A Behavioral Definition of States of the World

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Abstract

This paper elaborates an axiomatic treatment of the Subjective Expected Utility (SEU) model that dispenses with the assumption of an exogenous state space. Within a state-free description of uncertainty and alternatives, axioms for preferences are formulated and shown to characterize the existence of a subjective state space, a subjective probability and a utility function. In the representation, the individual appears to behave as if he used the state space to describe uncertainty and maximized SEU to make decisions. Moreover, the state space, probability and utility are unique in some appropriate sense.

Keywords: expected utility, subjective state space, causality, consequentialism
JEL classification: D81, D90

1 Introduction

In his axiomatic treatment of Subjective Expected Utility (SEU), Savage (1954) defines a state of the world as *a description of the world leaving no relevant aspect undescribed*. Though the set of states is subjective in the sense of depending upon the individual and representing the prior information available to him in the decision process, it is not derived from preferences and is in fact exogenously given. Subjectivity and exogeneity create some difficulties in applications of the theory. In a normative perspective, it is not clear whether the use of a state space is in itself part of the SEU recommendation and the Savage theory might remain of little help to individuals unwilling or unable to describe uncertainty through a state space.

What this paper calls into question is the ability of individuals to combine relevant aspects into well-defined states of the world. Think for instance of a portfolio manager and two assets whose future prices either go up or down. The manager still has to deal with the possible causal relationships between prices. If he believes that prices are causally independent, then his subjective state space should include the four states (up, up), $(up, down)$,

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$(down, up)$ and $(down, down)$. If he rather believes that whenever the price of the first asset goes up, the price of the second asset also does, then he should remove state $(up, down)$ from this state space. Finally, if he also believes in the reciprocal, he should further remove state $(down, up)$. Thus, the assumption of an exogenous state space presupposes a very clear understanding of causal relationships between relevant aspects that is perhaps unrealistic in sufficiently complex decision situations.

The individual's understanding of causality should also be correct. Even slight mistakes in the specification of states might indeed have undesirable effects. In the example, if the correct state space only contains (up, up), $(down, up)$ and $(down, down)$ and yet the manager also includes impossible state $(up, down)$ in his subjective state space, he will sometimes choose dominated actions: a share of asset 1 is dominated at each state by a share of asset 2 and yet, by symmetry, the inclusion of the impossible state makes the manager indifferent between the two alternatives. In contrast, if the correct state space includes all four possible states and yet the manager omits possible state $(up, down)$, he will believe unduly that asset 2 dominates asset 1, systematically prefer buying shares of the former and thus be led to choose underdiversified portfolios.

How exactly to acquire such a clear and correct understanding of causality? Assume that, for each relevant aspect, the individual knows a feasible experiment that always determines the way the uncertainty attached to this relevant aspect resolves. As long as the number of relevant aspects is finite, it is possible to perform all of these experiments at once and thus to observe directly the state that actually obtains. Repetitions of the latter compound experiment then provide at least a rough idea of the way causality works and, therefore, of the appropriate state space to use. If, after a sufficiently large number of repetitions, the manager never observes the configuration where the first price goes up while the second one goes down, he can legitimately start to believe that state $(up, down)$ is in fact impossible. But consider now complex decision situations where an infinity of relevant aspects exists. A state has to encode the resolution of uncertainty for infinitely many relevant aspects while, due to feasibility constraints, experiments can only measure the resolution of uncertainty for a finite number of them. It becomes impossible to observe empirically the actual state. In the example, even the daily observation of a finite number of prices cannot provide enough information to help the manager construct a state space when there is an infinity of assets. As a result, there appears to be an irreducible amount of arbitrariness in the choice of any specific state space and individuals might be reluctant to make their decisions on the basis of speculation about things they ignore and simply cannot observe.

This paper improves the axiomatic treatment of SEU by dispensing with this problematic assumption of an exogenous state space. More precisely, it elaborates a state-free description of uncertainty and alternatives (Section *2.1*) and focuses on what is here called a *Savage representation* of preference (Section *2.2*). Such a representation entails a subjective state space, a subjective probability and a utility function. In the representation, the individual behaves as if he used the subjective state space to formalize uncertainty and causality, and maximized SEU over that state space to make decisions. Behavioral axioms on preference are then formulated that consist not only of straightforward reformulations of the Savage ones but also of additional ones of a more logical and epistemic nature (Section *3.1*). These axioms are shown to characterize the existence of a *Savage representation* and the state space, probability and utility are shown to be unique in some appropriate sense (Section *3.2*). Finally, while the state space and the precise form of causality it captures are exogenous in Savage's SEU, they are here determined endogenously from behavior as a part of the representation of preference.

A key aspect of the paper is that preference is restricted to *feasible acts*. Intuitively, an alternative (or an act in the Savage terminology) is said to be feasible if it is always possible to perform experiments that determine the consequence it actually induces. The restriction of preference to feasible acts ends up providing a much compelling justification for the axioms that characterize Savage representations: what makes these axioms so compelling is precisely the fact that preference is, by definition, restricted to feasible acts. In the logic of the paper, a rational individual having to make choices between feasible acts is therefore expected to conform to these axioms and finally maximize SEU over some subjective state space. Thus, the use of a state space is justified upon axioms of rationality and becomes explicitly part of the SEU normative recommendation.

One may still worry about the possibility of dealing with causality upon a radically different mode. After all, the individual can always use the grand state space made of all theoretically conceivable states, redefine acts as distributions of consequences across states in the Savage fashion and formulate probability judgments on this grand state space. Thus, causality would be reduced to probabilistic correlation and would not require a new model as the one developed here. However, this would do nothing but move the problem on: if the individual is unable to say whether a given state is possible, it is hard to imagine that he can assign it a specific probability. Moreover, this approach would rely implicitly upon the assumption *consequentialism*. The latter says that any two acts that induce the same distribution of consequences across states should be indifferent. In the words of Savage, *if two different acts had the same consequences in every state of the world, there would from the present point of view be no point in considering them two different acts at all.* While certainly compelling at a first glance, consequentialism remains difficult to justify without a more precise definition of states of the world. In contrast, the present construction endogenizes, not only the state space, but also consequentialism with respect to that state space and thus provides an axiomatic justification of this assumption.

In its turn, consequentialism entails the existence of *Savage preferences* (preferences over the subjective state space). By construction, the latter necessarily conform to Savage's Sure Thing Principle (STP) as soon as they exist. It is indeed the same axiom that ends up implying all three of consequentialism, Savage preferences and their compliance with STP. This aspect of the construction is consistent with Hammond's (1988) consequentialist foundations of SEU and defense of STP. While Hammond formalizes his argument in the context of dynamic choice, the same intuition is here reformulated in a purely static framework and leads to the following conclusion: if preferences over a subjective state space are allowed to depart from STP, then it is not clear why they exist in the first place. This challenges the ambiguity literature which typically takes preferences as given and yet weakens STP to account for the distinction between risk and uncertainty in the sense of Knight (1921) and Ellsberg (1961). However, the negative conclusion on the possibility of ambiguity over the subjective state space only holds for behavior subject to feasibility constraints. Therefore, ambiguity might emerge in a natural way on the domain of unfeasibility. Indeed, since it is not always possible to determine experimentally the consequence they induce, unfeasible acts entail a genuine form of ambiguity that might be described through nonadditive probability as in Schmeidler (1989).

Literature provides various explicit constructions of states of the world. Kreps (1979) takes preference for flexibility, i.e. preference for larger menus of alternatives, to reveal the individual's subjective perception of his uncertain future preferences and identifies the latter with subjective states. Dekel, Lipman and Rustichini (2001) extend Kreps' analysis so as to obtain uniqueness of the state space. In contrast, preferences apply here to alternatives, and not menus, and subjective states are derived from betting behavior on the outcomes of *experiments*, a simplified version of Moore's (1999) *definite experimental projects*. Experiments stand for achievable operations that can always be performed on the random system and always return a definite outcome. Finally, feasible alternatives are defined as consequences that are contingent upon the outcomes of experiments.
In Blume, Easley and Halpern (2006), uncertainty is described through a propositional logic. The state space is obtained as the set of all truth assignments and is thus independent of preferences. On the contrary, preferences are used here to derive a *subjective implication relation* that incarnates the individual's subjective understanding of causality. Subjective implication determines indeed all the implications between the outcomes of experiments that are perceived by the individual. The state space is obtained as the set of assignments of outcomes to experiments that are consistent with subjective implication. Mathematically, subjective implication induces a Boolean algebra and the state space is constructed through the Stone (1936) representation theorem.
Lipman (1999) uses preferences to identify the correct logical implications that are recognized by the individual. Since he studies individuals who are not logically omniscient and fail to infer all logical implications of what they know, correct logical implications typically exist that are not reflected in preferences. Herein, the individual is logically omniscient: behavior is consistent with all the purely logical implications that are part of his description of uncertainty. But behavior is allowed to be consistent with additional and extra-logical implications. The latter represent the prior information that is available in the decision process. As in the portfolio example stated above, more prior information is expected to lead to belief in more subjective implications and therefore to smaller state spaces.

2 A framework for Savage representations

2.1 Framework

The behavioral definition of states of the world heavily relies upon the notions of *observability* and *feasibility*. Intuitively, an event is said to be observable if there exists an experiment that can always be effectively performed and always determines whether the event holds or not. An act is said to be feasible if its consequences are contingent upon observable events. The consequence induced by a feasible act is thus totally devoid of ambiguity: experiments always determine the specific consequence that a feasible act yields. To motivate this notion, think of two contracting parties who disagree about what they owe to each other. As long as their contract is feasible in this sense, it is always possible to determine experimentally who is right and who is wrong.

Observability is modeled as follows: an individual is assumed to describe the uncertainty generated by a random system in terms of the *experiments* he knows and considers to be relevant. Such an experiment represents an operation that can always be performed on the random system. Whenever effectively performed, it returns one and only one of two possible outcomes, conventionally called *yes* and *no*. The set $\mathcal{E}$ of experiments is endowed with some trivial elements and some algebraic operations:

(1) $y, n \in \mathcal{E}$ are trivial experiments,

(2) For each $e, f \in \mathcal{E}$, $e \circ f \in \mathcal{E}$ is the product experiment,

(3) For each $e \in \mathcal{E}$, $e' \in \mathcal{E}$ is the complementary experiment.

Experiments y and n are trivial in that they always return yes and no respectively. Performing them simply consists in doing nothing and assigning the outcomes yes to y and no to n. In addition, performing a product experiment of the form $e \circ f$ consists in performing e first and then f. Experiment $e \circ f$ returns yes if each of e and f does. Furthermore, a complementary experiment e' is obtained from e by switching the circumstances at which yes and no are replied. Thus, e' returns yes if and only if e returns no.

Each experiment $e \in \mathcal{E}$ induces two events: *e returns yes* and *e returns no*. Each event is observable since an experiment exists, by definition, that always determines whether it obtains or not. Such observable events represent the objective information that the individual could acquire by experimentation, observation or calculation and that is quite independent from his subjective speculation on uncertainty and causality.

In terms of the introductory example, there are two elementary relevant experiments: they determine whether the price of each asset goes up or not. To perform them effectively, the manager simply has to consult his information system. If the manager truly believes that, whenever the first price goes up, the second one also does, he is expected to be indifferent between a bet on the first price going up (ie. experiment 1 returning yes) and a bet on the two prices going up (ie. experiments 1 and 2 returning yes in that order).

In the Savage omelette example, there are again two elementary relevant experiments. The first one determines whether a given egg is rotten or not and the second one determines whether a rotten egg spoils a six-egg omelette or not. To perform them effectively, it is sufficient for the individual to break an egg to observe its actual inner state or to cook and taste a six-egg omelette with one rotten egg. If the individual truly thinks that using a rotten egg for a six-egg omelette spoils the omelette, then he is expected to be indifferent between a bet on experiment 1 returning yes and a bet on both experiments 1 and 2 returning yes in that order. Finally, the following example is used throughout the paper to illustrate various aspects of the argument.

Example Consider a countable collection of coins. An experiment can only involve a finite number of coins. For instance, $e =$ *examine coin 1 and then coin 3 and respond yes if and only if both coins land on heads* and $f =$ *examine coin 2 and then coin 4 and respond yes if and only if both coins land on heads* are experiments while *examine all coins labelled with an even number and respond yes if and only if they all land on heads* is not such an experiment since it relies upon an unfeasible operation. Then, $e \circ f$ is the experiment that consists in examining coins 1, 3, 2 and 4 in this order and returns yes if and only if all four coins land on heads. Experiment e' consists in examining coin 1 and then coin 3 and returns yes if and only if at least one of the two coins lands on tails.

Following Savage (1954), *a consequence is anything that may happen to the person.* Let $\mathcal{A}_0$ stand for the set of consequences. Adapting Blume, Easley and Halpern's (2006) notion of syntactic programs, the set $\mathcal{A}$ of acts is defined through finite combinations of experiments and consequences:
(1) Each consequence is an act,
(2) For any experiment $e \in \mathcal{E}$ and any acts $a, b \in \mathcal{A}$, $e(a, b) \in \mathcal{A}$ is also an act.
Each consequence is thus identified with the certain act that always yields this consequence. An act of the form $e(a, b)$ delivers the same consequences as a if e returns yes and the same ones as b if e returns no. The order of an act is defined as the number of steps used in its recursive construction. Additionally, acts can be seen as consequences that are contingent on the outcomes of experiments. Since experiments can always be effectively performed and always return a definite outcome, these acts are feasible in the sense that it is always possible to determine the consequence they are supposed to yield.
Finally, preferences $\succsim$ represent the behavior of the individual and apply to acts. As usual, $\succ$ and $\sim$ denote strict preference and indifference respectively. This paper thus focuses on preference given observability and feasibility constraints. As argued below in greater detail, the notions of observability and feasibility provide a much compelling justification for the axioms and the Savage representations presented in the next sections.

Example (cont.) Think of monetary consequences and let a_0 and b_0 stand for \$100 and \$0 respectively. The act $a = f(a_0, b_0)$ stands for a bet that pays \$100 if coins 2 and 4 are observed in this order to land on heads and \$0 otherwise. Similarly, $e(f(a_0, b_0), b_0)$ pays the same amount as a if coins 1 and 3 are observed in this order to land on heads and \$0 otherwise while $e \circ f(a_0, b_0)$ pays \$100 if coins 1, 3, 2 and 4 are observed in this order to land on heads and \$0 otherwise. An individual is naturally expected to be indifferent between the two latter acts, which illustrates Axiom B1 below.

2.2 Savage representations

This section presents Savage representations whose definition involves a state space, an interpretation, a probability measure and a utility function. A *state space* is a non-empty topological space Ω that is Hausdorff, compact and zero-dimensional. Let $\mathcal{L}(\Omega)$ denote the finite algebra of its clopen subsets (sets that are open and closed). Elements of Ω are called *states of the world* and elements of $\mathcal{L}(\Omega)$ are called *Savage events.*

Given a state space Ω, an *interpretation* is a mapping $K : \mathcal{E} \longrightarrow \mathcal{L}(\Omega)$ that makes the connection between states and experiments and satisfies the following conditions:
(1) $K : \mathcal{E} \longrightarrow \mathcal{L}(\Omega)$ is surjective,
(2) $\forall e, f \in \mathcal{E}$, $K(e \circ f) = K(e) \cap K(f)$,
(3) $\forall e \in \mathcal{E}$, $K(e') = K(e)^c$,
(4) $K(y) = \Omega$ and $K(n) = \varnothing$.
Thanks to the interpretation K, each state of the world encodes the outcomes of all experiments: the individual considers that experiment $e \in \mathcal{E}$ returns yes at state $\omega \in \Omega$ if and only if $\omega \in K(e)$. Put differently, $K(e)$ collects all states at which e returns yes and performing e amounts to determining whether $K(e)$ holds or not. The interpretation makes it thus possible to represent an experiment through the Savage event it determines. As a result, it also models the individual's subjective understanding of causal relationships between the outcomes of experiments. To see this, consider two experiments $e, f \in \mathcal{E}$ such that $K(e) \subseteq K(f)$. Then, f returns yes at any state at which e returns yes so that the individual believes that, whenever e returns yes, f also does.
In addition, surjectivity in condition (1) implies that each Savage event $E \in \mathcal{L}(\Omega)$ corresponds to some experiment $e \in \mathcal{E}$. Each Savage event E is thus observable in the sense that there is an experiment that always determines whether E holds or not. Conditions (2), (3) and (4) directly capture the content of the intuitive definitions of products, complements and trivial experiments given in the previous section. Since a product experiment returns yes if and only if each factor also does, the product is interpreted as the set-theoretic intersection. Since a complement returns yes if and only if the underlying experiment returns no, the complementation is interpreted as the set-theoretic complementation. Finally, since the trivial experiments y and n always return yes and no respectively, they are interpreted as the corresponding trivial subsets of the state space.
The Hausdorff requirement is here equivalent to the condition that for any two distinct states $\omega, \omega' \in \Omega$, there exists an experiment $e \in \mathcal{E}$ such that $\omega \in K(e)$ and $\omega' \in K(e')$. This means that performing e always results in one of the two states being refuted: if e returns yes, then ω' is refuted while, if e returns no, then ω is refuted. Therefore, the Hausdorff requirement ensures the existence of an experiment that always refutes one of two given distinct states. Moreover, due to compactness, a partition over Ω made of Savage events is necessarily finite. Consequently, any such partition is observable in the sense that it would be possible to use the experiments in $\mathcal{E}$ to construct another feasible experimental procedure that would always determine the one cell in the partition that obtains. At last, zero-dimensionality requires any open set to be a potentially infinite union of Savage events. An open set E is therefore necessarily of the form $E = \cup_{i \in I} K(e_i)$, where all e_i are experiments, and to say that E holds amounts to saying that there exists $i \in I$ such that e_i returns yes. Thus, an open set represents an existential statement on the outcomes of experiments. Whenever I is infinite, the statement is unobservable: E can certainly be verified in practice since it holds whenever some e_i is identified that returns yes, but it remains irrefutable in the sense that there is no experimental procedure likely to refute it. Dually, closed sets represent universal statements on the outcomes of experiments that are certainly refutable but may remain unverifiable. Due to the Hausdorff requirement, a

state is a typical example of such an unverifiable statement.

Example (cont.) The set $\Omega = \{heads,\ tails\}^{\mathbb{N}}$ provides an example of a state space. Open sets in Ω are defined as unions of products $\Pi_{n\geq 0}A_n$ where each A_n is a subset of $\{head,\ tails\}$ and all but finitely many A_n are equal to $\{heads,\ tails\}$. All clopen sets are of the form $E = \{\omega \in \Omega,\ \omega(i_1) = \alpha_1,\ ...,\ \omega(i_n) = \alpha_n\}$ where $(i_k)_{k=1}^n$ and $(\alpha_k)_{k=1}^n$ are finite sequences in $\mathbb{N}$ and $\{heads,\ tails\}$ respectively. A Savage event thus only determines the outcome of a finite number of coins. An interpretation K can be defined in the following way: for each experiment e, let i_1, ... , i_n be the labels of the finitely many coins involved by e and α_1, ... , α_n be the corresponding outcomes. Then, the set $K(e)$ is defined as the Savage event $\{\omega \in \Omega,\ \omega(i_1) = \alpha_1,\ ...,\ \omega(i_n) = \alpha_n\}$. Moreover, consider the closed subset $E = \{\omega \in \Omega,\ \forall k \geq 0,\ \omega(k) = heads\} \subseteq \Omega$ which corresponds to the situation where all coins land on heads. Such an event is certainly refutable for it is sufficient to exhibit one coin that lands on tails to refute it. But it remains unverifiable since it would take an infinite number of operations to verify it. Dually, the open subset $E^c = \{\omega \in \Omega,\ \exists k \geq 1,\ \omega(k) = tails\}$ is verified whenever a coin that lands on tails is exhibited but remains irrefutable in the sense that a finite number of operations cannot but fail to refute it.

A function f mapping states onto consequences is said to be a *Savage act* if it is measurable with respect to $\mathcal{L}(\Omega)$, i.e. for all $a_0 \in \mathcal{A}_0$, $\{f = a_0\} \in \mathcal{L}(\Omega)$ is a Savage event. Let $\mathcal{A}(\Omega)$ denote the set of Savage acts. Due to compactness, a Savage act has necessarily a finite range. For any Savage acts $f, g \in \mathcal{A}(\Omega)$ and any Savage event $E \subseteq \Omega$, let $f_E g \in \mathcal{A}(\Omega)$ denote the Savage act that is equal to f over E and to g over E^c. Each consequence $a_0 \in \mathcal{A}_0$ is identified with the constant Savage act that delivers a_0 at each state.
Given a state space Ω, an interpretation K over Ω can always be inductively extended into a surjective mapping $\varphi : \mathcal{A} \longrightarrow \mathcal{A}(\Omega)$ according to:
(1) $\forall a_0 \in \mathcal{A}_0$, $\varphi(a_0) = a_0$,
(2) $\forall e \in \mathcal{E}$, $\forall a, b \in \mathcal{A}$, $\varphi(e(a, b)) = \varphi(a)_{K(e)}\varphi(b)$.
The interpretation of experiments is thus extended into an interpretation of acts that takes the form of a functional preserving consequences and products of acts. Surjectivity makes it clear that each Savage act $f \in \mathcal{A}(\Omega)$ corresponds to some primitive act $a \in \mathcal{A}$ and is thus considered by the individual to be feasible. Alternatively, the feasibility of Savage acts follows from the requirement of measurability with respect to Savage events which, as explained above, are observable.

A *utility function* u is a function mapping consequences onto real numbers. A *probability measure* over Ω is here defined as a set function $\mathbb{P}$ mapping Savage events onto the unit interval of the reals and satisfying the following requirements:
(1) $\mathbb{P}(\Omega) = 1$,
(2) For all disjoint Savage events $E, F \in \mathcal{L}(\Omega)$, $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F)$,
(3) For all Savage events $E, F \in \mathcal{L}(\Omega)$, if $\mathbb{P}(E^c \cup F) = 1$, then $E \subseteq F$,
(4) For all $\epsilon > 0$, there exists a partition $(E_1, ..., E_n)$ of Ω made of Savage events such that $\mathbb{P}(E_i) < \epsilon$ for all $i \in [1, n]$.
Given that $\mathcal{L}(\Omega)$ is only a finite algebra of subsets, a probability measure is only required to be finitely additive. Condition (3) is a normalization requirement for subjective state

spaces. The equality $\mathbb{P}(E^c \cup F) = 1$ captures in a probabilistic way the idea that the individual thinks that Savage event E implies Savage event F. Then, condition (3) requires all such probabilistic implications to be already encoded in the state space itself. To motivate this requirement, recall the introductory portfolio example. It would certainly be possible to deal with the manager's understanding of uncertainty by assigning a null probability to all inconsistent states. But if such null states were allowed, there would be more than one state space to represent behavior. The normalization condition is precisely meant to forbid null states and leads to uniqueness in the derivation of the state space. Condition (4) finally appears in Kopylov (2007) and is the analogue of Savage's nonatomicity requirement.

Definition 1 A preference relation $\succsim$ is said to have a Savage representation if there exist a state space Ω, an interpretation K, a probability measure $\mathbb{P}$ and a nonconstant utility function u such that, for all $a, b \in \mathcal{A}$, $a \succsim b \iff \mathbb{E}_{\mathbb{P}} u(\varphi(a)) \geq \mathbb{E}_{\mathbb{P}} u(\varphi(b))$.

In a Savage representation, the individual behaves as if he used the state space and interpretation to formalize uncertainty and causality, and maximized SEU over that state space to make decisions. While the state space is exogenous in Savage (1954), its existence is here a property of preferences and behavior. However, Savage representations entail a few technical departures from the Savage version of SEU. The state space is indeed equipped here with a topological structure that induces notions of observable events and feasible acts. Probability is restricted to observable events, and preference is restricted to feasible acts. Moreover, the state space is normalized in the sense of condition (3) in the definition of probability and condition (4) modifies the requirement of nonatomicity.
In addition, Gilboa and Schmeidler (1994), Mukerji (1997) and Lipman (1999) show in different ways that non-SEU preferences over a certain state space can be reformulated as SEU preferences over larger state spaces. Their extended spaces typically lead to probability judgements on unobservable events and preferences on unfeasible acts. Due to observability and feasibility constraints, such SEU reformulations of non-SEU preferences are not possible here. More formally, it would be possible to define a weaker representation of preferences that would only require the existence of a state space Ω, an interpretation K and preferences $\succsim_\Omega$ over Savage acts such that $a \succsim b \iff \varphi(a) \succsim_\Omega \varphi(b)$, for all $a, b \in \mathcal{A}$. The proof of Theorem 1 would then support the following claim: if a weaker representation exists where induced preferences $\succsim_\Omega$ are SEU, then induced preferences must necessarily be SEU in all such weaker representations.

Definition 2 Two Savage representations supported by Ω_1, K_1, $\mathbb{P}_1$, u_1 and Ω_2, K_2, $\mathbb{P}_2$, u_2 are said to be equivalent if there exists a bijective, bimeasurable and bicontinuous mapping $\epsilon : \Omega_2 \longrightarrow \Omega_1$ and real numbers $\alpha \in \mathbb{R}_{++}$ and $\beta \in \mathbb{R}$ such that:
(1) For all $e \in \mathcal{E}$, $K_1(e) = \epsilon(K_2(e))$,
(2) For all $E \in \mathcal{L}(\Omega_1)$, $\mathbb{P}_1(E) = \mathbb{P}_2(\{\epsilon \in E\})$,
(3) For all $a_0 \in \mathcal{A}_0$, $u_1(a_0) = \alpha u_2(a_0) + \beta$.

The notion of equivalence for Savage representations plays an important role in the uniqueness of the subjective state space. Strictly speaking, one cannot hope to obtain uniqueness since it is always possible to relabel states in a given Savage representation and construct a second representation. Two Savage representations are said to be equivalent precisely when they are just a relabeling of each other. More precisely, there is equivalence when there exists a bijective mapping between the two state spaces that preserves both the topology and the algebra of observability. In addition, the interpretation and subjective probability in the one Savage representation must be deducible from their counterparts in the other Savage representation through this bijection, and utility functions must be positive affine transformation of each other.

3 Axioms for Savage representations

3.1 Axioms

This section presents the axioms that are used in the derivation of a Savage representation. They are split into three groups. Axioms in the first group essentially lead to the derivation of subjective probability and utility. They are well known in the literature since they are straightforward reformulations of the axioms of Savage (1954) and Kopylov (2007). In particular, A2 is the direct analogue of the Sure Thing Principle and its specific role is discussed in greater detail in the next section. Note also that the seventh axiom of Savage is not needed at all since Savage acts have necessarily a finite range in the state space to be constructed. In Axiom A3, $\mathcal{E}^*$ stands for the set of nonnull experiments. An experiment $e \in \mathcal{E}$ is said to be null if, for any acts $a, b, c \in \mathcal{A}$, acts $e(a, c)$ and $e(b, c)$ are indifferent.

A1 $\forall e \in \mathcal{E}$, $\forall a, b, c \in \mathcal{A}$, $a \succsim b$ or $b \succsim a$ and if $a \succsim b$ and $b \succsim c$, then $a \succsim c$
A2 $\forall e \in \mathcal{E}$, $\forall a, b, c, d \in \mathcal{A}$, $e(a, c) \succsim e(b, c) \iff e(a, d) \succsim e(b, d)$
A3 $\forall e \in \mathcal{E}^*$, $\forall a_0, b_0 \in \mathcal{A}_0$, $\forall c \in \mathcal{A}$, $a_0 \succsim b_0 \iff e(a_0, c) \succsim e(b_0, c)$
A4 $\forall e, f \in \mathcal{E}$, $\forall a_0, b_0, c_0, d_0 \in \mathcal{A}_0$ with $a_0 \succ b_0$ and $c_0 \succ d_0$, $e(a_0, b_0) \succsim f(a_0, b_0) \iff e(c_0, d_0) \succsim f(c_0, d_0)$
A5 $\exists a_0, b_0 \in \mathcal{A}_0$, $a_0 \succ b_0$
A6 $\forall a, b, c \in \mathcal{A}$ such that $a \succ b$, there is a finite family $(e_i)_{i=1}^n$ of experiments such that:
(1) $\forall i, j \in [1, n]$, $i \neq j \implies \forall a', b' \in \mathcal{A}$, $e_i \circ e_j(a', b') \sim b'$
(2) $\forall a', b' \in \mathcal{A}$, $e'_1 \circ ... \circ e'_n(a', b') \sim b'$
(3) $\forall i \in [1, n]$, $e_i(c, a) \succ b$ and $a \succ e_i(c, b)$

Axioms in the second group regulate the interaction between preferences over acts and the algebraic structure of the set of experiments. All these axioms can be justified through the rules for assigning outcomes to product and complement experiments that are presented in Section 2. For instance, Axiom B1 captures the behavioral content of the interpretation of the product: since a product replies yes if and only if each factor does, detaining a if $e \circ f$ returns yes and b otherwise is always indifferent to detaining $f(a, b)$ if e returns yes and b otherwise. For Axiom B2, since y stands for a trivial experiment that always returns yes, detaining a if y returns yes and b otherwise is expected to be indifferent to a. B2 also captures the same idea for experiment n. Since a complementary experiment returns yes if and only if the underlying experiment returns no, each act $e'(a, b)$ is somehow the same thing as $e(b, a)$ and B3 then requires each act $f(e'(a, b), c)$ to be indifferent to $f(e(b, a), c)$. Axioms B4, B5 and B6 capture a form of idempotence, commutativity and associativity for the product of experiments. Finally, B7 expresses the idea that a product experiment of the form $e \circ e'$ always returns no. In a Savage framework, these axioms are systematically satisfied since the acts that are compared in each of them are in fact equal to each other.

B1 $\forall e, f \in \mathcal{E}$, $\forall a, b \in \mathcal{A}$, $e \circ f(a, b) \sim e(f(a, b), b)$
B2 $\forall a, b \in \mathcal{A}$, $y(a, b) \sim a$ and $n(a, b) \sim b$
B3 $\forall e \in \mathcal{E}$, $\forall a, b, c \in \mathcal{A}$, $f(e'(a, b), c) \sim f(e(b, a), c)$
B4 $\forall e, f, g \in \mathcal{E}$, $\forall a, b \in \mathcal{A}$, $e \circ e(a, b) \sim e(a, b)$
B5 $\forall e, f, g \in \mathcal{E}$, $\forall a, b \in \mathcal{A}$, $e \circ f(a, b) \sim f \circ e(a, b)$
B6 $\forall e, f, g \in \mathcal{E}$, $\forall a, b \in \mathcal{A}$, $e \circ (f \circ g)(a, b) \sim (e \circ f) \circ g(a, b)$
B7 $\forall e \in \mathcal{E}$, $\forall a, b \in \mathcal{A}$, $e \circ e'(a, b) \sim b$

There are specific situations where the previous axioms cannot be expected to be satisfied. Consider, for instance, that the individual designs a state of the world at which the product experiment $e \circ f$ does not return yes and yet each of experiments e and f returns yes. This happens when the individual considers that the outcome of f depends upon whether e is effectively performed, a pattern that is reminiscent of the notion of contextuality in quantum physics. Then, Axiom B1 might fail to hold, since $e \circ f(a, b)$ and $e(f(a, b), b)$ could be considered at this state to yield respectively b and a. Similarly, consider this time an individual designing a state of the world at which none of experiments e and e' returns yes. This happens when the individual perceives unforeseen contingencies in the sense of Dekel, Lipman and Rustichini (2001), refuses to speculate about the possible causes of outcomes and wants his behavior to be consistent with his awareness of his ignorance. Then, Axiom B3 might fail to hold, since acts $f(e'(a, b), c)$ and $f(e(b, a), c)$ could be considered to yield respectively b and a at this state. However, since the experiments involved by these two cases violate the rules of assignment of outcomes to product and complement experiments, they do not really qualify as experiments in the sense of the paper and should not be allowed in the primitive set $\mathcal{E}$ in the first place. In this sense, it is really the fact that preference is restricted to feasible acts that justifies these axioms.

Axioms in the last group regulate the interaction between subjective implication and the algebraic structure of the set of experiments. Subjective implication $\leq$ is derived from primitive preferences $\succsim$ in the following way:

$$\forall e, f \in \mathcal{E},\ e \leq f \text{ if } \forall a, b \in \mathcal{A},\ e(a, b) \sim e \circ f(a, b)$$

Fix two experiments $e, f \in \mathcal{E}$. Whenever, for all acts $a, b \in \mathcal{A}$, the individual is indifferent between detaining a if $e \circ f$ returns yes and b otherwise and detaining a if e returns yes and b otherwise, he is said to behave as if he believed that the event *e returns yes* implies the event *f returns yes*, which is denoted by $e \leq f$. Note the formal analogy between the definition of subjective implication and that of *domination* in Kreps (1979). Subjective equivalence is denoted by $\approx$. Experiments are subjectively equivalent when the individual believes that they measure the same property of the random system.

Some of the subjective implications captured by $\leq$ can be said to be purely logical. Consider two experiments $e, f \in \mathcal{E}$ such that $e = e \circ f$. Then, the individual has to believe that the event *e returns yes* implies the event *f returns yes* in some obvious sense that does not involve preferences at all, but is a direct consequence of the idea that a product returns yes if and only if each factor also does. Observe that, by construction of subjective implication, $e = e \circ f$ necessarily results in $e \leq f$. As a result, the individual is logically omniscient in that his behavior is consistent with the purely logical implications that are part of his own description of uncertainty. To deal with failures of logical omniscience, it would be necessary to construct subjective implication in a different way. For instance, Lipman (1999) uses a primitive family of preferences representing dynamic behavior to derive his *information ordering*. See Lemma 4 for two equivalent definitions of subjective implication which are more directly comparable to this ordering.
Subjective implication is allowed to capture additional and extra-logical implications reflecting the prior information that is available in the decision process. But some consistency is required that is imposed through the remaining axioms. Axiom C1 is a richness condition for the set of experiments that delivers a form of distributivity. Axiom C2 requires complementation to reverse subjective implication. Thinking of subjective implication as the set-theoretic inclusion, it is easy to see that these axioms are again necessarily satisfied in a Savage framework.

C1 $\forall e, f, g \in \mathcal{E}$, if $e \circ f \leq g$, then there exist $h, k \in \mathcal{E}$ such that $e \leq h$, $f \leq k$ and $h \circ k \approx g$
C2 $\forall e, f \in \mathcal{E}$, $e \leq f \implies f' \leq e'$

Example (cont.) Assume that *coin 1 lands on heads* is subjectively equivalent to *coin 2 lands on heads*. By C2, *coin 1 lands on tails* must also be subjectively equivalent to *coin 2 lands on tails*. By B5, detaining a if *coin 1 lands on tails* and b otherwise must then be indifferent to detaining a if *coin 2 lands on tails* and b otherwise.

3.2 Axiomatic characterization of Savage representations

Theorem 1 uses the previous axioms to formulate a characterization of Savage representations. In the normative approach mentioned in the introduction, this result advises an individual who would adhere to the axioms to adopt a Savage representation and thus think of uncertainty, events and acts directly in terms of states of the world. In empirical applications, the theorem shows that these axioms represent simple testable predictions that can be derived from the SEU assumption over an unknown subjective state space. Whenever axioms are satisfied, it is even possible to reconstruct a Savage representation from preferences uniquely up to equivalence. Whenever axioms are violated, there cannot exist a state space over which behavior conforms to the Savage theory and a refutation of SEU is therefore obtained that is robust to misspecification of the state space.

Theorem 1
Preferences $\succsim$ conform to Axioms A1–6, B1–7 and C1–2 if and only if they have a Savage representation. Moreover, any two Savage representations of $\succsim$ are equivalent.

The construction of a Savage representation works in the following way: Lemmas 1 and 2 study the properties of subjective implication and show that the set of classes of subjective equivalence has the structure of a Boolean algebra. Then, the Stone theorem delivers a state space and an interpretation. In addition, Lemmas 3, 4 and 5 show that a certain condition called *consequentialism* is satisfied. Consequentialism requires any two primitive acts that induce the same Savage act through φ to be indifferent. See L5.3 for a more precise formulation. At last, Lemma 6 uses consequentialism to construct preferences $\succsim_\Omega$ over Savage acts and derives a probability measure and a utility function through the Kopylov (2007) version of SEU.

Theorem 1 is consistent with Hammond's (1988) consequentialist foundations of SEU and defense of the Sure Thing Principle (STP). These foundations shed light on the methodological problems in assuming a preference relation that would not satisfy Savage's STP: under a dynamic version of consequentialism, such preferences necessarily produce a form of inconsistency in dynamic behavior. See also Machina (1989), Hanany and Klibanoff (2007) or Al-Najjar and Weinstein (2009). In contrast, the proof of Theorem 1 captures the same intuition in a purely static framework. While A2 plays no role in the derivation of a subjective state space, the derivation of preferences $\succsim_\Omega$ over Savage acts heavily relies upon this axiom: A2 is indeed used to construct a family of preferences representing the dynamic behavior of the individual (Lemma 4) and show that the family satisfies both dynamic consistency (L5.1) and the dynamic version of consequentialism (L5.2) which together end up implying consequentialism (L5.3) and make it possible to construct preferences $\succsim_\Omega$ (L6.1). However, since A2 is the direct analogue of STP, preferences $\succsim_\Omega$ necessarily satisfy STP by construction. Put differently, whenever preferences over Savage acts are allowed to violate STP, then it is not clear why they exist in the first place.

The issue that is raised here might be overcome by an axiomatic characterization of a notion of *epistemic sophistication* that would play a similar role to Machina and Schmeidler's (1992) notion of *probabilistic sophistication*. The latter justifies the existence of preferences over lotteries that do not necessarily conform to the von Neumann and Morgenstern (1944) axioms expected utility under risk. Similarly, epistemic sophistication would require a state space Ω, an interpretation K and a non-necessarily SEU preference relation $\succsim_\Omega$ over Savage acts such that, for all $a, b \in \mathcal{A}$, $a \succsim b \iff \varphi(a) \succsim_\Omega \varphi(b)$. After all, the fact that the present construction fails to produce ambiguity sensitive preferences over the subjective state space does not mean that other constructions would also fail.

Alternatively, extensions of Savage representations beyond the domains of observability and feasibility might provide a different way to allow for ambiguity. Indeed, a Savage representation ends up producing restrictions on the likelihood plausibly assigned to unobservable events. The likelihood assigned to such an event has to remain below (resp. above) the subjective probability of any observable event that includes it (resp. it contains). These restrictions may fail to be strong enough to ensure additivity over the whole domain of unobservability. As a result, ambiguity and nonadditive probability might emerge as a way to describe subjective beliefs over unobservable events and behavior over unfeasible acts.

4 Appendix

4.1 Boolean algebras

A partially-ordered set (*poset*) is a set $\mathcal{L}$ together with a binary relation $\leq$ on $\mathcal{L}$ that is reflexive, antisymmetric and transitive. A lower semilattice is a poset that has a greatest lower bound (*meet*) for any nonempty finite subset. An upper semilattice is a poset that has a lowest upper bound (*join*) for any nonempty finite subset. A lattice is a poset that has both a meet and a join. The meet and join of two elements $x, y \in \mathcal{L}$ are denoted respectively by $x \wedge y$ and $x \vee y$.

A lower semilattice $\mathcal{L}$ is distributive if, for all $x, y, z \in \mathcal{L}$ such that $x \wedge y \leq z$, there exist $x', y' \in \mathcal{L}$ such that $x \leq x'$, $y \leq y'$ and $x' \wedge y' = z$. A lattice $\mathcal{L}$ is distributive if, for all $x, y, z \in \mathcal{L}$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. Any lattice is distributive as a semilattice if and only if it is distributive as a lattice.

A lattice isomorphism is a bijective application between two lattices that preserves the order as well as joins and meets. A lattice that is isomorphic to a distributive lattice is distributive itself.

A bounded lattice is a term $(\mathcal{L}, \leq, 0, 1)$ such that 0 (resp. 1) is the necessarily unique lowest (resp. greatest) element and is the identity element of the join (resp. the meet). A complemented lattice $\mathcal{L}$ is a bounded lattice for which, for each $x \in \mathcal{L}$, there is $x' \in \mathcal{L}$ such that $x \vee x' = 1$ and $x \wedge x' = 0$.

A Boolean algebra is defined as complemented and distributive lattice. A Boolean algebra isomorphism is a bijective application between two Boolean algebras that preserves the order as well as joins, meets and complements.

Typical examples of Boolean algebras are given by topological Boolean spaces, that is, Hausdorff, compact and zero-dimensional topological spaces. A topological space Ω is said to be Hausdorff if, for any distinct $\omega, \omega' \in \Omega$, there exist disjoint open sets $E, E' \subseteq \Omega$ such that $\omega \in E$ and $\omega' \in E'$. It is said to be compact if every open cover of Ω has a finite subcover. It is said to be zero-dimensional if every open set is the union of the clopen sets (sets that are both open and closed) it contains. Then, the set $\mathcal{L}(\Omega)$ of clopen subsets of Ω has the structure of a Boolean algebra for the standard set-theoretic operations.

Consider a Boolean algebra $\mathcal{L}$ and define its Stone space Ω as the set of all homomorphisms from $\mathcal{L}$ onto $\{0, 1\}$. Such an homomorphism is a mapping $\omega : \mathcal{L} \longrightarrow \{0, 1\}$ that preserves the Boolean operations. The Stone space Ω can be equipped with a topology constructed as the topology induced by Ω given the product topology on $\{0, 1\}^{\mathcal{L}}$ obtained from the discrete topology on each copy of $\{0, 1\}$. Let $\mathcal{L}(\Omega)$ be the finite algebra of its clopen subsets and define $K : \mathcal{L} \longrightarrow \mathcal{L}(\Omega)$ by $K(x) = \{\omega \in \Omega,\ \omega(x) = 1\}$, for all $x \in \mathcal{L}$. The Stone representation theorem shows that Ω is Hausdorff, compact and zero-dimensional and proves that K is a Boolean algebra isomorphism. As a result, all Boolean algebras can be represented through topological Boolean spaces. At last, observe that clopen subsets in Ω are exactly the sets of the form $\{\beta \in \Omega,\ \beta(x) = 1\}$, for a given (and necessarily unique) $x \in \mathcal{L}$.

4.2 Proof of Theorem 1

From now on, it is assumed that $\succsim$ conforms to axioms A1-6, B1-7 and C1-2. Recall that $\leq$ and $\approx$ denote subjective implication and equivalence respectively. Lemma 1 studies some properties of subjective implication.

Lemma 1
(L1.1) $\forall e \in \mathcal{E}$, $\forall a \in \mathcal{A}$, $e(a,a) \sim a$
(L1.2) $\forall e, f \in \mathcal{E}$, if $e \approx f$, then, $\forall a, b \in \mathcal{A}$, $e(a,b) \sim f(a,b)$
(L1.3) $\forall e, f, g \in \mathcal{E}$, $e \circ e \approx e$, $e \circ f \approx f \circ e$ and $e \circ (f \circ g) \approx (e \circ f) \circ g$
(L1.4) $\forall e \in \mathcal{E}$, $e'' \approx e$ and $e \circ e' \approx n$
(L1.5) $\forall e \in \mathcal{E}$, $e \circ y \approx e$, $e \circ n \approx n$ and $y' \approx n$
(L1.6) $\leq$ is reflexive and transitive
(L1.7) $\forall e, f \in \mathcal{E}$, $e \circ f \leq e$
(L1.8) $\forall e, f, g, h \in \mathcal{E}$, if $e \leq f$ and $g \leq h$, then $e \circ g \leq f \circ h$

Proof (*L1.1*) Consider an act $a \in \mathcal{A}$. By B7, $a \sim e \circ e'(a,a)$. By A1 and B1, $a \sim e(e'(a,a),a)$. By A1 and B3, $a \sim e(e(a,a),a)$. By A1 and B1, $a \sim e \circ e(a,a)$. Finally, by A1 and B4, $a \sim e(a,a)$. (*L1.2*) This is a direct consequence of B5. (*L1.3*) It is simple but fastidious to show L1.3, L1.4 and L1.5. For instance, consider an experiment $e \in \mathcal{E}$ and two acts $a, b \in \mathcal{A}$. Then, by B1, $(e \circ e) \circ e(a,b) \sim e \circ e(e(a,b),b)$ and, by A1 and B4, $(e \circ e) \circ e(a,b) \sim e(e(a,b),b)$. By A1 and B1, $(e \circ e) \circ e(a,b) \sim e \circ e(a,b)$, which shows $e \circ e \leq e$. Furthermore, by A1, B4 and B6, $e \circ (e \circ e)(a,b) \sim e(a,b)$, which shows $e \leq e \circ e$. Use A1, B1, B4, B5 and B6 to show commutativity and associativity in L1.3. (*L1.4*) Consider an experiment $e \in \mathcal{E}$ and two acts $a, b \in \mathcal{A}$. Then, by B1, $e \circ e''(a,b) \sim e(e''(a,b),b)$. By A1 and B3, $e \circ e''(a,b) \sim e(e(a,b),b)$. By A1, B1 and B4, $e \circ e''(a,b) \sim e(a,b)$, which shows $e \leq e''$. In addition, by B5, $e'' \circ e(a,b) \sim e \circ e''(a,b) \sim e(a,b)$. By A1, B2 and B3, $e'' \circ e(a,b) \sim e''(a,b)$, which shows $e'' \leq e$. For the second part of L1.4, to show that $n \leq e \circ e'$, it is sufficient to show that $n \leq e$, for any $e \in \mathcal{E}$. Fix an experiment $e \in \mathcal{E}$ and two acts $a, b \in \mathcal{A}$. Then, by B1, $n \circ e(a,b) \sim n(e(a,b),b)$. By A1 and B2, $n \circ e(a,b) \sim b \sim n(a,b)$, which shows $n \leq e$ for any $e \in \mathcal{E}$ and, therefore, $n \leq e \circ e'$. Moreover, proceeding as above and using B5, $(e \circ e') \circ n(a,b) \sim b$. By A1 and B7, $(e \circ e') \circ n(a,b) \sim e \circ e'(a,b)$, which shows $e \circ e' \leq n$ and thus delivers L1.4. (*L1.5*) Let us prove $e \circ y \approx e$ and $y' \approx n$ only. Fix two acts $a, b \in \mathcal{A}$. By A1, B1, B4 and B6, $e \circ (e \circ y)(a,b) \sim e \circ e(y(a,b),b) \sim e(y(a,b),b) \sim e \circ y(a,b)$. Then, by A1, B1, B2 and B5, $e \circ (e \circ y)(a,b) \sim y \circ e(a,b) \sim y(e(a,b),b) \sim e(a,b)$, which shows $e \leq e \circ y$. At last, use A1 and B5 to obtain $(e \circ y) \circ e(a,b) \sim e \circ y(a,b)$ and $e \circ y \leq e$. Proceed similarly for $e \circ n \approx n$. Moreover, it has already be shown that n subjectively implies any $e \in \mathcal{E}$ so that $n \leq y'$. Finally, since $y' \circ n \approx n$, and by L1.2, $y' \circ n(a,b) \sim n(a,b) \sim b$ for all $a, b \in \mathcal{A}$. But, using A1, B2, B3, $y'(a,b) \sim y(b,a) \sim b$. Therefore, $y' \circ n(a,b) \sim y'(a,b)$ and $y' \leq n$. (*L1.6*) Reflexivity is a direct consequence of B4. For transitivity, consider experiments $e, f, g \in \mathcal{E}$ such that $e \leq f$ and $f \leq g$. Then, for all $a, b \in \mathcal{A}$, $e(a,b) \sim e \circ f(a,b)$. In particular, for all $a, b \in \mathcal{A}$ and all $h \in \mathcal{E}$, $e(h(a,b),b) \sim e \circ f(h(a,b),b)$ and, by A1 and B1, $e \circ h(a,b) \sim (e \circ f) \circ h(a,b)$ ($\star$). Use A1, B1, B5 and B6 to show that $(e \circ f) \circ h(a,b)$ and $f \circ (e \circ h)(a,b)$ are always indifferent so that, by $(\star)$, $e \circ h(a,b) \sim f \circ (e \circ h)(a,b)$. In addition, the fact that $f \leq g$ implies, proceeding as above, that , for all $a, b \in \mathcal{A}$ and all $h \in \mathcal{E}$, $f \circ h(a,b) \sim (f \circ g) \circ h(a,b)$. In particular, one obtains $e \circ h(a,b) \sim f \circ (e \circ h)(a,b) \sim (f \circ g) \circ (e \circ h)(a,b)$. Still by A1, B1, B5 and B6, $(f \circ g) \circ (e \circ h)(a,b)$ and $(e \circ f) \circ (g \circ h)(a,b)$ can easily be shown to be indifferent so that $e \circ h(a,b) \sim (e \circ f) \circ (g \circ h)(a,b)$. Replace h with $g \circ h$ in $(\star)$ to obtain $e \circ h(a,b) \sim e \circ (g \circ h)(a,b)$ and, by B6, $e \circ h(a,b) \sim (e \circ g) \circ h(a,b)$. Finally, take $h = y$ and apply A1, L1.2 and L1.5 to obtain $e(a,b) \sim e \circ g(a,b)$ and conclude that $e \leq g$. (*L1.7*) It is sufficient to note that, by A1, B1, B4, B5 and B6, $(e \circ f) \circ e(a,b)$ and $e \circ f(a,b)$ are always indifferent. (*L1.8*) Proceeding as for transitivity, one obtains $e \circ (g \circ k)(a,b) \sim (e \circ f) \circ (g \circ k)(a,b)$ and $g \circ ((e \circ f) \circ k)(a,b) \sim (g \circ h) \circ ((e \circ f) \circ k)(a,b)$, for all $a, b \in \mathcal{A}$ and $k \in \mathcal{E}$. Given B4, B5 and B6, it is possible to show $e \circ (g \circ k)(a,b) \sim (e \circ g) \circ k(a,b)$, $(e \circ f) \circ (g \circ k)(a,b) \sim g \circ ((e \circ f) \circ k)(a,b)$ and $(g \circ h) \circ ((e \circ f) \circ k)(a,b) \sim ((e \circ g) \circ (f \circ h)) \circ k(a,b)$. Combining these various indifferences, one obtains $(e \circ g) \circ k(a,b) \sim ((e \circ g) \circ (f \circ h)) \circ k(a,b)$. Apply this to $k = y$ and use A1, L1.2 and L1.5 to obtain $e \circ g \leq f \circ h$.

Let $\mathcal{L}$ denote the set of classes of the equivalence relation $\approx$ on $\mathcal{E}$. The class of an experiment e is denoted $[e]$. For $x, y \in \mathcal{L}$, define $x \leq y$ if $e \leq f$ with $e \in x$ and $f \in y$ and let $x \wedge y = [e \circ f]$, $x' = [e']$ and $x \vee y = (x' \wedge y')'$ where $e \in x$ and $f \in y$. Lemma 1 together with C2 make these definitions meaningful. In addition, let 0 and 1 denote the classes of y and n respectively. In the next lemma, L2.1 shows that $\mathcal{L}$ is a Boolean algebra. In L2.2, Ω stands for the Stone space induced by $\mathcal{L}$. At last, let $K(x) = \{\omega \in \Omega, \omega(x) = 1\}$ and $K(e) = K([e])$, for all $e \in \mathcal{E}$ and $x \in \mathcal{L}$.

Lemma 2
(L2.1) $(\mathcal{L}, \leq, \wedge, \vee, 0, 1, \prime)$ is a Boolean algebra
(L2.2) Ω is a state space
(L2.3) $K : \mathcal{L} \longrightarrow \mathcal{L}(\Omega)$ is an isomorphism of Boolean algebras.
(L2.4) $K : \mathcal{E} \longrightarrow \mathcal{L}(\Omega)$ is an interpretation over Ω.

Proof (*L2.1*) By L1.6, $\mathcal{L}$ is a poset. Then, note that, by L1.3, $\wedge$ is necessarily idempotent, commutative and associative. In addition, by L1.7 and L1.8, each $x \wedge y$ is the greatest lower bound of x and y. Also note that L1.4, L1.5 and Axiom C2 directly imply $0' = 1$, $x'' = x$, $x \wedge x' = 0$, $x \vee x' = 1$ and $x \leq y \iff y' \leq x'$, for all $x, y \in \mathcal{L}$. These remarks imply that each $x \vee y$ is the lowest upper bound of x and y and, therefore, that $\mathcal{L}$ is a lattice. In addition, Axiom C1 directly implies that $\mathcal{L}$ is distributive as a semilattice and, therefore, distributive as a lattice as well. Since n subjectively implies each experiment e (see proof of Lemma 1), 0 is indeed the lowest element and, similarly, 1 is the greatest element. The lattice $\mathcal{L}$ is therefore bounded and previous results show that it is even a Boolean algebra. (*L2.2*) By definition, Ω is a topological Boolean space and, due to A5, it is nonempty. (*L2.3*) The fact that $K : \mathcal{L} \longrightarrow \mathcal{L}(\Omega)$ is an isomorphism simply reformulates the Stone representation theorem for Boolean algebras. (*L2.4*) Finally, $K : \mathcal{E} \longrightarrow \mathcal{L}(\Omega)$ is an interpretation since, by L2.3, it is surjective and preserves products, complements and trivial experiments.

Lemma 3

(L3.1) $\forall e, f \in \mathcal{E}$ such that $e \circ f \approx n$, $e(a,b) \sim f(b, e(a,b))$ $\forall a, b \in \mathcal{A}$
(L3.2) $e_n(a, e_{n-1}(a, ..., e_2(a, e_1(a,b))...)) \sim e(a,b)$, for any acts $a, b \in \mathcal{A}$, any finite sequence $(e_i)_{i=1}^n \in \mathcal{E}^n$ of experiments and any experiment $e \in \bigvee\{[e_i],\ i = 1...n\}$.

Proof *(L3.1)* Let $x = [e] \in \mathcal{L}$ and $y = [f] \in \mathcal{L}$. Then, $x \wedge y = 0$, $x \leq y'$ and $e \leq f'$. For any acts $a, b \in \mathcal{A}$, one has $e(a,b) \sim e \circ f'(a,b)$ and, by A1, B1 and B5, $e(a,b) \sim f'(e(a,b), b)$. Finally, A1, B2 and B3 imply $e(a,b) \sim f(b, e(a,b))$. *(L3.2)* Let us first prove the result in case $n = 2$. Note that $e_2 \leq e$ so that in particular $e_2(a, e_1(a,b)) \sim e_2 \circ e(a, e_1(a,b))$ for all $a, b \in \mathcal{A}$. Using A1, B1 and B3, one obtains $e_2(a, e_1(a,b)) \sim h(a,b)$ where $h = ((e_2 \circ e)' \circ e_1')'$. By L2.3, it is clear that $K([h]) = K([e])$ so that $h \approx e$ and, by L1.2, $h(a,b) \sim e(a,b)$. Hence the result. Let us now prove the result at rank $n+1$. Fix $a, b \in \mathcal{A}$. Denote $c = e_{n-1}(a, ..., e_2(a, e_1(a,b))...)$ and $d = e_{n-2}(a, ..., e_2(a, e_1(a,b))...)$ so that $c = e_{n-1}(a, d)$. Let $f \in [e_1] \vee ... \vee [e_{n-1}]$. In addition, fix $h_{n+1} \in [e_{n+1}] \vee [e_n]$ and $h_n \in [h_{n+1}] \vee [f]$. Thanks to the case $n = 2$, one has: $e_{n+1}(a, e_n(a,c)) \sim h_{n+1}(a,c)$. By induction hypothesis: $h_{n+1}(a,c) = h_{n+1}(a, e_{n-1}(a,d)) \sim h_n(a,b)$. By L2.3, $K([h_n]) = K([h_{n+1}] \vee [f]) = K([e])$ so that, by L1.2, $h_n(a,b) \sim e(a,b)$. Finally, $e_{n+1}(a, e_n(a,c)) \sim e(a,b)$, which shows the result.

For any $e \in \mathcal{E}$ and $a, b \in \mathcal{A}$, define $a \succsim_e b$ if there exists $c \in \mathcal{A}$ such that $e(a,c) \succsim e(b,c)$. This definition is meaningful thanks to A2.

Lemma 4
(L4.1) For each $e \in \mathcal{E}$, $\succsim_e$ is complete and transitive
(L4.2) $\forall e, f \in \mathcal{E}$, $e \leq f \iff \forall a, b \in \mathcal{A}$, $f(a,b) \sim_e a$
(L4.3) $\forall e, f \in \mathcal{E}$, if $e \approx f$, then $\forall a, b \in \mathcal{A}$, $a \succsim_e b \iff a \succsim_f b$
(L4.4) $\forall e, f \in \mathcal{E}$, $e \leq f \iff (\forall a, b \in \mathcal{A},\ a \succsim_e b \iff a \succsim_{e \circ f} b)$

Proof *(L4.1)* Completeness and transitivity follow from A1 and A2. *(L4.2)* If $e \leq f$, then $e(a,b) \sim e \circ f(a,b)$, for all $a, b \in \mathcal{A}$. By A1 and B1, $e(a,b) \sim e(f(a,b), b)$ and $a \sim_e f(a,b)$. This argument can be reversed thanks to A2. *(L4.3)* Since $e \approx f$, L1.2 implies $e(a,b) \sim f(a,b)$, for all a, b and, therefore, thanks to A1, if $e(a,c) \succsim e(b,c)$ for some c, then $f(a,c) \succsim f(b,c)$ for this c and reciprocally. *(L4.4)* Assume first that $e \leq f$ and $a \succsim_e b$. By L4.2, for any $c \in \mathcal{A}$, $f(a,c) \succsim_e f(b,c)$. By A2 and B1, $e \circ f(a,c) \succsim e \circ f(b,c)$ and, therefore, $a \succsim_{e \circ f} b$. Note that this argument can be reversed. Assume the characterizing condition. First, by A1, B1, B4, B5 and B6, for all $a, b \in \mathcal{A}$, $(e \circ f) \circ f(a,b) \sim e \circ f(a,b)$. By A1 and B1, $f(a,b) \sim_{e \circ f} a$ and, by the characterizing condition, $f(a,b) \sim_e a$. L4.4 then follows from L4.2.

Given the interpretation K obtained in Lemma 2, define $\varphi : \mathcal{A} \longrightarrow \mathcal{A}(\Omega)$ as in Definition 1. For any Savage event $E \in \mathcal{L}(\Omega)$ and acts $a, b \in \mathcal{A}$, define $a \succsim_E b$ if there exists $e \in \mathcal{E}$ such that $K(e) = E$ and $a \succsim_e b$. L4.3 makes this definition meaningful. Note that, for all acts $a, b \in \mathcal{A}$, $a \succsim_\Omega b$ if and only if $a \succsim b$.

Lemma 5
(L5.1) $\forall a, b \in \mathcal{A}$, for all measurable event $E \in \mathcal{L}(\Omega)$ and for all measurable partition $(E_1, ..., E_n)$ of E, if $\forall i \in [1, n]$, $a \succsim_{E_i} b$, then $a \succsim_E b$
(L5.2) $\forall a, b \in \mathcal{A}$, $\forall E \in \mathcal{L}(\Omega)$, $\varphi(a) = \varphi(b)$ on $E \Longrightarrow a \sim_E b$

(L5.3) $\forall a, b \in \mathcal{A}$, $\varphi(a) = \varphi(b) \Longrightarrow a \sim b$

Proof (*L5.1*) Define a family $(e_i)_{i=1}^n$ of experiments such that $K(e_i) = E_i$ for all $i \in [1, n]$ and let $e \in \mathcal{E}$ be such that $K(e) = E$. Necessarily, $i \neq j$ implies $e_i \circ e_j \approx n$ and $e \in \bigvee\{[e_i], i = 1...n\}$. L3.1 can be iteratively applied in the following way: $a \sim e_1(a, a) \succsim e_1(b, a) \sim e_2(a, e_1(b, a))$ thanks to A1, A2, L1.1 and L3.1 and assumption $a \succsim_{E_1} b$. Since $a \succsim_{E_2} b$, A2 and L3.1 further deliver $a \succsim e_2(b, e_1(b, a)) \sim e_3(e_1(b, a), e_2(b, e_1(b, a)))$. But note that $e_3 \leq e_1'$ so that, by L4.2, $e_1'(a, b) \sim_{e_3} a$. Therefore, by A2 and B3, $e_3(e_1(b, a), e_2(b, e_1(b, a))) \sim e_3(a, e_2(b, e_1(b, a))) \succsim e_3(b, e_2(b, e_1(b, a)))$. Finally, $a \succsim e_3(b, e_2(b, e_1(b, a)))$. By repeating this process, $a \succsim e_n(b, e_{n-1}(b, ..., e_2(b, e_1(b, a))...))$ and, by L3.2, $a \succsim e(b, a)$. By A1 and L1.1, $e(a, a) \sim e(b, a)$ so that $a \succsim_e b$ and $a \succsim_E b$. (*L5.2*) An induction on the set of acts delivers L5.2. Indeed, if both a and b are of order 0, then $a = \phi(a)$ and $\varphi(b) = b$ and, therefore, $a = b$ and $a \sim_E b$. If a and b are of order less than $n+1$, then $a = e(a', a'')$ and $b = f(b', b'')$ with a', a'', b', b'' of order less than n. Let $F_1 = E \cap K(e) \cap K(f)$. Clearly, F_1 is a Savage event and therefore of the form $F_1 = K(g)$, for some $g \in \mathcal{E}$. One necessarily has $g \leq e$ and $g \leq f$ so that, by L4.2, $a = e(a', a'') \sim_g a'$ and $b = f(b', b'') \sim_g b'$. Therefore, $a \sim_{F_1} a'$ and $b \sim_{F_1} b'$. Moreover, $\varphi(a') = \varphi(a) = \varphi(b) = \varphi(b')$ on F_1. By induction, $a' \sim_{F_1} b'$. By L4.1, $a \sim_{F_1} b$. Similarly, $a \sim_{F_2} b$, $a \sim_{F_3} b$ and $a \sim_{F_4} b$ with $F_2 = E \cap K(e') \cap K(f)$, $F_3 = E \cap K(e) \cap K(f')$ and $F_4 = E \cap K(e') \cap K(f')$. Finally, by L5.1, $a \sim_E b$. (*L5.3*) follows from L5.1 and L5.2.

Lemma 6
(L6.1) There exists preferences $\succsim_\Omega$ over $\mathcal{A}(\Omega)$ such that $a \succsim b \Longleftrightarrow \varphi(a) \succsim_\Omega \varphi(b)$, $\forall a, b \in \mathcal{A}$
(L6.2) $\succsim_\Omega$ satisfy all of the axioms of Kopylov (2007)

Proof (*L6.1*) Consider the preference relation $\succsim_\Omega$ over $\mathcal{A}(\Omega)$ defined for all Savage acts $f, g \in \mathcal{A}(\Omega)$ by $f \succsim_\Omega g$ if $a \succsim b$ for $a, b \in \mathcal{A}$ such that $\varphi(a) = f$ and $\varphi(b) = g$. L5.3 makes this meaningful. (*L6.2*) Axioms A1-6 imply that the Kopylov (2007) axioms all hold.

Consider that $\succsim$ has a Savage representation supported by Ω, K, u and $\mathbb{P}$ and let $\succsim_\Omega$ denote the SEU preference relation over $\mathcal{A}(\Omega)$ induced by u and $\mathbb{P}$. Say that two experiments $e, f \in \mathcal{E}$ are equivalent if $K(e) = K(f)$. Let $\mathcal{L}(K)$ denote the set of classes of equivalence. The class of a experiment e is denoted $[e]$. The interpretation K induces a mapping still denoted $K : \mathcal{L}(K) \longrightarrow 2^\Omega$ defined for all $x \in \mathcal{L}(K)$ by $K(x) = K(e)$ if $e \in x$. At last, $\mathcal{L}(K)$ has a natural order $\leq$ defined as follows: $\forall x, y \in \mathcal{L}(K)$, $x \leq y$ if $K(x) \subseteq K(y)$. The use of $\leq$ to denote both the latter partial order and subjective implication should not result in any confusion since L7.1 shows that these two notions coincide.

Lemma 7
(L7.1) $\forall e, f \in \mathcal{E}$, $K(e) \subseteq K(f) \Longleftrightarrow \forall a, b \in \mathcal{A}$, $e(a, b) \sim e \circ f(a, b)$
(L7.2) $\succsim$ necessarily satisfies A1 through A6, B1 through B4 and C1 through C5
(L7.3) $\mathcal{L}(K)$ has the structure of a Boolean algebra
(L7.4) $K : \mathcal{L}(K) \longrightarrow \mathcal{L}(\Omega)$ is an isomorphism of Boolean algebras

Proof (*L7.1*) If $K(e) \subseteq K(f)$, then, for all $a, b \in \mathcal{A}$, acts $e(a, b)$ and $e \circ f(a, b)$ induce the same Savage act through φ and necessarily $e(a, b) \sim e \circ f(a, b)$. If the latter condition holds for all $a, b \in \mathcal{A}$, let $E = K(e)$ and $F = K(f)$ so that $E \cap F = K(e \circ f)$ and fix $g, h \in \mathcal{A}(\Omega)$. Since φ is surjective by construction, let a and b be acts such that $g = \varphi(a)$ and $h = \varphi(b)$. By assumption, $g_E h \sim_\Omega g_{E\cap F} h$. Since u is nonconstant, this implies $\mathbb{P}(E \cap F) = \mathbb{P}(E)$. By condition (3) in the definition of a probability, one has $E \subseteq F$ and finally $K(e) \subseteq K(f)$. (*L7.2*) As a preliminary result, let us show that $\mathcal{L}(K)$ is a distributive lattice. For any $x, y \in \mathcal{L}(K)$, let $x \wedge y = [e \circ f]$, $x' = [e']$ and $x \vee y = (x' \wedge y')'$ where $e \in x$ and $f \in y$. Parts (2) and (3) of the definition of an interpretation make these definitions correct. In addition, let 0 and 1 denote the classes of n and y respectively. Since K preserves products and thus meets as well, each $x \wedge y$ is the greatest lower bound of x and y. Since K also preserves complements, it preserves joins as well and, therefore, each $x \vee y$ is the lowest upper bound of x and y. As a result, $\mathcal{L}(K)$ is a lattice and, since it is isomorphic, as a lattice, to the distributive lattice $\mathcal{L}(\Omega)$, it is distributive itself. To show L7.2, first note that preferences $\succsim_\Omega$ necessarily satisfy the condition that appears in L6.1. Then, the Kopylov axioms are all implied for $\succsim_\Omega$ and A1-6 directly follow from Kopylov's theorem 3.1. For B1-7, it is sufficient to note that the acts that are compared in these axioms induce pairwise the same Savage acts through φ. These axioms are therefore satisfied. To show Axiom C1, let $h \in [e] \vee [g]$ and $k \in [f] \vee [g]$. Then, clearly, $e \leq h$ and $f \leq k$. In addition, using distributivity, $[h] \wedge [k] = ([e] \vee [g]) \wedge ([f] \vee [g]) = ([e] \wedge [f]) \vee [g] = [g]$ so that $h \circ k \approx g$. To show Axiom C2, assume $e \leq f$. Then, by L7.1, $K(e) \subseteq K(f)$ and, therefore, $K(f') = K(f)^c \subseteq K(e)^c = K(e')$ so that, again by L7.1, $f' \leq e'$. (*L7.3*) There only remains to show that the distributive lattice $\mathcal{L}(K)$ is complemented. Since $K(n) = \emptyset$ and $K(y) = \Omega$, $K(n) \subseteq K(e) \subseteq K(y)$, for all $e \in \mathcal{E}$, and, therefore, $0 \leq x \leq 1$, for all $x \in \mathcal{L}(K)$. Then, 0 and 1 are indeed the lowest and greatest bound in the lattice. Note that $0' = 1$ and, for all $x \in \mathcal{L}(K)$, $x'' = x$, $x \wedge 1 = [e \circ y] = [e] = x$ and $x \vee 0 = (x' \wedge 0')' = (x' \wedge 1)' = x'' = x$ all follow from the definition of an interpretation and end up making $\mathcal{L}(K)$ a Boolean algebra. (*L7.4*) At last, K is bijective by construction and, in fact, a Boolean algebra isomorphism since it preserves the Boolean operations.

Proof of Theorem 1 (*Necessity of the axioms*) The necessity of axioms has been proved in L7.2. (*Sufficiency of the axioms*) Sufficiency uses Lemma 6. Since preferences $\succsim_\Omega$ conform to the Kopylov axioms, it follows from his Theorem 3.1 that there exists a nonconstant utility function u and a probability measure $\mathbb{P}$ over $(\Omega, \mathcal{L}(\Omega))$ such that $f \succsim_\Omega g \iff \mathbb{E}_\mathbb{P} u(f) \geq \mathbb{E}_\mathbb{P} u(g)$. To show condition (3) in the definition of a probability measure, let $E, F \subseteq \Omega$ be Savage events such that $E \subseteq F$ and $\mathbb{P}(F) = \mathbb{P}(E)$. With this notation, it is sufficient to show that $E = F$. Let $e, f \in \mathcal{E}$ be experiments such that $E = K(e)$ and $F = K(f)$. Since $E \subseteq F$, $e(a, b)$ and $e \circ f(a, b)$ induce the same Savage act through φ and, by L5.3, one has $e(a, b) \sim e \circ f(a, b)$, for all $a, b \in \mathcal{A}$. On the other hand, $f_E g$ and $f_F g$ are equal over an observable event of probability one so that $f_E g \sim_\Omega f_F g$ for all $f, g \in \mathcal{A}(\Omega)$ and, therefore, $e(a, b) \sim f(a, b)$, for all $a, b \in \mathcal{A}$. By A1 and B5, one has $f(a, b) \sim f \circ e(a, b)$, for all $a, b \in \mathcal{A}$ and, therefore, $f \leq e$ so that finally $e \approx f$ and $E = K(e) = K(f) = F$. Condition (4) is delivered by Kopylov's Theorem 3.1. At last, one has $a \succsim b \iff \varphi(a) \succsim_\Omega \varphi(b) \iff \mathbb{E}_\mathbb{P} u(\varphi(a)) \geq \mathbb{E}_\mathbb{P} u(\varphi(b))$, which ends the proof for the sufficiency of axioms. (*Uniqueness up to equivalence*) Assume first a Savage representation supported by Ω, K, u and $\mathbb{P}$. As shown in L7.3, $\mathcal{L}(K)$ is a Boolean algebra. Let Ω_0 be the associated Stone space. Consider, for each $\omega \in \Omega$, the mapping $\alpha_\omega : \mathcal{L}(K) \longrightarrow \{0,1\}$ defined by $\alpha_\omega(x) = 1$ iff $\omega \in K(x)$. The properties of an interpretation make sure that α_ω is an element of Ω_0. Let us now show that $\alpha : \Omega \longrightarrow \Omega_0$ is a bijective, bimeasurable and bicontinuous. Consider two states $\omega, \omega' \in \Omega$ such that $\omega \neq \omega'$. Since Ω is Hausdorff, there exists an open set $E \subseteq \Omega$ such that $\omega \in E$ and $\omega' \notin E$. Since Ω is zero-dimensional, E is the union of some clopen subsets. As a result, there is a clopen subset F that contains ω and not ω'. Then, let e be a experiment such that $K(e) = F$. One has $\alpha_\omega([e]) = 1$ and $\alpha_{\omega'}([e]) = 0$ so that α is indeed injective. For surjectivity, fix an element $\beta \in \Omega_0$ and consider the set of clopen subsets $\{K(x) \in \mathcal{L}(\Omega)$, for all $x \in \mathcal{L}(K)$ such that $\beta(x) = 1\}$. Any finite and nonempty subset of this set set has a nonzero lower bound. Given that Ω is assumed compact, the intersection of all its elements cannot be empty. Let ω denote an element in this intersection. If $\beta(x) = 1$, then $\omega \in K(x)$ and $\alpha_\omega(x) = 1$. If $\beta(x) = 0$, then $\beta(x') = 1$, $\alpha_\omega(x') = 1$ and $\alpha_\omega(x) = 0$. Therefore, $\beta = \alpha_\omega$. Let us now show that α is bimeasurable and bicontinuous. Fix a clopen subset E_0 of Ω_0. Necessarily, there exists a class $x \in \mathcal{L}(K)$ such that $E_0 = \{\beta \in \Omega_0,\ \beta(x) = 1\}$. Then, $\alpha^{-1}(E_0) = \{\omega \in \Omega,\ \alpha_\omega(x) = 1\} = K(x)$ is a clopen subset of Ω. In addition, let E be a clopen subset of Ω. Then, there exists a experiment e such that $K(e) = E$ and, therefore, a class $x \in \mathcal{L}(K)$ such that $K(x) = E$. Then, $(\alpha^{-1})^{-1}(E) = \{\alpha_\omega \in \Omega_0$, for all $\omega \in \Omega$ such that $\alpha_\omega(x) = 1\} = \{\beta \in \Omega_0,\ \beta(x) = 1\}$ is a clopen subset of Ω_0. Hence bimeasurability. Zero-dimensionality implies continuity and bicontinuity follows since Ω is compact and Ω_0 is Hausdorff. Assume now two Savage representations supported by Ω_1, K_1, u_1, $\mathbb{P}_1$ and Ω_2, K_2, u_2, $\mathbb{P}_2$ As a corollary of L7.1, two different Savage representations of the same preference relation necessarily induce the same Boolean algebra denoted $\mathcal{L}$. Let $\alpha^1 : \Omega_1 \longrightarrow \Omega_0$ and $\alpha^2 : \Omega_2 \longrightarrow \Omega_0$ be the homeomorphism obtained as above. Let $\epsilon : \Omega_2 \longrightarrow \Omega_1$ be the homeomorphism defined by $\alpha^1 \circ \epsilon = \alpha^2$. For all $\omega_2 \in \Omega_2$ and all $x \in \mathcal{L}$, $\omega_2 \in K_2(x) \iff \alpha^2_{\omega_2}(x) = 1 \iff \alpha^1_{\epsilon(\omega_2)}(x) = 1 \iff \epsilon(\omega_2) \in K_1(x)$. Finally, $K_1(e) = \epsilon(K_2(e))$, for all $e \in \mathcal{E}$. Let us now prove that $\varphi_2(a) = \varphi_1(a) \circ \epsilon$ by induction on the order of a. If the order of a is 0, then a is a consequence and $\varphi_2(a)(\omega_2) = a = \varphi_1(a)(\epsilon(\omega_2))$, $\forall \omega_2 \in \Omega_2$. Else, let $a = e(b,c)$ with b and c acts of order strictly less than the order of a. One has $\varphi_2(a) = \varphi_2(b)_{K_2(e)}\varphi_2(c)$. By induction, $\varphi_2(b) = \varphi_1(b) \circ \epsilon$ and $\varphi_2(c) = \varphi_1(c) \circ \epsilon$ so that $\varphi_2(a) = (\varphi_1(b) \circ \epsilon)_{K_2(e)}(\varphi_1(c) \circ \epsilon) = (\varphi_1(b)_{K_1(e)}\varphi_1(c)) \circ \epsilon = \varphi_1(a) \circ \epsilon$. Finally, consider $f, g \in \mathcal{A}(\Omega)$ and, since φ is surjective, let $a, b \in \mathcal{A}$ be acts such that $\varphi_1(a) = f$ and $\varphi_1(b) = g$. Then, $f \succsim_{\Omega_1} g \iff \varphi_1(a) \succsim_{\Omega_1} \varphi_1(b) \iff a \succsim b \iff \varphi_2(a) \succsim_{\Omega_2} \varphi_2(b) \iff \varphi_1(a) \circ \epsilon \succsim_{\Omega_2} \varphi_1(b) \circ \epsilon \iff f \circ \epsilon \succsim_{\Omega_2} g \circ \epsilon$. As a result, $\succsim_{\Omega_1}$ is SEU, not only with respect to u_1 and $\mathbb{P}_1$, but also with respect to u_2 and the image of $\mathbb{P}_2$ through ϵ. To conclude, it is sufficient to invoke Kopylov's (2007) theorem 3.1 in the case of a finite algebra, which provides a uniqueness argument for utility functions and subjective probabilities.

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