A Conceptual Introduction to Nelson's Mechanics

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Abstract

Nelson's programme for a stochastic mechanics aims to derive the wave function and the Schrödinger equation from natural conditions on a diffusion process in configuration space. If successful, this programme might have some advantages over the better-known deterministic pilot-wave theory of de Broglie and Bohm. The essential points of Nelson's strategy are reviewed, with particular emphasis on conceptual issues relating to the role of time symmetry. The main problem in Nelson's approach is the lack of strict equivalence between the coupled Madelung equations and the Schrödinger equation. After a brief discussion, the paper concludes with a possible suggestion for trying to overcome this problem.

1 Introduction

Within the foundations of quantum mechanics, Nelson's stochastic mechanics [1, 2] is generally less well-known than other approaches, and is often presented as a stochastic variant of de Broglie's [3] and Bohm's [4] pilot-wave theory. It is true that the two theories have striking similarities; in particular, in both theories the motions (in configuration space) are described using

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a velocity field of the form $\frac{1}{m}\nabla S$. This is the deterministic velocity field for a particle in de Broglie–Bohm theory, and the current velocity of the diffusion process in Nelson's theory. Nevertheless, there is a crucial conceptual difference between de Broglie–Bohm theory or its stochastic variants and Nelson's approach as it was originally conceived: in de Broglie–Bohm theory the function S is assumed from the start to be the phase of Schrödinger's wave function, which obeys the Schrödinger equation. Instead, Nelson's original aim was to assume only that the particles obey a diffusion process in the configuration space and to *derive* a wave function and the Schrödinger equation by imposing natural conditions on the diffusion. Nelson's original approach, in apparently not assuming the wave function to be part of the ontology of the theory, is thus highly unusual (as compared not only to de Broglie–Bohm but also to spontaneous collapse theories and especially to Everett), and in this respect might provide an interesting comparison to the otherwise very different approaches that seek to understand the quantum state in purely epistemic terms, including entanglement and other supposedly typical quantum features (Spekkens [5]), and that derive or seek to derive quantum mechanics from simple, usually information-theoretic, principles (for example Hardy [6] and Fuchs [7]).

In Section 2 we shall focus on this special feature of Nelson's approach and spell out in more detail the potential advantages of Nelson's original approach over de Broglie–Bohm theory (and its stochastic variants).

The main section of this paper, Section 3, is devoted to reviewing the approach itself in some detail. We shall try to isolate various issues of conceptual interest, in particular the role of time symmetry, which is not immediately intuitive in the context of stochastic processes. Indeed, the idea of irreducible indeterminism (and by extension that of probability) is often thought of in terms of an 'open future' and a 'fixed past', and transition probabilities are thought of as law-like in the forward direction of time but as merely epistemic in the backward direction. (For an explicit articulation of this position, see Arntzenius [8].) One can equally well, however, have a picture that does not privilege one direction of time, namely a picture of individual trajectories with the evolution along them subject to stochastic laws in both directions of time. Nelson uses such a picture, since (as we shall see) he takes it that the external forces acting on a system constrain in a law-like way both the forward and the backward transition probabilities of the process. (For a related discussion in the context of decoherent histories, see Bacciagaluppi [9]. I also hope to give a fuller philosophical discussion

of time symmetry in stochastic processes, both in general and with specific reference to Nelson's theory, in further publications.)

In Section 4 we then turn to the question of whether Nelson's approach, as stated so far, achieves its aim. As a matter of fact, it is rather well-known that it has not been entirely successful. The reason for this (as was detailed in print by Wallstrom [10]), is the lack of strict equivalence between the Schrödinger equation and the Madelung equations, seen as coupled equations for two abstract functions R and S. We assess the achievements of Nelson's programme in the light of these considerations.

In Section 5 we return to the comparison between Nelson's approach, as characterised in Section 4, and the Nelson-type variants of de Broglie–Bohm theory mentioned above. We conclude with some speculations on how it might still be possible, despite the known difficulties, to complete Nelson's original programme. [1]

2 Comparison with Pilot-wave Theory

The pilot-wave theory of de Broglie and Bohm is one of the better-known and best-understood approaches to the foundations of quantum mechanics. It is also one of the oldest. The mature version of de Broglie's theory was presented in October 1927 at the Fifth Solvay Congress [3]. The theory presented there is a new dynamics for systems of n particles (described in configuration space), where the motion of the i-th particle is determined by a velocity field of the form $\frac{1}{m_i}\nabla_i S(\mathbf{x}_1, \ldots, \mathbf{x}_n)$, S being the phase of the Schrödinger wave function. (S is defined in units of $\hbar$, that is, $\psi = Re^{iS/\hbar}$. One finds also the convention $\psi = Re^{iS}$, in which case the velocity field has the form $\frac{\hbar}{m}\nabla S$.)

At least as regards particle detections, it is clear that the theory can easily predict both interference and diffraction phenomena. Indeed, around the nodes of the modulus R of the wave function, the phase S will behave very irregularly, so one can at least qualitatively expect that the particles will be driven away from regions of configuration space where R is small. (Using second-order concepts, this corresponds to a large additional 'quantum potential' around the nodes.) Therefore, for instance, interference bands in

[1] I wish to record my special thanks to Shelly Goldstein for long and fruitful discussions of the main topics of this paper, in particular the kinematics of time reversal, the comparison with pilot-wave theory, and the work by Davidson.

the two-slit experiment should appear. And, indeed, it was the qualitative prediction of electron diffraction and its experimental detection that established the significance of de Broglie's matter waves even before the detailed theory of 1927 was worked out. Also quantitatively, as de Broglie remarks, the velocity field preserves the form of the particle distribution if at any time this is given by R^2.

Although de Broglie's paper and the related discussions include various applications, the measurement theory for quantum observables other than functions of position was worked out in general only when Bohm rediscovered and revived the theory a quarter of a century later [4]. Indeed, it ought to be puzzling at first how such a theory of particles in motion could have even qualitatively anything to do, for instance, with incompatible observables, with the projection postulate and with the rest of the full phenomenology of quantum mechanics. In modern terminology, what Bohm showed in general is that in situations such as measurements, the wave function of the *total* system decoheres in such a way that the non-interfering components are in fact separated in configuration space by regions with very small R (very large quantum potential), so that the representative point of the system is effectively trapped inside one of the components. This component alone (barring 'conspiratorial' reinterference) will be relevant at later times for the dynamics of the system, so that the particles behave as if the wave function had collapsed. Assuming that the behaviour of the particles comprising the apparatus is macroscopically different depending on which component is effectively guiding the (total) system, one can perform a selection of a corresponding sub-ensemble from an ensemble of (object) systems. Also quantitatively, if the original distribution is given by R^2, the particles comprising the sub-ensemble will be distributed according to the usual quantum mechanical Born rule (see [11] for a detailed discussion of this point). It is further straightforward to see that in the case of two (maximally) entangled particles, decoherence induced by a measurement on one of the particles forces the other particle and the relevant apparatus to produce the correlated outcome. Less than perfect correlations are quantitatively reproduced given an initial R^2 distribution.

At the individual level, de Broglie–Bohm theory is a new particle mechanics. At the statistical level, for (time-dependent) *equilibrium* ensembles, it reproduces the statistical predictions of quantum mechanics, and the analogy with the statistical mechanical underpinning of thermodynamics can be pursued in great detail. The theory is explicitly non-local and recovers in

this way the quantum mechanical violations of the Bell inequalities. The theory can be easily modified to include spin, and various generalisations aiming to cover QED and other field theories have been proposed. For fuller details, apart from the original papers mentioned above, the following textbook treatments are recommended: Holland [12], Bohm and Hiley [13], Dürr [14] and the forthcoming one by Valentini [15].

In Nelson's theory, as mentioned earlier, the current velocity for the distribution in configuration space is equal to the de Broglie–Bohm velocity. This means that although the particle trajectories in Nelson's theory in general will not be the same as in de Broglie–Bohm theory, the most probable trajectories will oscillate randomly around the typical trajectories of de Broglie–Bohm theory. In terms of recovering the predictions of quantum mechanics, and essentially of all the aspects just mentioned, Nelson's theory will thus share the features of de Broglie–Bohm theory. Often one is interested only in this aspect, and one can formulate Nelson-type theories that preserve the quantum distribution as defined by the Schrödinger wave, assumed as given, and one can study these as stochastic variants of de Broglie–Bohm (the present author is no exception [16,17]).

We shall now spell out the difference between Nelson's original approach and de Broglie–Bohm theory (or its stochastic variants), with reference to some commonly raised questions about de Broglie–Bohm. No attempt will be made to assess the merits or demerits of these supposed criticisms, which are only briefly sketched, but which are listed roughly in increasing order of the attention they deserve. It should also be emphasised that, by common consensus, the situation as regards non-locality is the same in Nelson as it is in de Broglie–Bohm (irrespective of the approach to Nelson chosen). Indeed, it appears that Nelson himself would have hoped that his theory should be fundamentally local, and that he abandoned it because of its non-local features [2]. The question of locality and non-locality, in particular of at what stage and how the non-locality is implemented, of course deserves a detailed discussion, but will not be considered in this paper.

A first group of questions raised about de Broglie–Bohm theory relates to the notion of equilibrium: (a) how to justify why particles in laboratory ensembles should be distributed according to R^2; (b) if particles in such ensembles are always distributed according to R^2, individual trajectories would appear to be unobservable (notice there is some tension between (a) and (b)). It should be clear that in a stochastic theory such as Nelson's (or stochastic variants of de Broglie–Bohm), there is no lingering question of justifying equi-

librium, while at the same time one will have ever so small but well-defined fluctuations.

The second group of questions focuses on the tension between the configuration-space perspective and the Hilbert-space perspective. As such it is a rather heterogeneous group, comprising: (a) the idea that de Broglie–Bohm theory breaks the symmetry of the Hilbert space (that is, that configurations play an unjustified privileged role in the theory); (b) objections to the fact that waves act on particles but are not acted back upon; (c) the status of 'empty waves', that is, of those components of the wave function that no longer contribute to guiding the particles after the relevant separation in configuration space due to decoherence; (d) the detailed arguments, recently put in print by Brown and Wallace [18], for the conceptual redundancy of particle positions and the pilot-wave picture if one admits the possibility of an Everett interpretation (this should be taken seriously as a criticism of de Broglie–Bohm).

If Nelson's approach is successful, the Hilbert-space concepts are all derived concepts. Arguably then: question (a) would disappear because the Hilbert space is not in fact fundamental; (b) because there would only appear to be waves acting on particles; similarly for (c); and (d) because the Everett interpretation is no longer motivated if the wave function is not fundamental. (If one holds functionalist views in philosophy of mind, however, even the existence of the decohering wave function as a derived feature will allow for a many-minds version of the Everett interpretation.)

Notice also that one can use configuration-space trajectories to formulate conditions that rule out parastatistics for indistinguishable particles [17, 19]. This is a case in which symmetry considerations at the level of the configuration space can play a non-trivial role.

Thirdly, it should be clear, even from the few remarks above, that the phenomenon of decoherence is crucial in order for de Broglie–Bohm theory to reproduce the phenomenology of the collapse of the wave function, and in fact of the whole 'classical regime' of quantum mechanics in the sense of the theory of decoherence. However, it appears to be merely a contingent fact that decoherence tends to produce separation of wave-function components in configuration space. If decoherence produced separation of components in momentum space, or if the pilot-wave kinematics and dynamics were defined with respect to momentum space, there would be no such distinguished regime in the theory, nor any effective collapse. While decoherence undoubtedly 'breaks the symmetry' of the Hilbert space, it does so purely contingently, and there appears to be no explanation why this matches the

fundamental choice of configuration space in pilot-wave theory.

In the non-relativistic particle case this coincidence is not so immediately striking, maybe because historically both de Broglie's guidance equation and Schrödinger's wave equation derived their respective forms from the optico-mechanical analogy (as spelled out, respectively, in de Broglie's thesis [20] and in Schrödinger's second paper on quantisation [21]). In the field-theory case, however, it is not obvious what the 'correct' configuration space of a pilot-wave theory should be, and it is striking that if a configuration space is chosen that does not match up with decoherence, the right phenomenology will not emerge. A further, methodological, disadvantage is that it appears that pilot-wave theory has to feed on the results of decoherence for any successful generalisation to field theories. (These criticisms are essentially due to Saunders [22].)

On the other hand, at least in the point-particle case, if Nelson's derivation is successful, the form of the quantum Hamiltonian is actually derived from the form of the particle dynamics (see Section 3 and the discussion in Section 4). The connection between decoherence and configurations is therefore immediate. Insofar as it appears that de Broglie–Bohm theory can be generalised to encompass the known field theories (see for instance Valentini [15,23]), and if these generalisations should indeed manage in conjunction with decoherence to recover the correct phenomenology, one can speculate that analogous Nelson-type theories might be able to explain also in the general case the otherwise mysterious connection between the choice of the configuration space and the form of the decoherence Hamiltonian.

A final, independent motivation for a renewed interest in Nelson's research programme has been provided by its use in the completely different context of quantum gravity, in a recent paper by Markopoulou and Smolin [24]. Here, the idea that the wave function is indeed a derived concept is again of crucial importance.

3 Nelson's Strategy

We now turn to the description and analysis of Nelson's strategy. In the main, we shall follow Nelson's original paper [1]. We have mentioned above that Nelson's theory is defined in terms of a diffusion process in the configuration space of the system. More precisely, Nelson suggests considering a stochastic

differential equation of the following form:

$$d\mathbf{x}(t) = \mathbf{b}(\mathbf{x}(t), t)dt + d\mathbf{w}(t). \tag{1}$$

The vector $\mathbf{b}(\mathbf{x}(t), t)$ is called the mean (forward) velocity (see also (33) below). $\mathbf{w}(t)$ is a Wiener process; the $d\mathbf{w}(t)$ are Gaussian with mean 0, independent of the $d\mathbf{x}(s)$ for $s \leq t$, and

$$E_t[dw_i(t)dw_j(t)] = 2\nu\delta_{ij}dt, \tag{2}$$

$\nu > 0$ being the diffusion coefficient and E_t denoting the expectation value at time t. For simplicity (and since we are not discussing non-locality) we can specialise to the case of a single particle, but the formalism and derivation are quite general.

Formally, (1) is the same equation as in the Einstein–Smoluchowski theory of Brownian motion. Nelson, however, emphasises that the context is different: the Einstein–Smoluchowski theory describes macroscopic Brownian motion in a fluid in the limit of infinite friction, while Nelson postulates the equation for elementary particles in the vacuum. The corresponding Fokker–Planck equation for the distributions has the form

$$\frac{\partial \rho}{\partial t} = -\mathrm{div}(\mathbf{b}\rho) + \nu\Delta\rho. \tag{3}$$

Nelson's aim is to impose natural constraints on the diffusion process such that (1) takes the form

$$d\mathbf{x} = \left(\frac{1}{m}\nabla S + \nu\frac{\nabla R^2}{R^2}\right)dt + d\mathbf{w} \tag{4}$$

(possibly, as will be discussed in more detail below, up to additional terms in $\mathbf{b}$ that will not contribute to the divergence in (6)), where the functions S and R satisfy the so-called Hamilton–Jacobi–Madelung equation,

$$\frac{\partial S}{\partial t} = -\frac{1}{2m}(\nabla S)^2 - V + \frac{\hbar^2}{2m}\frac{\Delta R}{R}. \tag{5}$$

If one inserts $\rho = R^2$ into the Fokker–Planck equation (3), with $\mathbf{b}$ defined as in (4), one obtains

$$\frac{\partial R^2}{\partial t} = -\mathrm{div}\left(\frac{1}{m}(\nabla S)R^2\right). \tag{6}$$

Equation (6) has the form of the usual quantum mechanical continuity equation. Equations (5) and (6) are called the Madelung equations and can be standardly derived from the Schrödinger equation setting $\psi = Re^{iS/\hbar}$. If one can find natural conditions on (1) such that (5) and (6) hold, this may suggest that it is (1) that is fundamental, the Schrödinger equation being only a convenient mathematical way of writing (5) and (6). Of course the problem can be trivialised by imposing as condition precisely that the current velocity be given through the gradient of the Schrödinger wave function, or some such *ad hoc* condition. A non-trivial solution to the problem should not make reference to the Schrödinger wave and equation in the formulation of the relevant conditions.

We shall see below, mainly in Section 3.2, how Nelson proposes to do this. At first sight, however, this strategy appears conceptually puzzling. It appears that the distribution R^2 itself contributes to determining the particle trajectories via (4). But how can an individual trajectory be affected by the distribution of other particles in an ensemble? This, however, is a red herring. Take a stochastic differential equation of the form (1) and consider *any* solution ρ of the corresponding Fokker–Planck equation (3). At least formally, one can always associate with ρ a so-called *osmotic velocity*

$$\mathbf{u}_\rho := \nu \frac{\nabla \rho}{\rho}, \tag{7}$$

and we can define a corresponding *current velocity*,

$$\mathbf{v}_\rho := \mathbf{b} - \mathbf{u}_\rho, \tag{8}$$

so that

$$d\mathbf{x} = \left(\mathbf{v}_\rho + \nu \frac{\nabla \rho}{\rho} \right) dt + d\mathbf{w}. \tag{9}$$

The Fokker–Planck equation for this ρ then reduces to a continuity equation:

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= -\mathrm{div} \left[\left(\mathbf{v}_\rho + \nu \frac{\nabla \rho}{\rho} \right) \rho \right] + \nu \Delta \rho = \\ &= -\mathrm{div}(\mathbf{v}_\rho \rho), \end{aligned} \tag{10}$$

so that $\mathbf{v}_\rho$ is indeed the current velocity corresponding to the distribution ρ. If $\mathbf{u}_\rho$ and $\mathbf{v}_\rho$ are not too singular, the corresponding stochastic differential equation has indeed well-defined solutions (see also the related discussion by

Carlen [25]). This choice of representation, in which some distribution enters explicitly, does not affect the time evolution of an individual particle as given by (1): one could as well use a different solution ρ' of (3) and write

$$d\mathbf{x} = \left(\mathbf{v}_{\rho'} + \nu \frac{\nabla \rho'}{\rho'} \right) dt + d\mathbf{w}. \tag{11}$$

We see that reference to some specific distribution can be thought of as defining a convenient way of writing the mean velocity $\mathbf{b}$. As we shall presently see, however, the choice of the distribution ρ is connected to the choice of a *time reversal* of equation (1).

3.1 Time reversal of diffusion processes

Take again the stochastic differential equation (1):

$$d\mathbf{x} = \mathbf{b}dt + d\mathbf{w}. \tag{12}$$

Such an equation essentially describes only the forward transition probabilities of a stochastic process (in the sense of a measure over all possible trajectories), and the process is in general underdetermined by equation (1). Indeed, suppose an arbitrary Markov process is defined on a time interval $[t_1, t_2]$. Given all the forward transition probabilities from a time s to a time $t > s$ for any $s, t \in [t_1, t_2]$, one can define as many processes as one has possible initial distributions at time t_1. Therefore, the backward transition probabilities that one can obtain for a process (by conditionalising on future states) are also underdetermined. As opposed to a deterministic equation, (1) has no well-defined time reversal, and specifying such a time-reversal will amount to a further condition on the process.[2]

A time reversal of (1) will be a diffusion with the same diffusion coefficient ν (since the size of the fluctuations does not depend on the time direction), that is, it will have the form

$$d\mathbf{x} = \mathbf{b}' d(-t) + d\mathbf{w}_*, \tag{13}$$

[2]Notice that if one considers the process to be defined on $]-\infty, +\infty[$, and if one invokes the asymptotic properties of such processes, then the convergence of the distribution in both directions of time will generally enforce a unique distribution ρ and therefore a unique time reversal. Since however ρ and $\mathbf{b}$, though related, are still unspecified, the rest of our discussion in Section 3.2 will apply. We choose our mode of presentation because it is closer to Nelson's own and because it is more self-contained.

with some suitable mean velocity $\mathbf{b}'$, and where the $d\mathbf{w}_*(t)$ are again Gaussian with mean 0, and with

$$E_t\left[dw_{*i}(t)dw_{*j}(t)\right] = 2\nu\delta_{ij}dt, \tag{14}$$

but now the $d\mathbf{w}_*(t)$ are independent of the $d\mathbf{x}(s)$ for $s \geq t$. The Fokker–Planck equation corresponding to (13) is

$$\frac{\partial\rho}{\partial(-t)} = -\mathrm{div}(\mathbf{b}'\rho) + \nu\Delta\rho. \tag{15}$$

Now let us choose a representation (9) of (1):

$$d\mathbf{x} = \left(\mathbf{v}_\rho + \nu\frac{\nabla\rho}{\rho}\right)dt + d\mathbf{w}. \tag{16}$$

The time reversal of (16) will have the form (13), as mentioned, but more specifically, since $\rho(t)$ is manifestly invariant under time reversal, it will take the form

$$d\mathbf{x} = \left(\mathbf{v}'_\rho + \nu\frac{\nabla\rho}{\rho}\right)d(-t) + d\mathbf{w}_*, \tag{17}$$

with a suitable current velocity $\mathbf{v}'_\rho$. But now, analogously to (10), ρ will satisfy also the continuity equation

$$\frac{\partial\rho}{\partial(-t)} = -\mathrm{div}(\mathbf{v}'_\rho\rho). \tag{18}$$

If (18) is meant to be the time reversal of (10), then we should further set

$$\mathbf{v}'_\rho = -\mathbf{v}_\rho, \tag{19}$$

that is, we have uniquely fixed $\mathbf{b}'$.

One generally defines $\mathbf{b}_* := -\mathbf{b}'$ and writes

$$d\mathbf{x} = \mathbf{b}_*dt + d\mathbf{w}_* \tag{20}$$

rather than (13). The corresponding Fokker–Planck equation (15) becomes

$$\frac{\partial\rho}{\partial t} = -\mathrm{div}(\mathbf{b}_*\rho) - \nu\Delta\rho \tag{21}$$

(the so-called backward Fokker–Planck equation).

We see that there is indeed a one-to-one correspondence between the choice of a solution ρ to the Fokker–Planck equation in the representation (9) of (1) and the choice of a time reversal for (1).

Now, if we make such a choice and write down the pair of stochastic differential equations

$$d\mathbf{x} = \left(\mathbf{v}_\rho + \nu\frac{\nabla\rho}{\rho}\right)dt + d\mathbf{w} \tag{22}$$

and

$$d\mathbf{x} = \left(\mathbf{v}_\rho - \nu\frac{\nabla\rho}{\rho}\right)dt + d\mathbf{w}_*, \tag{23}$$

this fixes ρ uniquely as the single-time distribution of the process, even if we take the process to be defined only on an interval $[t_1, t_2]$. The intuitive reason is that any distribution has to get closer to ρ in both directions of time, but then it has to be ρ for all times. (In the following, we shall drop the index ρ from $\mathbf{v}_\rho$ and $\mathbf{u}_\rho$.) Since the process is Markov, this also fixes uniquely the entire process. The equations (22) and (23) thus fix the process uniquely, while it is underdetermined if only one of the two equations is given.

Lest one be worried by the fact that the process is entirely fixed, let us emphasise again that a stochastic process is defined as a probability measure over a space of trajectories. Which trajectory is actually realised is a contingent matter, and so is the actual single-time distribution in any collection of sub-systems. Therefore, fixing the distribution at the level of the process does not fix the distribution in any actual such collection; rather, if one will, it is just a reflection of the fact that transitions along trajectories are governed in both directions of time by stochastic differential equations. What is fixed are the *probabilities* for distributions of sub-systems, and an actual distribution of sub-systems that is far from 'equilibrium' can be interpreted as a fluctuation, which is most likely both to evolve towards equilibrium in the time direction we label 'future' and to have evolved from equilibrium in the time direction we label 'past'.

3.2 Dynamical time symmetry and derivation of the Madelung equations

The framework for the derivation of the Madelung equations is now in place. Nelson takes the pair of equations

$$d\mathbf{x} = (\mathbf{v} + \mathbf{u})dt + d\mathbf{w}, \qquad d\mathbf{x} = (\mathbf{v} - \mathbf{u})dt + d\mathbf{w}_*, \tag{24}$$

where $\mathbf{u}$ is expressible as $\nu\dfrac{\nabla\rho}{\rho}$ in terms of the common distribution of the two evolutions.

Thus far, (24) is merely a kinematical representation of a diffusion process in time-reversible notation: $\mathbf{v}$ and $\mathbf{u}$ (or ρ) are quite arbitrary, if coupled through the continuity equation

$$\frac{\partial\rho}{\partial t} = -\mathrm{div}(\mathbf{v}\rho). \tag{25}$$

The question of whether the process is time-symmetric is a dynamical question. We need to determine the process by some further dynamical law that together with (25) will fix $\mathbf{v}$ and $\mathbf{u}$ (or equivalently, $\mathbf{b}$ and $\mathbf{b}_*$). One can both imagine laws that will do this in a time-symmetric way (for instance, $\mathbf{v} = \dfrac{1}{m}\nabla S$ and $\mathbf{u} = \dfrac{\hbar}{2m}\dfrac{\nabla R^2}{R^2}$, where $\psi = Re^{iS/\hbar}$ satisfies a time-symmetric equation such as the Schrödinger equation), and laws that will do it in a time-asymmetric way ($\mathbf{v}$ and $\mathbf{u}$ the same as above, but ψ satisfies some time-asymmetric equation, as are some non-linear variants of the Schrödinger equation; compare for instance [26]). Nelson's aim is to recover the time-symmetric Schrödinger equation, so he will essentially seek to impose constraints on $\mathbf{v}$ and $\mathbf{u}$ in the form of a time-symmetric dynamical law (which however should not make reference to Schrödinger's equation in its formulation!).

Before discussing that, however, let us consider in more detail the continuity equation (25). This will turn into an equation of the same form as the usual quantum continuity equation if we define $R^2 := \rho$ and impose

$$\mathbf{v} = \frac{1}{m}\nabla S + \tilde{\mathbf{v}}, \tag{26}$$

where $\tilde{\mathbf{v}}$ is a term satisfying

$$\mathrm{div}(\tilde{\mathbf{v}}R^2) = 0. \tag{27}$$

That is, $\tilde{\mathbf{v}}$ does not contribute to the divergence. This can be expressed equivalently also as

$$\tilde{\mathbf{v}} = \frac{\mathrm{rot}(\mathbf{t})}{R^2}, \tag{28}$$

for some function $\mathbf{t}$, or, using $\mathbf{u} = \nu\dfrac{\nabla\rho}{\rho}$, as

$$\mathrm{div}(\tilde{\mathbf{v}}) + \frac{1}{\nu}\tilde{\mathbf{v}}\cdot\mathbf{u} = 0. \tag{29}$$

(The treatment of such additional terms is somewhat different in Nelson's original approach and in Nelson-type pilot-wave theories; see below, Section 5. Notice also that Guerra and Morato [27] use a variational principle to define the dynamics of Nelson's theory. This approach would appear to justify setting $\mathbf{v}$ equal to a gradient, thus excluding the extra term $\tilde{\mathbf{v}}$ altogether.) Under these constraints then,

$$\frac{\partial R^2}{\partial t} = -\mathrm{div}\left(\frac{1}{m}\nabla S R^2\right), \tag{30}$$

as desired.

Given the definition of R and the condition (26), we return to the question of whether we can impose a dynamical law that will fix $\mathbf{v}$ and $\mathbf{u}$ such that S and R now obey also the Hamilton–Jacobi–Madelung equation (5). At this point, Nelson considers the so-called forward and backward stochastic derivatives of the process, which he defines, respectively, as

$$D\mathbf{x}(t)\Big|_{\mathbf{x}(t)=\mathbf{x}} = \lim_{\varepsilon\to 0^+} E_t\left[\frac{\mathbf{x}(t+\varepsilon)-\mathbf{x}(t)}{\varepsilon}\Big|\mathbf{x}(t)=\mathbf{x}\right] \tag{31}$$

and

$$D_*\mathbf{x}(t)\Big|_{\mathbf{x}(t)=\mathbf{x}} = \lim_{\varepsilon\to 0^+} E_t\left[\frac{\mathbf{x}(t-\varepsilon)-\mathbf{x}(t)}{-\varepsilon}\Big|\mathbf{x}(t)=\mathbf{x}\right]. \tag{32}$$

Here $E_t[\,.\,|\mathbf{x}(t)=\mathbf{x}]$ is the expectation value at time t conditional on the value of the process being $\mathbf{x}$. Thus, the definition of $D\mathbf{x}(t)$ involves the forward transition probabilities from time t to times $t+\varepsilon$, while the definition of $D_*\mathbf{x}(t)$ involves the backward transition probabilities from time t to times $t-\varepsilon$. One sees easily that

$$D\mathbf{x}(t)\Big|_{\mathbf{x}(t)=\mathbf{x}} = \mathbf{b}(\mathbf{x},t) \qquad \text{and} \qquad D_*\mathbf{x}(t)\Big|_{\mathbf{x}(t)=\mathbf{x}} = \mathbf{b}_*(\mathbf{x},t) \tag{33}$$

(thus justifying the terminology of mean velocities). Also, if $\nu = 0$ and therefore $\mathbf{u} = 0$, both $D\mathbf{x}(t)$ and $D_*\mathbf{x}(t)$ equal $\mathbf{v}$, and the stochastic derivatives reduce to the usual derivative.

Nelson points out that applying D or D_* to an arbitrary random variable $f(\mathbf{x}(t), t)$, one obtains

$$Df(\mathbf{x}(t),t)\Big|_{\mathbf{x}(t)=\mathbf{x}} = \left[\frac{\partial}{\partial t} + \mathbf{b}(\mathbf{x},t)\cdot\nabla + \nu\Delta\right] f(\mathbf{x},t), \tag{34}$$

and

$$D_*f(\mathbf{x}(t),t)\Big|_{\mathbf{x}(t)=\mathbf{x}} = \left[\frac{\partial}{\partial t} + \mathbf{b}_*(\mathbf{x},t)\cdot\nabla - \nu\Delta\right] f(\mathbf{x},t). \tag{35}$$

He introduces the quantity $\mathbf{a} := \frac{1}{2}(D\mathbf{b}_* + D_*\mathbf{b})$, which he calls the second stochastic derivative of $\mathbf{x}$ (or mean acceleration). One can easily calculate the quantity $\mathbf{a}$ by applying (34) and (35) to the components of $\mathbf{b}_*$ and $\mathbf{b}$. This yields

$$\begin{aligned}\mathbf{a} &= \frac{1}{2}\left[\frac{\partial}{\partial t} + \mathbf{b}\cdot\nabla + \nu\Delta\right]\mathbf{b}_* + \frac{1}{2}\left[\frac{\partial}{\partial t} + \mathbf{b}_*\cdot\nabla + \nu\Delta\right]\mathbf{b} = \\ &= \frac{1}{2}\frac{\partial}{\partial t}(\mathbf{b}+\mathbf{b}_*) + \frac{1}{2}(\mathbf{b}\cdot\nabla)\mathbf{b}_* + \frac{1}{2}(\mathbf{b}_*\cdot\nabla)\mathbf{b} - \frac{1}{2}\nu\Delta(\mathbf{b}-\mathbf{b}_*) \qquad (36)\end{aligned}$$

(understood componentwise), or, using $\mathbf{b} = \mathbf{v} + \mathbf{u}$ and $\mathbf{b}_* = \mathbf{v} - \mathbf{u}$,

$$\begin{aligned}\mathbf{a} &= \frac{\partial\mathbf{v}}{\partial t} + \frac{1}{2}[(\mathbf{v}+\mathbf{u})\cdot\nabla](\mathbf{v}-\mathbf{u}) + \frac{1}{2}[(\mathbf{v}-\mathbf{u})\cdot\nabla](\mathbf{v}+\mathbf{u}) - \nu\Delta\mathbf{u} = \\ &= \frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} - (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u}. \qquad (37)\end{aligned}$$

Let us provisionally ignore the term $\tilde{\mathbf{v}}$ in (26) and insert $\mathbf{v} = \frac{1}{m}\nabla S$ and $\mathbf{u} = \nu\frac{\nabla\rho}{\rho} = \nu\nabla\ln\rho$ into (37), yielding

$$\mathbf{a} = \frac{1}{m}\nabla\frac{\partial S}{\partial t} + \frac{1}{m^2}(\nabla S\cdot\nabla)\nabla S - \nu^2(\nabla\ln\rho\cdot\nabla)\nabla\ln\rho - \nu^2\Delta\nabla\ln\rho. \tag{38}$$

One can use the fact that

$$(\nabla S\cdot\nabla)\nabla S = \frac{1}{2}\nabla(\nabla S)^2, \tag{39}$$

and similarly that

$$\begin{aligned}(\nabla \ln \rho \cdot \nabla)\nabla \ln \rho + \Delta \nabla \ln \rho &= \frac{1}{2}\nabla(\nabla \ln \rho)^2 + \nabla \Delta \ln \rho = \\ &= 2\nabla \left[\frac{(\nabla R)^2}{R^2} + \Delta \ln R \right] \end{aligned} \tag{40}$$

(where one also uses $\ln \rho = 2 \ln R$). Since

$$\Delta \ln R = \frac{R\Delta R - (\nabla R)^2}{R^2}, \tag{41}$$

(40) becomes

$$(\nabla \ln \rho \cdot \nabla)\nabla \ln \rho + \Delta \nabla \ln \rho = 2\nabla \frac{\Delta R}{R}, \tag{42}$$

and (38) simplifies to

$$m\mathbf{a} = \nabla \left[\frac{\partial S}{\partial t} + \frac{1}{2m}(\nabla S)^2 - 2m\nu^2 \frac{\Delta R}{R} \right]. \tag{43}$$

In order to obtain the Hamilton–Jacobi–Madelung equation we can therefore impose

$$\nu = \frac{\hbar}{2m} \tag{44}$$

and

$$m\mathbf{a} = -\nabla V, \tag{45}$$

where V is the external potential. (The integration constant from (43) will induce only an irrelevant global gauge transformation.)

If we set $\mathbf{v} = \frac{1}{m}\nabla S + \tilde{\mathbf{v}}$, we obtain some additional terms in (38), which we must set equal to zero. Explicitly, we have to impose

$$\frac{\partial \tilde{\mathbf{v}}}{\partial t} + (\tilde{\mathbf{v}} \cdot \nabla)\frac{1}{m}\nabla S + (\frac{1}{m}\nabla \mathbf{S} \cdot \nabla)\tilde{\mathbf{v}} + (\tilde{\mathbf{v}} \cdot \nabla)\tilde{\mathbf{v}} = 0. \tag{46}$$

The crucial condition on the process that allows Nelson to recover the Madelung equations (apart from the assumption that $\mathbf{v}$ is essentially a gradient) is (45): a law-like statement (indeed, a stochastic analogue of Newton's second law!) that is entirely time-symmetric and involves both the forward and the backward transition probabilities of the process in an essential way.

Postulating (45) is very suggestive, of course, but it is not peculiar to Nelson's theory. As he points out, (45) holds in the Ornstein–Uhlenbeck theory of Brownian motion, and his theory thus combines the kinematics of the Einstein–Smoluchowski theory with the dynamics of the Ornstein–Uhlenbeck theory.

Notice also that while choosing (45) as the crucial condition on $\mathbf{b}$ and $\mathbf{b}_*$ yields precisely the Hamilton–Jacobi–Madelung equation, just about any condition would yield an equation coupling $\mathbf{v}$ and $\mathbf{u}$ in a non-trivial way. One especially interesting condition is $mD\mathbf{b} = -\nabla V$ (which does not involve mixing forward and backward transition probabilities). If $\mathbf{v}$ is a gradient (not in general), it is straightforward to show that

$$mD\mathbf{b} = mD_*\mathbf{b}_* = m\frac{1}{2}(D\mathbf{b} + D_*\mathbf{b}_*). \tag{47}$$

This yields

$$\frac{\partial S}{\partial t} = -\frac{1}{2m}(\nabla S)^2 - V - 2m\nu^2\frac{\Delta R}{R}. \tag{48}$$

For $\nu = \dfrac{\hbar}{2m}$, this is exactly the Hamilton–Jacobi–Madelung equation, except for the sign of the quantum potential, and it corresponds to a non-linear and non-linearisable wave equation. Notice, however, that by taking a real linear combination of conditions (45) and (47) of the form

$$\alpha m\frac{1}{2}(D\mathbf{b}_* + D_*\mathbf{b}) + \beta m\frac{1}{2}(D\mathbf{b} + D_*\mathbf{b}_*) = -\nabla V, \tag{49}$$

with $\alpha + \beta = 1$ and $\alpha - \beta > 0$, one can again obtain the Hamilton–Jacobi–Madelung equation by imposing

$$\nu = \frac{1}{\sqrt{\alpha - \beta}}\frac{\hbar}{2m}. \tag{50}$$

The diffusion coefficient ν can therefore take any positive value for an appropriate choice of α and β. Such a result was first derived by Davidson [28].

4 Inequivalence of the Madelung equations and the Schrödinger equation

The conditions to be imposed on the system of stochastic differential equations (24) in order to relate them to the Madelung equations, as we have seen,

are that $m\mathbf{a} = -\nabla V$ and that $\mathbf{b}+\mathbf{b}_*$ should be a gradient (or a slightly more general expression where the additional term satisfies some supplementary conditions). It is, however, not the case that the Madelung equations for two functions S and R imply the Schrödinger equation for the corresponding function $\psi = Re^{iS/\hbar}$ (although the converse is true). This, as was emphasised by Wallstrom [10], requires a supplementary condition on S, in fact one that appears rather *ad hoc*.

The problem is that $\mathbf{v} = \dfrac{1}{m}\nabla S$ defines $\mathbf{v}$ locally as the gradient of a function, but does not specify whether S is a single-valued or multi-valued function. The equivalence of the Madelung equations and the Schrödinger equation, however, depends extremely sensitively on the multi-valuedness properties of S. Indeed, both the assumption of single-valuedness and that of multi-valuedness (without further constraints) are problematic.

If S is assumed to be single-valued, then one can derive a Schrödinger equation from the Madelung equations, but this case does not capture the full generality of the Schrödinger equation, namely it does not include wave functions with angular momentum, for which S is indeed multi-valued. If instead one allows S to have an arbitrary multi-valued behaviour, then, as Wallstrom shows by example, there are solutions of the Madelung equations that do not correspond to *any* solution of the relevant Schrödinger equation.

The dilemma is between allowing too few or too many solutions of the Madelung equations than those necessary to recover all and only solutions of the Schrödinger equation. In order to do so, one has to assume that S is generally multi-valued, but that the difference in value acquired along a closed curve is restricted to an integer multiple of the Planck constant h. Obviously, this means assuming from the start that S is the phase of a complex function.

As Wallstrom puts it, in order to derive the Schrödinger equation from the Madelung equations, one has to impose a quantisation condition just like the Bohr–Sommerfeld condition of the old quantum theory. More neutrally, one could say that, *given* the non-trivial assumption that the current velocity of the process is given by some 'phase waves', Nelson shows there are natural conditions under which the complex function defining the waves obeys a Schrödinger equation and the distribution of the particles is given by the squared amplitude of this complex function, as well as fixing the form of the Hamiltonian.

This may not quite be Nelson's original aim, but it is a striking and

non-trivial result. To illustrate just how striking it is one might resort to a historical fable. One could imagine that de Broglie had a yet younger brother, Édouard, who, starting from his brother's ideas on phase waves determining particle motions, connected them with ideas about Brownian motion and arrived to the Schrödinger equation a couple of years ahead of its actual discovery.

5 Conclusion

In Section 1, we mentioned that Nelson's stochastic mechanics is often presented as a theory in which one assumes the Schrödinger wave and Schrödinger equation as given. In these theories one constructs certain diffusion processes in such a way that the (asymptotic) distribution of the process is always given by R^2. In the previous section, we have suggested that Nelson's own approach to his theory can rather be seen as one in which one assumes that the phase of a complex function (defined on configuration space) describes the current velocity of a diffusion process, and one formulates conditions directly on the process that turn out to imply that this complex function obeys the Schrödinger equation. Some of the differences between the two approaches are worth spelling out in more detail.

If one assumes that the complex function ψ obeys the Schrödinger equation, then one can define $\mathbf{v}$ and $\mathbf{u}$ directly in terms of S and R under fewer constraints. In particular, while in Nelson's approach the diffusion coefficient ν is equal to $\dfrac{\hbar}{2m}$, in the alternative approach it can be arbitrary. This is not an essential difference, however, since, as mentioned, Davidson [28] has shown that it is possible to have Nelson-style derivations of the Madelung equations using diffusion coefficients other than $\dfrac{\hbar}{2m}$. Another constraint that no longer applies is the supplementary condition (46) on the additional velocity term $\tilde{\mathbf{v}}$.

More importantly, the Nelson-type pilot-wave theories need not assume the time-reversed equation (23) as an equation of the theory, nor the associated unique distribution ρ as the distribution for the particles. This line is taken explicitly for instance by Bohm and Hiley [13]. Therefore, such stochastic variants of de Broglie–Bohm theory are not committed to considering the time reversed transitions as law-like (and they often omit them entirely from the presentation). For large times, both approaches to the theory will agree

that there is a well-defined probability measure over the positions (namely the asymptotic distribution R^2). For early times, however, before the forward transitions have driven the distribution close to the asymptotic one, the stochastic variants of de Broglie–Bohm theory are not committed to any particular particle distribution. Of course one can use the R^2-measure to define which trajectories should be considered as typical also for early times. This, however, should not be interpreted as a law of nature about the distributions to be expected in the world. Rather, it has to be argued for independently, as in the case of deterministic theories such as classical statistical mechanics or de Broglie–Bohm theory itself.

An approach that takes the Schödinger wave and equation as given, of course, is open to any of the criticisms listed in Section 2 as applying to de Broglie–Bohm theory (insofar as one accepts them, and with the qualified exception of the criticisms relating to equilibrium). The approach in which one assumes the existence of phase waves without assuming the Schrödinger equation, however, falls short of Nelson's original aim, and does not seem to fare much better. Indeed, if one assumes the existence of some phase waves and therefore the corresponding complex waves, and shows that these waves obey the Schrödinger equation, then in terms of ontological commitment this approach is just as problematic.

In order to make a difference to the interpretational debate in the sense sketched in Section 2, Nelson's approach would have to be developed truly along the lines of Nelson's original aim. I wish to conclude this paper by sketching a speculative strategy that might lead to meeting Wallstrom's criticism, and which I believe is new.

This strategy relies on the idea that by varying the potential V, one should be able to eliminate those nodes of R around which S accumulates terms other than hn. More precisely, by varying V, one wishes to make the complement of the nodal set of R simply connected in a neighbourhood of a certain time t. If this is true (the nodes dictated by antisymmetrisation of fermion wavefunctions are ineliminable, but the complement set of these nodes is simply connected if space has at least three dimensions), one could allow S to be multi-valued, but it would have to be single-valued in a neighbourhood of t, so that equivalence with the Schrödinger equation is ensured in that neighbourhood. Equivalence then follows for all times, even for those times when S might become multi-valued: the multi-valuedness will be automatically restricted to that obtainable when a Schrödinger wave acquires a multi-valued phase, as when one imparts angular momentum to a system.

Thus, one would allow S to be multi-valued but discount the 'non-Schrödinger' solutions of the Madelung equations on the grounds that they are not globally well-defined in time, at least if V is time-dependent.

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