the elasticity of the inflation factor w.r.t. future consumption c_{t+1}:

$$\varepsilon_{\pi c} \equiv \tfrac{d\pi_{t+1}}{dc_{t+1}}\tfrac{c_{t+1}}{\pi_{t+1}} = -\tfrac{1}{\sigma}$$

We observe that an increase in the expected price p_{t+1} reduces the real balances $m_{t+1} \equiv M_{t+1}/p_{t+1}$ and, according to equation (37), the future consumption. The decrease of c_{t+1} determines an increase of π_{t+1} according to (37). However, such increase is required to be large enough in order to make the initial prophecy of higher inflation self-fulfilling. This happens if and only if the elasticity of intertemporal substitution σ in the left-hand side of (5.5) is sufficiently low, as shown, among the others, by Cooley and Hansen (1989).

It is worth noting that these conclusions are similar the results of Ferraris and Watanabe (2011) obtained in a monetary search model à la Lagos and Wright (2005) with credit constraint. They show indeed that when agents are at their borrowing limits, fluctuations in capital occur through cycles of high orders, chaos and sunspot equilibria.

5.6 A numerical illustration

Let us fix the parameters as follows according to yearly data: $s = 2/3$, $A = 1$, $\Delta = 9/10$, $\theta_0 = 1/2$, $n_0 = n_1 = 1$. Notice that s takes the usual value of endogenous growth literature (see Mankiw *et al.* (1992)). In addition, for now, we assume $\beta_0 = 0.98$, $\beta_1 = 0.95$, $\sigma = 0.2$ and we consider a plausible money growth jointly with a moderate credit sensitivity to collaterals, namely $(\eta, \gamma) = (1.02, 0.01)$. It is worth noting at this point that choosing an elasticity of intertemporal substitution $\sigma = 0.2$ allows both to focus on a locally indeterminate BGP, since we get a bifurcation value $\sigma^* \approx 0.24$, and to consider a plausible value for this parameter according to Campbell (1999), Kocherlakota (1996) and Vissing-Jorgensen (2002).

According to this calibration, and as shown in Proposition 6, we find effects of the money growth factor η on the welfare levels of the two classes, that are the opposite to those obtained in the first regime where patient agents hold no money:

$$\frac{\partial U_0}{\partial \eta} < 0 \text{ and } \frac{\partial U_1}{\partial \eta} > 0$$

The negative impact on patient agents comes from the negative effect of η on their initial consumption c_{00}. This happens because now patient agents bear an opportunity cost to finance future consumption with money. This cost appears to be larger than the benefit of additional money transfers from the monetary authority. Conversely, impatient agents have a positive impact on their initial consumption c_{10} as they bear a relatively lower opportunity cost of holding money. It is worth noting however that the effect on the total welfare of the economy is positive (i.e. $n_0 \partial U_0/\partial \eta + n_1 \partial U_1/\partial \eta > 0$). As mentioned previously, considering that the more patient agents are richer as they hold the total stock of capital, this result shows that a higher monetary growth rates allows at the same time to increase total welfare and to decrease the inequality measured as the lag between the stationary utility level of the two types of agents.

As shown by Proposition 6, the welfare impact of credit sensitivity to collaterals is given by

$$\frac{\partial U_0}{\partial \gamma} > 0 \text{ and } \frac{\partial U_1}{\partial \gamma} > 0$$

For both agents, increasing the credit sensitivity to collaterals allows to decrease the amount of money holding and thus the associated opportunity cost. This allows to increase consumption and thus welfare.

Let us now focus on Proposition 10 which clarifies the role of monetary policy and regulation on the occurrence of local indeterminacy and growth cycles. Indeed, η and γ have an impact on σ^* and, thus, affect the indeterminacy range $(0, \sigma^*)$. Monetary policy (regulation) plays a stabilizing role if η (γ) has a negative impact on σ^*, that is the room for indeterminacy and expectation-driven fluctuations shrinks. Since the signs of derivatives $\partial\sigma^*/\partial\eta$ and $\partial\sigma^*/\partial\gamma$ are in general ambiguous, to address the stabilization issue, we simulate in the following the shape of the function $\sigma^*(\eta, \gamma)$.

Consider, for instance, a growth rate of 2%, that is $g^* = 1.02$. We can easily check that all the restrictions of the model are satisfied. We derive from this calibration that the flip bifurcation value for the elasticity of intertemporal substitution in consumption is $\sigma^* \approx 0.282$, again a plausible value according to Campbell (1999), Kocherlakota (1996) and Vissing-Jorgensen (2002).

Focussing on the impact of η and γ on the existence of local indeterminacy, we plot in Figure 5 the bifurcation value σ^* as a function of η and γ around $(\eta, \gamma) = (1.02, 0.01)$.

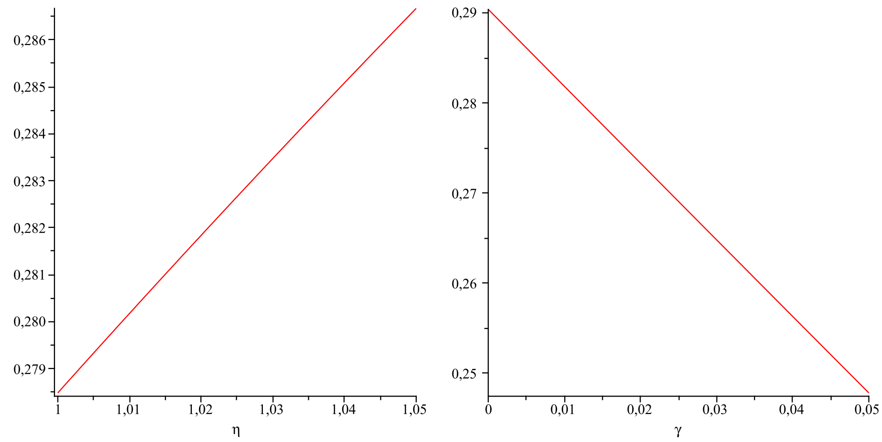

Figure 5: Monetary policy (η) and regulation (γ)

This numerical exercise shows that the indeterminacy range $(0, \sigma^*)$ expands with η and shrinks with γ as σ^* is an increasing function with respect to η and a decreasing function with respect to γ. Indeed, the higher the monetary growth (η), the higher the inflation rate and the opportunity cost of holding money (nominal interest rate), the heavier the burden on liquidity-constrained agents. Thus, the larger the credit market imperfection, the more likely the occurrence of endogenous fluctuations because of the local indeterminacy. Conversely the sensitivity of credit grants to collaterals (γ) reduces the market imperfection, because, when γ increases, collaterals matter more and the liquidity constraint becomes lighter for patient agents.

We can then conclude from these numerical exercise that, if the goal of the government is to avoid growth cycles, any increase of money supply should be followed by an increase of the sensitivity of credit grants to collaterals. In the light of the recent crisis, our results could be interpreted as a macroeconomic policy device: since central banks are increasing the money supply through quantitative easing policies, raising the weight of collaterals for all agents could also appear as a mean to rule out destabilizing effects.

6 Conclusion

In this paper, we have considered a Ramsey model with heterogenous agents and liquidity constraint. Heterogenous discount rates imply a social segmentation in the long run: patient agents end up to hold the total stock of capital. Money is needed to finance future consumption when the agents have an insufficient amount of collaterals. In particular, impatient agents hold money.

In this economy, two regimes may arise according to the possibility that patient agents hold balances or do not. We have studied the role of monetary policy (η) and market regulation (γ) in both the cases.

(1) Patient agents buy on credit and money is needed only by workers.

Money growth improves the patient agents' welfare (they receive balances), but worsens that of impatient agents, who bear the opportunity cost of holding money (the higher inflation, the higher the interest rate). Since patient agents are richer than impatient ones as they own the total amount of capital, this conclusion can be interpreted as an increase of inequalities as the difference between the utility level along the BGP of the two types of agents is increasing with η. Moreover, the equilibrium jumps on the BGP from the initial condition and there is no transitional dynamics.

(2) Patient agents needs money to consume. Monetary growth (η) is growth-enhancing because under a higher inflation, patient agents renounce to balances, consume less and accumulate more capital. The higher sensitivity of credit grant to collaterals (γ) also promote economic growth, because patient agents accumulate more capital to collateralize the consumption purchases and to reduce money demand. However, in general, money growth and credit sensitivity to collaterals have an ambiguous effect on the initial consumptions of patient and impatient agents and the resulting effect on welfare remains ambiguous. We have then shown using sufficient conditions on structural parameters and a numerical simulation that an expansionary monetary policy can increase the total welfare along the BGP but through a negative impact on the patient agents' utility who now bear the opportunity cost of holding money. Patient agents are still richer than impatient ones as they own the total amount of capital but this conclusion can be interpreted as a decrease of inequalities measured as the lower difference between the utility level along the BGP of the two types of agents. On the contrary, increasing the credit sensitivity to collaterals improves welfare for both agents.

Since in this second regime there is a transitional dynamics we have provided conditions for local indeterminacy and growth cycles. Stabilization becomes an important concern in the authority's agenda and two policy instruments can be used: the growth rate of money and the sensitivity of credit grants to collaterals. We have shown that the indeterminacy range widens with η and shrinks with γ. Indeed, a higher monetary growth raises the inflation rate and the opportunity cost of money holding, making the burden of liquidity constraint heavier and the credit market imperfection larger. Conversely the sensitivity of credit grants to collaterals γ reduces the market imperfection: when γ increases, collaterals matter more and the capitalists' liquidity constraint weakens. If the goal of the government is to avoid growth cycles, any increase of money supply should be accompanied by an increase of the sensitivity of credit grants to collaterals.

7 Appendix

7.1 Proof of Proposition 1

Focus on the equilibrium budget constraints (10) and (20). Substituting $k_{0t} = k_t n_1 / n_0$ and (15) in (10), and (21) and (15) in (20) we obtain, respectively:

$$n_1 \left[k_{t+1} - (\Delta + r)\, k_t\right] + n_0 c_{0t} \quad = \quad n_1 \theta_0 \left(m_{1t+1} \pi_{t+1} - m_{1t}\right) \qquad (38)$$

$$\theta_1 m_{1t} + (1-\theta_1)\, m_{1t+1}\pi_{t+1} \quad = \quad w_t \tag{39}$$

Replacing (18) in (38) and (39) gives

$$n_1\left[k_{t+1} - (\Delta + r)\, k_t\right] + n_0 c_{0t} \quad = \quad n_1\theta_0\,(\eta - 1)\, m_{1t} \tag{40}$$

$$\left[\theta_1 + (1-\theta_1)\,\eta\right] m_{1t} \quad = \quad (1-s)\, A k_t \tag{41}$$

since $w_t = (1-s)\, Ak_t$. Dividing (41) by k_t, we obtain constant ratios:

$$(x_{0t}, x_{1t}) = \left(0, \tfrac{(1-s)A}{1+(\eta-1)\theta_0}\right)$$

Dividing (40) by k_t, we get

$$\tfrac{c_{0t}}{k_t} = \tfrac{n_1}{n_0}\left[\Delta + sA + \tfrac{(1-s)A(\eta-1)\theta_0}{1+(\eta-1)\theta_0} - g_{t+1}\right]$$

The patient agents' Euler equation (4.1) becomes $c_{0t+1}/c_{0t} = \left[\beta_0\,(\Delta + sA)\right]^{\sigma}$. The only possible equilibrium (compatible with the nonnegativity of quantities and the transversality condition) is the BGP: $k_t = k_0 g^{*t}$ and $c_{0t} = c_{00} g^{*t}$ with $g^* = \left[\beta_0\,(\Delta + sA)\right]^{\sigma}$. Under Assumption 6, we have $g^* \geq 1$. Moreover, the transversality condition (11) becomes

$$\lim_{t\to+\infty}\left(\beta_0 {g^*}^{\frac{\sigma-1}{\sigma}}\right)^t = 0$$

and holds under Assumption 6. There is no transition and the initial consumption of the patient agent jumps on the BGP: (7.1) gives

$$c_{00} = k_0 \tfrac{n_1}{n_0}\left[\Delta + sA + \tfrac{(1-s)A(\eta-1)\theta_0}{1+(\eta-1)\theta_0} - g^*\right]$$

Under Assumption 6, we get $c_{00} > 0$.

The consumption of impatient agents along the BGP is given by $c_{1t} = c_{10} g^{*t}$. From (19) evaluated at time $t = 0$, we have

$$n_1\,(g^* - \Delta) + n_0\tfrac{c_{00}}{k_0} + n_1\tfrac{c_{10}}{k_0} = n_1 A$$

and, using (7.1), we get:

$$c_{10} = \tfrac{(1-s)Ak_0}{1+(\eta-1)\theta_0}$$

■

7.2 Proof of Proposition 2

We compute the derivatives of the initial consumption levels given in Proposition 1 for both the agents:

$$\tfrac{\partial c_{00}}{\partial\eta} = \theta_0\tfrac{(1-s)Ak_0}{[1+(\eta-1)\theta_0]^2}\tfrac{n_1}{n_0} > 0 \text{ and } \tfrac{\partial c_{10}}{\partial\eta} = -\theta_0\tfrac{(1-s)Ak_0}{[1+(\eta-1)\theta_0]^2} < 0$$

Using the expression of welfare along the BGP (see (24)), we derive

$$\tfrac{\partial U_i}{\partial\eta} = \tfrac{c_{i0}^{-1/\sigma}}{1-\beta_i {g^*}^{1-1/\sigma}}\tfrac{\partial c_{i0}}{\partial\eta}$$

and the results follow. We finally observe that, if $\eta > 1$, then $\partial c_{00}/\partial\theta_0 > 0$ and $\partial c_{10}/\partial\theta_0 < 0$. Since, from (24), we have

$$\tfrac{\partial U_i}{\partial\theta_0} = \tfrac{c_{i0}^{-1/\sigma}}{1-\beta_i {g^*}^{1-1/\sigma}}\tfrac{\partial c_{i0}}{\partial\theta_0}$$

the results follow. ■

7.3 Proof of Proposition 3

Reconsider the equilibrium budget constraints. Replacing $k_{0t} = k_t n_1/n_0$, (15) and (27) in (25), and (21), (15) and (27) in (20) we obtain, respectively:

$$\begin{aligned} & \left[(1-\theta_0)\, n_0 m_{0t+1} - \theta_0 n_1 m_{1t+1}\right] \pi_{t+1} + \theta_0 \left(n_0 m_{0t} + n_1 m_{1t}\right) \\ = \; & n_1 \left[(\Delta + r)\, k_t - k_{t+1}\right] + n_0 \left(m_{0t} - c_{0t}\right) \end{aligned} \quad (42)$$

$$\begin{aligned} & \left[(1-\theta_1)\, n_1 m_{1t+1} - \theta_1 n_0 m_{0t+1}\right] \pi_{t+1} + \theta_1 \left(n_0 m_{0t} + n_1 m_{1t}\right) \\ = \; & n_1 w_t \end{aligned} \quad (43)$$

In the second regime (capitalists hold money), the CIA constraint is binding:

$$c_{0t} = m_{0t} + \gamma k_t n_1/n_0 \quad (44)$$

and (42) reduces to

$$\begin{aligned} & \left[(1-\theta_0)\, n_0 m_{0t+1} - \theta_0 n_1 m_{1t+1}\right] \pi_{t+1} + \theta_0 \left(n_0 m_{0t} + n_1 m_{1t}\right) \\ = \; & n_1 \left[(\Delta + r - \gamma)\, k_t - k_{t+1}\right] \end{aligned} \quad (45)$$

(18) becomes

$$n_0 m_{0t} + n_1 m_{1t} = \left(n_0 m_{0t+1} + n_1 m_{1t+1}\right) \pi_{t+1}/\eta \quad (46)$$

Replacing (46) in (45) and (43), we get, respectively,

$$\begin{aligned} & \left[n_0 m_{0t+1} + \tfrac{1-\eta}{\eta}\theta_0 \left(n_0 m_{0t+1} + n_1 m_{1t+1}\right)\right] \pi_{t+1} \\ = \; & n_1 \left[(\Delta + sA - \gamma)\, k_t - k_{t+1}\right] \\ & \left[n_1 m_{1t+1} + \tfrac{1-\eta}{\eta}\theta_1 \left(n_0 m_{0t+1} + n_1 m_{1t+1}\right)\right] \pi_{t+1} \\ = \; & n_1 (1-s)\, A k_t \end{aligned}$$

Under Assumption 4 and constraint (44), patient agents' Euler equation (5.1) becomes

$$\tfrac{n_0 m_{0t+1} + \gamma n_1 k_{t+1}}{n_0 m_{0t} + \gamma n_1 k_t} = \left(\beta_0 \tfrac{\pi_t}{\pi_{t+1}} \tfrac{\Delta + r - \gamma}{1 - \gamma \pi_t}\right)^{\sigma}$$

Dividing (7.3), (7.3) and (7.3) by k_{t+1} and replacing (4.3), we obtain at time t:

$$\left[n_0 x_{0t} + \frac{1-\eta}{\eta}\theta_0 \left(n_0 x_{0t} + n_1 x_{1t}\right)\right] \pi_t = n_1 \left(\frac{\Delta + sA - \gamma}{g_t} - 1\right) \quad (47)$$

$$\left[n_1 x_{1t} + \frac{1-\eta}{\eta}\theta_1 \left(n_0 x_{0t} + n_1 x_{1t}\right)\right] \pi_t = \frac{n_1 (1-s)\, A}{g_t} \quad (48)$$

$$\frac{n_0 x_{0t} + \gamma n_1}{n_0 x_{0t+1} + \gamma n_1} \left(\beta_0 \frac{\pi_t}{\pi_{t+1}} \frac{\Delta + sA - \gamma}{1 - \gamma \pi_t}\right)^{\sigma} = g_{t+1} \quad (49)$$

Dividing (47) and (48) side by side, we get (3). From (46) and (7.3), we have (28). Deriving π_t from (48) and replacing π_t and π_{t+1} in (49), we obtain (28). ■

7.4 Proof of Proposition 4

Equation (28) becomes at the steady state:

$$\eta n_1 x_1 + (1-\eta)\,\theta_1\,(n_0 x_0 + n_1 x_1) = n_1\,(1-s)\,A \tag{50}$$

Replacing (50) in the Euler equation (28) evaluated at the steady state, gives

$$g = \gamma\eta + \beta_0\,(\Delta + sA - \gamma)\,g^{\frac{\sigma-1}{\sigma}} \equiv \varphi\,(g) \tag{51}$$

an implicit equation which determines the balanced growth rate g^*. Replacing (50) in (3) evaluated at the steady state, gives

$$\eta n_0 x_0 + (1-\eta)\,\theta_0\,(n_0 x_0 + n_1 x_1) = n_1\,(\Delta + sA - \gamma - g^*) \tag{52}$$

Equations (50) and (52) write as a linear system

$$\begin{bmatrix} \eta + (1-\eta)\,\theta_0 & (1-\eta)\,\theta_0 \\ (1-\eta)\,\theta_1 & \eta + (1-\eta)\,\theta_1 \end{bmatrix} \begin{bmatrix} n_0 x_0 \\ n_1 x_1 \end{bmatrix} = n_1 \begin{bmatrix} \Delta + sA - \gamma - g^* \\ (1-s)\,A \end{bmatrix}$$

and can be solved with respect to (x_0, x_1). Noting that $\theta_0 + \theta_1 = 1$, we obtain

$$x_0 = \frac{(\Delta + sA - \gamma - g^*)\,[1 + (\eta-1)\,\theta_0] + (1-s)\,A(\eta-1)\theta_0}{\eta}\frac{n_1}{n_0} \tag{53}$$

$$x_1 = \frac{(1-s)\,A\,[1 + (\eta-1)\,\theta_1] + \theta_1\,(\eta-1)\,(\Delta + sA - \gamma - g^*)}{\eta} \tag{54}$$

where g^* is given by (51).

The asset ratio is given by $m/k = n_0 x_0 + n_1 x_1 = n_1\,(A + \Delta - \gamma - g^*)$. The consumption of patient agents along the BGP is given by $c_{0t} = c_{00} g^{*t}$ and the binding cash-in-advance along the BGP implies: $c_{00} = (x_0 + \gamma n_1/n_0)\,k_0$. Similarly, the consumption of impatient agents along the BGP is given by $c_{1t} = c_{10} g^{*t}$ and the binding cash-in-advance along the BGP implies: $c_{10} = m_0 = x_1 k_0$.

The existence of a balanced (monetary) equilibrium requires the positivity of variables (and nominal interest rate) and the transversality condition to be satisfied.

(1) Positivity of variables x_0 and x_1 needs, respectively:

$$g^* < \Delta + sA - \gamma + \frac{(1-s)\,A(\eta-1)\theta_0}{1 + (\eta-1)\,\theta_0} \equiv g_1 \tag{55}$$

$$g^* < \Delta + sA - \gamma + \frac{(1-s)\,A[1 + (\eta-1)\theta_1]}{(\eta-1)\,\theta_1} \equiv g_2 \tag{56}$$

Inequality (55) implies (56) as $g_2 > g_1$. Moreover, if $g^* < g_1$, we get $c_{00} > 0$ and $c_{10} > 0$.

(2) We need to have a positive nominal interest rate to get a monetary equilibrium, as money is required to be a dominated asset. We define the nominal interest factor as $i_t \equiv \pi_t\,(\Delta + sA)$ and we require the zero lower bound for the interest rate: $i_t - 1 > 0$ (a productive economy exists if money is a dominated asset). Since patient agents hold the stock of capital in the long run, we need condition (5.1) to be satisfied with equality at the steady state, that is:

$$g^* = \left(\beta_0 \frac{\Delta+sA-\gamma}{1-\gamma\pi}\right)^\sigma$$

Noting that $i > 1$ implies $\Delta + sA > 1/\pi$. We obtain from (7.4): $g^* \geq (\beta_0/\pi)^\sigma$. Since, along the balanced growth path, (17) becomes $\eta = \pi g^*$, the inequality which ensures a positive nominal interest rate writes

$$\eta > \beta_0 g^{*(\sigma-1)/\sigma} \tag{57}$$

(3) The Balanced Growth Transversality Condition (BGTC) requires

$$\beta_0 g^{*(\sigma-1)/\sigma} < 1 \tag{58}$$

Noting that $g^* = \gamma\eta + (\Delta + sA - \gamma)\,\beta_0 g^{*(\sigma-1)/\sigma}$, we get, more explicitly, $g^* < \Delta + sA + \gamma(\eta - 1) \equiv g_3$. Note that, when $\eta = 1$, we get $g_3 > g_1$. Moreover, under Assumption 5, (58) implies (57).

To summarize, we conclude from all these claims that there exists $\bar{\eta}_1 > 1$ such that when $\eta \in [1, \bar{\eta}_1)$, any BGP $g^* < g_1$ satisfies the positivity constraints of all the variables, the positivity of the nominal interest rate and the BGTC.

Let us prove now the existence and uniqueness of the BGP g^* that is a solution of equation (51), namely $g^* = \varphi(g^*)$. We are looking for a solution such that $g^* \in [1, g_1)$. We need to ensure that $g_1 > 1$ or, equivalently,

$$\Delta + sA - \gamma + \frac{(1-s)A(\eta-1)\theta_0}{1+(\eta-1)\theta_0} > 1$$

Under Assumption 7.1, this inequality holds.

Consider first the simplest case $\sigma = 1$ yielding $g^* = \gamma\eta + (\Delta + sA - \gamma)\,\beta_0$. Under Assumption 8 we get $g^* \geq 1$. Under Assumption 9 with $\sigma = 1$, inequality $g^* < g_1$ holds for $\eta = 1$. Then, there exists $\bar{\eta}_2 > 1$ such that, when $\eta \in [1, \bar{\eta}_2)$, $g^* = \gamma\eta + (\Delta + sA - \gamma)\,\beta_0 < g_1$.

Let us now consider the case $\sigma < 1$. We get, $\varphi'(g) < 0$ for every $g > 0$. Moreover, we have $\lim_{g\to 0^+} \varphi(g) = +\infty$ and $\lim_{g\to +\infty} \varphi(g) = \gamma\eta$. Graphically we get the following figure.

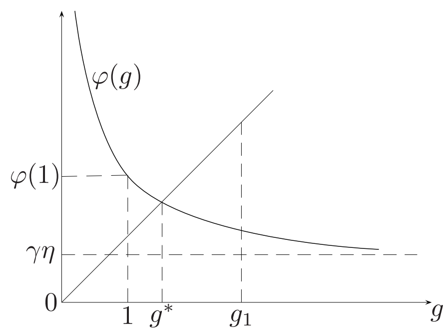

Figure 6: Existence and uniqueness of $g^* \in (1, g_1)$ when $\sigma < 1$.

We immediately obtain the existence and the uniqueness of a solution g^* of equation (51). Moreover, under Assumption 8, we get $g^* \geq 1$ while Assumption 9 implies $\varphi(g_1) < g_1$ and thus $g^* < g_1$ when $\eta = 1$. Then there exists $\bar{\eta}_3 > 1$ such that when $\eta \in [1, \bar{\eta}_3)$, $g^* < g_1$.

Let us finally consider the case $\sigma > 1$. We get, $\varphi'(g) > 0$, $\varphi''(g) < 0$ for every $g > 0$. Moreover, we have $\lim_{g\to 0^+} \varphi(g) = \gamma\eta > 0$, $\lim_{g\to+\infty} \varphi(g) = +\infty$ and $\lim_{g\to+\infty} \varphi'(g) = 0 < 1$. Graphically we get the following figure.

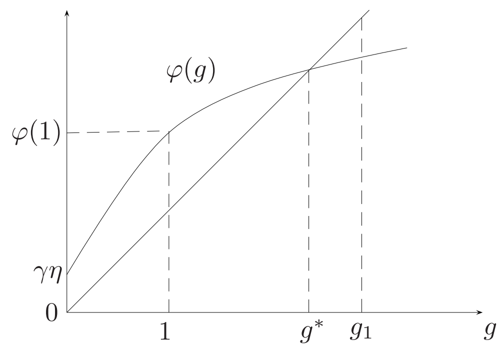

Figure 7: Existence and uniqueness of $g^* \in (1, g_1)$ when $\sigma > 1$.

The existence and the uniqueness of a solution g^* of equation (51) follow. Moreover, under Assumption 8 we get $g^* \geq 1$ while Assumption 9 implies $\varphi(g_1) < g_1$ and thus $g^* < g_1$ when $\eta = 1$. Then there exists $\bar{\eta}_4 > 1$ such that when $\eta \in [1, \bar{\eta}_4)$, $g^* < g_1$.

Taking $\bar{\eta} \equiv \min\{\bar{\eta}_1, \bar{\eta}_2, \bar{\eta}_3, \bar{\eta}_4\}$, we obtain the results. ■

7.5 Proof of Proposition 5

Focus first on (29). Totally differentiating with respect to γ, η, g and noting that

$$\beta_0 g^{*1-1/\sigma} = \frac{g^* - \gamma\eta}{\Delta + sA - \gamma}$$

we get

$$g^* \frac{\pi(\Delta + sA) - 1}{\Delta + sA - \gamma} d\gamma + \gamma d\eta + \frac{\gamma\pi(1-\sigma) - 1}{\sigma} dg^* = 0$$

Setting $d\gamma = 0$ in (7.5), we get the partial derivative (5): the impact of monetary policy on the growth rate is always positive.

Setting $d\eta = 0$ in (7.5), we obtain (5). The impact of regulation is positive $(\partial g^*/\partial\gamma > 0)$ because the nominal interest rate is positive: $\pi(\Delta + sA) > 1$. ■

7.6 Proof of Proposition 6

Consider the expression (24). The effect of η on U_0 is given by

$$\frac{\partial U_0}{\partial \eta} = \frac{c_{00}^{1-1/\sigma}}{1 - \beta_0 g^{*1-1/\sigma}} \left[\frac{1}{g^*} \frac{\partial g^*}{\partial \eta} \frac{\beta_0 g^{*1-1/\sigma}}{1 - \beta_0 g^{*1-1/\sigma}} + \frac{1}{c_{00}} \frac{\partial c_{00}}{\partial \eta} \right]$$

The trade-off between the initial consumption and the growth rate is a usual feature of the endogenous growth models. The utility of patient agents increases because of the positive impact on the growth rate:

$$\frac{1}{g^*} \frac{\partial g^*}{\partial \eta} \frac{\beta_0 g^{*1-1/\sigma}}{1 - \beta_0 g^{*1-1/\sigma}} > 0$$

(see the inequality (5) and the transversality condition: $1 - \beta_0 g^{*1-1/\sigma} > 0$) and may decrease because of the ambiguous effect on the initial consumption. Indeed, using (32), we get

$$\frac{\partial c_{00}}{\partial \eta} = k_0 \frac{n_1}{n_0} \left[\frac{(1-s)A\theta_0 - \theta_1(\Delta + sA - g^* - \gamma)}{\eta^2} - \frac{1+(\eta-1)\theta_0}{\eta} \frac{\partial g^*}{\partial \eta} \right] \tag{59}$$

which can be positive or negative.

Consider the term between brackets in equation (7.6). Note from Proposition 5 that if γ is close enough to zero, the derivative $\partial g^*/\partial \eta$ is also close to zero and $\partial U_0/\partial \eta$ has the same sign as $\partial c_{00}/\partial \eta$. Consider now the term between brackets in equation (59). If θ_0 is also close enough to zero, then $\partial c_{00}/\partial \eta < 0$ so that $\partial U_0/\partial \eta < 0$. Therefore, there exist $\hat{\gamma} > 0$ and $\hat{\theta}_0 \in (0,1)$ such that, when $\gamma \in (0, \hat{\gamma})$ and $\theta_0 \in (0, \hat{\theta}_0)$, then $\partial U_0/\partial \eta < 0$.

The impact of γ on U_0 is given by

$$\frac{\partial U_0}{\partial \gamma} = \frac{c_{00}^{1-1/\sigma}}{1-\beta_0 g^{*1-1/\sigma}} \left[\frac{1}{g^*} \frac{\partial g^*}{\partial \gamma} \frac{\beta_0 g^{*1-1/\sigma}}{1-\beta_0 g^{*1-1/\sigma}} + \frac{1}{c_{00}} \frac{\partial c_{00}}{\partial \gamma} \right]$$

There exists a trade-off between the initial consumption and the growth rate. The utility of patient agents increases because of the positive impact on the growth rate:

$$\frac{1}{g^*} \frac{\partial g^*}{\partial \gamma} \frac{\beta_0 g^{*1-1/\sigma}}{1-\beta_0 g^{*1-1/\sigma}} > 0$$

and can decrease because of the ambiguous effect on the initial consumption. Indeed, using (32), we get

$$\frac{\partial c_{00}}{\partial \gamma} = k_0 \frac{n_1}{n_0} \left[\frac{\eta - 1}{\eta} \theta_1 - \frac{\partial g^*}{\partial \gamma} \frac{1+(\eta-1)\theta_0}{\eta} \right] \tag{60}$$

which may be positive or negative.

Consider the term between brackets in equation (7.6). From equations (30) and (32) in Proposition 4 we derive

$$c_{00} = k_0 \frac{n_1}{n_0} \frac{(\eta-1)[\gamma\theta_1 + (1-s)A\theta_0] + (\Delta + sA - g^*)[1+(\eta-1)\theta_0]}{\eta}$$

Substituting this expression into (60) gives

$$\frac{1}{c_{00}} \frac{\partial c_{00}}{\partial \gamma} = \frac{(\eta-1)\theta_1 - \frac{\partial g^*}{\partial \gamma}[1+(\eta-1)\theta_0]}{(\eta-1)[\gamma\theta_1 + (1-s)A\theta_0] + (\Delta + sA - g^*)[1+(\eta-1)\theta_0]} \tag{61}$$

Substituting (61) into (7.6) then yields after simplifications

$$\begin{aligned}\frac{\partial U_0}{\partial \gamma} &= \frac{c_{00}^{1-1/\sigma}}{1-\beta_0 g^{*1-1/\sigma}} \left[\frac{(\eta-1)\theta_1}{(\eta-1)[\gamma\theta_1 + (1-s)A\theta_0] + [1+(\eta-1)\theta_0][\Delta + sA - g^*]} \right. \\ &\quad \left. + \frac{\partial g^*}{\partial \gamma} \frac{\beta_0 g^{*1-1/\sigma}[(\eta-1)[\gamma\theta_1 + (1-s)A\theta_0] + [1+(\eta-1)\theta_0](\Delta + sA)] - g^*[1+(\eta-1)\theta_0]}{g^*\left[1-\beta_0 g^{*1-1/\sigma}\right]\{(\eta-1)[\gamma\theta_1 + (1-s)A\theta_0] + [1+(\eta-1)\theta_0][\Delta + sA - g^*]\}} \right]\end{aligned}$$

Consider then equation (29) that can be written as

$$\beta_0 g^{*1-1/\sigma}(\Delta + sA) = g^* - \gamma\left[\eta - \beta_0 g^{*1-1/\sigma}\right]$$

Substituting this expression into (7.6) finally gives

$$\begin{aligned}\frac{\partial U_0}{\partial \gamma} &= \frac{c_{00}^{1-1/\sigma}}{1-\beta_0 g^{*1-1/\sigma}} \left[\frac{(\eta-1)\theta_1}{(\eta-1)[\gamma\theta_1 + (1-s)A\theta_0] + [1+(\eta-1)\theta_0][\Delta + sA - g^*]} \right. \\ &\quad \left. + \frac{\partial g^*}{\partial \gamma} \frac{\beta_0 g^{*1-1/\sigma}(\eta-1)[\gamma\theta_1 + (1-s)A\theta_0] - \gamma[1+(\eta-1)\theta_0]\left[\eta - \beta_0 g^{*1-1/\sigma}\right]}{g^*\left[1-\beta_0 g^{*1-1/\sigma}\right]\{(\eta-1)[\gamma\theta_1 + (1-s)A\theta_0] + [1+(\eta-1)\theta_0][\Delta + sA - g^*]\}} \right]\end{aligned}$$

Therefore, there exists $\tilde{\gamma} > 0$ such that when $\gamma \in (0, \tilde{\gamma})$, then $\partial U_0/\partial \gamma > 0$.

Focus now on the welfare of impatient agents. The impact of η on U_1 is given by

$$\frac{\partial U_1}{\partial \eta} = \frac{c_{10}^{1-1/\sigma}}{1-\beta_1 g^{*1-1/\sigma}} \left[\frac{1}{g^*} \frac{\partial g^*}{\partial \eta} \frac{\beta_1 g^{*1-1/\sigma}}{1-\beta_1 g^{*1-1/\sigma}} + \frac{1}{c_{10}} \frac{\partial c_{10}}{\partial \eta} \right]$$

As for agents of type 0, there exists a trade-off between the initial consumption and the growth rate. The utility of impatient agents increases because of the positive impact on the growth rate:

$$\frac{1}{g^*} \frac{\partial g^*}{\partial \eta} \frac{\beta_1 g^{*1-1/\sigma}}{1-\beta_1 g^{*1-1/\sigma}} > 0$$

(inequality (5) and transversality condition: $1 - \beta_1 g^{*1-1/\sigma} > 0$) and may decrease because of the ambiguous effect on the initial consumption. Indeed, using (33) and (59), we get

$$\frac{\partial c_{10}}{\partial \eta} = -k_0 \frac{\partial g^*}{\partial \eta} - \frac{n_0}{n_1} \frac{\partial c_{00}}{\partial \eta} \tag{62}$$

which can be positive or negative.

Consider the term between brackets in equation (7.6). Recall that if γ is close enough to zero, the derivative $\partial g^*/\partial \eta$ is also close to zero and $\partial U_1/\partial \eta$ has the same sign as $\partial c_{10}/\partial \eta$. Consider then equation (62). If $\partial g^*/\partial \eta$ is close to zero, $\partial c_{00}/\partial \eta$ and $\partial c_{10}/\partial \eta$ work in opposite directions. Therefore, there exist $\bar{\gamma} \in (0, \hat{\gamma}]$ and $\bar{\theta}_0 \in (0, \hat{\theta}_0]$ such that, if $\gamma \in (0, \bar{\gamma})$ and $\theta_0 \in \left(0, \bar{\theta}_0\right)$, then $\partial U_1/\partial \eta > 0$.

Focus finally on $\partial U_1/\partial \gamma$. We have

$$\frac{\partial U_1}{\partial \gamma} = \frac{c_{10}^{1-1/\sigma}}{1-\beta_1 g^{*1-1/\sigma}} \left[\frac{1}{g^*} \frac{\partial g^*}{\partial \gamma} \frac{\beta_1 g^{*1-1/\sigma}}{1-\beta_1 g^{*1-1/\sigma}} + \frac{1}{c_{10}} \frac{\partial c_{10}}{\partial \gamma} \right]$$

There exists a trade-off between the initial consumption and the growth rate. The utility of impatient agents increases because of the positive impact on the growth rate:

$$\frac{1}{g^*} \frac{\partial g^*}{\partial \gamma} \frac{\beta_1 g^{*1-1/\sigma}}{1-\beta_1 g^{*1-1/\sigma}} > 0$$

and can decrease because of the ambiguous effect on the initial consumption. Indeed, using (33), we find

$$\frac{\partial c_{10}}{\partial \gamma} = -\theta_1 \frac{\eta - 1}{\eta} \left(1 + \frac{\partial g^*}{\partial \gamma}\right) k_0 < 0 \tag{63}$$

Consider the term between brackets in equation (7.6). From equations (31) and (33) in Proposition 4 we derive

$$c_{10} = k_0 \frac{(1-s)A[1+(\eta-1)\theta_1] + \theta_1(\eta-1)(\Delta + sA - \gamma - g^*)}{\eta}$$

Substituting this expression into (63) gives

$$\frac{1}{c_{10}} \frac{\partial c_{10}}{\partial \gamma} = -\frac{\theta_1(\eta-1)\left(1+\frac{\partial g^*}{\partial \gamma}\right)}{(1-s)A[1+(\eta-1)\theta_1] + \theta_1(\eta-1)(\Delta+sA-\gamma-g^*)} \tag{64}$$

Substituting (64) into (7.6) then yields after simplifications

$$\begin{aligned} \frac{\partial U_1}{\partial \gamma} &= \frac{c_{10}^{1-1/\sigma}}{1-\beta_1 g^{*1-1/\sigma}} \left[\frac{-(\eta-1)\theta_1}{(1-s)A[1+(\eta-1)\theta_1] + (\eta-1)\theta_1[\Delta+sA-\gamma-g^*]} \right. \\ &+ \left. \frac{\partial g^*}{\partial \gamma} \frac{\beta_1 g^{*1-1/\sigma}\{(1-s)A[1+(\eta-1)\theta_1]+(\eta-1)\theta_1[\Delta+sA-\gamma]\} - g^*(\eta-1)\theta_1}{g^*\left[1-\beta_1 g^{*1-1/\sigma}\right]\{(1-s)A[1+(\eta-1)\theta_1]+(\eta-1)\theta_1[\Delta+sA-\gamma-g^*]\}} \right] \end{aligned}$$

Therefore, there exists $\tilde{\theta}_1 \in (0,1)$ such that when $\theta_1 \in (0, \tilde{\theta}_1)$, then $\partial U_1/\partial \gamma > 0$. ■

7.7 Proof of Lemma 7

Equation (50) reduces to

$$\eta\left(1-z_0\right)+\left(1-\eta\right)\theta_1=\tfrac{n_1(1-s)A}{n_0x_0+n_1x_1}$$

Linearizing equation (28) and using (7.7), we obtain

$$\left(z_0-a_0\right)\tfrac{dx_{0t+1}}{x_0}-\left(z_0-a_0\right)\tfrac{dx_{1t+1}}{x_1}=z_0\tfrac{dx_{0t}}{x_0}+\left(1-z_0\right)\tfrac{dx_{1t}}{x_1}$$

In order to linearize the second equation, we apply the elasticities (5.5). Under equation (50), equation (28), evaluated at the steady state, writes:

$$1-\gamma\eta/g^*=\beta_0\left(sA+\Delta-\gamma\right)g^{*-1/\sigma} \tag{65}$$

After some computations, we obtain using equations (7.7) and (65):

$$\begin{aligned}&\left(a_0+\varepsilon_0-\tfrac{y_0+\varepsilon_0}{\sigma}\right)\tfrac{dx_{0t+1}}{x_0}+\left(1-a_0+\varepsilon_1-\tfrac{\varepsilon_1}{\sigma}\right)\tfrac{dx_{1t+1}}{x_1}\\ =\;&\left[h\left(a_0+\varepsilon_0\right)-\tfrac{y_0}{\sigma}\right]\tfrac{dx_{0t}}{x_0}+h\left(1-a_0+\varepsilon_1\right)\tfrac{dx_{1t}}{x_1}\end{aligned}$$

where h is given by (34).

System (7.7)-(7.7) is represented by the following Jacobian matrix:

$$\begin{aligned}J\;=\;&\begin{bmatrix}z_0-a_0 & a_0-z_0\\ a_0+\tfrac{\sigma-1}{\sigma}\varepsilon_0-\tfrac{y_0}{\sigma} & 1-a_0+\tfrac{\sigma-1}{\sigma}\varepsilon_1\end{bmatrix}^{-1}\\ &\begin{bmatrix}z_0 & 1-z_0\\ h\left(a_0+\varepsilon_0\right)-\tfrac{y_0}{\sigma} & h\left(1-a_0+\varepsilon_1\right)\end{bmatrix}\end{aligned} \tag{66}$$

Straightforward computations yield the expression of J as given by (35). From (3), noticing that

$$g^*=\Delta+sA-\gamma-\left(1-s\right)A\tfrac{\eta n_0x_0+(1-\eta)\theta_0(n_0x_0+n_1x_1)}{\eta n_1x_1+(1-\eta)\theta_1(n_0x_0+n_1x_1)}$$

and using $\theta_0+\theta_1=1$ and (50), we compute the elasticities ε_0 and ε_1:

$$\varepsilon_1=-\varepsilon_0=\tfrac{x_1}{g^*}\tfrac{\eta z_0}{\eta(1-z_0)+(1-\eta)\theta_1}$$

Replacing (7.7) in (66), we get (35). We then easily derive the determinant and the trace:

$$D=\tfrac{y_0(1-z_0)+h\sigma(z_0-a_0-\varepsilon_0)}{(\sigma-y_0)(z_0-a_0)}\text{ and }T=1+D+\tfrac{1+(h-1)\sigma}{(\sigma-y_0)(z_0-a_0)}\varepsilon_0$$

Note finally that from (30)-(31) we get

$$n_0x_0+n_1x_1=n_1\left(\Delta+A-\gamma-g^*\right)$$

It follows from (7.7) that

$$\eta\left(1-z_0\right)+\left(1-\eta\right)\theta_1=\tfrac{(1-s)A}{\Delta+A-\gamma-g^*}>0$$

and thus $\varepsilon_1=-\varepsilon_0>0$ for any $\eta>1$. ■

7.8 Proof of Lemma 8

We have proved that $\varepsilon_0 < 0$ for any $\eta > 1$. Moreover, we know that $h > 1$. Finally it is easy to derive that $z_0 - a_0 > 0$. We conclude that for any $\sigma \geq 0$

$$\tfrac{1+(h-1)\sigma}{(\sigma-y_0)(z_0-a_0)}\varepsilon_0 \neq 0$$

so that $T \neq 1 + D$ and any bifurcation with an eigenvalue equal to 1 is ruled out. Moreover, when $\beta_0 = \beta_0^*$ with β_0^* as given by (5.5), we get $g^* = 1$ and from the expressions of D and T, we derive that, when $\sigma = \bar{\sigma} \equiv y_0$, then $D = \pm\infty$ and $T = \pm\infty$. By continuity, there exists $\epsilon > 0$ such that for any $g^* \in [1, 1+\epsilon)$, the result holds. ■

7.9 Proof of Lemma 9

Fix now the steady state g^* by scaling the parameter β_0 according to (5.5). The origin of the line Σ is given by

$$\begin{aligned} D_0 &= \tfrac{1-z_0}{a_0-z_0} = 1 - \tfrac{\theta_1+(1-\theta_1)\eta}{\eta z_0} < 0 \\ T_0 &= 1 + D_0 + \tfrac{\varepsilon_0}{y_0(a_0-z_0)} = 1 + D_0 + \tfrac{x_1}{y_0(1-z_0)} \end{aligned}$$

Then, $D_0 < T_0 - 1$. Note now that D_0 is an increasing function of η. Therefore, if $D_0 > -1$ when $\eta = 1$, then $D_0 > -1$ for any $\eta > 1$. Thus, using (5.5), (53) and (54) with $g^* = \eta = 1$, we get $D_0 > -1$ under Assumption 7.2. The endpoint is given by

$$\begin{aligned} D_\infty &\equiv \lim_{\sigma\to+\infty} D(\sigma) = h\tfrac{z_0-a_0-\varepsilon_0}{z_0-a_0} = h\left(1 + \tfrac{\varepsilon_0}{a_0-z_0}\right) = h\left(1 + \tfrac{x_1}{1-z_0}\right) > 1 \\ T_\infty &\equiv \lim_{\sigma\to+\infty} T(\sigma) = 1 + D + \tfrac{1+(h-1)\sigma}{(\sigma-y_0)(z_0-a_0)}\varepsilon_0 = 1 + h + \tfrac{\varepsilon_0}{a_0-z_0} \\ &= 1 + h + \tfrac{x_1}{z_1} = 1 + D_\infty - (h-1)\tfrac{x_1}{1-z_0} > 2 \end{aligned}$$

Then, $D_\infty > 1$, $T_\infty > 2$ and $D_\infty > T_\infty - 1$. The slope of Σ is given by

$$\begin{aligned} S &= \tfrac{D'(\sigma)}{T'(\sigma)} = \tfrac{-[h(z_0-a_0-\varepsilon_0)+(1-z_0)]y_0}{-([h(z_0-a_0-\varepsilon_0)+(1-z_0)]y_0+[(h-1)y_0+1]\varepsilon_0)} \\ &= \left[1 + \tfrac{\frac{1}{y_0}+h-1}{\frac{1-z_0}{\varepsilon_0}-h\left(1+\frac{a_0-z_0}{\varepsilon_0}\right)}\right]^{-1} = \left[1 + \tfrac{\frac{1}{y_0}+h-1}{\frac{1-z_0}{\varepsilon_0}-h\left(1+\frac{1-z_0}{x_1}\right)}\right]^{-1} \end{aligned}$$

and does not depend directly on σ. We know that $\varepsilon_0 < 0$ and $h > 1$: thus $S < 0$ or $S > 1$. We finally notice that

$$\begin{aligned} D'(\sigma) &= \tfrac{\varepsilon_0}{a_0-z_0}\tfrac{y_0}{(\sigma-y_0)^2}\left[\tfrac{1-z_0}{\varepsilon_0} - h\left(1 + \tfrac{a_0-z_0}{\varepsilon_0}\right)\right] \\ &= \tfrac{x_1 y_0}{g(1-z_0)(\sigma-y_0)^2}\left[\tfrac{1-z_0}{\varepsilon_0} - h\left(1 + \tfrac{1-z_0}{x_1}\right)\right] \\ &= \tfrac{S}{1-S}\tfrac{x_1}{1-z_0}\tfrac{1+(h-1)y_0}{(\sigma-y_0)^2} \end{aligned}$$

We conclude that $D'(\sigma) < 0$. ■

References

[1] Barbar, R. and S. Bosi (2010), Collaterals and Macroeconomic Volatility, *Research in Economics* **64**, 146-161.

[2] Becker, R. A. (1980), On the Long-Run Steady State in a Simple Dynamic Model of Equilibrium with Heterogeneous Households, *Quarterly Journal of Economics* **95**, 375-382.

[3] Becker, R. A. and C. Foias (1987), A Characterization of Ramsey Equilibrium, *Journal of Economic Theory* **41**, 173-184.

[4] Becker, R. A. and C. Foias (1994), The Local Bifurcation of Ramsey Equilibrium, *Economic Theory* **4**, 719-744.

[5] Bosi, S. and T. Seegmuller (2010), On the Ramsey Equilibrium with Heterogeneous Consumers and Endogenous Labor Supply, *Journal of Mathematical Economics* **46**, 475-492.

[6] Boucekkine, R., O. Licandro, L. Puch and F. del Rio (2005), Vintage Capital and the Dynamics of the AK Model, *Journal of Economic Theory* **120**, 39-72.

[7] Boucekkine, R. and R. Ruiz-Tamarit (2008), Special Functions for the Study of Economic Dynamics : the Case of the Lucas-Uzawa Model, *Journal of Mathematical Economics* **44**, 33-54.

[8] Campbell, J. (1999), Asset Prices, Consumption and the Business Cycle, in Taylor J.B. and Woodford M. (eds.), *Handbook of Macroeconomics*, North-Holland, Amsterdam, 1231-1303.

[9] Carlstrom, C. T. and T. S. Fuerst (2003), Money Growth Rules and Price Level Determinacy, *Review of Economic Dynamics* **6**, 263-75.

[10] Cooley T. F. and G. D. Hansen (1989), The Inflation Tax in a Real Business Cycle Model, *American Economic Review* **79**, 492-511.

[11] Grandmont, J.-M., P. Pintus and R. de Vilder (1998), Capital-labour Substitution and Competitive Nonlinear Endogenous Business Cycles, *Journal of Economic Theory* **80**, 14-59.

[12] Grandmont J.-M. and Y. Younès (1972), On the Role of Money and the Existence of a Monetary Equilibrium, *Review of Economic Studies* **39**, 355-372.

[13] Ferraris, L. and M. Watanabe (2008), Collateral Secured Loans in a Monetary Economy, *Journal Economic Theory* **143**, 405-424.

[14] Ferraris, L. and M. Watanabe (2011), Collateral Fluctuations in a Monetary Economy, *Journal Economic Theory* **146**, 1915-1940.

[15] Fostel, A. and J. Geanakoplos (2008), Leverage Cycles and the Anxious Economy, *American Economic Review* **98**, 1211-1244.

[16] Geanakoplos, J. and W. Zame (2007), Collateralized Asset Markets, *Yale University Discussion Paper.*

[17] Kiyotaki, N. and J. Moore (1997), Credit Cycles, *Journal of Political Economy* **105**, 211-248.

[18] Kocherlakota, N. (1996), The Equity Premium: It's Still a Puzzle, *Journal of Economic Literature* **36**, 42-71.

[19] Lagos, R. and R. Wright (2005), A Unified Framework for Monetary Theory and Policy Analysis, *Journal of Political Economy* **113**, 463-484.

[20] Le Van, C. and Y. Vailakis (2003), Existence of a Competitive Equilibrium in a One Sector Growth Model with Heterogeneous Agents and Irreversible Investment, *Economic Theory* **22**, 743-771.

[21] Lucas R. and N. Stokey (1987), Money and Interest in a Cash-In-Advance Economy, *Econometrica* **55**, 491-513.

[22] Mankiw N., D. Romer and D. Weil (1992), A Contribution to the Empirics of Economic Growth, *Quarterly Journal of Economics* **107**, 407-437.

[23] Romer P. (1986), Increasing Returns and Long-Run Run Growth, *Journal of Political Economy* **94**, 1002-1037.

[24] Stockman A. (1981), Anticipated Inflation and the Capital Stock in a Cash-In-Advance Economy, *Journal of Monetary Economics* **8**, 387-393.

[25] Svensson L. (1985), Money and Asset Prices in a Cash-In-Advance Economy, *Journal of Political Economy* **93**, 919-944.

[26] Vissing-Jorgensen, A. (2002), Limited Asset Market Participation and the Elasticity of Intertemporal Substitution, *Journal of Political Economy* **110**, 825-853.

[27] Woodford M. (1986), Stationary Sunspot Equilibria in a Finance Constrained Economy, *Journal of Economic Theory* **40**, 128-137.