History of mathematical education in ancient, medieval and pre modern India (within the Chapter: Mathematics Education in Oriental Antiquity and Middle Ages)

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History of mathematics education in Oriental Antiquity and Middle Ages

This chapter is devoted to the history of mathematics education in Asia during the ancient and medieval periods. As no systematic account of the history of all Asian countries can be given here, this chapter will focus only, on the one hand, on China (more precisely, the political formations that existed in what is today Chinese territory) and certain states that came under its direct cultural influence (what are today Korea, Japan, Vietnam); and on the other hand, on India.

It should be said at once that even the chronological boundaries of the period under investigation turn out to be difficult to establish precisely and uniformly for all the territories studied: in discussing the Indian subcontinent, it turns out to be convenient in this chapter to examine the so-called premodern period, from the thirteenth to the eighteenth centuries, as a single entity, while for China and Japan the premodern period will be defined in a different way, beginning in the seventeenth century, and discussed in another chapter.

This chapter consists of two sections, written respectively by Alexei Volkov and Agathe Keller.

Mathematical Education in India

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1. Introduction

Very little is known of the context in which much of ancient India's scholarly knowledge burgeoned. Part of this ignorance springs precisely from the fact that very little is known about elementary, higher or specialized education in ancient and medieval India. For ancient and medieval mathematics in the Indian sub-continent, most of the studied textual sources are in Sanskrit, a brahmanical language which became the scholarly language of an educated cosmopolitan elite. Sources in vernacular languages could provide information on the contents and means of mathematical knowledge transmission in wider circles, but concerning mathematics and astronomy, little-studied and very late texts provide only meager testimonies. Also, archeology has until now given us little information on how mathematics was taught. In the following section, to sketch an uncertain image of mathematical education in India during the period in question, we have structured this section along the three historical periods relevant to the history of mathematics in India: Vedic period (ca. 2500 BCE-500 BCE), Classical and Medieval period (-500 BCE-XIIth century), and Premodern period (XIIIth-XVIIIth century). However, in fact, the continuous use of ancient texts throughout this period mingled with our ignorance serves to combine it all into one continuous mode of transmission.

2. Vedic India (ca. 2500 BCE-500 BCE)

The oldest texts that have come down to us from the Indian subcontinent, the Vedas (ca. 2500-1700 BCE), gave rise to a set of scholarly commentaries that define the knowledge that should be imparted to a brahman and how this knowledge should be transmitted. According to such texts, the high caste male (and in rare cases, female) should live through four life stages, one being the state of *brahmacārin* or student. The pupil (*śiṣya*) and his teacher or master (*ācārya, guru*) practiced restraint and yogic exercises to develop an inner energy (*tapas*) that was believed to be central to good learning. Given that important religious texts were oral texts, knowledge was seen as being acquired through heard (*śruti*) and remembered (*smṛti*) texts. An important part of the education of a brahman consist~~s~~ed of learning how to recite specific parts of the Vedas. This implies not only knowing these texts by heart but also knowing how to chant them according to strict metrical rules. In some cases, the recitation involved knowing how to chant Vedic verses following many systematic combinations of their syllables: first in order, than inverting one verse/syllable after another, then reciting it backwards, and so on, so that the recitation itself could be seen as an application of a systematical «mathematical» combination.

To become a student, one had to find a teacher who would accept to perform an *upanayana* ceremony, by which the teacher became symbolically pregnant with his student, who was usually between 8 and 12 years old. Women were not normally made to study, but there are known exceptions. Education then lasted at least twelve years. The teaching season opened with an *upakarman* ceremony on a full-moon day in July or August and ended for five to six months. To end the period of apprenticeship, a ceremony was held in which the student offered a present to his teacher. Teaching was probably not performed by an individual preceptor. Texts describe the benefit for a student of having several teachers. Further, small "assemblies" (*śakha, charaṇa, pariṣad,* etc.) were founded around the transmission of a certain number of texts and specific interpretations, housing students and teachers together. Students had the duty to tend the houses, fires, and cattle of these "assemblies" or of his teachers. Obviously such "assemblies" could gather several members of the same extended family and, reciprocally, such education could be undertaken within a family unit. It was possible for a student to stay for his entire life.

The verb *adhī-* is usually used to indicate how a student should learn the Vedic text (it is referred to sometimes as «one's own lesson» *svādhyāya*), meaning both «learning by heart» and also «seeking». Vedic auxiliaries (*vedāṅga*) proclaimed that they served the purpose of seeking the meaning of the Vedic text. They consisted of five topics: Phonetics, Metrology, Etymology, Ritual, and Astral Science (*jyotiṣa*). Astral science as propounded by the *Jyotiṣavedāṅga* («Vedic auxiliary on Asterisms») (ca 1200 BCE), while not properly a mathematicised astronomy from the perspective of planet movements, still contained procedures involving elementary arithmetical operations, such as the Rule of Three. The *Śulbasūtra*s («Rules of the Chord») (8th to 5th century B.C.E.) were sub-parts of larger ritual texts, each belonging to different schools and ascribed to different authors (Baudhāyana, Apāstamba, Kātyāyana, Mānava). Secondary literature on the history of mathematics groups all *Śulbasūtra*s together. They indeed share many rules and topics in common. It is uncertain how such texts should be understood in relation to mathematical education.They describe the construction of Vedic ritual altars and delimit ritual grounds. They provide algorithms to construct with strings and poles, with oriented geometrical figures (square, rectangles, right triangles, etc.) having a given size. Procedures describe how to transform one ~~a~~ figure into another with the same area or a given part of this area (a rectangle into a square, an isosceles triangle into a square, etc.). Rules for constructing altars of given shapes (some very complex, such as a hawk with open wings) with a fixed number of bricks are given as well. Each *Śulbasūtra* often gives several separate procedures for a same aim. It also contains rules with a general scope, such as a procedure for the Pythagorean Theorem. However, texts give little information on how to transmit these rules. Each separate text discusses several ritual schools, thus exposing contradictory data on the length of this altar or the size of that brick. The knowledge imparted by the *Śulbasūtras* was probably intended to be mastered by the *adhvaryu* priest, one of the well-versed priests in charge of Vedic sacrifices. According to a hypothesis offered by Chattopadhyaya (1986), the paving knowledge of the *Śulbasūtras* would have been inherited from the artisans who had constructed the remarkable prehistoric cities of the Indus civilizations. This supposes, most provocatively, a transmission of knowledge from non-brahmins to brahmins. The knowledge imparted in the *śulbasūtra*s is still brought to life today by a brahmin cast of Kerala, the Nambudiris, who regularly perform vedic sacrifices. If priests know the theory of how altars and sacrificial areas should be delimited and constructed, the actual setting can be ~~is~~ constructed by specialized artisans of a lower caste but by specialised brahmins as well. Thus contemporary anthropological evidence could support Chattopadhyaya's hypothesis of a collaboration between artisans and priests, although that would involve the improbable supposition that over thousands of years, roles and channels of transmissions remained unchanged or that, in this respect, random, the situation happens to be the same at the beginning of the XXIst century as it was three thousand years ago. Thus little is known about the Vedic education of low-caste. Given the occupational definitions of castes, it is often imagined that the education was taken charge of either by guilds or by a family.

Separate independent schools for astral science probably developed during the end of the Vedic period. Religious sects contesting Vedic values proliferated, among them appearing Buddhism and Jainism. Each would eventually develop a non-Sanskrit scholarly literature. No mathematical or astronomical buddhist text has been transmitted from that time, although buddhist texts refer to astronomy and even evoke counting devices of the kind of an abacus. The four branches of Jaina canonical texts include principles of mathematics (*gaṇitānuyoga*), arithmetic (*saṃkhyāna*), and astral science (*jyotiṣa*). Such texts are known to us in later forms as compilations made during the Classical and Medieval ages. They form a separate part of the history of mathematics in India, although in constant dialog with Hindu lore. We are uncertain, however, how these texts were integrated into the curriculum of the monks. They may have been at some specific moment in time part of the curricula, but this does not mean they were constantly studied.

Thus from the Vedic period is first transmitted a vision of how education is carried out: transmission is oral and seems to be done—with only few exceptions—within each individual's own caste and profession. Scholarly mathematics seems to have been used essentially for ritual, cosmological, and religious purposes: to determine the moment of sacrifice, explain the universe, and help build proper sacrificial altars and grounds.

Also, scholarly knowledge developed into a standard given form: as treatises described as being said orally whose aphorism (*sūtra*s) should be as simple and concise as possible. They were sometimes studied with their written commentaries, often by another author. This form would have an everlasting imprint on the texts transmitted to us in the Classical and Medieval ages of India.

2. Classical and Medieval India (-500 BCE-XIIth century CE)

After the burgeoning of the late Vedic period, one thousand years of silence followed in the transmission of Sanskrit texts in astral science and mathematics. Then, two kinds of Sanskrit mathematical texts come to light. The first type larger in quantity consists in mathematical chapters of astronomical treatises. The second type is the self-proclaimed "mathematics for worldly affairs" (*loka vyavahāra*), which were often related to Jain lore.

Both kinds of texts are in the form of versified rules, more or less aphoristic, that transmit definitions and procedures. It is likely that these rules did not aim to describe precisely an algorithm to be carried out or a defined object, but rather to coin the important and remarkable elements worth memorizing: the rest were completed either by wit or with the help of a commentary. Thus, secondary literature has often considered these treatises as student manuals. The commentary is consequently seen as something akin to «teacher's notes». These commentaries in prose can stage dialogs, where it is tempting to read a representation of an actual act of teaching. Indeed, the words to coin how these texts are transmitted and conveyed allude to a broad educational context. They thus refer to themselves as showing (*pradṛś-*), indicating (*upadiś-*), explaining (*pratipād-*); all these verbs can also mean «to teach».

In the Vth century C.E., a mathematical astronomy appeared as an already completed body of knowledge through two treatises that were self-proclaimed compilations: the «Five astronomical treatises» (*Pañcasiddhānta)* of Varāhamihira (476 CE) and the *Āryabhaṭīya* of Āryabhaṭa (499). The latter devoted a chapter to mathematics, giving a definition of the place value notation, evoking derivations of sines, and providing different elements of arithmetics, algebra, and indeterminate analysis. These topics were subsequently constantly re-explored. The *Āryabhaṭīya* gave rise to schools, commentaries, and criticisms—a steady tradition of Sanskrit scholarly astral science including mathematics.

Most of the texts devoted to "worldly mathematics" (also called «board mathematics» -*pāṭīgaṇita,* probably a reference to the slab on which computations and drawings could be carried out) have survived and reached us by chance. This was the case of the *Bakhshālī Mansucript* (bearing the name of the town where it was excavated), of uncertain date (8th to 12th century); it was found by a peasant digging a hole in his field near Peshwar (now in present-day Pakistan) in 1881. Sometimes, we have only a single unique manuscript belonging to one library or collection, such as Ṭhakkura Pheru's *Gaṇitasārakaumudī*, «Moonlight of the essence of mathematics» (ca. 1310) or the Paṭan manuscript (later than the XIVth century), both Jain texts known in a unique recension. The most famous of all such texts were those compiled by Śrīdhāra (ca. 9th century): the *Triśatika* («Three Hundred [Rules]») and the *Pāṭīgaṇita* («Board Mathematics»). Many of these texts, often self-proclaimed compilations, testify to vernacular lore of all kinds and have strong ties with the Jain tradition, such as the very popular Sanskrit *Gaṇitasārasaṃgraha* («Collection of the essence of mathematics») by Mahāvīra (ca. fl. 850).

Both mathematical traditions refer to each other as separate but dependent forms. By the XIIth century, a synthesis of both traditions was attempted in certain circles. This is probably the case with Bhāskarācarya's (b. 1114) mathematical texts, the *Līlāvatī* («With Fun» or maybe the name of the girl to whom the examples in this text were addressed), which was devoted to arithmetics, and the *Bījagaṇita* («Seed

Mathematics»), devoted to algebra; both were integrated as chapters in his astronomical treatise, the *Siddāntaśiromāṇi* («Crest-jewl (among) astronomical treatises»).

2. 1. Teaching elementary mathematics

Very little testimony exists on what would have been the elements of mathematics taught to any child to whom an education was given.

Medieval non-mathematical texts offer only fleeting information on the mathematical education of non-brahmins. Thus Kauṭilya's *Arthaśāstra* («Manual of Statecraft», ca. 100 BCE-100 CE), a sort of Indian machiavelic law manual for Kings, gives information on regular education. We learn from this text that after the ceremony of tonsure, a child was taught writing and arithmetic (*saṃkhyāna*).

The future king was encouraged to learn accounting, so as not to be easily swindled. Further, the *Arthaśāstra* included a detailed list of measuring units and their shifting values; evidently, it was important for a King to master conversions.

The Pali Buddhist canon (through the *Gaṇakamoggallānasutta* of the *Majjhimanikāya*) compilated in the first centuries before C. E. described a brahmin calculator (*gaṇaka*) who took in live-in pupils (*antevāsin*). He started by teaching them how to count to a hundred.

As we will see, scholarly mathematical texts, whether devoted to «worldly mathematics» or to astral science, provide a classification of operations and topics. It is noteworthy that but for one exception, no text describes how one should carry out addition or subtraction with the decimal place value notation. More generally, the algorithms put forth in scholarly mathematical texts~~,~~ seem to take for granted that additions and subtractions on higher numbers, multiplications, divisions, squares, and cubes of digits were known. We do not know if children were made to learn multiplication tables, but this could have been the case; as later (modern) manuscripts of vernacular tables of multiplication, squares, and cubes are known.

According to Hayashi 2001, *saṃkhyāna* could have also included «a sort of statistical estimate of the quantity of nuts, crops, etc.»

2.2. Board mathematics: a prerequisite for mathematical astronomy?

Texts of "worldly mathematics" probably testify to what would have been a general mathematical culture—not an elementary one, but not one of high-brow Sanskrit mathematical knowledge either. Much like the mathematical chapters of astronomical texts, these texts present a structure of elementary and/or fundamental operations (*parikarman*) and of "practices" (*vyavahāra*), which express an implicit theory of how such mathematics is organized. They also provide a structured list in which knowledge can be coined and memorized. Bhāskarācarya gives what has in the secondary literature become a sort of canonical subdivision of topics. Arithmetic (*rāśigaṇita*) was subdivided into eight operations (*parikarman)*~~,~~ and eight practices (*vyavahāra)* that included topics we would probably classify in geometry and trigonometry*:*

The operations were:	The practices were:
1. Addition (*saṅkalita*)	1. Mixtures (*miśraka*)
2. Subtraction (*vyavakalita*)	2. Series (*śreḍhī*)
3. Multiplication (*pratyutpanna*)	3. Figures (*kṣetra*)
4. Division (*bhāgahāra*)	4. Excavations (*khata*)
5. Square (*varga*)	5. Stacks (*citi*)
6. Square-Root (*vargamūla*)	6. Sawings (*krakacika*)
7. Cube (*ghana*)	7. Grains (*rāśi*)
8. Cube-Root (*ghanamūla*)	8. Shadow (*chāyā*)

Each operation was defined for integers (actually referred to as regular numbers, *saṅkhyā*), fractions (*bhinna*), and zero (*śunya*). Such an organisation was articulated with algebra (*bījagaṇita*). For Bhāskarācarya, this topic was structured by six rules (*vidha*) which correspond to the six first operations of arithmetics. These rules were applied to positives (*dhana*) and negatives (*ṛna*), zero, undeterminates (e.g. «unknowns», *avyakta*), and surds (*karaṇī*). They were also defined as providing sub-operations for rules concerning different topics such as indeterminate linear *(kuṭṭaka)* and quadratic (*vargaprakṛti)* problems, and equations with one or more unknowns (*samakaraṇa*). Many variations of the number and contents of both operations and practices for arithmetic (usually rules of proportions including the Rule of Three (*trairāśika*) are included as operations), and types of equations for algebra, are known. However, such an ordered structure that is found in all mathematical texts handed down to us could have been of the progressive learning of mathematics. As we will see in the section on higher education in mathematics, algebra may have been viewed as a more advanced topic in mathematics. Variations then of structures could account for the fact that despite belonging to a common Sanskrit mathematical culture, each school and each family had its own particularities.

The question of the nature of practices (*vyavahāra*) and how they relate to vocational training, for instance, is still very much open to research. Hayashi (2001) provides elements showing that accountants and calculators, as scribes, seemed to have been regularly needed for all sorts of administrative activities. One can thus imagine that these professionals needed to have solid mathematical preparation. However, these worldly mathematics did not provide any rules for accounting. Indeed, each "practice" was a scholarly topic. Bhāskara (628 C.E) thus evokes a series of scholars who before him had worked, composed, and compiled treatises, as Sanskrit scholarly knowledge self- proclaims itself:

> Or else, the scope of mathematics is vast, there are eight practices Mixtures, Series, Figures, Excavations, Piles of brick, Sawings, Mounds of grain, and Shadows. (...) For each, rules and books were made and compiled by master Maskari, Pūraṇa, Mudgala, and others. (Keller 2007, 30)

In all cases, whether explicitly or implicitly, these topics seemed to have been at least partially mastered by one who was to know astronomy.

Thus at the outset of the mathematical chapter (*gaṇitādhyāya*) of his astronomical treatise *Brahmasputasiddhānta*, Brahmagupta (629 C.E) claims:

> BSS.12.1 Whoever knows separately the twenty operations (*parikarman*) beginning with addition, and the eight practices (*vyavahāra*) ending with shadows, he is a calculator (*gaṇaka*)[1]. (Plofker 2007, 421)

Prthudakasvamin, a IXth century commentator, specifies: «He has the rank (*adhikārin*) to learn the Sphere». In other words, to learn mathematical astronomy—and specifically the movement of planets, which is the topic called *gola* (the Sphere)—one first had to master the different operations of «board mathematics».

Rules and problems given in mathematical texts can contribute to an idea of what was required of a good mathematician—that is, the skills that one learning mathematics should acquire.

2.3. Riddle culture

The Sanskrit elite culture no doubt valued playfulness, which could have also been part of the mathematical pedagogy. Texts of «board mathematics» often contain versified mathematical problems that can be seen as mathematical riddles. Their authorship is often doubtful: similar type problems are known to have traveled from text to text. Further, all mathematical commentaries of astronomical treatises contain list of versified examples, some of which echo those of «board mathematics» texts. They often used the vocative and encouraged rapidity and wit. For example, here is a problem of computations of the purity of gold melting given in the *Līlavātī*.

> L.102. Example: Parcels of gold weighing severally ten, four, two and four *māṣas*, and of fineness thirteen, twelve, eleven and ten respectively, being melted together, tell me quickly, merchant who art conversant with the computation of gold, what is the fineness of the mass? If the twenty *māṣas* of gold be reduced to sixteen by refining, tell me instantly the touch of the purified mass. Or, if its purity when refined be of sixteen, prithee, what is the number to which the twenty *māṣas* are reduced?[2]

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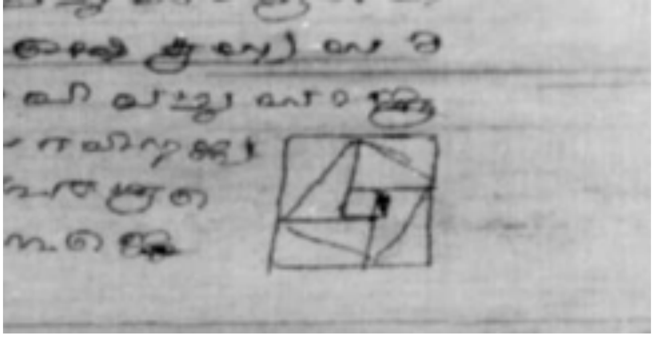
[1] Note that Brahmagupta counts more operations than *Bhāskarācarya*: he adds the Rules of Three, Five, Nine, and Eleven; the inverse Rule of Three; Five rules to reduce fractions; Barter and Exchange; and Rules to sell living beings.

[2] If xi is the fineness (or "touch") of the ith piece of gold, and yi its purity, x and y their respective value for the melted mass of gold, then xy=∑xiyi. Therefore, x=∑/y, and y=∑/x.



Statement	Touch	13	12	11	10
	Weight	10	4	2	4

> Answer:
> After melting, fineness 12 .Weight 20
> After refining, the weight being sixteen *māṣas*; the touch is 15. The touch being sixteen, the weight is 15[3]. (Colebrooke 1817, 46-47)

Such gold-melting problems can be found in all known «board mathematics» texts, including those mentioned above, the *Bakhshālī Mansucript*, the *Gaṇitasārasaṃgraha* and the *Pāṭīgaṇita,* the *Gaṇitasārakaumudī,* and the *Patan manuscript*.

The playfulness of the problem can be seen in the ever-ceasing variations of weight or fineness of the gold. Indeed, the problem seemed to deal with an algorithm of adding and dividing rather than with the apprentice of a jeweler or of a superintendent of coins (as described by the rules for alligation in the *Arthaśāstra,* a text on goverment rule). Such problems, as we can see, addressed a reader or a listener and evoke his or her qualities, the most common one being training in being quick when the rule exemplified in the problem is known.

The problem is set down in a sort of numerical table on a working surface: first, products are taken within the pairs forming each column; then, columns are summed before dividing, respectively, either by the intended fineness or weight. Such graphic dispositions may have been carried out on a working surface, such as a dust board or slate with chalk. They could have also served the purpose of representing mental computations to be carried out.

Indeed, if the problems evoked the witty quick one who did not need to write computations, the commentaries sometimes hit on the dull-minded one who was too slow and for which an extended explanation was needed. Bhāskara thus described a diagram, probably to illustrate the Pythagorean Theorem, that he will draw for the dull minded.

> When one has sketched an equi-quadrilateral field and divided [it] in eight, one should form four rectangles whose breadth and length are three and four and whose diagonals are five. There, in the same way, stands in the middle a field whose sides are the diagonals of the [four] rectangles which were the selected quadrilaterals [and] which is an equi-quadrilateral field. And the square of the diagonal of a rectangular field there, is the area in the interior equi-quadrilateral field. (...) And a field is sketched in order

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[3] 13*10+12*4+11*2+10*4= 240 =xy. Therefore in the first case x=240/20=12. In the second case, y=240/16=15. In the third case, x=240/15=16.

to convince a dull-minded one (*duḥvigdha*). (Fig.1). (Keller 2006 vol.1, 15)

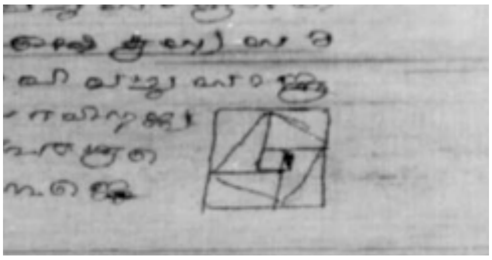

Fig. 1. Mss Burnell 517 British Library

However, mathematical texts did describe tools for drawing diagrams (pairs of compasses, ropes, and chalk) or solid objects made of clay to explain computations involving solids. They often evoked the explanation of an expert to actually "show" (teach, explain, prove) the rule.

2.4. Knowing how to apply the general rule

Riddles, solved problems, and commentaries were all directed towards a hermeneutic act, the proper interpretation of a rule. Bhāskara, the seventh century commentator who used drawing for the dull rules, saw the rule provided by the treatise he commented upon as a seed (*bīja*) to be grown. In other words, the rule was a general statement that could encompass many different rules. This no doubt was part of a more general scholarly conception of how a rule should be coined. Looking at this endeavor from the point of view of acquired skills, it is clear that one had to know how to apply such general rules to many different cases. Thus consider, for instance, the rule for summing (*saṅkalita*), such as the one given in the *Brahmaspuṭhasiddhānta:*

> BSS. 12.2 Of two quantities the denominators and the numerators when multiplied by the opposite denominators have the same divisor. In addition, the numerators are added together; in subtraction the difference of the numerators is to be computed. (adapted from Plofker 2007, 421)

We immediately read this rule as concerning fractions. Pṛthudhakasvamin, the IXth century commentator, specified that integer numbers can be seen as having a denominator equal to one. Such a rule can thus also be applied to integers. He further extended the rule to series, echoing other rules "for addition" given in worldly mathematical texts. Indeed, through a solved example, he provides an interpretation of the rule as a procedure to sum arithmetic progressions.

2.5. Higher Education in mathematics and mathematical astronomy

In the shift from the Vedic period to the Classical and Middle Ages, mutations in the organization of state and local royalties showed the development of learned settlements in association with religious complexes (*maṭha*s), giving rise to what may have been the oldest universities of the ancient world. Such universities were known to have cultivated the study of vernacular languages not restricting themselves to scholarly Sanskrit. Chinese travelers and buddhist monks have left us testimonies of great buddhist centers of learning, such as Nalanda, in the Eastern Gangetic plain, but they do not mention the teaching of mathematics or astronomy in this context. Thus Fa-Hien (fl. 399 CE) described Pāṭaliputra and its hundreds of monks. The Vth century astronomer Āryabhaṭa mentioned that he learned astronomy in Kusumpura and presented this as a title of glory. His VIIth century commentator Bhāskara glossed this name with Pāṭaliputra, while others here recognized the name of Kurukṣetra, another centre of higher learning.

Astral science, and within it mathematical astronomy, was understood as a scholarly topic with divine origin and a long stream of forefathers, from gods (often Brahma) to seers (men with supernatural powers), to great scholars and gurus.

Varāhamihira in his *Bṛhatsaṃhitā* recorded the names of a large number of astronomers, among whom can be mentioned Garga who will often be recalled as well as a certain number of foreigners such as the recognisable Romaka (Roman?).

Sūryadeva Yajvan, a XIIth century commentator of Āryabhaṭa, described this process of transmission. The field of astral science was first seen or intuited (*dṛś-*) by Brahma. He then founded the discipline or composed a treatise which he taught to a great scholar, who in turn synthesized it and wrote a new treatise, and taught it to his followers. Compiling, reducing/synthesizing, and transmitting then were the activities/steps which a student used to become a teacher.

We have seen that a minimum of mathematical knowledge was apparently required to study mathematical astronomy. On the other hand, mathematics was often brought up as the ultimate step in mastering astronomy, as it enabled one to ground astronomical rules. For instance, after having evoked what was necessary for preparing a calculator, or a mathematician, on the outset of his algebraic chapter, Brahmagupta states:

> BSS.18.2 « [One becomes] a master (*ācārya*) among experts (*vidyā*) of the doctrine (*tantra*) by knowing the pulverizer, zero, negatives, positives, unknowns, elimination of the middle [term], [reduction to one] colour/unknown, *bhāvita* (multiplication) and the square nature. (adapted from Plofker 2007, 428)

Indeed, for many late medieval authors, algebra was a tool to ground arithmetics. With the Rule of Three and the Pythagorean Theorem, algebra served as the basis of another skill, that of understanding and then producing explanations for Rules.

2.6. Criticizing

Some of the wry irony of treatises and commentaries support the idea that debates were likely very virulent in astral science and mathematics in India during the classical and medieval times.

Thus, the ultimate scholar, no longer a student, was most probably one who could engage in debates with his teacher, but not criticize him. In Bhāskara's medieval commentary of the *Āryabhaṭīya,* the author was compared as he started his treatise to a warrior on the battlefield raising his sword. The same commentater later comments on a compound (*samavṛttiparidhi*, the circumference of an evenly circular [figure]) which can be understood as either referring to a disk or a circle. He notes specifically about the first meaning, nonsensical within the context:

> This very analysis [of the compound] has been taught by master Prabhākara. Because he is a *guru*, we are not blaming him.

This suggest that such an explanation of the compound is wrong. The early *Arthaśāstra* states thus:

> *5.5* A science imparts discipline to one, whose intellect has (the qualities of) desire to learn, listening, learning, retention, thorough understanding, reflection, rejection and intentness on truth, not to any other person.

It thus defines the ideal scholar by the qualities that a good student needs, including an eventual rejection of bad rules. It also adds:

> 5.6 But training and discipline in the sciences (are acquired) by (accepting) the authoritativeness of the teachers in the respective sciences.

This tension of criticizing and respecting forefathers will be carried over into the premodern period.

3. Premodern India (XIIIth-XVIIth century)

By the beginning of the early modern period, information on what the secondary literature calls astronomical «schools» became available. Most Sanskrit mathematical manuscripts handed to us by tradition~~,~~ were copied at the end of this period and afterwards. Through their history and those of family libraries, family lineages running through centuries in astral science (including mathematical astronomy) were revealed. Thus, the XIIth century astronomer and mathematician Bhāskarācarya belonged to a family of astronomers. His sons and nephews were also well-known court astrologers and composed astronomical texts.

The most famous school was initiated by Mādhava (c. 1340-1425) near the town today known as Kochi. Nīlakaṇṭha (1445-1545) was from a Nambuttiri familly in Trikkantiyur on the coast of South Kerala. He thus travelled to learn his mathematics from Parameśvara's (fl.ca.1430) son Dāmodara (fl.ca.1460) at Ālattūr (Aśvatthagrāma), Kerala. The "Kerala school" produced astronomers who ~~that~~ were also specialists in other fields of knowledge. This can be seen as "premodern" features of scholarly Sanskrit India, where traditional boundaries were breaking down, and scholars writing in vernacular languages sought new structures of knowledge. Nīlakaṇṭha, known also as a philosopher, had many pupils as Śaṅkara (fl. 1550) who produced an important commentary on both the *Līlāvatī* and Nīlakaṇṭha's works: although substatially renewing mathematical tools and theories, the authors positioned themselves in a continuous tradition.

The main turn in premodern mathematics seemed to be the emphasis on providing proofs (*upapatti*) of the rules coined by canonical authors. The "Kerala school" was indeed known for its endeavor at correcting and grounding Āryabhaṭa's parameters, leading to its development of the Taylor series. Commentators of Bhāskarācarya wrote books with algebraical "proofs" of his arithmetical rules. More advanced students no doubt had to learn these explanations first before creating their own grounding of mathematical algorithms.

In the premodern period we know of rival schools and families. Thus, Divākara (fl.ca.1530), a Gujarati astronomer of the XVIth century, went to study near Benares with Gaṇeśa (b. 1507), who was famous for his commentary of the *Līlāvatī* (among other texts) which included many proofs. Divākara 's descendants became important astrologers in this city, although they came under attack by a rival family of astrologers, notably for their acceptance of Islamic astronomical theories.

Indeed, the premodern period also saw the arrival and installation in most of North India of Muslims from central Asia, Afghanistan, and Persia. They arrived with an Arabic and Persian culture grounded in mathematics and astronomy. However, more even than the mathematical education of Hindu, Buddhist, and Jains, the mathematical education of Indian Muslims is, to my knowledge, mainly uncharted territory. For Muslims of the Indian subcontinent, as well as likely elsewhere in Muslim countries, elementary mathematical education was developed in Madrassas. Under the reign of Sultan Fīrūz Shāh Tughluq (1305-1388) who commissioned many scientific texts and translations, it is known that a number of Madrassas were opened to encourage literacy and numeracy. Arabic and Persian mathematics flourished in the Indian subcontinent: many manuscripts of such texts can be found in Indian libraries, notably ~~in~~ those of the Asiatic Society of Mumbai and Calcutta. Certainly, centers of learning in the Indian subcontinent were devoted to studying these texts. Further, during this period, astrolabes and table-texts influenced by Arabic and Persian literature were translated into Sanskrit, and in turn many Sanskrit texts were translated into Persian. Court patronages were available for both Hindu and Muslim astrologers and mathematicians: a close study of the courts of such Moghol rulers would probably yield much information on how mathematics and astral sciences were taught. Note that Jain monks were known to have been active players, enabling the cross-fertilization of both mathematical and astronomical traditions. Jain mathematicians then from the late Vedic period to the premodern seemed to have been crucial actors of mathematical activity in the Indian sub-continent. Investigating how mathematics could have been taught to them is yet another venue for further research.

Conclusion

All texts are intended to impart information: one can read them as instructional and thus, often too quickly, try to reconstruct from them a classroom picture. Our knowledge is overall uncertain, but one might nonetheless imagine an assembly where pupils, teachers, and scholars at different stages of learning worked together, and this representation should extend from the Vedic period to the late modern period. This is a sort of dream, a fantasy such as the mirage of Jai Singh's XVIIIth century court in which we would like to imagine Jesuit priests, Hindu pandits, and Arabic and Persian scholars translating Euclid together. But as the stories goes,[4] while this vision is but a mirage, it is also an incentive for further research.

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[4] A translation into Sanskrit was made from al-Tūsī's Persian version but obviously Euclid's text was not understood by the pandits who undertook this work.

References

Chattopadhyaya, Debiprasad (ed.) . 1986. *History of Science and Technology in Ancient India, the beginnings*. New-Dehli: Firma Klm private limited.

Colebrooke, Henry T. 1817.. *Algebra, with Arithmetic and mensuration*. London: J. Murray..

Datta, Bibhutibhussan. & Singh, Avadesh N. 1935. *History of Hindu Mathematics*, Lahore: Motilal Banarsidass .

Hayashi, Takao. Mathematics (*gaṇita*). 2001. In *Storia della Scienza*, volume II.II.X. 1-7, 772–790. Rome: Istituto della Enciclopedia Italiana..

Hayashi, Takao. 1995. *The Bakhshālī Manuscript. An ancient Indian mathematical treatise*. Groningen. Egbert Forsten.

Malamoud, Charles. 1989. *Cuire le Monde*, La Découverte, Paris.

Mookerji, K. Radha. 1947. *Ancient Indian Education, Brahmanical and Buddhist*. New Dehli: Motilal Banarsidass.

Narasimha, Rodham. 2007. Epistemology and language in indian astronomy and mathematics. *Journal of Indian Philosophy*, 35:521–541,.

Pingree, David. 1981. *Jyotiḥśāstra : astral and mathematical literature*. Wiesbaden:Harrassowitz.

Pingree, David. 2003. «The Logic of Non-Western Science: Mathematical Discoveries in Medieval India», in *Daedalus*, Vol. 132, No. 4,, 45-53.

Plofker, Kim. 2007 Mathematics in India in *Mathematics in Egypt, Mesopotamia, China, India and Islam*, Katz, Victor (ed.), 385–514. Princeton: Princeton University Press...

Rangacarya, M. 1912. *The Gaṇitasārasaṃgraha of Mahāvīra, edited with translation and notes*. Madras : Madras Goverment Publication.

Sarma, K. V. 1985. *Vedāṅga Jyotiṣa of Lagadha, edited and translated*. New Dehli: INSA.

Scharfe, Harmut. 2002. *Education in ancient India*. Leiden: Brill.

Sen, S. N; Bag, A. K. 983. *The śulbasūtras*. New-Dehli: INSA 1.Keller, Agathe. 2006. *Expounding the mathematical seed, Bhāskara and the mathematical chapter of the Āryabhaṭīya*, Basel: Birkhauser.

Keller, Agathe. 2007. Qu'est ce que les mathématiques? Les réponses taxinomiques de Bhāskara un commentateur, mathématicien et astronome du VIIème siècle in *Sciences et Frontières*, ,29-61. Bruxelles: Kimé.

Shukla, K. S. 1959 . *Pāṭīgaṇita of Śrīdharācarya*. Lucknow: Lucknow University.

Shukla, K. S. 1976. *Āryabhaṭīya of Āryabhaṭa, with the commentary of Bhāskara I and Someśvara*. New-Dehli: Indian National Science Academy.

Somayajipad, C.V, Staal, Frits and Nambudiri, I.T. 1983. *Agni, the Vedic ritual of the fire altar*. Berkeley: Asian Humanities Press,.