Stochastic dominance, risk and disappointment: a synthesis

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Stochastic dominance, risk and disappointment: a synthesis.

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ABSTRACT: The theory of disappointment of Loomes and Sugden [1986] has never been given an axiomatics. This article, where a theory of disappointment is derived from a simple axiomatics, makes up for this omission. The new theory is close to that of Loomes and Sugden although the functional representing the preferences of the decision-maker is now lottery-dependent. Actually, preferences exhibit four properties of interest : (a) risk-averse and risk prone investors actually behave differently; (b) risk is defined in a consistent way with risk aversion; (c) the functional is nothing but the opposite to a convex measure of risk (Föllmer and Schied [2002]) when constant marginal utility is assumed and (d) violations of the second-order stochastic dominance property are allowed for when monetary values are taken into account (but not when when "utils" are substituted for them). Moreover, the preorder induced by stochastic dominance over utils is as "close" to the preorder of preferences as possible and utility functions may be elicited through experimental testing.

JEL classification: D81. KEY-WORDS: disappointment, risk-aversion, expected utility, risk premium, stochastic dominance, subjective risk.

RESUME: La théorie de la déception proposée par Loomes et Sugden [1986] n'a jamais été pourvue d'une axiomatisation. Ce document de travail comble cette lacune. On y présente une théorie très proche de celle de Loomes et Sugden où la fonctionnelle représentant les préférences est "loterie-dépendante". Cette théorie possède quatre propriétés très intéressantes : (a) les investisseurs ayant de l'aversion pour le risque se comportent vraiment différemment de ceux qui sont neutres vis-à-vis du risque (b) le risque est défini de façon cohérente avec l'aversion pour le risque (c) la fonctionnelle représentant les préférences n'est autre que l'opposé d'une mesure convexe de rique à la Föllmer et Schied [2002], si l'utilité marginale de la richesse est constante et (d) la dominance stochastique de second ordre est respectée non pour les valeurs monétaires mais pour les seules utilités. De plus, le préodre induit par la dominance stochastique est alors aussi proche que possible du préordre des préférences et l'on peut déterminer la fonction d'utilité élémentaire à partir de tests empiriques.

JEL classification: D81. MOTS-CLES: déception, aversion pour le risque, risque subjectif, prime de risque, utilité espérée.

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1 *Introduction*

It is often emphasized, in psychological literature, that (a) disappointment (elation) is experimented, once a decision has been taken, when the chosen option turns out to be worse (better) than expected (see *e.g.* Mellers [2000]), (b) that it is the most frequently experimented emotion (see Weiner *et alii* [1979]) and (c) that disappointment is the most powerful among the negative emotions which are experimented (see Schimmack and Diener [1997]).[1] Moreover, as Frijda [1994] points out, "actual emotion, affective response, anticipation of future emotion can be regarded as the primary source of decisions". To sum up, it is most likely that expected elation/disappointment plays an important role in decision-making.

This role was first formalized independently by Bell [1985] and Loomes and Sugden [1986]. Despite its earliness, their approach has revealed surprisingly close to the analyses mentioned above. Indeed, consider a random outcome whose possible values are monetary prizes. Then, according to Loomes and Sugden, winning a prize gives rise to a satisfaction which can be split into two elements: *(i)* the utility from winning the prize with certainty and *(ii)* elation (or disappointment) which depends on the difference between this utility and a reference level which is often viewed as a "prior expectation" of the utility from the random outcome. As a corollary, the utility from the random outcome is an average of the utilities of the prizes –namely the expected utility of the investor's wealth– plus an average of elations or disappointments –namely expected elation/disappointment–. This additional term is positive (negative) when the investor experiments elation (disappointment).

Despite their psychological relevance, the models of Bell [1985] and of Loomes and Sugden [1986] have been somewhat neglected in the economic literature, probably because they lack an axiomatic framework. In the meantime, other disappointment models have been developed. However, they also lack an axiomatic basis. An exception is Gul [1991].[2]

Our theory of disappointment is a fully choice-based decision-making under risk. It is derived from a set of testable axioms. It corresponds to a set of models which will be called **LS**-models, because they are close to the disappointment model of Loomes and Sugden [1986]. The new theory is endowed with **four important properties**: (i) it allows for an attitude towards risk which does not require any assumption about marginal utility, (ii) it yields coherent definitions of risk and risk aversion, (iii) it is compatible with Artzner's *et alii* [1997] approach of measures of risk and (iv) it allows for many behavioural anomalies.

To understand what property (i) actually means, recall that an investor is generally assumed to be sensitive to the utility of his wealth. For instance, when the expected utility theory (henceforth **EU** theory) is valid, the investor's welfare is a probability weighted average of the utilities of the possible outcomes. As a consequence, he is risk-averse (prone) if his elementary utility function is

[1] There is a lot of empirical evidence which supports this view (see Van Dijk and Van der Pligt [1996], Zeelenberg *et alii* [2002], Van Dijk *et alii* [2003]).

[2] See below.

concave (convex). Anyway, in any case, the investor takes into account nothing but an average of the results of a gamble to the results of which he is sensitive. Hence, whatever his attitude towards risk (risk-aversion, risk-proneness or neutrality), he actually behaves in the same way. This is a well-known paradox.

By contrast, according to Loomes and Sugden, an investor will be disappointment-averse (prone) if and only if (henceforth *iff*) his welfare includes expected disappointment (elation) in addition to the expected utility of his terminal wealth.[3] In particular, when marginal utility is constant, he is risk-averse (prone) *iff* his welfare includes expected disappointment (elation) in addition to his expected wealth, and, consequently, the paradox vanishes.

Little attention has been paid, up to now, to properties (*ii*) and (*iii*) in the litterature. Property (*ii*) means that risk may be defined in a consistent way with risk aversion. This occurs in Loomes and Sugden's approach[4] since any risk premium may be split into elementary risk premia, each of which may be identified to the product of a quantity of risk by a specific risk-aversion. Property (*iii*) is met when constant marginal utility is assumed: the certainty equivalent of a lottery is then the opposite to a convex measure of risk *à la* Föllmer and Schied [2002]. Property (*iv*) consists in allowing for violations of the independence and/or second-order stochastic dominance properties since both types of violations are commonly observed in experimental tests.

Finally the four above properties constitute a strong incentive for favouring the use of **LS**-models. To our knowledge, they are not met simultaneously in any other fully-axiomatized model of decision-making under risk.

An important additional reason is that this approach makes possible the **elicitation of the utility function** of an investor. Indeed, it will soon become apparent that, among the normalized convex or concave elementary utility functions, there exists one of them which is the most likely to be that of the decision-maker[5] and, what is more, which may be elicited from a sequence of binary choices.

The rest of this article is organized as follows: Section 2 is devoted to the study of a particular case: that of constant marginal utility. In Section 3, stochastic dominances are revisited and the definition of a rational investor is clarified. In Section 4, the axiomatization of a general theory of disappointment is developed. Section 5 deals with the elicitation property and Section 6 concludes.

[3] A disappoinment premium is but a component of a global risk premium. The two premia coincide when constant marginal utility is assumed. Similarly, global risk aversion includes disappointment aversion (See below).

[4] When the reference level in the functional coincides with the expected utility of the investor's wealth.

[5] Since it makes the preorder induced by second order "subjective" stochastic dominance as "close" as possible to that of the investor's preferences. See Section 3.

2 *The case of constant marginal utility.*

In this section, we focus on the particular case of constant marginal utility. Some technical definitions will be first recalled. Next the definitions of stochastic dominances and that of risk will be reexamined. A simplified axiomatics will then be set, leading to a representation theorem, where the functional is lottery dependent. Finally, the functional will be particularized so as to characterize the behaviour of risk-averse investors with constant marginal utility, which will be endowed with some interesting properties.

2.1 Preliminary definitions

In this article, the decision-maker faces a problem of risky choice over a set $\mathfrak{X}=\{X,Y,Z,...\}$ of random variables mapping a set of "states of nature" Ω on to a set $\mathcal{C}$ of outcomes. By assumption, the set $\mathcal{C}$ is bounded and, consequently, may be identified, without loss of generality, to an interval $[a,b]$ of $\mathbb{R}$. The outcomes, *i.e.* the elements of $[a,b]$ are identified to monetary prizes.

Let $\mathcal{F}$ be a set of "events" (*i.e.* a σ-algebra on Ω) and $P(.)$ a probability measure over $\mathcal{F}$, which, by assumption, is known with certainty. Any random variable $X\in\mathfrak{X}$ is then endowed with a probability distribution[6]. The subset of the probability distributions will be labelled $\mathfrak{L}$. A probability distribution will be identified to its cumulative distribution function (henceforth *c.d.f.*). The *c.d.f.* of X is labelled $F_X(x)$ and its expected value $\mathbf{E}[X]$. Hence, we get that:

$$F_X(x)=P(\omega\in\Omega\mid X(\omega)\leq x)$$

When the set of events is finite, a random variable $X\in\mathfrak{X}$ has a finite support $\{x_1,x_2,...,x_N\}$ where $x_1<x_2<...<x_N$; it will be labelled

$$X=[x_1,p_1;x_2,p_2;...;x_N,p_N]$$

where $p_n=P(X=x_n)\geq 0$ and $\sum_{n=1}^{N}p_n=1$. The subset of random variables with finite support will be denominated $\mathfrak{X}^f$. It is a subset of $\mathfrak{X}$. The corresponding subset of probability distributions will be labelled $\mathfrak{L}^f$.

A decision-maker has a preference relation on $\mathfrak{X}$. His preferences will be denoted $\succsim$, with $\succ$ (strict preference) and $\sim$ (indifference). For instance it is equivalent to state that X is weakly (strongly) preferred to Y or to write $X\succsim Y$ $(X\succ Y)$. Actually, one may consider the set of the corresponding probability distributions (or *c.d.f.*'s) and write instead $F_X\succsim F_Y$ $(F_X\succ F_Y)$. Indeed, as usually done in models dealing with decision under risk, it will be implicitly assumed that all the random variables generating the same probability distribution are indifferent.[7] Because of this assumption, the same symbol ($\succsim$) will be used, throughout this paper, to denote the preference relations on $\mathfrak{X}$ and those on $\mathfrak{L}$

[6] Which is defined by the following formula where $\mathcal{B}_{[a,b]}$ is the Borel algebra of the bounded interval $[a,b]$: for all $A\in\mathcal{B}_{[a,b]}$, $P_X(A)=P(\omega\in\Omega\mid X(\omega)\in A)$

[7] *i.e.*: $F_X=F_Y\Longrightarrow X\sim Y$

The random variable whose outcome is x with certainty, will be denominated δ_x. The certainty equivalent of $X \in \mathfrak{X}$ is the certain outcome which is indifferent to X. It is labelled $\mathbf{c}(X)$ (*i.e.* $X \sim \delta_{\mathbf{c}(X)}$).

Let $\lambda \in [0,1]$. One may define the $(\lambda, 1-\lambda)$-probability mixture of two random variables, Y and Z, as a random variable X whose *c.d.f.* F_X is the $(\lambda, 1-\lambda)$-convex combination of F_Y and F_Z.[8] It is labelled: $X = \lambda Y \oplus (1-\lambda) Z$.

More generally let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, ...\lambda_N)$ where $\lambda_i > 0$ and $\sum_{i=1}^{N} \lambda_i = 1$. One may define the $\boldsymbol{\lambda}$-probability mixture of $\{X_i\}$ where $X_i \in \mathfrak{X}$ for $i = 1, ..., N$, as a random variable $X \in \mathfrak{X}$ whose probability distribution F_X is such that $F_X = \sum_{i=1}^{N} \lambda_i F_{X_i}$, *i.e.* for any $x \in [a,b]$, $F_X(x) = \sum_{i=1}^{N} \lambda_i F_{X_i}(x)$. It will be labelled: $X = \lambda_1 X_1 \oplus \lambda_2 X_2 \oplus \, ... \, \oplus \lambda_N X_N$. As a consequence, the set $\mathfrak{X}$ ($\mathfrak{L}$) constitutes a *mixture set*, *i.e.* a convex subset with respect to $\oplus$ $(+)$.

Finally, in the rest of this article, **the word "lottery" (the phrase "simple lottery") will refer indifferently to a random variable belonging to $\mathfrak{X}$ ($\mathfrak{X}^f$) or to its probability distribution belonging to $\mathfrak{L}$ ($\mathfrak{L}^f$).**

2.2 Definitions of stochastic dominance and risk aversion.

Stochastic dominances are partial pre-ordering relations over $\mathfrak{X}$. They include first-order stochastic (henceforth **FOS**) and second-order stochastic (henceforth **SOS**) dominances. The partial pre-order induced by **FOS** (**SOS**) dominance will be labelled $\mathcal{D}_1$,($\mathcal{D}_2$) –with $\underline{\mathcal{D}}_{\mathbf{1}}$ ($\underline{\mathcal{D}}_{\mathbf{2}}$) for strict dominance–. Stochastic dominances may be characterized by the following equivalences:

$$X \; \mathcal{D}_1 \; Y \Longleftrightarrow F_X(x) \leq F_Y(x) \; for \; any \; x \in [a,b]$$

and

$$X \; \mathcal{D}_2 \; Y \Longleftrightarrow \int_a^x \left(F_X(t) - F_Y(t)\right) dt \leq 0 \; for \; any \; x \in [a,b]$$

According to Rothschild and Stiglitz [1970], a decision-maker is *strongly risk-averse* if he prefers X to any mean preserving spread of X.[9] However, the definition of risk aversion which is used in this article is more restrictive than that of Rothschild and Stiglitz. Indeed, we set the following definition

Definition 1 (strict risk aversion). *A decision-maker is strictly risk-averse iff he prefers X to Y whenever X **SOS** dominates Y, what formally reads:*

$$X \; \mathcal{D}_2 \; Y \Longrightarrow X \succsim Y \tag{1}$$

Clearly strict risk aversion implies strong risk aversion which in its turn implies weak risk aversion.[10] Finally, it is equivalent to say: (i) no violations of the **SOS** dominance property may occur or (ii) the decision-maker is strictly risk-averse.

[8] In other words, for any $x \in [a,b]$, we get: $F_X(x) = \lambda F_Y(x) + (1-\lambda) F_Z(x)$, where $\lambda \in [0,1]$.

[9] Recall that Y is a mean preserving spread of X *iff* (a) $\mathbf{E}[Y] = \mathbf{E}[X]$ and (b) $X \; \mathcal{D}_{\mathbf{2}} \; Y$.

[10] Recall that a decision-maker is *weakly risk-averse* if he prefers $\delta_{\mathbf{E}[X]}$ to X.

Now let $\mathfrak{X}_{\mathbf{x}}$ be the subset of lotteries exhibiting the same expected value $\mathbf{x} \in [a, b]$, what formally reads:

$$\mathfrak{X}_{\mathbf{x}} \overset{def}{=} \{X \in \mathfrak{X} \mid \mathbf{E}[X] = \mathbf{x}\}$$

The degenerate lottery $\delta_{\mathbf{x}}$ belongs to $\mathfrak{X}_{\mathbf{x}}$ and so does the following binary lottery:

$$X_{\mathbf{x}} \overset{def}{=} [a, 1 - \pi(\mathbf{x}); b, \pi(\mathbf{x})]$$

where $\pi(\mathbf{x}) \overset{def}{=} (\mathbf{x} - a)/(b - a)$. Clearly $\delta_{\mathbf{x}}$ dominates any lottery X by **SOS** dominance belonging to $\mathfrak{X}_{\mathbf{x}}$ and any lottery X belonging to $\mathfrak{X}_{\mathbf{x}}$ dominates $X_{\mathbf{x}}$ by **SOS** dominance. Hence, when a decision-maker is assumed to be strictly risk-averse, any lottery $X \in \mathfrak{X}_{\mathbf{x}}$ is such that:

$$\delta_{\mathbf{x}} \succeq X \succeq X_{\mathbf{x}} \tag{2}$$

In other words, the following proposition holds:

Proposition 1. *When a decision-maker is strictly risk-averse, then $\delta_{\mathbf{x}}$ ($X_{\mathbf{x}}$) is a maximal (minimal) element in the subset $\mathfrak{X}_{\mathbf{x}}$.*

Proof. For any $X \in \mathfrak{X}_{\mathbf{x}}$ we have $\delta_{\mathbf{x}}\ \mathcal{D}_2\ X$ and $X\ \mathcal{D}_2\ X_{\mathbf{x}}$.□

2.3 The axiomatics of a simplified theory of disappointment

We now develop a simplified theory of disappointment which will be generalized later (See Section 4).The first two axioms of our simplified theory are but those of **EU** theory.

Axiom 1. (ordering of $\succsim$). *The binary relation $\succsim$ is a total preorder over $\mathfrak{X}$.*

Axiom 2. (continuity of $\succsim$). *For any lottery $X \in \mathfrak{X}$ the sets $\{Z \in \mathfrak{X} \mid Z \succsim X\}$ and $\{Z \in \mathfrak{X} \mid X \succsim Z\}$ are closed in the topology of weak convergence.*

The above axioms need no special comment. Simply recall that they imply the Archimedean property.[11] As a consequence, given any lottery $X \in \mathfrak{X}_{\mathbf{x}}$, there exists one real number $\lambda \in [0, 1]$ such that X is indifferent to the following simple lottery:

$$\mathbf{X}_{\mathbf{x}}^{\lambda} \overset{def}{=} \lambda\delta_{\mathbf{x}} \oplus (1 - \lambda) X_{\mathbf{x}} = [a, (1 - \lambda)(1 - \pi(\mathbf{x})); \mathbf{x}, \lambda; b, (1 - \lambda)\pi(\mathbf{x})]$$

which is the $(\lambda, 1 - \lambda)$-mixing of $\delta_{\mathbf{x}}$ and $X_{\mathbf{x}}$. Note that $\mathbf{X}_{\mathbf{x}}^{0} = X_{\mathbf{x}}$ and $\mathbf{X}_{\mathbf{x}}^{1} = \delta_{\mathbf{x}}$.

Next, as first shown by Debreu [1954], Axioms 1 and 2 imply that there exists a continuous utility functional mapping $\mathfrak{X}$ on to an interval of $\mathbb{R}$ which represents the investor's preferences. It is defined up to a strictly continuous and increasing transformation.

[11] *i.e.* if $X, Y, Z \in \mathfrak{X}$ and $X \succsim Y \succsim Z$, then there exists $\lambda \in [0, 1]$ such that $\lambda X \oplus (1 - \lambda) Z \sim Y$.

To get stronger results one (or more) additional axiom(s) must be set. In **EU** theory, a third axiom, namely the independence axiom, is set. Unfortunately, it is well known that the **EU** theory is not capable of predicting some empirically observed patterns such as the Allais paradox. However, the independence axiom may still hold on rather large subsets of $\mathfrak{X}$. This is likely to be the case if the investor is sensitive to elation/disappointment.

To see this, consider a decision-maker with constant marginal utility.[12] As a consequence he cares for monetary outcomes. Elation (disappointment) will then occur when the realized outcome x is higher (lower) than a reference level $\mathbf{x}$. The reference level may be viewed as a "prior expectation" of the value of the lottery, which is likely to be an average of *ex-post* outcomes, for instance their expected value ($\mathbf{x} = \mathbf{E}[X]$). Elation or disappointment will then depend on the spread $x - \mathbf{E}[X]$, *i.e.* the value of a lottery will be the algebraic sum of its expected value and of the expectation of a function of the spread, $\mathbf{E}[\mathcal{E}(x - \mathbf{E}[X])]$.[13] As a consequence, the value of the $(\lambda, 1-\lambda)$-mixing of two lotteries exhibiting the same expected value $\mathbf{x}$, will be the $(\lambda, 1-\lambda)$-convex combination of the values of the two considered lotteries. In other words, certainty equivalents will combine linearly for lotteries exhibiting the same expected value. Hence, the following axiom will be set.

Axiom 3. (linearity of certainty equivalents over $\mathfrak{X}_{\mathbf{x}}$) *The certainty equivalent of the $(\lambda, 1-\lambda)$-mixing of two lotteries exhibiting the same expected value is the $(\lambda, 1-\lambda)$-convex combination of their certainty equivalents, what formally reads:*

$$\forall X, Y \in \mathfrak{X}_{\mathbf{x}},\ \forall \lambda \in [0,1],\ \mathbf{c}(\lambda X \oplus (1-\lambda) Y) = \lambda \mathbf{c}(X) + (1-\lambda) \mathbf{c}(Y)$$

The axiom clearly implies that the independence property is met over any subset of lotteries exhibiting the same expected value. It also implies the following result:

Proposition 2. *Under Axioms 1 to 3, the total preorder of preferences of a decision-maker $\succsim$ may be represented over $\mathfrak{X}_{\mathbf{x}}$ by a continuous real-valued function $\mathcal{U}_{\mathbf{x}}(.)$ which is linear, i.e.:*

$$\forall X, Y \in \mathfrak{X}_{\mathbf{x}},\ \forall \lambda \in [0,1],\ \mathcal{U}_{\mathbf{x}}(\lambda X \oplus (1-\lambda) Y) = \lambda \mathcal{U}_{\mathbf{x}}(X) + (1-\lambda) \mathcal{U}_{\mathbf{x}}(Y) \quad (3)$$

Furthermore $\mathcal{U}_{\mathbf{x}}(.)$ is defined up to an affine and positive transformation.

Proof. Since the subset $\mathfrak{X}_{\mathbf{x}}$ is a mixture set, the same proof as the one given in Fishburn [1970] may be used.□

Actually, a stronger result is available, as shown in the next proposition:

Proposition 3 (representation theorem for $\succeq$ over $\mathfrak{X}_{\mathbf{x}}$). *Under Axioms 1 to 3, the functional $\mathcal{U}_{\mathbf{x}}(.)$ may be defined as:*

$$\mathcal{U}_{\mathbf{x}}(X) \overset{def}{=} \int_a^b u_{\mathbf{x}}(x) dF_X(x) \quad (4)$$

[12] Note that this assumption will soon be relaxed.
[13] Which is generally negative.

where $u_{\mathbf{x}}(.)$ is a continuous and increasing function mapping $[a,b]$ on to $[u_{\mathbf{x}}(a), u_{\mathbf{x}}(b)]$ which is defined up to an affine and positive transformation.

Proof. If only simple lotteries were considered (*i.e.* if the representation theorem were stated for lotteries belonging to the subset $\mathfrak{X}^f \cap \mathfrak{X}_{\mathbf{x}}$), the three above axioms would clearly be sufficient for the above proposition to hold. However, this result holds even when the whole set $\mathfrak{X}_{\mathbf{x}}$ is taken into account. The proof is given in Appendix 1. Note that neither a dominance axiom nor a monotonicity axiom need then to be set.$\square$

From now on, we set the following normalization conditions for $\mathcal{U}_{\mathbf{x}}(.)$:

$$\mathcal{U}_{\mathbf{x}}(\delta_{\mathbf{x}}) = \mathbf{x} \overset{def}{=} \mathbf{c}(\delta_{\mathbf{x}}) \ \ and \ \ \mathcal{U}_{\mathbf{x}}(X_{\mathbf{x}}) = \gamma_{\mathbf{x}} \overset{def}{=} \mathbf{c}(X_{\mathbf{x}}) \tag{5}$$

where $\gamma_{\mathbf{x}} \overset{def}{=} \mathbf{c}(X_{\mathbf{x}})$ and which imply the following ones for $u_{\mathbf{x}}(.)$:

$$u_{\mathbf{x}}(\mathbf{x}) = \mathbf{x} \ \ and \ \ \pi(\mathbf{x})\, u_{\mathbf{x}}(b) + (1-\pi(\mathbf{x}))\, u_{\mathbf{x}}(a) = \gamma_{\mathbf{x}} \tag{6}$$

and conversely.

Now let $X \in \mathfrak{X}_{\mathbf{x}}$ and λ be defined by:

$$X \sim \mathbf{X}_{\mathbf{x}}^{\lambda} \overset{def}{=} \lambda\delta_{\mathbf{x}} \oplus (1-\lambda)\, X_{\mathbf{x}}$$

As a consequence, we get that:

$$\mathbf{c}(X) = \mathbf{c}\left(\mathbf{X}_{\mathbf{x}}^{\lambda}\right) = \mathbf{c}(\lambda\delta_{\mathbf{x}} \oplus (1-\lambda)X_{\mathbf{x}})$$

Then, from Axiom 3, we get that:

$$\mathbf{c}(X) = \lambda\mathbf{c}(\delta_{\mathbf{x}}) + (1-\lambda)\,\mathbf{c}(X_{\mathbf{x}})$$

and, from (5) that:

$$\mathbf{c}(X) = \lambda\mathcal{U}_{\mathbf{x}}(\delta_{\mathbf{x}}) + (1-\lambda)\,\mathcal{U}_{\mathbf{x}}(X_{\mathbf{x}}).$$

Finally, from Proposition 2, we get that

$$\mathbf{c}(X) = \mathcal{U}_{\mathbf{x}}(\lambda\delta_{\mathbf{x}} \oplus (1-\lambda X_{\mathbf{x}})) = \mathcal{U}_{\mathbf{x}}\left(\mathbf{X}_{\mathbf{x}}^{\lambda}\right) = \mathcal{U}_{\mathbf{x}}(X)$$

or more generally that:

$$\mathcal{U}_{\mathbf{E}[X]}(X) = \mathbf{c}(X) \tag{7}$$

and the following proposition holds:

Proposition 4. (representation theorem for $\succsim$ over $\mathfrak{X}$) *Under Axioms 1 to 3, the preorder of preferences $\succsim$ may be represented over $\mathfrak{X}$ by the following lottery-dependent functional*:

$$\mathcal{U}(X) = \mathcal{U}_{\mathbf{E}[X]}(X) \overset{def}{=} \int_a^b u_{\mathbf{E}[X]}(x)dF_X(x) \tag{8}$$

where $u_{\mathbf{E}[X]}(.)$ is a continuous and increasing function mapping $[a,b]$ on to $\left[u_{\mathbf{E}[X]}(0), u_{\mathbf{E}[X]}(1)\right]$ which satisfies the normalization conditions (6).

Proof. It is a direct consequence of (7).□

One may consider $u_z(x)$ as a function of z and x and set $f\left(z,x\right) \overset{def}{=} u_z(x)$. The functional (8) then reads:

$$\mathcal{U}\left(X\right) \overset{def}{=} \int_a^b f(\mathbf{E}\left[X\right], x) dF_X(x) \tag{9}$$

where $f\left(z,x\right)$ is is strictly increasing with respect to x and meets the following normalizing conditions:

$$f\left(z,z\right) = z \ ; \ \pi\left(z\right) f\left(z,b\right) + \left(1-\pi\left(z\right)\right) f\left(z,a\right) = \mathbf{c}\left(X_z\right)$$

where $\pi\left(z\right) \overset{def}{=} \left(z-a\right) / \left(b-a\right)$ and $z \in [a,b]$.

Finally, we shall say, from now on, that strictly risk-averse investors obeying Axioms 1 to 3 are rational risk-averse decision-makers with constant marginal utility.

Unfortunately the above functional remains still far general. Hence, in the next subsection we particularize $f\left(z,x\right)$ to provide an operational specification. This will be done through assessing an additional condition to preferences: the translation invariance of risk premia.

2.4 Translation invariance of risk premia

Consider the risk premium $\mathbf{RP}(X) \overset{def}{=} \mathbf{E}[X] - \mathbf{c}\left(X\right)$ of an arbitrary lottery $X \in \mathfrak{X}$. It is commonly assumed that risk premia are translation-invariant, *i.e.*,

$$\mathbf{RP}(X+x) = \mathbf{RP}(X)$$

or, equivalently:

$$\mathcal{U}\left(X+x\right) = \mathbf{c}\left(X+x\right) = \mathbf{c}\left(X\right) + x = \mathcal{U}\left(X\right) + x$$

The following lemma gives a necessary and sufficient condition for $\mathbf{RP}(.)$ to exhibit the invariance property.

Lemma 1. *Let $f(.,.)$ be a derivable function with respect to any of its two variables. Then $f(.,.)$ is endowed with the translation invariance property iff:*

$$f(z,x) = x + \mathcal{E}\left(x-z\right)$$

where $\mathcal{E}\left(.\right)$ is strictly increasing and meets the requirement: $\mathcal{E}\left(0\right) = 0$.

Proof. It is given in Appendix 1.□

Now, since we focus on strictly risk-averse investors, any non-degenerate lottery $X \in \mathfrak{X}$, will exhibit a strict negative risk premium, *i.e.* we shall have, if X is not degenerate:

$$\mathbf{E}\left[\mathcal{E}\left(X - \mathbf{E}\left[X\right]\right)\right] < 0 \tag{10}$$

A function $\mathcal{E}(.)$ meeting the above requirement will be called a **regular function**. Strictly concave functions are regular functions since a sufficient condition for (10) to hold is to assume that $\mathcal{E}(.)$ is strictly concave. This is a direct consequence of Jensen's inequality.

Finally, the preferences of a risk-averse rational decision-maker with constant marginal utility and translation-invariant risk premia may be represented by the following functional:

$$\mathcal{U}(X) = \mathbf{E}[X] + \int_a^b \mathcal{E}(x - \mathbf{E}[X]) dF_X(x) \tag{11}$$

where $\mathcal{E}(.)$ is a strictly increasing regular function. If $\mathcal{E}(.)$ is concave, it may be viewed as the opposite to a **convex measure of risk** (in the sense of Föllmer and Schied [2002]), since one may set:

$$\mathbf{r}(X) = -\mathcal{U}(X) = -\mathbf{E}[X] - \int_a^b \mathcal{E}(x - \mathbf{E}[X]) dF_X(x)$$

where $\mathbf{r}(X)$ is the measure of risk of X.[14]

The interest of the above result is that it allows for grounding a convex measure of risk on a theory of the behaviour of economic agents towards risk. The risk controller is then assumed to be a risk-averse rational decision-maker with constant marginal utility and translation-invariant risk premia, which is commonly admitted. Moreover, (11) may also be rewritten as:

$$\mathbf{RP}(X) \stackrel{def}{=} \mathbf{E}[X] - \mathbf{c}(X) = \mathbf{E}[X] - \mathcal{U}(X) = -\int_a^b \mathcal{E}(x - \mathbf{E}[X]) dF_X(x)) \tag{12}$$

where $\mathbf{RP}(X)$ is the risk premium of X. Hence, the risk premium may be split into into elementary premia, which can be viewed as the contributions of the variance, the skewness, the kurtosis ... of a lottery to the total risk premium which is demanded by an investor. Indeed, if $\mathcal{E}(.)$ is "smooth enough", one may write:

$$\mathbf{RP}(X) = -\textstyle\sum_{p=2}^{+\infty} \mathbf{E}\left[(X - \mathbf{E}[X])^p\right] \mathcal{E}^{(p)}\left(\mathbf{E}[X]\right)/p! \tag{13}$$

The total risk premium is now an infinite sum of elementary premia, each of which is proportional to the product of two terms: the pth order centered moment of the random variable X, *i.e.* $\mathbf{E}\left[(X - \mathbf{E}[X])^p\right]$, and the pth order derivative of $\mathcal{E}(.)$ taken at point $z = \mathbf{E}[X]$. Any even moment is nothing but a quantity of a "symmetric" risk and its coefficient must be negative if the investor is risk-averse, whatever the considered definition of risk. An odd moment may be viewed as a quantity of an "asymmetric" risk and its coefficient must be positive if the investor is risk-averse. Finally, Equation (13) may be viewed as a **theoretical grounding of the multimoment approach of the Capital Asset Pricing Model**.

[14] The proof of this statement may be found in Chauveau and Thomas [2014]: *Valuing non-quoted CDS with consistent default probabilities*, unpublished manuscript.

To sum up we have developed a fully choice-based theory of disappointment which clearly may give rise to many applications in Finance. However, the assumption of constant marginal utility is somewhat too restrictive and, consequently, we now turn to the general case of variable marginal utility. The corresponding theory will be developped in Section 4, including a new axiomatics which is but a slightly modified version of the present one.

As a preliminary to the presentation of the generalized theory, the concept of stochastic dominance will first be revisited in the next Section (Section 3). Indeed, since a rational investor is likely to be sensitive to nothing but to the utilities of outcomes[15], the consistency of his behaviour should be checked for with a test of stochastic dominance over utils rather than over monetary outcomes. Hence we shall focus on **subjective stochastic dominances** where utils are substituted for monetary values.

3 *Subjective stochastic dominances and utility functions*

As indicated in the next definition, the subjective first order (second order) stochastic (henceforth **SFOS** (**SSOS**)) dominance property is just the same as the usual **FOS** (**SOS**) dominance property except that the utility of any outcome, namely $u(x)$, is substituted for its monetary value, namely x, in the corresponding tests.

3.1 Subjective stochastic dominances

Definition 2. (Subjective first-order and second-order stochastic dominance). *Let* $(X_1, X_2) \in \mathfrak{X} \times \mathfrak{X}$, *let* $u(.)$ *be an elementary utility function and let* $Y_i = u(X_i)$ *for* $i = 1, 2$. *It is equivalent to state that* X_1 ***SFOS (SSOS)*** *dominates* X_2 *or that* Y_1 *dominates* Y_2 *by* ***FOS (SOS)*** *stochastic dominance, i.e.:*

$$X_1 \ \mathcal{D}_1^u \ X_2 \overset{def}{\Leftrightarrow} Y_1 \ \mathcal{D}_1 \ Y_2 \qquad (X_1 \ \mathcal{D}_2^u \ X_2 \overset{def}{\Leftrightarrow} Y_1 \ \mathcal{D}_2 \ Y_2),$$

where ***SFOS (SSOS)*** *dominance is denominated* $\mathcal{D}_1^u$ $(\mathcal{D}_{\mathbf{2}}^u)$.

Clearly the definition of subjective stochastic dominance depends on the considered elementary utility function $u(.)$. However, looking at levels of outcomes may sometimes be equivalent to looking at utilities. This happens to be the case when **FOS** dominance is considered. Indeed, **FOS** dominance is a property which is conservative through the change of random variable: $Y = u(X)$.[16] By contrast, this result is no longer valid, when **SOS** dominance is considered. Actually, the following characterization of **SSOS** dominance holds:

[15] See above Section 2.
[16] The proof of this statement is trivial.

Proposition 5. (characterization of subjective second-order stochastic dominance). *Let* $(X_1, X_2) \in \mathfrak{X} \times \mathfrak{X}$. *Let* $u(.)$ *be a n.u. function and let* $Y_i = u(X_i)$ *for* $i = 1, 2$. *It is equivalent to state:*

(a) $X_1 \ \mathcal{D}_2^u \ X_2$ *or*

(b) $\int_a^z u'(x)(F_{X_1}(x) - F_{X_2}(x))dx \leq 0$ *for any* $z \in [a, b]$

Proof. It is given in Appendix 1.□

Finally, the relations between subjective and standard stochastic dominances may be summed up as follows:

(a) if the utility function is strictly increasing, then the **SFOS** dominance property is met *iff* the (standard) **FOS** dominance property is met.

(b) the **SSOS** dominance property may be met with $u(.)$ and violated with $v(.)$ where $u(.)$ and $v(.)$ are two elementary utility functions. In particular, the **SOS** and the **SSOS** dominance properties are not necessarily met simultaneously.

(c) if constant marginal utility is assumed, (*i.e.* if $u(x) = x$), then subjective and standard stochastic dominances coincide.

3.2 Consistent utility functions

We now turn to a new concept: that of consistent utility functions and we set the following definitions:

Definition 3. (normalized utility functions).

A normalized utility function (henceforth n.u. function) is a continuously derivable and strictly increasing function $u(.)$ *mapping* $[a, b]$ *on to* $[0, 1]$. *The set of n.u. functions will be denoted* $\mathbb{U}$.

Definition 4. (consistency/inconsistency).

A n.u. function $u(.)$ *is consistent if the preorder induced by* ***SSOS*** *dominance is consistent with the total preorder induced by preferences iff the following implication is met:*

$$X_1 \mathcal{D}_2^u X_2 \implies X_1 \succsim X_2$$

where $(X_1, X_2) \in \mathfrak{X} \times \mathfrak{X}$.

A n.u. function $u(.)$ *is inconsistent, if the preorder induced by* ***SSOS*** *dominance contradicts the total preorder induced by preferences, i.e. if there exists at least one pair of lotteries* $(X_1, X_2) \in \mathfrak{X} \times \mathfrak{X}$ *such that simultaneously:*

$$X_1 \mathcal{D}_2^u X_2 \quad and \quad X_2 \succ X_1$$

The above definition implies that a *n.u.* function is either consistent or inconsistent and that the preorder induced by **SSOS** dominance is partial. From now on, the subset of inconsistent (consistent) *n.u.* functions will be labelled $\mathbb{U}_{\mathrm{I}}$ ($\mathbb{U}_{\mathrm{C}}$).

We now give an example of a consistent *n.u.* function: the utility function of an investor obeying **EU** theory. Indeed, we have:

$$\mathbf{E}[u(X)] = \int_a^b u(x) F_X(x) dx = 1 - \int_a^b u'(x) F_X(x) dx$$

and, consequently:

$$\mathbf{E}\left[u\left(X_{1}\right)\right]-\mathbf{E}\left[u\left(X_{2}\right)\right]=\int_{a}^{b}u'\left(x\right)\left(F_{X_{2}}\left(x\right)-F_{X_{1}}\left(x\right)\right)dx$$

or, equivalently:

$$X_{1}\succsim X_{2}\Longleftrightarrow X_{1}\ \mathcal{D}_{2}^{u}\ X_{2}$$

Other examples of consistent *n.u.* functions will be given in Appendix 2.[17] Now, as a preliminary to the definition the canonical utility function of an investor, we show that there exists a link between the concavity of a *n.u.* function utility and the occurence of violations of the **SSOS** dominance property.

3.3 Concavity of *n.u.* functions and violations of the subjective second-order stochastic dominance property.

Let $\mathbb{X}_{\mathbf{2}}^{u+}$ ($\mathbb{X}_{\mathbf{2}}^{u-}$) consist in the subset of pairs of lotteries (X_1, X_2) over which the two preorders, $\mathcal{D}_{2}^{u}$ and $\succsim$, coincide (disagree). A *n.u.* function $u\left(.\right)$ is all the more a good candidate for characterizing the tastes of the considered investor, that $\mathbb{X}_{\mathbf{2}}^{u+}$ is larger and $\mathbb{X}_{\mathbf{2}}^{u-}$ smaller. Actually, one may define a binary relation over the preorders induced by **SSOS** dominance as indicated in the next definition.

Definition 5. *The preorder $\mathcal{D}_{2}^{u}$ is closer to the total preorder $\succsim$ than the preorder $\mathcal{D}_{2}^{v}$ iff either:*
(a) $\mathbb{X}_{\mathbf{2}}^{u-}\subset\mathbb{X}_{\mathbf{2}}^{v-}$ or:
(b) $\mathbb{X}_{\mathbf{2}}^{u-}=\mathbb{X}_{\mathbf{2}}^{v-}$ and $\mathbb{X}_{\mathbf{2}}^{v+}\subseteq\mathbb{X}_{\mathbf{2}}^{u+}$

From now on, the binary relation " $\mathcal{D}_{2}^{u}$ is closer to $\succsim$ than $\mathcal{D}_{2}^{v}$ " will be denominated " $\mathcal{D}_{2}^{u}\ \mathcal{C}l\ \mathcal{D}_{2}^{v}$ ". It is obviously a preorder. It is partial since there may exist two *n.u.* functions $u\left(.\right)$ and $v\left(.\right)$ such that neither $\mathbb{X}_{\mathbf{2}}^{u-}\subset\mathbb{X}_{\mathbf{2}}^{v-}$ nor $\mathbb{X}_{\mathbf{2}}^{v-}\subset\mathbb{X}_{\mathbf{2}}^{u-}$. Since we assume that investors are rational – *i.e.* we rule out violations of **SSOS** dominance–, we focus on consistent *n.u.* functions. If two *n.u.* functions, $u\left(.\right)$ and $v\left(.\right)$, are consistent, then, by definition,

$$\mathbb{X}_{\mathbf{2}}^{u-}=\mathbb{X}_{\mathbf{2}}^{v-}=\varnothing$$

Now consider a pair $\left(u\left(.\right),v\left(.\right)\right)$ of consistent *n.u.* functions. The preorder $\mathcal{D}_{2}^{u}$ induced by $u\left(.\right)$ will be "closer" to the total preoder of preferences than the preorder $\mathcal{D}_{2}^{v}$ induced by $v\left(.\right)$ *iff* $\mathbb{X}_{\mathbf{2}}^{v+}\subseteq\mathbb{X}_{\mathbf{2}}^{u+}$. In other words, we have the following equivalence for $u\left(.\right),v\left(.\right)\in\mathbb{U}_{\mathrm{C}}$:

$$\mathcal{D}_{2}^{u}\ \mathcal{C}l\ \mathcal{D}_{2}^{v}\Longleftrightarrow\mathbb{X}_{\mathbf{2}}^{v+}\subseteq\mathbb{X}_{\mathbf{2}}^{u+}$$

Actually, we are looking for a *n.u.* function $\mathbf{u}\left(.\right)\in\mathbb{U}_{\mathrm{C}}^{*}$, such that, among the consistent n.u. functions inducing preorders by **SSOS** dominance, $\mathcal{D}_{2}^{\mathbf{u}}$ would be the **closest** to $\preceq$, *i.e.* such that:

$$\forall u\left(.\right)\in\mathbb{U}_{\mathrm{C}},\ \mathcal{D}_{2}^{\mathbf{u}}\ \mathcal{C}l\ \mathcal{D}_{2}^{u}\tag{14}$$

[17] They will include the functionals of LS models.

Clearly, if $\mathcal{D}_2^{\mathbf{u}}$ meets the above condition, it will meet the following one:

$$\forall u(.) \in \mathbb{U},\ \ \mathcal{D}_2^{\mathbf{u}}\ \mathcal{C}l\ \mathcal{D}_2^{u}$$

since a preorder $\mathcal{D}_2^u$ induced by a consistent $n.u.$ function $u(.)$ is closer to $\succsim$ than any preorder $\mathcal{D}_2^v$ induced by an inconsistent $n.u.$ function $v(.)$. Unfortunately, no function $\mathbf{u}(.)$ will satisfy (14), unless some additional restrictions are put to the subset of $n.u.$ functions which is taken into account.

Let $\mathbb{U}^*$ be the subset of concave or convex $n.u.$ functions whereas $\mathbb{U}_{\mathrm{I}}^*$ ($\mathbb{U}_{\mathrm{C}}^*$) will denominate the subset of inconsistent (consistent) concave or convex $n.u.$ functions. We now show that if $u(.)$ and $v(.)$ are two $n.u.$ functions such that $u(.)$ is more concave than $v(.)$ then $\mathcal{D}_2^u$ is closer to $\succsim$ than $\mathcal{D}_2^v$.

Proposition 6. *Let $u(.)$ and $v(.)$ be two $n.u.$ functions such that $u(.)$ is more concave (i.e. less convex) than $v(.)$. Then, the following implication will hold:*

$$X_1\ \mathcal{D}_2^v\ X_2 \Rightarrow X_1\ \mathcal{D}_2^u\ X_2$$

and so will the following inclusions:

$$\mathbb{X}_{\mathbf{2}}^{v+} \subseteq \mathbb{X}_{\mathbf{2}}^{u+}\ ;\ \mathbb{X}_{\mathbf{2}}^{v-} \subseteq \mathbb{X}_{\mathbf{2}}^{u-}$$

Proof. It is given in Appendix 1.□

Clearly the above proposition means that concavifying utility functions increases the number of comparable pairs of lotteries, given that two lotteries X_1 and X_2 are comparable iff either $X_1\ \mathcal{D}_2^u\ X_2$ or $X_2\ \mathcal{D}_2^u\ X_1$.

. Actually, it increases both the size of the subset of the pairs of comparable lotteries which do not violate (standard) **SOS** dominance and that of the subset of the pairs of comparable lotteries which do violate (standard) **SOS** dominance. We are thus led to focus on concave or convex $n.u.$ functions.

3.4 Canonical utility functions.

We may now show that there exists a concave or convex $n.u.$ function which dominates the others in that it makes the two preorders $\mathcal{D}_2^{\mathbf{u}}$ and $\succsim$ never disagree and coincide on a maximum number of pairs of lotteries.

Proposition 7. (canonical utility function). *There exists a unique $n.u.$ function $\mathbf{u}(.)$ such that any concave or convex $n.u.$ function $u(.)$ which is more concave than $\mathbf{u}(.)$ is inconsistent. Function $\mathbf{u}(.)$ is concave or convex and it will be called, from now on, the investor's canonical utility function. The preorder $\mathcal{D}_{\mathbf{2}}^{\mathbf{u}}$ is the closest to the preorder of preferences $\succsim$ among the preorders $\mathcal{D}_2^u$ where $u(.)$ is a concave or convex $n.u.$ function.*

Proof. The proof is given in Appendix 1.□

Proposition 7 means that among the consistent concave/convex $n.u.$ functions there exists a function which is more concave than the others and which is as close to $\succsim$ as possible.

Finally, if the $n.u.$ function of an investor is his canonical utility function, no **SSOS** dominance violations may occur. The investor is well rational. Conversely, if he is rational, then the canonical utility function is the most likely to be the actual investor's *n.u.* function. We shall later show how to **elicit the canonical utility function of a decision-maker** from binary choices over simple lotteries (See Section 5).

4 *The axiomatics*

A fully choice-based theory of decision-making under risk is now presented. Recall that a rational decision-maker should be sensitive to the utility of an outcome rather than to its monetary value. Hence, he may be viewed as making a risky choice among random variables whose **consequences are valued in utils rather than in dollars**, *i.e.* among random variables mapping Ω on to a set Γ which may now be identified to $[0,1]$. The set of these random variables will be labelled $\mathfrak{U}$ and that of their probability distributions $\mathfrak{D}$. Preferences are now defined over $\mathfrak{U}$ (or $\mathfrak{D}$) and the corresponding binary relation is labelled $\trianglerighteq$.

Since the canonical utility function $\mathbf{u}(.)$ is a one-to-one mapping of $[a,b]$ on to $[0,1]$, it may also be viewed as a one-to-one mapping of the set $\mathfrak{X}$ on to $\mathfrak{U}$. For any element $Y \in \mathfrak{U}$ there exists a unique random variable $X \in \mathfrak{X}$, such that $X = \mathbf{u}^{-1}(Y)$ or, equivalently, such that $Y = \mathbf{u}(X)$.[18] When Y is valued in utils, X is valued in monetary units. Similarly, there exists a one-to-one mapping of $\mathfrak{L}$ on to $\mathfrak{D}$ which is defined according to the below equalities:

$$G_Y = F_X \circ \mathbf{u}^{-1} \iff Y = \mathbf{u}(X)$$

where $G_Y(.)$ $(F_X(.))$ is the *c.d.f.* of Y (X).[19] To make things clearer, we shall say that:

(a) an element of $\mathfrak{U}$ is a $\mathbf{u}$-lottery whose certainty equivalent is $\mathfrak{c}(Y)$ and whose expected value is $\mathbb{E}[Y] = \int_0^1 y dG_Y(y)$

(b) an element of $\mathfrak{X}$ is a $\mathbf{m}$-lottery whose certainty equivalent is $\mathbf{c}(X)$) and whose expected value is $\mathbf{E}[X] = \int_a^b x dF_X(x)$.

The preference relation $\succsim$ over $\mathfrak{X}$ induces a preference relation $\trianglerighteq$ over $\mathfrak{U}$ which is defined by the following equivalence:

$$Y_1 \trianglerighteq Y_2 \iff \mathbf{u}^{-1}(Y_1) \succsim \mathbf{u}^{-1}(Y_2) \tag{15}$$

for any $Y_1, Y_2 \in \mathfrak{U}$. Conversely, let $\trianglerighteq$ be a preference relation over $\mathfrak{U}$, when consequences are valued in utils. Then it induces a preference relation $\succsim$ over $\mathfrak{X}$ which is defined by the following equivalence:

$$X_1 \succsim X_2 \iff \mathbf{u}(X_1) \trianglerighteq \mathbf{u}(X_2) \tag{16}$$

for any $(X_1, X_2) \in \mathfrak{X} \times \mathfrak{X}$. The two binary relations $\succsim$ and $\trianglerighteq$ are consistent if any of the above equivalences holds, which is now assumed. When it is endowed with the binary relation $\trianglerighteq$, the subset $\mathfrak{U}$ exhibits the same properties as those of the set $\mathfrak{X}$ when it is endowed with the binary relation $\succsim$.

[18] The denomination $Y = \mathbf{u}(X)$ will mean that, for any $s \in \mathcal{F}$, $Y(s) = \mathbf{u}(X(s))$.

[19] Recall that $F_Y = F_X \circ \mathbf{u}^{-1}$ means $F_Y(y) = F_X\left(\mathbf{u}^{-1}(y)\right)$ for any $y \in [0,1]$.

4.1 The general axiomatics.

The problem which is now addressed is that of setting a general axiomatics. As a preliminary, we focus on preferences over **u**-lotteries.

4.1.1 Preferences over u-lotteries

A strictly risk-averse decision-maker is now characterized by the following implication:

$$\forall Y_1, Y_2 \in \mathfrak{U}, \quad Y_1 \; \mathcal{D}_2 \; Y_2 \Longrightarrow Y_1 \trianglerighteq Y_2 \tag{17}$$

Let $\mathfrak{U}_{\mathbf{y}} \overset{def}{=} \{Y \in \mathfrak{U} | \; \mathbb{E}[Y] = \mathbf{y}\}$. Proposition 1 may then be restated as indicated below:

Proposition 8. *When a decision-maker is strictly risk-averse, then $\delta_{\mathbf{y}}$ $(Y_{\mathbf{y}})$ is a maximal (minimal) element in the subset $\mathfrak{U}_{\mathbf{y}}$, i.e.*

$$Y \in \mathfrak{U}_{\mathbf{y}}, \implies \delta_{\mathbf{y}} \trianglerighteq Y \trianglerighteq Y_{\mathbf{y}}$$

Proof. It is analogous to that of Proposition 1.□

Since **SSOS** dominance with monetary units is equivalent to **SOS** dominance with utils (See Section 3), (17) is equivalent to the following implication:

$$\forall Y_1, Y_2 \in \mathfrak{U}, \quad \mathbf{u}^{-1}(Y_1) \;\; D_2^u \;\; \mathbf{u}^{-1}(Y_2) \Longrightarrow Y_1 \trianglerighteq Y_2 \tag{18}$$

From Proposition 8 and from (16) ((15)) we get that $\delta_{\mathbf{u}^{-1}(\mathbf{y})}$ $(X_{\mathbf{u}^{-1}(\mathbf{y})})$ is a maximal (minimal) element in the subset $\mathcal{X}_{\mathbf{y}} \overset{def}{=} \{X \in \mathfrak{X} \mid \mathbf{E}[\mathbf{u}(X)] = \mathbf{y}\}$ *i.e.*:

$$X \in \mathcal{X}_{\mathbf{y}} \implies \delta_{\mathbf{u}^{-1}(\mathbf{y})} \succsim X \succsim X_{\mathbf{u}^{-1}(\mathbf{y})}$$

We may now substitute for Axioms 1 to 3 the following axioms:

Axiom 1' (ordering of $\trianglerighteq$). *The binary relation $\trianglerighteq$ is a total preorder over $\mathfrak{U}$.*

Axiom 2' (continuity of $\trianglerighteq$). *For any lottery $Y \in \mathfrak{U}$ the sets $\{Z \in \mathfrak{U} \mid Z \trianglerighteq Y\}$ and $\{Z \in \mathfrak{U} \mid Y \trianglerighteq Z\}$ are closed in the topology of weak convergence.*

Axiom 3'. (linearity of certainty equivalents over $\mathfrak{U}_{\mathbf{y}}$)

The certainty equivalent of the $(\lambda, 1-\lambda)$-mixing of two u-lotteries which exhibit the same expected value is the $(\lambda, 1-\lambda)$-convex combination of their certainty equivalents, what formally reads:

$$\forall Y_1, Y_2 \in \mathfrak{U}_{\mathbf{y}}, \; \forall \lambda \in [0,1], \; \mathfrak{c}(\lambda Y_1 \oplus (1-\lambda) Y_2) = \lambda \mathfrak{c}(Y_1) + (1-\lambda) \mathfrak{c}(Y_2)$$

Finally, setting Axioms 1' to 3' implies the following results, which are analogous to those already presented in Section 2.

Proposition 9. (representation theorem for $\trianglerighteq$ over $\mathfrak{U}$). *Under Axioms 1' to 3', the preorder of preferences of a strictly averse decision-maker –labelled $\trianglerighteq$– may be represented over $\mathfrak{U}$ by the following lottery-dependent functional:*

$$\mathcal{V}(Y) \overset{def}{=} \int_0^1 v_{\mathbb{E}[Y]}(z) dG_Y(z) \tag{19}$$

where $v_{\mathbb{E}[Y]}(.)$ is a continuous and increasing function mapping $[0,1]$ on to itself which satisfies the following normalization conditions:

$$v_{\mathbb{E}[Y]}(\mathbb{E}[Y]) = \mathbb{E}[Y] \quad and \quad \mathbb{E}[Y]\, v_{\mathbb{E}[Y]}(1) + (1-\mathbb{E}[Y])\, v_{\mathbb{E}[Y]}(0) = \mathfrak{c}\left(Y_{\mathbb{E}[Y]}\right)$$

Proof. It is the same as the proof given in Section 2 for Proposition 4.□

Finally, one can use the one-to-one correspondance $\mathbf{u} : [a,b] \to [0,1]$ to get the next result:

Proposition 10. (representation theorem for $\succsim$ over $\mathfrak{X}$). *Under Axioms 1' to 3", the preorder of preferences $\succsim$ of a strictly averse decision-maker may be represented over the set $\mathfrak{X}$ of m-lotteries by the following lottery-dependent functional*:

$$\mathcal{U}(X) \stackrel{def}{=} \int_a^b \mathfrak{u}_{\mathbf{E}[\mathbf{u}(X)]}(x) dF_X(x)$$

where $F_X(x) \stackrel{def}{=} G_Y(\mathbf{u}(x))$ and where $\mathfrak{u}_{\mathbf{E}[\mathbf{u}(X)]}(x) \stackrel{def}{=} v_{\mathbf{E}[\mathbf{u}(X)]}(\mathbf{u}(x))$ is a continuous and increasing function mapping $[a,b]$ on to $[0,1]$ which meets the following normalization conditions:

$$\mathfrak{u}_{\mathbf{E}[\mathbf{u}(X)]}(\mathbf{u}^{-1}(\mathbf{E}[\mathbf{u}(X)])) = v_{\mathbf{E}[\mathbf{u}(X)]}(\mathbf{E}[\mathbf{u}(X)]) = \mathbf{E}[\mathbf{u}(X)] \tag{20}$$

$$\mathbf{E}[\mathbf{u}(X)]\, \mathfrak{u}_{\mathbf{E}[\mathbf{u}(X)]}(a) + (1-\mathbf{E}[\mathbf{u}(X)])\, \mathfrak{u}_{\mathbf{E}[\mathbf{u}(X)]}(b) = \mathfrak{c}\left(X_{\mathbf{E}[\mathbf{u}(X)]}\right) \tag{21}$$

Proof. Recall that $\mathbf{E}[\mathbf{u}(X)] = \mathbb{E}[Y]$ that $dF_X(t) = \mathbf{u}'(x)\, dG_Y(\mathbf{u}(x))$, that $\mathbf{u}'(.)$ is strictly positive and that $\int_a^b \mathbf{u}'(x)\, dt = \mathbf{u}(b) - \mathbf{u}(a) = 1$.□

The above results (See propositions 9 and 10) have been obtained from a set of axioms dealing with preferences over **u**-lotteries. They can be obtained differently, from a set of axioms dealing with preferences over **m**-lotteries.

4.1.2 Back to preferences over m-lotteries

We now consider, as in Section 2, that consequences are valued in monetary units and we have to set axioms for preferences over m-lotteries so as to get the same results as those implied by Axioms 1' to 3'. Axioms 1' and 2' are clearly equivalent to Axioms 1 and 2 which, as a consequence, will be maintained. By contrast, Axioms 3 and 3' are not equivalent and, consequently, a new axiom must now be substituted for Axiom 3:

Axiom 3". *The utility of the certainty equivalent of the $(\lambda, 1-\lambda)$-mixing of two m-lotteries which exhibit the same expected utility is the $(\lambda, 1-\lambda)$-convex combination of the utilities of their certainty equivalents, what formally reads:*

$$\forall X_1, X_2 \in \mathcal{X}_{\mathbf{y}},\ \forall \lambda \in [0,1],\ \mathbf{u}(\mathbf{c}(\lambda X_1 \oplus (1-\lambda) X_2)) = \lambda \mathbf{u}(\mathbf{c}(X_1)) + (1-\lambda)\, \mathbf{u}(\mathbf{c}(X_2))$$

Axiom 3" is equivalent to Axiom 3' and we get the following result:

Proposition 11. (representation theorem for $\succsim$ over $\mathfrak{X}$). *Under Axioms 1, 2, and 3", the preorder of preferences $\succsim$ of a strictly averse decision-maker may be represented over $\mathfrak{X}$ by the following lottery-dependent functional*:

$$\mathcal{U}(X) \stackrel{def}{=} \int_a^b \mathfrak{u}_{\mathbf{E}[\mathbf{u}(X)]}(x) dF_X(x) \tag{22}$$

where $\mathfrak{u}_{\mathbf{E}[\mathbf{u}(X)]}(.)$ *is a continuous and increasing function mapping* $[a, b]$ *on to* $[0, 1]$ *which satisfies the normalization conditions* (20).

Proof. Proposition 11 is clearly equivalent to Proposition 10.□

It will be convenient to set $f(z, x) \stackrel{def}{=} \mathfrak{u}_z(x)$. The above functional then reads:

$$\mathcal{U}(X) \stackrel{def}{=} \int_a^b f(\mathbf{E}[\mathbf{u}(X)], \mathbf{u}(x)) dF_X(x) \tag{23}$$

where $\mathbf{u}(.)$ is the canonical utility function and where $f(z, x)$ is strictly increasing with respect to x and and meets, as in Section 2, the two following normalizing conditions which are derived from (20) and (21).

$$f(\mathbf{E}[\mathbf{u}(X)], \mathbf{E}[\mathbf{u}(X)]) = \mathbf{E}[\mathbf{u}(X)]$$

$$f(\mathbf{E}[\mathbf{u}(X)], \mathbf{u}(\gamma_\mathbf{x})) = \mathbf{E}[\mathbf{u}(X)] f(\mathbf{E}[\mathbf{u}(X)], \mathbf{u}(b)) + (1 - \mathbf{E}[\mathbf{u}(X)]) f(\mathbf{E}[\mathbf{u}(X)], \mathbf{u}(a))$$

Finally, as in Section 2, we now particularize the functional (22) to get a more operational specification.

4.2 LS-models

One may now assume that risk premia are translation-invariant when they are expressed in utils, *i.e.* that the following property is met:

$$\boldsymbol{RP}(\mathbf{u}(X) + u) = \boldsymbol{RP}(\mathbf{u}(X))$$

or, equivalently:

$$\int_a^b f(\mathbf{E}[\mathbf{u}(X) + u], \mathbf{u}(x) + u) dF_X(x) = \int_a^b f(\mathbf{E}[\mathbf{u}(X)], \mathbf{u}(x)) dF_X(x)$$

Recall that, under reasonable mathematical assumptions (See Section 2 above), a necessary and sufficient condition for $\boldsymbol{RP}(.)$ to exhibit the invariance property is that: $f(z, x) = z + \mathcal{E}(z - x)$ where function $\mathcal{E}(.)$ characterizes the investor's behaviour towards disappointment/elation. Hence the functional now reads:

$$\mathcal{U}(X) = \mathbf{E}[\mathbf{u}(X)] + \int_a^b \mathcal{E}(\mathbf{u}(x) - \mathbf{E}[\mathbf{u}(X)]) dF_X(x) \tag{24}$$

where $[a, b]$ is the support of lotteries, $\mathbf{u}(.)$ is an *n.u.* function and $\mathcal{E}(.)$ is a continuous and strictly increasing function which fulfills the self explanatory condition $\mathcal{E}(0) = 0$.[20]

We, here again, focus on strictly risk-averse decision-makers. Any non degenerate-lottery $X \in \mathfrak{X}$, will then exhibit a negative disappointment premium, *i.e.* we shall have, for $X \notin \Delta$:

$$\mathbf{E}[\mathcal{E}(\mathbf{u}(X) - \mathbf{E}[\mathbf{u}(X)])] < 0$$

[20] No elation/disappointment is experimented if the actual outcome coincides with its expected value.

As in Section 2, a function $\mathcal{E}(.)$ meeting the above requirement will be called a regular function.[21]

Finally, the preferences of a strictly risk-averse decision-maker with translation-invariant risk premia may be represented by a regular functional such as (24). A strictly risk-averse decision-maker whose behaviour obeys Axioms 1' to 3' and whose risk premia are translation-invariant will be called a **rational risk-averse decision-maker**. His behaviour is characterized by the functional (24) where $\mathcal{E}(.)$ is a strictly increasing and regular function satisfying the requirement $\mathcal{E}(0) = 0$. A functional such as $\mathcal{U}(.)$ which is defined by (24) and which meets the above requirements will be called, from now on, a **LS-functional** and will characterize **LS-models**. Note that the *n.u.* function of a **LS**-functional is consistent.[22]

LS-models are very close to models *à la* Loomes and Sugden. Indeed the original functional of Loomes and Sugden [1986] reads:[23]

$$\mathcal{U}(X) = \mathbf{E}[\mathbf{u}(X)] + \int_a^b \mathcal{E}(\mathbf{u}(x) - \overline{\mathbf{u}})dF_X(x)$$

Now recall that **EU** theory is often violated by experiments and that no general agreement has yet been found about the explaining power of its challengers, *i.e.* Non-**EU** theories. Hence it is interesting to point out that, because of its flexibility, the functional (24) is **compatible with many of the anomalies of financial theory**. An example is given in Appendix 2.

Finally, we, here again, get a decomposition of the risk premium

$$\boldsymbol{RP}(X) \overset{def}{=} \mathbf{E}[\mathbf{u}(X)] - \mathbf{u}(\mathbf{c}(X))$$

into elementary premia, which can be viewed as the contributions of the variance, the skewness, the kurtosis ... of the utility of a lottery to the total risk premium which is demanded by an investor. This was Allais' [1979] original intuition. If $\mathcal{E}(.)$ is smooth enough, one may now write:

$$\boldsymbol{RP}(X) = -\textstyle\sum_{p=2}^{+\infty}\mathbf{E}\left[(\mathbf{u}(X) - \mathbf{E}[\mathbf{u}(X)])^p\right]\dfrac{\mathcal{E}^{(p)}(\mathbf{E}[\mathbf{u}(X)])}{p!}$$

Anyway, the above results are of interest if $\mathbf{u}(.)$ can be elicited. Before we address this question (See Section 5) we give a presentation of some links existing between **LS**-models and some other disappointment models.

4.3 Overview of the related literature

In these models disappointment or elation are measured somewhat differently: Delquie and Cillo [2006] use all the outcomes of the lottery; Grant and Kajii

[21] Recall that strictly concave functions are regular functions since a sufficient condition for (10) to hold is to assume that $\mathcal{E}(.)$ is strictly concave.

[22] A technical condition, namely $\sup \mathcal{E}'(z) \leq 1$, will be imposed to $\mathcal{E}(.)$. See Appendix 2.

[23] and our axiomatization can be viewed as grounding their approach when $\overline{\mathbf{u}} = \mathbf{E}[\mathbf{u}(X)]$.

[1998] adapt the setting of the rank-dependent expected utility model (Quiggin [1982] among others) to highlight the dependence on the best possible outcome; Jia *et alii* [2001] generalize Bell's [1985] approach and advocate the use of its expected value. Indeed, they consider the expected value of the lottery as the reference point for measuring disappointment. Their preference functional can be defined as:

$$\mathcal{U}_{JDB}(X) = \int_a^b \left(1 + d\mathbf{1}_{[x<\mathbf{E}(X)]} - e\mathbf{1}_{[x>\mathbf{E}(X)]}\right) x dF_X(x) \tag{25}$$

where d and e are two positive parameters. The above functional is nothing but a particular case of (23). To see this point, just set:

$$\mathbf{u}(x) = x \text{ and } f(z,x) = \left(1 + d\mathbf{1}_{[x<z]} - e\mathbf{1}_{[x>z]}\right) x$$

LS-models may also be viewed as particular case of lottery dependent utility (henceforth **LDU**) models which were first developed by Becker and Sarin [1987]. The preference functional of a **LDU** model is then:[24]

$$\mathcal{U}_{LDU}(X) = \textstyle\sum_{k=1}^{K} p_k v\left(h(X), x_k\right) \tag{26}$$

where $v[.,.]$ is a function defined over $[a,b] \times \mathbb{R}^+$ and whose values belong to $[0,1]$ and where $h(.)$ is a function defined over $\mathfrak{X}$ and whose values belong to R. The functional (26) may be derived from three axioms which have been provided by Becker and Sarin [1987]: total ordering, continuity and monotonicity. Their first two axioms are those of **EU** theory and the third one is nothing but the **SOS** dominance principle. However, as pointed out by Starmer [2000] "the basic model is conventional theory for minimalists as, without further restriction, it has virtually no empirical content."[25] Finally, almost any non-**EU** model can be viewed as a **LDU** model, once an appropriate functional form of $v(.,.)$ has been chosen.

Becker and Sarin then particularize their model assessing $h(X)$ to be linear with respect to the probabilities p_k, that is they set $h(X) = \sum_{k=1}^{K} h_k p_k$ and they define a function $H(.)$ such that $H(x_k) = h_k$. To sum up, the authors set:

$$h(X) = \textstyle\sum_{k=1}^{K} H(x_k)\, p_k = \mathbf{E}\left[H[X]\right] \tag{27}$$

and the new model then belongs to a subset of **LDU** models called lottery-dependent expected utility models (henceforth LDEU models). The functions $h(.)$, or $H(.)$, may be chosen arbitrarily but they have to be specified before testable implications of the model be derived. As a consequence, **LDEU** models are not choice-based.

Schmidt [2001] considers somewhat more general models called "lottery-dependent convex utility models" (henceforth LDCU models). A condition less restrictive than (27) is fulfilled by LDCU models. It reads:

$$h(X_i) = \lambda \text{ and } \alpha_i \geq 0 \text{ and } \textstyle\sum_{i=1}^{N} \alpha_i = 1 \Rightarrow h\left(\sum_{i=1}^{N} \alpha_i X_i\right) = \lambda \tag{28}$$

[24] To make short only simple lotteries are taken into account.

[25] Starmer: *Developments in Non Expected Utility Theory*, JEL, p. 345.

Four axioms are necessary to develop this class of models. The first two axioms (total ordering and continuity) are, again, those of **EU** theory. The author then substitutes for the independence axiom two new axioms: the first one, called the lottery dependent independence axiom, states that the independence property is met over any subset $\mathfrak{X}_\lambda$ of lotteries fulfilling (28). However, to derive **LDCU** models, $\mathcal{U}(X)$ has to be linear in every subset $\mathfrak{X}_\lambda$ for all λ. A linear $\mathcal{U}(.)$ is obtained *iff* there exists a sequence of functions $\{\varphi_\lambda, \lambda \in [\mathcal{U}(\delta(0)), \mathcal{U}(\delta(1))]\}$ where $\varphi_\lambda : \mathbb{R} \to \mathbb{R}^+$ is continuous and strictly increasing so that:

$$\forall \lambda \in [\mathcal{U}(\delta(0)), \mathcal{U}(\delta(1))], \mathcal{U}(X) = \varphi_\lambda [u_\lambda(X)] \ if \ X \in \mathfrak{X}_\lambda$$

This result is guaranteed by an additional axiom which is called the linearity axiom and which enables him to select one particular function $\mathcal{U}(.)$ from all the candidates.

By contrast with **LS**-models, Schmidt's approach is not choice-based since using the axioms implies that we know, on *a priori* grounds, the function $h(.)$. One cannot characterize the subsets $\mathfrak{X}_\lambda$ from experiments and the relation $h(X) = \lambda$ looks like an *ad hoc* condition whose meaning is not very clear.

Finally, it must be mentioned that Gul [1991] developed an **implicit expected utility model of disappointment** where the certainty equivalent of the lottery plays the role of reference level. It is fully axiomatized but Gul's theory is not endowed with the four properties we are interested in.

An additional interesting property of **LS**-models is their elicitation property: indeed the canonical utility function of a decision -maker may then be elicited from the choices he makes when he faces a sequence of random outcomes. Hence, the next section is devoted to a presentation of the elicitation propery of **LS**-models.

5 *The elicitation property.*[26]

Some preliminary definitions and results will be first given.

5.1 Preliminary definitions and results.

Recall that, unlike **EU** models, **LS**-models are not endowed with a global independence property. In particular, some couples of indifferent lotteries ($(X_1, X_2) \in \mathfrak{X} \times \mathfrak{X}$ with $X_1 \sim X_2$) are such that $\lambda X_1 \oplus (1-\lambda) X_2 \nsim X_1$ for some values of $\lambda \in [0,1]$. However, there also exist, in these models, some couples of indifferent lotteries which do exhibit the betweenness property: they will be called, from now on, strongly indifferent lotteries.

Definition 6 (strong indifference). *Two lotteries X_1 and X_2 are strongly indifferent iff they meet the following requirement:*

$$\forall \lambda \in [0,1], \ \lambda X_1 \oplus (1-\lambda) X_2 \sim X_1 \tag{29}$$

[26] This section is but a rewriting of: Chauveau Th., N. Nalpas [2010]. *Disappointment models: an axiomatic approach*, CES workingpaper, 2010.102

Strong indifference may be characterized in the following way:

Proposition 12. *In **LS**-models, two lotteries X_1 and X_2 are strongly indifferent iff they exhibit the same certainty equivalent and the same expected utility, what formally reads:*

$$X_1 \approx X_2 \Longleftrightarrow \mathbf{c}\left(X_1\right) = \mathbf{c}\left(X_2\right) \quad \textit{and} \quad \mathbf{E}\left[\mathbf{u}\left(X_1\right)\right] = \mathbf{E}\left[\mathbf{u}\left(X_2\right)\right]$$

Proof. It is given in Appendix 1.□

The binary relation "X_1 and X_2 are strongly indifferent" will be labelled "$X_1 \approx X_2$". It is obviously reflexive and symmetric. From Proposition 12 we get that it is also transitive and, consequently, it is an equivalence relation over $\mathfrak{X}$. Finally, note that strong indifference implies indifference in the usual sense which will be called, from now on, *weak indifference*. A related new concept needs now to be introduced: that of strong equivalents.

Definition 7. (strong equivalents). *Let $X \in \mathfrak{X}$ be an arbitrary lottery and let*

$$\underline{X}_p^x \stackrel{def}{=} [a, 1-p; x, p] \quad ; \quad \overline{X}_q^y \stackrel{def}{=} [y, q; b, 1-q]$$

where $x, y \in \,]a, b[$. Then, if X and $\underline{X}_p^x$ ($\overline{X}_q^y$) are strongly indifferent, $\underline{X}_p^x$ ($\overline{X}_q^y$) is the left (right) strong equivalent of X.

The above definition will make sense only if any lottery is endowed with a unique couple of strong equivalents. As indicated in the next proposition, this happens to be the case.

Proposition 13. *In **LS**-models, a lottery $X \in \mathfrak{X}$, has exactly one left and one right strong equivalent.*

Proof. It is given in Appendix 1.□

5.2 The elicitation property.

We now turn to the elicitation property. The first step of the argument is as follows: let $\overline{X}_q^y$ be the right strong equivalent of $\underline{X}_p^x$ *i.e.* let:

$$\overline{X}_q^y \approx \underline{X}_p^x$$

Then, the difference between the expected utility of $\underline{X}_p^x$ and that of $\underline{X}_q^y$ is $1-q$.[27] Indeed, we get that:

$$\mathbf{E}\left[\mathbf{u}(\underline{X}_p^x)\right] - \mathbf{E}\left[\mathbf{u}(\underline{X}_q^y)\right] = \mathbf{E}\left[\mathbf{u}(\overline{X}_q^y)\right] - \mathbf{E}\left[\mathbf{u}(\underline{X}_q^y)\right] = \left((1-q) + qu\left(y\right)\right) - qu\left(y\right) = 1-q$$

The second step consists in defining a sequence of binary lotteries, $\{\underline{X}_{p_n}^{x_n}\}_{n\in\mathbb{N}}$ as indicated below:

$$x_0 = w \;;\; p_0 = \pi \;\text{ and }\; \overline{X}_{p_{n+1}}^{x_{n+1}} \approx \underline{X}_{p_n}^{x_n} \tag{30}$$

Note that, if $\overline{X}_q^y$ is the right strong equivalent of $\underline{X}_p^x$, then $y < x$. As a consequence, $\{x_n\}_{n\in\mathbb{N}}$ is a decreasing sequence. Moreover, it is such that the

[27] Recall that $\underline{X}_q^y \stackrel{def}{=} [a, b-q; y, q]$

difference between the expected utilities of two consecutive binary lotteries, $\underline{X}_{p_n}^{x_n}$ and $\underline{X}_{p_{n+1}}^{x_{n+1}}$, is equal to the second weight $(1-p_{n+1})$ of the right strong equivalent of $\underline{X}_{p_n}^{x_n}$, what formally reads:

$$\mathbf{E}\left[\mathbf{u}(\underline{X}_{p_n}^{x_n})\right] - \mathbf{E}\left[\mathbf{u}(\underline{X}_{p_{n+1}}^{x_{n+1}})\right] = 1 - p_{n+1}$$

and, consequently, the expected utility of the initial simple lottery is the sum of the expected utility of any element of the sequence and of the accumulation of the second weights of the right strong equivalents what formally reads:

$$\pi u(w) - \mathbf{E}\left[u(\underline{X}_{p_n}^{x_n})\right] = \mathbf{E}\left[u(\underline{X}_{p_0}^{x_0})\right] - \mathbf{E}\left[u(\underline{X}_{p_n}^{x_n})\right] = \sum_{i=1}^{n}(1-p_i)$$

Alternatively, one could consider the following sequence of binary lotteries:

$$y_0 = w \ ; \ q_0 = \pi \ ; \ \underline{X}_{q_{n+1}}^{y_{n+1}} \approx \overline{X}_{1-q_n}^{y_n}) \tag{31}$$

The elements of the sequence $\{\overline{X}_{1-q_n}^{y_n}\}_{n\in\mathbb{N}}$ are endowed with the following property:

$$\mathbf{E}\left[\mathbf{u}(\underline{X}_{q_{n+1}}^{y_{n+1}})\right] - \mathbf{E}\left[\mathbf{u}(\underline{X}_{q_n}^{y_n})\right] = 1 - q_{n+1} \Longrightarrow \pi\mathbf{u}(w) = \mathbf{E}\left[\mathbf{u}(\underline{X}_{q_n}^{y_n})\right] - \sum_{i=1}^{n}(1-q_i)$$

From now on, the sequences $\{\underline{X}_{p_n}^{x_n}\}_{n\in\mathbb{N}}$ and $\{\overline{X}_{1-q_n}^{y_n,1}\}_{n\in\mathbb{N}}$, will be called the canonical sequences generated by *(w,π)*. The first (second) one is the left canonical sequence (right canonical sequence). As shown below, they respectively converge, in **LS**-models, towards $\delta(0)$ or $\delta(1)$.

Proposition 14. *Let $(w,\pi) \in]a,b[\times]0,1[$. Consider the canonical sequences of binary lotteries generated by (w,π). Then, in **LS**-models where investors are disappointment averse, the left (right) canonical sequence is decreasing[28] (increasing[29]) and converges towards $\delta(a)$[30] ($\delta(b)$[31]). Moreover we have the following equalities:*

$$\mathbf{u}(w) = \left(\sum_{i=1}^{\infty}(1-p_i)\right)/\pi = \left(1 - \sum_{i=0}^{\infty}(1-q_i)\right)/\pi \tag{32}$$

Proof. It is given in Appendix 1.□

Finally, in **LS**-models, the set of lotteries $\mathfrak{X}$ is well endowed with the *elicitation property*, *i.e.* the value of $\mathbf{u}(w)$ can be elicited with as much accuracy as desired for any outcome $w \in]a,b[$. Indeed, one may choose an arbitrary probability $\pi \in]0,1[$ and build, from the answers of an investor facing lotteries of the $\underline{X}_p^x$ type and/or of the $\overline{X}_q^y$ type, the two canonical sequences generated by (w,π). An accurate ranging of $\mathbf{u}(w)$ may be obtained since we have the inequalities:

$$0 < \left(\sum_{i=1}^{n}(1-p_i)\right)/\pi \leq \mathbf{u}(w) \leq \left(1 - \sum_{i=0}^{n}(1-q_i)\right)/\pi \tag{33}$$

[28] That is $\underline{X}_{p_m}^{x_m}$ is preferred to $\underline{X}_{p_n}^{x_n}$ if $n \geq m$

[29] That is $\overline{X}_{q_n}^{y_n}$ is preferred to $\overline{X}_{q_n}^{y_n,b}$ if $n \geq m$

[30] Equivalently, one can say that $\{x_n\}_{n\in\mathbb{N}}$ is a decreasing sequence of real numbers converging towards a and that $\{p_n\}_{n\in\mathbb{N}}$ is a sequence of real numbers converging towards 1.

[31] Equivalently, one can say that $\{y_n\}_{n\in\mathbb{N}}$ is an increasing sequence of real numbers converging towards b and that $\{q_n\}_{n\in\mathbb{N}}$ is a sequence of real numbers converging towards 1.

6 *Concluding remarks*

In this paper, a fully choice-based theory of disappointment has been developed that can be viewed as an axiomatic foundation of models *à la* Loomes and Sugden [1986]. **LS**-models are endowed with many interesting properties which have been presented above. Note that the results do not depend on the assumption that the set of possible outcomes is bounded. Indeed, extensions to $\mathbb{R}$ itself are straightforward. Moreover, under the assumption of constant relative risk aversion(s), one can easily implement the above approach to value any financial asset.[32] The results do not either depend on the assumption of the concavity or the convexity of the canonical utility function over the support $[0,1]$ of the lotteries. Such an assumption may be relaxed and an straightforward generalization of the above results consists in taking into account smooth *n.u.* functions whose graph includes a concave and a convex section or, more generally, N successive concave and convex sections.

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7 Appendix 1 (Proofs)

Proof.of Lemma 1.

Let $f(z,x) \overset{def}{=} g(x, z-x)$. Invariance by translation implies that:

$$\int_a^b g(x+\Delta x, z-x)\, dF_X(x) = \int_a^b g(x, z-x)\, dF_X(x) + \Delta x$$

or:

$$\int_a^b [g(x+\Delta x, z-x) - g(x, z-x)]\, dF_X(x) = \Delta x$$

or, equivalently:

$$\int_a^b [g(x+\Delta x, z-x) - g(x, z-x) - \Delta x]\, dF_X(x) = 0$$

Since the above equality is valid for any *c.d.f.* $F_X(.)$ and any values of x, z and Δx, we get that:

$$g(x+\Delta x, z-x) - g(x, z-x) - \Delta x = 0$$

and that:

$$\lim_{\Delta x \to 0} \frac{g(x+\Delta x, z-x) - g(x, z-x)}{\Delta x} = \frac{\partial g}{\partial x}(x, z-x) = 1$$

Finally, the following equality is checked:

$$g(x, z-x) = x + \mathcal{E}(z-x)$$

□

Proof of Proposition 3

We are looking for a utility function $u_{\mathbf{x}}(.)$ over $[a,b]$, satisfying the expected utility representation, *i.e.* meeting the following requirement:[33]

$$\mathbf{E}[u_{\mathbf{x}}(x)] = \mathcal{U}_{\mathbf{x}}(X) \tag{34}$$

We shall successively, consider the case when X:

(**I**) a binary lottery Z belonging to any of two special subsets of $\mathfrak{X}_{\mathbf{x}}$. whose exact definition is given below.

(**II**) an arbitrary binary lottery Z belonging to $\mathfrak{X}_{\mathbf{x}}$ (*i.e.* $Z = [y, x; 1-\pi, \pi]$ with $\pi x + (1-\pi)y = \mathbf{x}$)

(**III**) an arbitrary simple lottery Z belonging to $\mathfrak{X}_{\mathbf{x}}$ (*i.e.* $Z \in \mathfrak{X}_{\mathbf{x}} \cap \mathfrak{X}^f$)

(**IV**) an arbitrary lottery belonging to $\mathfrak{X}_{\mathbf{x}}$. (*i.e.* $Z \in \mathfrak{X}_{\mathbf{x}}$)

I. The first step consists in considering *two particular subsets* of binary lotteries which belong to $\mathfrak{X}_{\mathbf{x}}$ and include either the outcome a or the outcome b among their outcomes. More formally, they are defined as indicated below:

[33] Recall that the degenerate lottery δ_z does not belong to $\mathfrak{X}_{\mathbf{x}}$ unless $z = \mathbf{x}$. Hence we must make up for this drawback and define a utility function from its expected utility representation over non degenerate lotteries..

Group **A**. The elements of this group are defined as indicated below:

$$\underline{X}_p^x \overset{def}{=} [a, 1-p; x, p] \,.$$

where $x \in [\mathbf{x}, b]$. Since by assumption $\underline{X}_p^x \in \mathfrak{X}_{\mathbf{x}}$ we get that:

$$px + a\,(1-p) = \mathbf{x}. \tag{35}$$

Group **B.** The elements of this group are defined as indicated below:

$$\overline{X}_q^y \overset{def}{=} [y, q; b, 1-q] \,.$$

where $y \in [a, \mathbf{x}]$. Since, by assumption $\overline{X}_q^y \in \mathfrak{X}_{\mathbf{x}}$ we get that:

$$qy + b\,(1-q) = \mathbf{x}. \tag{36}$$

Now recall that, by definition,

$$X_{\mathbf{x}} \overset{def}{=} [a, 1-\pi(\mathbf{x})\,; b, \pi(\mathbf{x})] \,,$$

where $x \in [\mathbf{x}, 1]$ and

$$\pi(\mathbf{x}) \overset{def}{=} (\mathbf{x} - a) \,/\, (b-a)$$

Moreover, let $\lambda \in [0, 1]$ be defined by the following equivalence:

$$\boldsymbol{X}_{\mathbf{x}}^{\lambda} \sim \underline{X}_p^x$$

where:

$$\boldsymbol{X}_{\mathbf{x}}^{\lambda} \overset{def}{=} \lambda\delta_{\mathbf{x}} \oplus (1-\lambda)\, X_{\mathbf{x}}$$

Finally, we are looking for a utility function $u_{\mathbf{x}}(.)$ satisfying the following expected utility representation (34):

$$pu_{\mathbf{x}}(x) + (1-p)\,u_{\mathbf{x}}\,(a) = \lambda\mathbf{x} + (1-\lambda)\,u_{\mathbf{x}}(\gamma_{\mathbf{x}}) \tag{37}$$

where $\gamma_{\mathbf{x}}$ is the certainty equivalent of $X_{\mathbf{x}} = \underline{X}_{\pi(\mathbf{x})}^b$. Let:

$$u_{\mathbf{x}}(x) = \frac{x-a}{\mathbf{x}-a}\left[\lambda\mathbf{x} + (1-\lambda)\,\gamma_{\mathbf{x}}\right] - \frac{x-\mathbf{x}}{\mathbf{x}-a}u_{\mathbf{x}}\,(a) \tag{38}$$

Then, for $x = \mathbf{x}$, we get that $p = 1$, $\lambda = 1$, and

$$u_{\mathbf{x}}\,(\mathbf{x}) = \mathbf{x} \tag{39}$$

Similarly, for $x = b$, we get that $p = \pi(\mathbf{x})$, $\lambda = 0$, and that:

$$u_{\mathbf{x}}(b) = \frac{b-a}{\mathbf{x}-a}\gamma_{\mathbf{x}} - \frac{b-\mathbf{x}}{\mathbf{x}-a}u_{\mathbf{x}}\,(a) \tag{40}$$

B. We now consider the second group of binary lotteries. From (36) we get that:

$$qy + b(1-q) = \mathbf{x}$$

or, equivalently:

$$q = \frac{b-\mathbf{x}}{b-y} \Longleftrightarrow 1-q = \frac{\mathbf{x}-y}{b-y}$$

Next, let $\mu \in [0,1]$ be now defined as

$$\mathbf{X}_{\mathbf{x}}^{\mu} \sim \overline{X}_q^y;$$

we are now looking for a utility function meeting the following requirement:

$$\mathbf{E}\left[u_{\mathbf{x}}(X)\right] = \mathcal{U}_{\mathbf{x}}(\mathbf{X}_{\mathbf{x}}^{\mu}) \tag{41}$$

or:

$$qu_{\mathbf{x}}(y) + (1-q)\,u_{\mathbf{x}}(b) = \mu\mathbf{x} + (1-\mu)\,\gamma_{\mathbf{x}} \tag{42}$$

or, equivalently:

$$\frac{b-\mathbf{x}}{b-y}u_{\mathbf{x}}(y) + \frac{\mathbf{x}-y}{b-y}u_{\mathbf{x}}(b) = \mu\mathbf{x} + (1-\mu)\,\gamma_{\mathbf{x}}$$

or, finally:

$$u_{\mathbf{x}}(y) = \frac{b-y}{b-\mathbf{x}}\left[\mu\mathbf{x} + (1-\mu)\,\gamma_{\mathbf{x}}\right] - \frac{\mathbf{x}-y}{b-\mathbf{x}}u_{\mathbf{x}}(b) \tag{43}$$

Hence for $y = \mathbf{x}$, we get that $q = 1$, $\mu = 1$, and, again, equation (39). For $y = a$, we get that $q = 1 - \pi(\mathbf{x})$, $\mu = 0$, and:

$$u_{\mathbf{x}}(a) = \frac{b-a}{b-\mathbf{x}}\gamma_{\mathbf{x}} - \frac{\mathbf{x}-a}{b-\mathbf{x}}u_{\mathbf{x}}(b) \tag{44}$$

Clearly equations (40) and (44) are the same and may be rewritten as:

$$\mathbf{E}\left[u_{\mathbf{x}}(X)\right] = \mathbf{x}u_{\mathbf{x}}(b) + (1-\mathbf{x})\,u_{\mathbf{x}}(a) = \gamma_{\mathbf{x}} \tag{45}$$

The normalization conditions of the utility function are then reduced to (39) and (45). Finally, note that we have the following relations:

$$\mathcal{U}_{\mathbf{x}}\left(\underline{X}_p^x\right) = \frac{\mathbf{x}-a}{x-a}u_{\mathbf{x}}(x) + \frac{x-\mathbf{x}}{x-a}u_{\mathbf{x}}(a)$$
$$\mathcal{U}_{\mathbf{x}}\left(\overline{X}_q^y\right) = \frac{\mathbf{x}-y}{b-y}u_{\mathbf{x}}(b) + \frac{b-\mathbf{x}}{b-y}u_{\mathbf{x}}(x)$$
$$\mathcal{U}_{\mathbf{x}}(X_{\mathbf{x}}) = \gamma_{\mathbf{x}} \quad ; \ \mathcal{U}_{\mathbf{x}}(\delta_{\mathbf{x}}) = \mathbf{x}$$

II We now show that *the above result holds for all binary lotteries belonging to* $\mathfrak{X}_{\mathbf{x}}$. Indeed, let $Z = [y, x; 1-\pi, \pi]$ and assume that $Z \in \mathfrak{X}_{\mathbf{x}}$. As a consequence, we get that

$$\pi x + (1-\pi)\,y = \mathbf{x} \tag{46}$$

Consider the two following compound lotteries:

$$\alpha \underline{X}_p^x \oplus (1-\alpha)\, \overline{X}_q^y$$

and

$$\beta Z \oplus (1-\beta)\, X_{\mathbf{x}}$$

where $\alpha, \beta, \in [0,1]$. The two lotteries have the same support $\{0, x, y, 1\}$ and will coincide[34] *iff* they exhibit the same probabilities, *i.e. iff*:

$$\begin{aligned} \alpha (1-p) &= (1-\beta)(1-\pi(\mathbf{x})) \\ (1-\alpha) q &= (1-\pi)\beta \\ \alpha p &= \pi \beta \\ (1-\alpha)(1-q) &= \pi(\mathbf{x})(1-\beta) \end{aligned} \tag{47}$$

Actually there are only *two independent equations* among the four above because (a) the probabilities sum to one –we may therefore leave aside the last equation– and (b) since p and q both depend on $\mathbf{x}$,[35], combining the three remaining equations gives an equation which is nothing but (46).

From the linearity of $\mathcal{U}_{\mathbf{x}}(.)$ we also get that:

$$\begin{aligned} \mathcal{U}_{\mathbf{x}}\left(\alpha \underline{X}_p^x \oplus (1-\alpha)\, \overline{X}_q^y\right) &= \alpha \mathcal{U}_{\mathbf{x}}\left(\underline{X}_p^x\right) + (1-\alpha)\, \mathcal{U}_{\mathbf{x}}\left(\overline{X}_q^y\right) \\ \mathcal{U}_{\mathbf{x}}\left(\beta Z \oplus (1-\beta)\, X_{\mathbf{x}}\right) &= \beta \mathcal{U}_{\mathbf{x}}(Z) + (1-\beta)\, \mathcal{U}_{\mathbf{x}}(X_{\mathbf{x}}) \end{aligned}$$

and, consequently:

$$\begin{aligned} \mathcal{U}_{\mathbf{x}}(Z) &= \beta^{-1}\left[\alpha \mathcal{U}_{\mathbf{x}}\left(\underline{X}_p^x\right) + (1-\alpha)\, \mathcal{U}_{\mathbf{x}}\left(\overline{X}_q^y\right) - (1-\beta)\, \mathcal{U}_{\mathbf{x}}(X_{\mathbf{x}})\right] \\ &= \frac{\alpha}{\beta} \mathcal{U}_{\mathbf{x}}\left(\underline{X}_p^x\right) + \frac{(1-\alpha)}{\beta} \mathcal{U}_{\mathbf{x}}\left(\overline{X}_q^y\right) - \frac{(1-\beta)}{\beta} \mathcal{U}_{\mathbf{x}}(X_{\mathbf{x}}) \end{aligned}$$

and, substituting α and β for their values in (47) we get that:

$$\mathcal{U}_{\mathbf{x}}(Z) = \frac{\pi}{p} \mathcal{U}_{\mathbf{x}}\left(\underline{X}_p^x\right) + \frac{1-\pi}{q} \mathcal{U}_{\mathbf{x}}\left(\overline{X}_q^y\right) - \frac{(1-q)(1-\pi)}{q\pi(\mathbf{x})} \mathcal{U}_{\mathbf{x}}(X_{\mathbf{x}})$$

or equivalently:

$$\begin{aligned} \mathcal{U}_{\mathbf{x}}(Z) = \pi u_{\mathbf{x}}(x) + (1-\pi)\, u_{\mathbf{x}}(y) + \pi \frac{x-\mathbf{x}}{\mathbf{x}-a} u_{\mathbf{x}}(a) \\ + (1-\pi) \frac{\mathbf{x}-y}{b-\mathbf{x}} u_{\mathbf{x}}(b) - \frac{(1-q)(1-\pi)}{q\pi(\mathbf{x})} u_{\mathbf{x}}(\gamma_{\mathbf{x}}) \end{aligned}$$

---

[34] *i.e.* we have:

$$\alpha \underline{X}_p^d \oplus (1-\alpha)\, \overline{X}_q^c = \beta Z \oplus (1-\beta)\, X_{\mathbf{x}}$$

[35] They are linked together by the following formula:

$$q = \frac{d(1-p)}{d-c}$$

and, finally:

$$\begin{aligned}\mathcal{U}_{\mathbf{x}}(Z) &= \mathbf{E}\left[u_{\mathbf{x}}(Z)\right] + \frac{\mathbf{x}-y}{x-y}\frac{x-\mathbf{x}}{(b-\mathbf{x})}u_{\mathbf{x}}(a) + \frac{x-\mathbf{x}}{b-\mathbf{x}}\frac{\mathbf{x}-y}{b-\mathbf{x}}u_{\mathbf{x}}(b) \\ &- \frac{\mathbf{x}-y}{b-\mathbf{x}}\frac{x-\mathbf{x}}{x-y}\frac{b-a}{\mathbf{x}-a}\left[\pi(\mathbf{x})u_{\mathbf{x}}(b) + (1-\pi(\mathbf{x}))u_{\mathbf{x}}(a)\right] \\ &= \mathbf{E}\left[u_{\mathbf{x}}(Z)\right] + \frac{\mathbf{x}-y}{x-y}\frac{x-\mathbf{x}}{(b-\mathbf{x})}u_{\mathbf{x}}(a) + \frac{x-\mathbf{x}}{b-\mathbf{x}}\frac{\mathbf{x}-y}{b-\mathbf{x}}u_{\mathbf{x}}(b) \\ &- \left[\frac{\mathbf{x}-y}{b-\mathbf{x}}\frac{x-\mathbf{x}}{x-y}u_{\mathbf{x}}(b) + \frac{\mathbf{x}-y}{x-y}\frac{x-\mathbf{x}}{\mathbf{x}-a}u_{\mathbf{x}}(a)\right] \\ &= \mathbf{E}\left[u_{\mathbf{x}}(Z)\right]\end{aligned}$$

III Next, from a straightforward induction method, we can derive, for any simple lottery whose expected value is $\mathbf{x}$:

$$\mathbf{E}\left[u_{\mathbf{x}}(Z)\right] = \sum p_Z(x)\, u_{\mathbf{x}}(x) = \mathcal{U}_{\mathbf{x}}(Z) \;\; if \;\; Z \in \mathfrak{X}_{\mathbf{x}} \cap \mathfrak{X}^f$$

Thus, the utility of a simple lottery equals the expected utility of its consequences. However the induction argument is valid only if the number of possible consequences is *finite*. Hence, It remains to be shown that Axiom 3 implies expected utility maximization on the set $\mathfrak{X}_{\mathbf{x}}$.

IV To obtain the expected utility representation over the *entire subset*,[36] one may proceed as indicated below. Any lottery $X \in \mathfrak{X}_{\mathbf{x}}$ whose *c.d.f.* $F_X(.)$ is continuous may be viewed as the limit of two sequences of simple lotteries whose expected value is $\mathbf{x}$ and which either **SSOD** dominate X or are **SSOD** dominated by X. To see this, recall that we have:

$$\delta_{\mathbf{x}} \succsim X \succsim X_{\mathbf{x}}$$

and consider the *two sequences of simple lotteries* $\{X_n^*\}_{n\in\mathbb{N}}$ and $\{X_n^{**}\}_{n\in\mathbb{N}}$ where X_n^* and X_n^{**} both belong to $\mathfrak{X}_{\mathbf{x}}$ and which are defined as indicated below:

Step 1. Lotteries X_1^* and X_1^{**} are such that:

$$\begin{aligned}F_{X_1^*}(x) &= F_X\left(x_1^2\right) \text{ if } x \in \left[x_1^1, x_1^3\right[\;\; ; F_{X_1^*}(x) = 1 \text{ if } x = x_1^3 \\ F_{X_1^{**}}(x) &= 0 \text{ if } x \in \left[x_1^1, x_1^2\right[\;\; ; F_{X_1^{**}}(x) = 1 \text{ if } x \in \left[x_1^2, x_1^3\right]\end{aligned}$$

where:

$$x_1^1 = a \;\; ; \;\; x_1^2 = \mathbf{E}[X] = \mathbf{x} \;\; ; x_1^3 = b$$

$$\mathbf{E}[X] = \int_a^b x dF_X(x) \Longleftrightarrow F_X(\mathbf{E}[X]) = \frac{b - \mathbf{E}[X]}{b-a}$$

[36] Note that we do not need any additional axiom (dominance and/or monotonicity) since these further axioms are are implied by the property of strict risk aversion.

Step 2. Lotteries X_2^* and X_2^{**} are defined as indicated below:

$$F_{X_2^*}(x) \;= F_X\left(x_2^2\right) \text{ if } x \in \left[x_2^1, x_2^3\right[\;;\; F_{X_2^*}(x) \;= F_X\left(x_2^4\right) \text{ if } x \in \left[x_2^3, x_2^5\right[\;;\; F_{X_2^*}(x) = 1 \text{ if } x = x_2^5$$
$$F_{X_2^{**}}(x) = 0 \text{ if } x \in \left[x_2^1, x_2^2\right[\;;\; F_{X_2^{**}}(x) = F_X\left(x_2^3\right) \text{ if } x \in \left[x_2^2, x_2^4\right[\;;\; F_{X_2^{**}}(x) = 1 \text{ if } x \in \left[x_2^4, x_2^5\right]$$

where:

$$x_2^1 \overset{def}{=} x_1^1 = a \;;\; x_2^3 \overset{def}{=} x_1^2 = \mathbf{E}[X] = \mathbf{x} \;;\; x_2^5 \overset{def}{=} x_1^3 = b$$

and where x_2^2 and x_2^4 are defined by the following conditions:

$$\int_{x_1^1}^{x_1^2} x dF_X(x) = x_2^2 F_X\left(x_2^3\right) = x_2^2(F_X\left(x_1^2\right) - F_X\left(x_1^1\right))$$
$$\int_{x_1^2}^{x_1^3} x dF_X(x) = x_2^4\left(1 - F_X\left(x_1^2\right)\right) = x_2^4\left(F_X\left(x_1^3\right) - F_X\left(x_1^2\right)\right)$$

and so on...

....

Step n. Finally, $\{X_n^*\}_{n\in\mathbb{N}}$ and $\{X_n^{**}\}_{n\in\mathbb{N}}$ are defined *recursively* by the following equations:

$$x_{n+1}^{2i-1} = x_n^i \qquad \text{if } i = 1, (2)^n + 1 \tag{48}$$
$$x_{n+1}^{2i} = \left(\int_{x_n^i}^{x_n^{i+1}} x dF_X(x)\right)/(F_X(x_n^{i+1}) - F_X(x_n^i)) \quad \text{if } i = 1, (2)^n \tag{49}$$

$$F_{X_{n+1}^*}(x) = F_X(x_{n+1}^{2i}) \quad \text{if } x \in \left[x_{n+1}^{2i-1}, x_{n+1}^{2i+1}\right[\text{ and } i = 1, (2)^n \tag{50}$$
$$F_{X_{n+1}^{**}}(x) = F_X(x_{n+1}^{2i+1}) \quad \text{if } x \in \left[x_{n+1}^{2i}, x_{n+1}^{2i+2}\right[\text{ and } i = 1, (2)^n \tag{51}$$

given that we have $F_{X_{n+1}^*}(x_{n+1}^{(2)^n+1}) = F_{X_{n+1}^{**}}(x_{n+1}^{(2)^n+1}) = 1$ and $F_{X_{n+1}^{**}}(x) = F_X(x_n^1) = 0$ if $x \in \left[x_{n+1}^1, x_{n+1}^2\right[$.

Whatever the value of x, the *two sequences of simple lotteries which we have built converge towards* X. To see this, first recall that the sequence whose general term is $\left|F_{X_n^{**}}(x) - F_{X_n^*}(x)\right|$ converges towards a limit $\ell_x \geq 0$ since it is a *decreasing sequence of positive real numbers*. Hence, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that:

$$n \geq N \Rightarrow \ell_x + \varepsilon \leq \left|F_{X_n^*}(x) - F_{X_n^{**}}(x)\right| \leq \ell_x \tag{52}$$

Actually ℓ_x *must be zero* otherwise a contradiction would appear. To see this, assume that $\ell_x > 0$ and consider the subdivision $\left\{x_N^i\right\}_{i=1,(2)^{N-1}+1}$. Assume *provisionnally* that $x \in \left[x_N^{2k}, x_{N-1}^{k+1}\right[$. From (50) and (51) we get that:

$$F_{X_N^*}(x) = F_X(x_N^{2k}) \textit{ and } F_{X_N^{**}}(x) = F_X(x_N^{2k+1}) = F_X(x_{N-1}^{k+1})$$

From (52) we get that:

$$F_{X_N^*}(x) - F_{X_N^{**}}(x) = F_X(x_N^{2k}) - F_X(x_N^{2k+1}) \geq \ell_x \tag{53}$$

Consider the next subdivision $\{x^i_{N+1}\}_{i=1,(2)^{N+1}+1}$ and assume, again for instance, that $x \in [x^{2k}_N, x^{4k}_{N+1}[$. From (50) and (51) we get that:

$$F_{X^*_{N+1}}(x) = F_X(x^{4k}_{N+1}) \text{ and } F_{X^{**}_{N+1}}(x) = F_X(x^{4k+1}_{N+1}) = F_X(x^{2k}_N)$$

From (52) we get that:

$$F_{X^*_{N+1}}(x) - F_{X^{**}_{N+1}}(x) = F_X(x^{4k+1}_{N+1}) - F_X(x^{2k}_N) \geq \ell_x$$

Let $y \in [x^{4k}_{N+1}, x^{2k+1}_N[$. From (50) and (51) we get that:

$$F_{X^*_{N+1}}(y) = F_X(x^{4k}_{N+1}) \; and \; F_{X^{**}_{N+1}}(y) = F_X(x^{4k+1}_{N+1}) = F_X(x^{2k+1}_N)$$

From (52) we get that:

$$F_{X^*_{N+1}}(y) - F_{X^{**}_{N+1}}(y) = F_X(x^{4k}_{N+1}) - F_X(x^{2k+1}_N) \geq \ell_y$$

Now, since we have $F_X(x^{2k}_N) - F_X(x^{4k}_{N+1}) \geq 0$ and $F_X(x^{4k}_{N+1}) - F_X(x^{2k+1}_N) \geq 0$ and using (53) we get that:

$$\begin{aligned}
\left|F_X(x) - F_{X^*_n}(x)\right| &= \left|F_X(x^{2k}_N) - F_X(x^{2k+1}_N)\right| \\
&= \left|F_X(x^{2k}_N) - F_X(x^{4k}_{N+1})\right| + \left|F_X(x^{4k}_{N+1}) - F_X(x^{2k+1}_N)\right| \\
&\geq \ell_x + \ell_y
\end{aligned}$$

As a consequence we cannot have $\left|F_X(x) - F_{X^*_n}(x)\right| \in [\ell_x - \varepsilon, \ell_x]$ for any infinitesimal quantity ε unless $\ell_y = 0$. Such a conclusion clearly *does not depend on the initial choice of x and y.* Finally, the two sequences both converge towards X *i.e.* we get that :

$$\lim_{n\to\infty} X^{**}_n = \lim_{n\to\infty} X^*_n = X \tag{54}$$

Since $\mathcal{U}_{\mathbf{x}}(.)$ is continuous we also get that:

$$\lim_{n\to\infty} \mathcal{U}_{\mathbf{x}}(X^{**}_n) = \mathcal{U}_{\mathbf{x}}(X) = \lim_{n\to\infty} \mathcal{U}_{\mathbf{x}}(X^*_n) \tag{55}$$

Now recall that, by definition, the two sequences of simple lotteries which we have built are such that:
(a) X^*_n (X^{**}_n) is a mean prserving spread (henceforth **MPS**) of X^*_{n-1} (X^{**}_{n+1}), and X^{**}_n belong to $\mathfrak{X}_{\mathbf{x}}$
(b) X^*_n is a **MPS** of X which, in its turn, is a **MPS** of X^{**}_n.
Hence, we get that:

$$\delta_{\mathbf{x}} \succsim X^{**}_1 \succsim ... \succsim X^{**}_n \succsim X \succsim X^*_n \succsim ... \succsim X^*_1 \succsim X_{\mathbf{x}}$$

or, equivalently:

$$\mathcal{U}_{\mathbf{x}}(\delta_{\mathbf{x}}) \geq \mathcal{U}_{\mathbf{x}}(X^{**}_1) \geq ... \geq \mathcal{U}_{\mathbf{x}}(X^{**}_n) \geq \mathcal{U}_{\mathbf{x}}(X) \geq \mathcal{U}_{\mathbf{x}}(X^*_n) \geq ... \geq \mathcal{U}_{\mathbf{x}}(X^*_1) \geq \mathcal{U}_{\mathbf{x}}(X_{\mathbf{x}}) \tag{56}$$

or, alternatively:

$$\mathcal{U}_{\mathbf{x}}(\delta_{\mathbf{x}}) \geq \mathbf{E}[u_{\mathbf{x}}(X_1^{**})] \geq ... \geq \mathbf{E}[u_{\mathbf{x}}(X_n^{**})] \geq ..\mathcal{U}_{\mathbf{x}}(X) ... \geq \mathbf{E}[u_{\mathbf{x}}(X_n^{*})] \geq ... \geq \mathbf{E}[u_{\mathbf{x}}(X_1^{*})] \geq \mathcal{U}_{\mathbf{x}}(X_{\mathbf{x}}) \tag{57}$$

and, finally:

$$\lim_{n\to\infty} \mathbf{E}[u_{\mathbf{x}}(X_n^{**})] = \mathcal{U}_{\mathbf{x}}(X) = \lim_{n\to\infty} \mathbf{E}[u_{\mathbf{x}}(X_n^{*})] \tag{58}$$

Now we have to show that $\mathcal{U}_{\mathbf{x}}(X) = \mathbf{E}[u_{\mathbf{x}}(X)]$. To do so, just do as before given that the subdivisions are now defined as indicated below:

$$\begin{aligned} u_{n+1}^{2i-1} &= u_n^i \\ \int_{u_n^i}^{u_n^{i+1}} udG_U(u)) &= u_n^{2i}((G_U(u_n^{i+1}) - G_U(u_n^i)) \quad \text{for } i = 1, (2)^n \end{aligned}$$

where $u = u_{\mathbf{x}}(x)$ $u_n^i = u_{\mathbf{x}}\left(x_n^i\right)$,, $G_U(u) = G_U(u_{\mathbf{x}}(x)) = F_X(x)$ and $\int_0^1 udG_U(u) = \int_a^b u_{\mathbf{x}}(x)\, dF_X(x)$.

$$\begin{aligned} G_{U_{n+1}^{*}}(u) &= G_U(u_{n+1}^{2i}) \quad \text{for } u \in \left[u_{n+1}^{2i-1}, u_{n+1}^{2i+1}\right[\text{ and } i = 1, (2)^n \\ G_{U_{n+1}^{**}}(u) &= G_U(u_{n+1}^{2i+1}) \quad \text{for } u \in \left[u_{n+1}^{2i}, u_{n+1}^{2i+2}\right[\text{ and } i = 1, (2)^n \end{aligned}$$

Finally, the two sequences $\{\mathbf{E}[u_{\mathbf{x}}(X_n^{*})]\}_{n\in\mathbb{N}}$ and $\{\mathbf{E}[u_{\mathbf{x}}(X_n^{**})]\}_{n\in\mathbb{N}}$ have the same limit $\mathcal{U}_{\mathbf{x}}(X)$ which is but $\mathbf{E}[u_{\mathbf{x}}(X)]$.$\square$

Proof.of Proposition 5.

Let $Y_i = u(X_i)$ for $i = 1, 2$. Because of the definition of **SSOS** dominance, it is equivalent to state:

(a) X_1 $\mathcal{D}_2^u$ X_2, or

(b) Y_1 $\mathcal{D}_2$ Y_2, or, equivalently,

(c) $\int_0^v [F_{Y_1}(t) - F_{Y_2}(t)]\, dt \leq 0$ for $v \in [0, 1]$.

Now, by assumption, $u(.)$ is strictly increasing and, consequently $u^{-1}(.)$ is well defined (and also strictly increasing). Hence the following equality is met:

$$\int_0^v \left[F_{Y_1}(t) - F_{Y_2}(t)\right] dt = \int_a^{u^{-1}(v)} (F_{X_1}(x) - F_{X_2}(x))u'(x)\, dx$$

and, consequently, condition (c) is equivalent to the following one:

$$\int_0^z u'(x)\,(F_{X_1}(x) - F_{X_2}(x))dx \leq 0 \ \mathit{for\ any}\ z \in [0, 1] \quad \square \tag{59}$$

Proof of Proposition 6.

As a preliminary, recall that $u(.)$ is more concave than $v(.)$ *iff* $u \circ v^{-1}(.)$ is concave *i.e.* if there exists $g(.)$ mapping $[0, 1]$ on to itself and such that:$u(x) = g \circ v(x)$ with $g'(.) > 0$ and $g''(.) < 0$

The proof is grounded on the following calculations:

Let $\Delta \overset{def}{=} \int_a^z u'(x)\left(F_{X_1}(x) - F_{X_2}(x)\right) dt$, we get that:

$$\begin{aligned}
\Delta &= \int_a^z g'(v(x))\, v'(x)\left(F_{X_1}(x) - F_{X_2}(x)\right) dx \\
&= \left[g'(v(x)) \int_a^x v'(t)\left(F_{X_1}(t) - F_{X_2}(t)\right) dt\right]_a^z \\
&- \int_a^z g''(v(x))\, v'(x)\left[\int_a^x v'(t)\left(F_{X_1}(t) - F_{X_2}(t)\right) dt\right] dx
\end{aligned}$$

or:

$$\begin{aligned}
\Delta &= g'(v(z)) \int_a^z v'(t)\left(F_{X_1}(t) - F_{X_2}(t)\right) dt \\
&- \int_a^z g''(v(x))\, v'(x)\left[\int_a^x v'(t)\left(F_{X_1}(t) - F_{X_2}(t)\right) dt\right] dx
\end{aligned}$$

Finally, we get the following equivalences and/or implications which hold *for any* z:

$$\int_a^z v'(t)\left(F_{X_1}(t) - F_{X_2}(t)\right) dt < 0 \Rightarrow \int_a^z u'(x)\left(F_{X_1}(x) - F_{X_2}(x)\right) dx < 0$$

or:

$$X_1 \precsim_{\mathbf{2}}^{v} X_2 \Rightarrow X_1 \precsim_{\mathbf{2}}^{u} X_2 \quad \Leftrightarrow \quad \mathbb{X}_{\mathbf{2}}^{v} \subset \mathbb{X}_{\mathbf{2}}^{u}$$

and, as a consequence:

$$\mathbb{X}_{\mathbf{2}}^{v-} \subseteq \mathbb{X}_{\mathbf{2}}^{u-} \text{ and } \mathbb{X}_{\mathbf{2}}^{v+} \subseteq \mathbb{X}_{\mathbf{2}}^{u+}$$

□

Proof of Proposition 7.

Let $\mathbb{U}^* \subset \mathbb{U}$ denote the subset of concave or convex $n.u.$ functions and let $\mathbb{U}_{\mathrm{I}}^*$ ($\mathbb{U}_{\mathrm{C}}^*$) be the subset of inconsistent (consistent) concave or convex $n.u.$ functions. We have $\mathbb{U}^* = \mathbb{U}_{\mathrm{I}}^* \cup \mathbb{U}_{\mathrm{C}}^*$ and $.\mathbb{U}_{\mathrm{I}}^* \cap \mathbb{U}_{\mathrm{C}}^* = \{\mathbf{f}(.)\}$ where $\mathbf{f}(.)$ is the $n.u.$ affine function defined by $\mathbf{f}(x) = (x-a)/(b-a)$. Two cases may occur, according to the fact that the (standard) **SOS** dominance property is violated or not.

A. We first assume that the **SOS** dominance property is not violated, $i.e.$ $\mathbb{U}_{\mathrm{C}}^* \neq \varnothing$. As a consequence, there exists at least one concave function which is consistent. It is the $n.u.$ affine function $\mathbf{f}(.)$.

1. A first subcase is when $\mathbb{U}_{\mathrm{C}}^* = \{\mathbf{f}(.)\}$; Proposition 7 is then clearly valid.

2. We now leave aside this trivial subcase and assume that $\mathbb{U}_{\mathrm{C}}^*$ includes at least one strictly concave $n.u.$ function.

Let $\mathbb{H} = \bigcap_{u \in \mathbb{U}_{\mathrm{I}}^*} hypo(u)$. where $hypo(u)$ is the strict hypograph of $u \in \mathbb{U}_{\mathrm{I}}^*$.Since the hypographs are convex, so is $\mathbb{H}$ and so is its "northern" frontier which may be defined from the following equality: $hypo(\mathbf{u}) \overset{def}{=} \bigcap_{u \in \mathbb{U}_{\mathrm{I}}^*} hypo(u)$.

Clearly function $\mathbf{u}(.)$ is concave. Since the hypographs $hypo(u)$ are open, we do not know yet whether $\mathbb{H}$ is closed $-i.e$ whether $\mathbf{u}(.)$ belongs to $\mathbb{H}$ and, consequently is consistent– or not. Finally, we are going to prove directly that $\mathbb{X}_{\mathbf{2}}^{\mathbf{u}-} = \varnothing$. The proof is three-step.

(a) *The first step consists in defining a consistent concave n.u. function* $u(.)$ *which is close to* $\mathbf{u}(.)$. Now let $u(.)$ be defined by the following equality:

$$u(x) \overset{def}{=} \mathbf{u}(x) - y(x)$$

where

$$y(x) = \eta \left(\frac{x-a}{b-a} \right) - \eta \left(\frac{x-a}{b-a} \right)^2$$

Clearly, $y(x) \geq 0$ for $x \in [a,b]$, $y'(x) \geq 0$ for $x \in [a, a+(b-a)/2]$, $y'(x) \leq 0$ for $x \in [a+(b-a)/2,]\, b$, $y(a) = y(b) = 0$, $a + (b-a)/2 = Arg \max [y(x)]$ and $\max [y(x)] = \eta/4$. A sufficient condition for $u(.)$ to be concave is that:

$$\eta < \frac{1}{2}(b-a)^2 \inf_{x \in [a,b]} (-\mathbf{u}''(x))$$

Moreover, $u(.)$ will be strictly increasing if $u'(x)$ is strictly positive. A sufficient condition for this is that:

$$\eta < (b-a) \inf_{x \in [a,b]} (\mathbf{u}'(x))$$

and, finally, $u(.)$ is concave and strictly increasing if the real number η satisfies the below inequality:

$$\eta < \min \left[\frac{1}{2}(b-a)^2 \inf_{x \in [a,b]} (-\mathbf{u}''(x)), (b-a) \inf_{x \in [a,b]} (\mathbf{u}'(x)) \right] \tag{60}$$

Since $u(a) = \mathbf{u}(a) = 0$ and $u(b) = \mathbf{u}(b) = 1$, $u(.)$ is normalized and since $\eta > 0$, the hypograph of $\mathbf{u}(.)$ strictly includes that of $u(.)$. As a consequence, may not be inconsistent otherwise we would have $hypo(u) \subset hypo(\mathbf{u})$ and simultaneously $u(.) \in \mathbb{U}_{\mathrm{I}}^{*}\mathrm{N}$. This would contradict the fact that $\mathbb{H} = \bigcap_{u \in \mathbb{U}_{\mathrm{I}}^{*}} hypo(u)$. Finally$u(.)$ is well consistent. Finally, the function $u(.)$ is a concave $n.u.$ function if *(60)* is met.

(b) *The second step consists in looking for an upper bound for the following difference*:

$$\Delta = \left| \int_a^z \mathbf{u}'(x)(F_{X_1}(x) - F_{X_2}(x))dx - \int_a^z u'(x)(F_{X_1}(x) - F_{X_2}(x))dx \right|$$

Integrating by parts yields:

$$\begin{aligned} \Delta &= \left| \int_a^z (\mathbf{u}'(x) - u'(x))(F_{X_1}(x) - F_{X_2}(x))dx \right| \\ &= \left| (\mathbf{u}(z) - u(z))(F_{X_1}(z) - F_{X_2}(z)) + \int_a^z (\mathbf{u}'(x) - u'(x))(dF_{X_1}(x) - dF_{X_2}(x)) \right| \end{aligned}$$

and, consequently:

$$\Delta \leq \left|\left(\mathbf{u}\left(z\right)-u\left(z\right)\right)\left(F_{X_1}\left(z\right)-F_{X_2}\left(z\right)\right)\right|+\left|\int_a^z\left(\mathbf{u}'\left(x\right)-u'\left(x\right)\right)\left(dF_{X_1}\left(x\right)-dF_{X_2}\left(x\right)\right)\right| \tag{61}$$

The first term is bounded indicated as below:

$$\left|\left(\mathbf{u}\left(z\right)-u\left(z\right)\right)\left(F_{X_1}\left(z\right)-F_{X_2}\left(z\right)\right)\right| \leq \left|\left(\mathbf{u}\left(z\right)-u\left(z\right)\right)\right| \leq \sup_{z\in[a,b]}\left|\mathbf{u}'\left(z\right)-u'\left(z\right)\right|$$

We now show that he second term may be bounded as indicated below

$$\left|\int_a^z\left(\mathbf{u}'\left(x\right)-u'\left(x\right)\right)\left(dF_{X_1}\left(x\right)-dF_{X_2}\left(x\right)\right)\right| \leq 2\sup_{z\in[a,b]}\left|\mathbf{u}'\left(z\right)-u'\left(z\right)\right|$$

Indeed, we have

$$\left|\int_a^z\left(\mathbf{u}'\left(x\right)-u'\left(x\right)\right)\left(dF_{X_1}\left(x\right)-dF_{X_2}\left(x\right)\right)\right| \leq \begin{array}{c}\left|\int_a^z\left(\mathbf{u}'\left(x\right)-u'\left(x\right)\right)dF_{X_1}\left(x\right)\right| \\ +\left|\int_a^z\left(\mathbf{u}'\left(x\right)-u'\left(x\right)\right)dF_{X_2}\left(x\right)\right|\end{array}$$

and, for $i=1,2.$:

$$\begin{aligned}\left|\int_a^z\left(\mathbf{u}'\left(x\right)-u'\left(x\right)\right)dF_{X_i}\left(x\right)\right| &\leq \sup_{z\in[a,b]}\left|\mathbf{u}'\left(z\right)-u'\left(z\right)\right|\int_a^z dF_{X_i}\left(x\right) \\ &\leq \sup_{z\in[a,b]}\left|\mathbf{u}'\left(z\right)-u'\left(z\right)\right|\end{aligned}$$

Finally, an upper bound of is given by the following inequality:

$$\Delta \leq 3\sup_{z\in[a,b]}\left|\mathbf{u}'\left(z\right)-u'\left(z\right)\right|$$

Now, recall that $\sup_{z\in[a,b]}\left|\mathbf{u}'\left(z\right)-u'\left(z\right)\right| = \sup_{z\in[a,b]}\left|\eta\left(\frac{z-a}{b-a}\right)-\eta\left(\frac{z-a}{b-a}\right)^2\right| = \eta/4.$ As a consequence, we get that

$$\Delta \leq 3\eta/4$$

(c) *The last step consists in showing that if* $\mathbf{u}\left(.\right)$ *were not consistent, then we would get a contradiction.* Indeed if $\mathbf{u}\left(.\right)$ were not consistent there would exist two lotteries X_1 and X_2 such that $X_1 \preceq X_2$ and, simultaneously, there would exist $z\in[a,b]$, such that $\int_a^z\mathbf{u}'\left(x\right)\left(F_{X_1}\left(x\right)-F_{X_2}\left(x\right)\right)dx > 0$. In other words, there would exist a strictly positive real number ϵ such that

$$\int_a^z\mathbf{u}'\left(x\right)\left(F_{X_1}\left(x\right)-F_{X_2}\left(x\right)\right)dx \geq \epsilon > 0$$

Since $u\left(.\right)$ is consistent, we must have $\int_a^z u'\left(x\right)\left(F_{X_1}\left(x\right)-F_{X_2}\left(x\right)\right)dx < 0$ and, consequently: we get that:

$$\Delta = \int_a^z\mathbf{u}'\left(x\right)\left(F_{X_1}\left(x\right)-F_{X_2}\left(x\right)\right)dx + \left|\int_a^z u'\left(x\right)\left(F_{X_1}\left(x\right)-F_{X_2}\left(x\right)\right)dx\right| \geq \epsilon$$

and, finally:

$$\epsilon \leq 3\eta/4$$

Hence, if η is small enough, *i.e.* if $\eta < 4\epsilon/3$, we get a contradiction and, finally, $\mathbf{u}(.)$ is well consistent.

B. We now assume that the **SOS** dominance property is violated, *i.e.* $\mathbb{U}_{\mathrm{C}}^{*} = \varnothing$. No concave *n.u.* functions may be consistent. By contrast, the subset of convex *n.u.* functions is never empty since it always includes the following function: $\underline{u}(x) = 0$ for $x \in [a, b[$ and $\underline{u}(b) = 1$. The rest of the proof is analogous to the above one.□

Proof of Proposition 12.

The first part of the proof consists in proving that, in **LS**-models, two lotteries X_1 and X_2 which have the same expected utility $\overline{\mathbf{u}}$ and the same certainty equivalent $\mathbf{c}$, are strongly equivalent.

Let X_1 and X_2 exhibit the same expected utility $\overline{\mathbf{u}}$ and the same certainty equivalent $\mathbf{c}$. From (24) we get, for $i = 1, 2$:

$$\mathbf{u}(\mathbf{c}) = \overline{\mathbf{u}} + \textstyle\sum_{n=1}^{N} p_n^i \left(\mathcal{E}\left(\mathbf{u}(x_n) - \overline{\mathbf{u}}\right)\right)$$

where $X_i = \left[x_1, p_1^i; ...; x_N, p_N^i\right]$ $(i = 1, 2)$ and where $\overline{\mathbf{u}} = \sum_{n=1}^{N} p_n^i \mathbf{u}(x_n)$. As a consequence, we have:

$$\textstyle\sum_{n=1}^{N} p_n^1 \mathcal{E}\left(\mathbf{u}(x_n) - \overline{\mathbf{u}}\right) - \sum_{n=1}^{N} p_n^2 \mathcal{E}\left(\mathbf{u}(x_n) - \overline{\mathbf{u}}\right) = 0 \tag{62}$$

Now, consider the compound lottery

$$X_\lambda \overset{def}{=} \lambda X_1 \oplus (1-\lambda) X_2 = \left[x_1, \lambda p_1^1 + (1-\lambda) p_1^2; ...; x_N, \lambda p_N^1 + (1-\lambda) p_N^2\right]$$

Its expected utility is:

$$\mathbf{E}\left[\mathbf{u}(X_\lambda)\right] = \textstyle\sum_{n=1}^{N} (\lambda p_n^1 + (1-\lambda) p_n^2) \mathbf{u}(x_n) = \overline{\mathbf{u}}$$

From (24) we also get that:

$$\mathbf{u}(\mathbf{c}(X_\lambda)) = \overline{\mathbf{u}} + \textstyle\sum_{n=1}^{N} \left(\lambda p_n^1 + (1-\lambda) p_n^2\right) \mathcal{E}\left(\mathbf{u}(x_n) - \overline{\mathbf{u}}\right)$$

where $\mathbf{c}(X_\lambda)$ is the certainty equivalent of X_λ and, finally:

$$\mathbf{u}(\mathbf{c}(X_\lambda)) - \mathbf{u}(\mathbf{c}) = \lambda \left(\textstyle\sum_{n=1}^{N} p_n^1 \mathcal{E}\left(\mathbf{u}(x_n) - \overline{\mathbf{u}}\right) - \sum_{n=1}^{N} p_n^2 \mathcal{E}\left(\mathbf{u}(x_n) - \overline{\mathbf{u}}\right)\right) = 0$$

The proof of the converse is as follows. We must show that if X_1 and X_2 are strongly equivalent –*i.e.* if they have the same certainty equivalent and if they exhibit the betweenness property–, then they exhibit the same expected utility. To do so, we consider two discrete lotteries:

$$X_i = \left[x_1, p_1^i; ...; x_N, p_N^i\right] \quad i = 1, 2$$

and their probability mixture:

$$\lambda X_1 \oplus (1-\lambda) X_2 = \left[x_1, \lambda p_1^1 + (1-\lambda) p_1^2; ...; x_N, \lambda p_N^1 + (1-\lambda) p_N^2\right]$$

where $\lambda \in [0,1]$.

We assume that they have the same certainty equivalent. Hence, we have, for $i = 1, 2$:

$$\mathbf{u}(c) = \mathbf{u}(\mathbf{c}(X_i)) = \overline{\mathbf{u}}_i + \sum_{n=1}^{N} p_n^i \mathcal{E}\left(\mathbf{u}_n^i\right) \tag{63}$$

where:

$$\overline{\mathbf{u}}_i = \sum_{n=1}^{N} p_n^i \mathbf{u}(x_n) \text{ and } \mathbf{u}_n^i = \mathbf{u}(x_n) - \overline{\mathbf{u}}_i \tag{64}$$

Now, recall that, by definition, we have:

$$\begin{aligned}\mathbf{u}(\mathbf{c}(\lambda X_1 \oplus (1-\lambda) X_2)) &= \lambda \overline{\mathbf{u}}_1 + (1-\lambda)\overline{\mathbf{u}}_2 \\ &\quad + \sum_{n=1}^{N}\left[\lambda p_n^1 + (1-\lambda) p_n^2\right]\mathcal{E}\left(\lambda \mathbf{u}_n^1 + (1-\lambda)\mathbf{u}_n^2\right)\end{aligned}$$

and, from (63) and (64), we get that:

$$\begin{aligned}\lambda \mathbf{u}(\mathbf{c}(X_1)) + (1-\lambda)\mathbf{u}(\mathbf{c}(X_2)) &= \lambda \overline{\mathbf{u}}_1 + (1-\lambda)\overline{\mathbf{u}}_2 \\ &\quad + \sum_{n=1}^{N} \lambda p_n^1 \mathcal{E}\left(\mathbf{u}_n^1\right) + \sum_{n=1}^{N}(1-\lambda) p_n^2 \mathcal{E}\left(\mathbf{u}_n^2\right)\end{aligned}$$

Now, from the betweenness property we get that: $\mathbf{u}(\mathbf{c}(\lambda X_1 \oplus (1-\lambda) X_2))$ $= \lambda \mathbf{u}(\mathbf{c}(X_1)) + (1-\lambda)\mathbf{u}(\mathbf{c}(X_2))$, and, consequently:

$$\begin{array}{c}\sum_{n=1}^{N} p_n^1 \lambda \mathcal{E}\left(\mathbf{u}_n^1\right) \\ +\sum_{n=1}^{N} p_n^2 (1-\lambda)\mathcal{E}\left(\mathbf{u}_n^2\right)\end{array} = \lambda\left(\sum_{n=1}^{N} p_n^1 \mathcal{E}\left(\begin{array}{c}\lambda \mathbf{u}_n^1 \\ +(1-\lambda)\mathbf{u}_n^2\end{array}\right)\right) + (1-\lambda)\left(\sum_{n=1}^{N} p_n^2 \mathcal{E}\left(\begin{array}{c}\lambda \mathbf{u}_n^1 + \\ (1-\lambda)\mathbf{u}_n^2\end{array}\right)\right)$$

or, equivalently:

$$\begin{array}{c}\sum_{n=1}^{N} p_n^1 \lambda \mathcal{E}\left(\mathbf{u}_n^1\right) + \sum_{n=1}^{N} p_n^2 (1-\lambda)\mathcal{E}\left(\mathbf{u}_n^1\right) \\ +\sum_{n=1}^{N} p_n^2 (1-\lambda)\left(\mathcal{E}\left(\mathbf{u}_n^2\right) - \mathcal{E}\left(\mathbf{u}_n^1\right)\right)\end{array} = \sum_{n=1}^{N}\left[\lambda p_n^1 + (1-\lambda) p_n^2\right]\mathcal{E}\left(\lambda \mathbf{u}_n^1 + (1-\lambda)\mathbf{u}_n^2\right)$$

and, finally:

$$\sum_{n=1}^{N}\left[\lambda p_n^1 + (1-\lambda) p_n^2\right]\left[\mathcal{E}\left(\mathbf{u}_n^1\right) - \mathcal{E}\left(\begin{array}{c}\lambda \mathbf{u}_n^1 + \\ (1-\lambda)\mathbf{u}_n^2\end{array}\right)\right] = \sum_{n=1}^{N} p_n^2 (1-\lambda)\left(\mathcal{E}\left(\mathbf{u}_n^1\right) - \mathcal{E}\left(\mathbf{u}_n^2\right)\right)$$

$$\sum_{n=1}^{N}\left[\begin{array}{c}\lambda p_n^1 + \\ (1-\lambda) p_n^2\end{array}\right]\left[\begin{array}{c}\mathcal{E}(\mathbf{u}(x_n) - \overline{\mathbf{u}}_1) \\ -\mathcal{E}(\mathbf{u}(x_n) - \overline{\mathbf{u}}_\lambda)\end{array}\right](1-\lambda)^{-1} = \sum_{n=1}^{N} p_n^2 \left(\begin{array}{c}\mathcal{E}(\mathbf{u}(x_n) - \overline{\mathbf{u}}_1) \\ -\mathcal{E}(\mathbf{u}(x_n) - \overline{\mathbf{u}}_2)\end{array}\right)$$

$$\sum_{n=1}^{N} \varpi_n(\lambda)\left(\mathbf{u}_n^1 - \mathbf{u}_n^2\right)\mathcal{E}'\left(\begin{array}{c}\mathbf{u}(x_n) - \overline{\mathbf{u}}_1 \\ +\theta_n(\lambda)\left(\mathbf{u}_n^1 - \mathbf{u}_n^2\right)\end{array}\right) = \sum_{n=1}^{N} p_n^2 \left(\mathbf{u}_n^1 - \mathbf{u}_n^2\right) \times \mathcal{E}'\left(\begin{array}{c}\mathbf{u}(x_n) - \overline{\mathbf{u}}_1 \\ +\zeta_n\left(\mathbf{u}_n^1 - \mathbf{u}_n^2\right)\end{array}\right)$$

$$(\overline{\mathbf{u}}_1 - \overline{\mathbf{u}}_2)\left\{\sum_{n=1}^{N} \varpi_n(\lambda)\mathcal{E}'\left(\begin{array}{c}\mathbf{u}(x_n) - \overline{\mathbf{u}}_1 \\ +\theta_n(\lambda)(\overline{\mathbf{u}}_1 - \overline{\mathbf{u}}_2)\end{array}\right)\right\} = (\overline{\mathbf{u}}_1 - \overline{\mathbf{u}}_2) \times \left\{\sum_{n=1}^{N} p_n^2 \mathcal{E}'\left(\begin{array}{c}\mathbf{u}(x_n) - \overline{\mathbf{u}}_1 \\ +\zeta_n(\overline{\mathbf{u}}_1 - \overline{\mathbf{u}}_2)\end{array}\right)\right\}$$

$$(\overline{\mathbf{u}}_1 - \overline{\mathbf{u}}_2) F(\lambda) = (\overline{\mathbf{u}}_1 - \overline{\mathbf{u}}_2)\Lambda$$

Since $F(\lambda)$ cannot be equal to Λ for any value of λ, we must have $\overline{\mathbf{u}}_1 - \overline{\mathbf{u}}_2 =$ $0.\square$

Proof of Proposition 13.

The lotteries $\underline{X}_p^x$ and X will be strongly indifferent *iff*:

$$\begin{aligned} p\mathbf{u}(x) &= \pi \\ p\mathbf{u}(x) + p\mathcal{E}(\mathbf{u}(x) - p\mathbf{u}(x)) + (1-p)\mathcal{E}(-p\mathbf{u}(x)) &= \gamma \end{aligned}$$

where:

$$\begin{aligned} \pi &\overset{def}{=} \mathbf{u}(\mathbf{z}(X)) = \mathbf{E}[\mathbf{u}(X)] \in [0,1] \\ \gamma &\overset{def}{=} \mathbf{u}(\mathbf{c}(X)) = \pi + \mathbf{E}[\mathcal{E}(\mathbf{u}(X) - \pi)] \in [\mathbf{u}_\pi, \pi] \\ \mathbf{u}_\pi &= \pi + \pi\mathcal{E}(1-\pi) + (1-\pi)\mathcal{E}(-\pi\mathbf{u}(x)) \end{aligned}$$

The above system may be rewritten as indicated below:

$$\begin{aligned} \mathbf{u}(x) &= \pi/p \\ \pi + p\mathcal{E}((\pi/p) - \pi) + (1-p)\mathcal{E}(-\pi) &= \gamma \end{aligned}$$

the first equation is checked *iff* $x = \mathbf{u}^{-1}(\pi/p)$. The second equation has a unique solution because the function

$$\varphi(p) = \pi + p\mathcal{E}((\pi/p) - \pi) + (1-p)\mathcal{E}(-\pi)$$

is strictly increasing and maps$[\pi, 1]$ over $[\mathbf{u}_\pi, \pi]$. Indeed we have:

$$\begin{aligned} \varphi'(p) &= \mathcal{E}((\pi/p) - \pi) - \mathcal{E}(-\pi) - \pi/p\mathcal{E}'((\pi/p) - \pi) \\ &= (\mathcal{E}(-\pi) + (\pi/p)\mathcal{E}'(\theta(\pi/p) - \pi)) - \mathcal{E}(-\pi) - \pi/p\mathcal{E}'((\pi/p) - \pi) \\ &= (\pi/p)[\mathcal{E}'(\theta(\pi/p) - \pi) - \mathcal{E}'((\pi/p) - \pi)] \end{aligned}$$

and $\varphi'(p)$ is well negative since $\mathcal{E}(.)$ is concave, and that, consequently, $\mathcal{E}'(.)$ is decreasing.□

Proof of Proposition 14.

If x_{n+1} were greater than x_n, $\overline{X}_{p_{n+1}}^{x_{n+1}}$ would exhibit **FOS** dominance over $\underline{X}_{p_n}^{x_n}$. Hence, x_{n+1} is lower than x_n and $\{x_n\}_{n\in\mathbb{N}}$ is a decreasing sequence. It is also bounded below by a. Consequently, it converges towards a limit $\ell \geq a$. Next, note that the two strongly indifferent simple lotteries $\underline{X}_{p_n}^{x_n}$ and $\overline{X}_{p_{n+1}}^{x_{n+1}}$have the same expected utility, *i.e.*, we have:

$$p_n u(x_n) = p_{n+1} u(x_{n+1}) + (1 - p_{n+1}) \quad for \quad n = 0, 1, ... \tag{65}$$

and, consequently:

$$\pi u(w) = p_n u(x_n) + \textstyle\sum_{i=1}^n (1 - p_i) \quad for \quad n = 1, 2, ...$$

The above equality implies $S_n \overset{def}{=} \sum_{i=1}^n (1 - p_i) \leq \pi u(w)$. Since$\{S_n\}_{n\in\mathbb{N}^*}$ is an increasing sequence, it converges towards a limit $\Sigma \leq \pi u(w)$. As a consequence, $S_n - S_{n-1} = (1 - p_{n.}) \to 0$, *i.e.* $p_{n.} \to 1$. Moreover, since we have:

$\underline{X}_{p_{n+1}}^{x_n+1} \prec \overline{X}_{p_{n+1}}^{x_{n+1}} \sim \underline{X}_{p_n}^{x_n}$, the sequence of binary lotteries $\left\{\underline{X}_{p_n}^{x_n}\right\}_{n\in\mathbb{N}}$ is decreasing and converges towards $\underline{X}_1^{\ell} = \delta(\ell)$. Similarly, $\{\overline{X}_{p_n}^{x_n}\}_{n\in\mathbb{N}^*}$ converges towards $\overline{X}_1^{\ell} = \delta(b-\ell)$.

We now show that $\ell = a$. The proof is by contradiction. Indeed assume $\ell > a$. Then, since $\underline{X}_{p_n}^{x_n} \succ \delta(\ell)$, there exists a binary lottery $\underline{X}_{p_n}^{x_n^*}$ such that $\ell < x_n^* < x_n$, and $\underline{X}_{p_n}^{x_n^*} \sim \delta(\ell)$. Let x_{n+1}^* and p_{n+1}^* be defined by $\overline{X}_{p_{n+1}^*}^{x_{n+1}^*} \approx \underline{X}_{p_n}^{x_n^*}$. Since $\{\overline{X}_{p_n}^{x_n}\}_{n\in\mathbb{N}^*}$ converges towards $\delta(\ell)$, there exists an integer N, such that $m \geq N \Rightarrow \ell \leq x_m < x_{n+1}^*$ and $p_m \geq p_{n+1}^*$. This implies that $\overline{X}_{p_{n+1}^*}^{x_{n+1}^*}$ should be preferred to the $\overline{X}_{p_m}^{x_m}$s and, consequently, that $\delta(\ell)$ should be preferred to the $\overline{X}_{p_m}^{x_m}$s, which contradicts the fact that $\{\overline{X}_{p_n}^{x_n}\}_{n\in\mathbb{N}}$ is decreasing and converges towards $\delta(\ell)$. Hence $\ell = a$ and $\{S_n\}_{n\in\mathbb{N}}$ converges towards $\Sigma = \pi u(w)$. As a consequence, equality (31) is checked.$\square$

8 *Appendix 2.*

8.1 About the common ratio effect.

As shown in Loomes and Sugden's article [1986], several restrictions on the shape of $\mathcal{E}(.)$ allow to predict both the common ratio effect and the isolation effect together with the preservation of the **FOS** dominance principle. Indifference curves in the Marschak-Machina triangle may also have a mixed fanning shape[37] which is considered as a desirable property (See *e.g.* Starmer [2000]). As an example, consider the problems 3 and 4 from Kahneman and Tversky [1979].

Problem 3: Choice between lottery A and lottery B where:

A=$[0.2, 0.8; 0, 4000]$ and B=$[1; 3000]$

Problem 4: Choice between lottery C and lottery D where:

C=$[0.8, 0.2; 0, 4000]$ and D=$[0.75, 0.25; 0, 3000]$

Most people choose lottery B in problem 3 and lottery C in problem 4. Hence, to offer a good representation of their preferences the following inequalities must hold:

$$\mathcal{U}(\text{A})/\mathcal{U}(\text{B}) \leq 1 \text{ and } \mathcal{U}(\text{C})/\mathcal{U}(\text{D}) \geq 1 \tag{66}$$

The paradox is as follows: consider an arbitrary utility function $u(.)$ such that $u(0) = 0$. Then "B preferred to A" implies that

$$u(3000) > 0.8u(4000) \tag{67}$$

whereas "C preferred to D" implies that

$$0.2u(4000) > 0.25u(3000)$$

and, consequently:

$$0.8u(4000) > u(3000) \tag{68}$$

Finally, (67) contradicts (68) and the pattern of preferences is not compatible with **EU** theory. Now to make clearly apparent the potential of a **LS** functional for management applications, we focus on a particular case where the functional reads:

$$\mathcal{U}(X) = \int_a^b \mathbf{u}(x)\left[1 - \mathfrak{a}\left(\mathbf{u}(x) - \mathbf{E}\left[\mathbf{u}(X)\right]\right)\right] dF_X(x)$$

where $\mathbf{u}(.)$ is a *n.u.* function and $\mathfrak{a}$ a positive parameter that controls for disappointment aversion. Without loss of generality, we normalize $\mathbf{u}(.)$ as follows:

$$\mathbf{u}(0) = 0 \text{ and } \mathbf{u}(4000) = 1$$

Using our axiomatics, we get that:

$U(\text{A}) = 0.8(1 - 0.2A\left[0.8\right])$and $U(\text{B}) = u(3000)$;

$U(\text{C}) = 0.2(1 - 0.8A\left[0.2\right])$and $U(\text{D}) = 0.25(1 - 0.75u(3000)\mathfrak{a}\left[0.25\mathbf{u}(3000)\right])u\left(3000\right)$

[37] Namely, a fanning out shape in the lower right corner of the triangle diagram and a fanning in shape in the upper left of the triangle diagram.

We can rewrite the inequalities (66) above as follows:

$$0.8(1-0.2\mathfrak{a}\left[0.8\right])\leq\mathbf{u}(3000)\leq0.8\frac{(1-0.8\mathfrak{a}\left[0.2\right])}{(1-0.75\mathbf{u}(3000)\mathfrak{a}\left[0.25\mathbf{u}(3000)\right])}$$

or, alternatively as the two following conditions:

$$\mathfrak{a}\left[0.8\right]\geq\frac{0.8-\mathbf{u}(3000)}{0.16}$$

$$1.25\mathbf{u}(3000)-1\leq0.9375\mathbf{u}^{2}(3000)\mathfrak{a}\left[0.25\mathbf{u}(3000)\right]-0.8\mathfrak{a}\left[0.2\right]$$

By assumption, $\mathfrak{a}\left[.\right]$is positive and it is a decreasing function of $E\left[\mathbf{u}\left(X\right)\right]$. Hence the first condition is automatically reached, whereas the second condition can always be achieved if $\mathfrak{a}(.)$is decreasing enough. For instance, with a linear utility function ($u(3000)=0.75$), the Allais paradox is solved with the following choice of parameters which fully satisfy the sufficient condition for **FOS** dominance consistency :

$\mathfrak{a}(0.8)=0.3125;\mathfrak{a}(0.2)=0.35$

and

$\mathfrak{a}(0.25u(3000))=\mathfrak{a}(0.1875)=0.5425$

8.2 Consistency of the *n.u.* function of a LS functional.

In the proof, X_1 is assumed to **SSOS** dominate to the second-order X_2 *i.e.*

$$\lambda_1-\lambda_2=-\int_a^b u'\left(t\right)\left(F_{X_1}\left(t\right)-F_{X_2}\left(t\right)\right)dt\geq0$$

where

$$\lambda_i\overset{def}{=}\mathbf{E}\left[X_i\right]=\int_a^b u\left(x\right)dF_{X_i}=1-\int_a^b u'\left(t\right)F_{X_i}\left(t\right)dt$$

Now let:

$$\Delta\mathcal{U}\overset{def}{=}\mathcal{U}\left(X_1\right)-\mathcal{U}\left(X_2\right)\geq0$$

The difference between the two functionals is:

$$\begin{aligned}\Delta\mathcal{U}&=\int_a^b\left(u\left(x\right)+\mathcal{E}\left(u\left(x\right)-\lambda_1\right)\right)dF_{X_1}\left(x\right)-\int_a^b\left(u\left(x\right)+\mathcal{E}\left(u\left(x\right)-\lambda_2\right)\right)dF_{X_2}\left(x\right)\\&=\left(\lambda_1-\lambda_2\right)+\int_a^b\mathcal{E}\left(u\left(x\right)-\lambda_1\right)dF_{X_1}\left(x\right)-\int_a^b\mathcal{E}\left(u\left(x\right)-\lambda_2\right)dF_{X_2}\left(x\right)\end{aligned}$$

or, equivalently as:

$$\Delta\mathcal{U}=T1+T2$$

where:

$$T1\overset{def}{=}\left(\lambda_1-\lambda_2\right)+\int_a^b\left(\mathcal{E}\left(u\left(x\right)-\lambda_1\right)-\mathcal{E}\left(u\left(x\right)-\lambda_2\right)\right)dF_{X_2}\left(x\right)$$

and:

$$T2 \stackrel{def}{=} \int_a^b \mathcal{E}\left(u\left(x\right)-\lambda_1\right)\left(dF_{X_1}\left(x\right)-dF_{X_2}\left(x\right)\right)$$

Using a Taylor development in $T1$ yields:

$$T1 = \left(\lambda_1-\lambda_2\right) - \int_a^b \left(\lambda_1-\lambda_2\right)\mathcal{E}'\left(u\left(x\right)-\lambda_1+\theta_1\left(\lambda_1-\lambda_2\right)\right)dF_{X_2}\left(x\right)$$

with $\theta_1 \in [0,1]$ and integrating $T2$ by parts yields:

$$T2 = \left[\mathcal{E}\left(u\left(x\right)-\lambda_1\right)\left(F_{X_1}\left(x\right)-F_{X_2}\left(x\right)\right)\right]_a^b - \int_a^b \mathcal{E}'\left(u\left(x\right)-\lambda_1\right)u'\left(x\right)\left(F_{X_1}\left(x\right)-F_{X_2}\left(x\right)\right)dx$$

or, equivalently:

$$T2 = \mathcal{E}\left(-\lambda_1\right)\left(F_{X_1}\left(a\right)-F_{X_2}\left(a\right)\right) - \int_a^b \mathcal{E}'\left(u\left(x\right)-\lambda_1\right)u'\left(x\right)\left(F_{X_1}\left(x\right)-F_{X_2}\left(x\right)\right)dx$$

and, finally:

$$\begin{aligned} T2 = {} & \mathcal{E}\left(-\lambda_1\right)\left(F_{X_1}\left(a\right)-F_{X_2}\left(a\right)\right) - \mathcal{E}'\left(1-\lambda_1\right)\int_a^b u'\left(t\right)\left(F_{X_1}\left(t\right)-F_{X_2}\left(t\right)\right)dt \\ & + \int_a^b \mathcal{E}''\left(u\left(x\right)-\lambda_1\right)u'\left(x\right)\left[\int_a^x u'\left(t\right)\left(F_{X_1}\left(t\right)-F_{X_2}\left(t\right)\right)dt\right]dx \end{aligned}$$

Clearly, the condition $[1-\sup\mathcal{E}'(z)] \geq 0$ implies that $sign\left(T1\right) = sign\left(\lambda_1-\lambda_2\right)$ and that $T1$ is positive. The second term $T2$ is also positive, since it is the sum of three positive terms: indeed the first term, which reads $\mathcal{E}\left(-\lambda_1\right)\left(F_{X_1}\left(a\right)-F_{X_2}\left(a\right)\right)$, is positive because $\mathcal{E}\left(-\lambda_1\right)$ is negative (since $\mathcal{E}\left(0\right)=0$ and that $\mathcal{E}'\left(.\right)>0$) and that $\left(F_{X_1}\left(a\right)-F_{X_2}\left(a\right)\right)$ is also negative (from **SSOS** dominance). The second term is positive because $\mathcal{E}'\left(1-\lambda_1\right)$ is positive, and that the integral $\int_a^b u'\left(t\right)\left(F_{X_1}\left(t\right)-F_{X_2}\left(t\right)\right)dt$ is negative (again from **SSOS** dominance). The last term is positive because $\mathcal{E}''\left(u\left(x\right)-\lambda_1\right)$ is negative, $u'\left(x\right)$ is positive and $\int_a^x u'\left(t\right)\left(F_{X_1}\left(t\right)-F_{X_2}\left(t\right)\right)dt$ is negative (from **SSOS** dominance). Finally, $\Delta\mathcal{U} \stackrel{def}{=} \mathcal{U}\left(X_1\right)-\mathcal{U}\left(X_2\right)$ is positive and, consequently X_1 is well preferred to X_2.$\square$