6 Appendix 7: results with a more general production function

This appendix considers the two types of equilibria that are studied in the paper - equilibrium with a non-exploding bubble, equilibrium with an exploding bubble- with more general assumptions on the production technology. For the first type of bubble, two results are shown. With a Cobb-Douglas production function, uncertainty does not affect the existence of a long run bubbly equilibrium. Therefore, pessimism will no more be favorable to the existence of bubbles. With a CES production function, the results of the paper obtained for an infinite elasticity of substitution remains true if this elasticity is high enough. Bubbly equilibria may exist and are favored by pessimism.

For the second type of bubbles, assuming a CES production function with an elasticity of substitution greater than 1, two results of the paper are kept: pessimism favors the existence of small stochastic bubbles whereas optimism favors the existence of big stochastic bubbles.

6.1 Case 1: non-exploding bubbles

6.1.1 The general model

Using a general production function, it is necessary to set a more general model in which the value of the bubble is conditional to the realization of the states 1 and 2.

Assume a general production function

$$F\left[K_t, L_t, A_t(\sigma_t)\right]$$

with A_t a random variable that can take 2 values A^1 or A^2 depending on the state $\sigma_t = 1$ or 2. The value of the bubble also depends on the state and can take in period t one of the two values p_t^1 or p_t^2.

Along a competitive equilibrium, the wage and the capital gross rate of return are given by:

$$\begin{aligned} w_t^i &= w^i(K_{t-1}) = F_L'\left[K_{t-1}, 1, A^i\right] \\ R_t^i &= R^i(K_{t-1}) = F_K'\left[K_{t-1}, 1, A^i\right] \end{aligned}$$

for period t and state i.

The budget constraints are now:

$$c_t^i + s_t^i + p_t^i x_t^i = w_t^i \text{ if state } i \text{ occurs in } t \tag{47}$$

$$d_{t+1}^1 = R_{t+1}^1 s_t^i + p_{t+1}^1 x_t^i \tag{48}$$

$$d_{t+1}^2 = R_{t+1}^2 s_t^i + p_{t+1}^2 x_t^i \tag{49}$$

It is possible to eliminate s_t^i and x_t^i from (48) and (49)

$$s_t^i = \frac{\frac{d_{t+1}^1}{p_{t+1}^1} - \frac{d_{t+1}^2}{p_{t+1}^2}}{\frac{R_{t+1}^1}{p_{t+1}^1} - \frac{R_{t+1}^2}{p_{t+1}^2}} \tag{50}$$

$$x_t^i = \frac{\frac{d_{t+1}^1}{R_{t+1}^1} - \frac{d_{t+1}^2}{R_{t+1}^2}}{\frac{p_{t+1}^1}{R_{t+1}^1} - \frac{p_{t+1}^2}{R_{t+1}^2}} \tag{51}$$

to obtain the intertemporal budget constraint:

$$c_t^i + d_{t+1}^1 \frac{\frac{1}{p_{t+1}^1}\left(1 - p_t^i \frac{R_{t+1}^2}{p_{t+1}^2}\right)}{\frac{R_{t+1}^1}{p_{t+1}^1} - \frac{R_{t+1}^2}{p_{t+1}^2}} + d_{t+1}^2 \frac{\frac{1}{p_{t+1}^2}\left(p_t^i \frac{R_{t+1}^1}{p_{t+1}^1} - 1\right)}{\frac{R_{t+1}^1}{p_{t+1}^1} - \frac{R_{t+1}^2}{p_{t+1}^2}} = w_t \tag{52}$$

Assuming that the equilibrium is such that $d_{t+1}^1 > d_{t+1}^2$, the maximization program of the function (5) under the constraint (52) gives:

$$c_t^i = \frac{w_t^i}{1+\beta} \tag{53}$$

$$d_{t+1}^1 \frac{\frac{1}{p_{t+1}^1}\left(1 - p_t^i \frac{R_{t+1}^2}{p_{t+1}^2}\right)}{\frac{R_{t+1}^1}{p_{t+1}^1} - \frac{R_{t+1}^2}{p_{t+1}^2}} = \frac{\pi_1 \beta w_t^i}{1+\beta} \tag{54}$$

$$d_{t+1}^2 \frac{\frac{1}{p_{t+1}^2}\left(p_t^i \frac{R_{t+1}^1}{p_{t+1}^1} - 1\right)}{\frac{R_{t+1}^1}{p_{t+1}^1} - \frac{R_{t+1}^2}{p_{t+1}^2}} = \frac{(1-\pi_1)\beta w_t^i}{1+\beta} \tag{55}$$

From the two last expressions and (50), it is possible to determine s_t^i :

$$s_t^i = \frac{\beta w_t^i}{1+\beta}\left(\frac{\pi_1}{1 - p_t^i \frac{R_{t+1}^2}{p_{t+1}^2}} - \frac{1-\pi_1}{p_t^i \frac{R_{t+1}^1}{p_{t+1}^1} - 1}\right)$$

To simplify, the coefficient $\beta/(1+\beta)$ is now denoted by a.

The intertemporal equilibrium conditions can be obtained from $K_t = s_t^i$ and (47) with $x_t^i = 1$ and (53). If state 1 occurs in t:

$$\begin{aligned} K_t &= aw_t^1 \left(\frac{\pi_1}{1 - p_t^1 \frac{R_{t+1}^2}{p_{t+1}^2}} - \frac{1-\pi_1}{p_t^1 \frac{R_{t+1}^1}{p_{t+1}^1} - 1} \right) \\ K_t + p_t^1 &= aw_t^1 \end{aligned}$$

If state 2 occurs in t:

$$\begin{aligned} K_t &= aw_t^2 \left(\frac{\pi_1}{1 - p_t^2 \frac{R_{t+1}^2}{p_{t+1}^2}} - \frac{1-\pi_1}{p_t^2 \frac{R_{t+1}^1}{p_{t+1}^1} - 1} \right) \\ K_t + p_t^2 &= aw_t^2 \end{aligned}$$

It is possible to give an expression of the dynamics that only depends on K_t. In eliminating p_t^1 and p_t^2 of the equations, one gets: if state 1 occurs in t,

$$K_t = aw^1(K_{t-1}) \left(\frac{\pi_1}{1 - \frac{(aw^1(K_{t-1})-K_t)R^2(K_t)}{(aw^2(K_t)-K_{t+1})}} - \frac{1-\pi_1}{\frac{(aw^1(K_{t-1})-K_t)R^1(K_t)}{(aw^1(K_t)-K_{t+1})} - 1} \right) \quad (56)$$

if state 2 occurs in t,

$$K_t = aw^2(K_{t-1}) \left(\frac{\pi_1}{1 - \frac{(aw^2(K_{t-1})-K_t)R^2(K_t)}{(aw^2(K_t)-K_{t+1})}} - \frac{1-\pi_1}{\frac{(aw^2(K_{t-1})-K_t)R^1(K_t)}{(aw^1(K_t)-K_{t+1})} - 1} \right) \quad (57)$$

It is evident that $K_t = aw^1(K_{t-1})$ is a solution of (56), and that $K_t = aw^2(K_{t-1})$ is a solution of (57). These solutions correspond to the absence of bubble. The existence of a bubbly solution is obtained if (56) and (57) have also another solution such that $p_t^i = aw_t^i - K_t > 0 \ \forall t \ \forall i$. Assuming a bubbly solution and simplifying by $aw_t^i - K_t$, a new system of equations is obtained.

If state 1 occurs in t,

$$\begin{aligned} 1 = & \frac{\pi_1 R^2(K_t) aw^1(K_{t-1})}{\left[aw^1(K_{t-1}) - K_t\right] R^2(K_t) - (aw^2(K_t) - K_{t+1})} \\ & + \frac{(1-\pi_1) R^1(K_t) aw^1(K_{t-1})}{\left[aw^1(K_{t-1}) - K_t\right] R^1(K_t) - (aw^1(K_t) - K_{t+1})} \end{aligned} \quad (58)$$

if state 2 occurs in t,

$$1 = \frac{\pi_1 R^2(K_t) aw^2(K_{t-1})}{\left[aw^2(K_{t-1}) - K_t\right] R^2(K_t) - \left(aw^2(K_t) - K_{t+1}\right)} + \frac{(1-\pi_1)\, R^1(K_t) aw^2(K_{t-1})}{\left[aw^2(K_{t-1}) - K_t\right] R^1(K_t) - \left(aw^1(K_t) - K_{t+1}\right)} \tag{59}$$

6.1.2 The case of a Cobb-Douglas production function

In this part, the existence of bubbly equilibria is studied with two possible assumptions on the production technology.
Case 1: standard Cobb-Douglas technology

$$F\left[K_{t-1}, L_t, A_t(\sigma_t)\right] = K_{t-1}^{\alpha} L_t^{1-\alpha} A_i \tag{60}$$

with $A_1 > A_2$. At equilibrium, the wage and the capital gross rate of return are given by:

$$\begin{aligned} w_t^i &= (1-\alpha) A_i K_{t-1}^{\alpha} \\ R_t^i &= \alpha A_i K_{t-1}^{\alpha-1} \end{aligned}$$

Case 2: Cobb-Douglas technology with a technical progress *à la Romer* (1986). In this case, $A_t(\sigma_t)$ is an externality that is equal ex-post to $A_t(\sigma_t) = A_i K_{t-1}^{1-\alpha}$. At equilibrium, the wage and the capital gross rate of return are given by:

$$\begin{aligned} w_t^i &= (1-\alpha) A_i K_{t-1} \\ R_t^i &= \alpha A_i \end{aligned}$$

A common property of these two different assumptions will play an important rule in the study:

$$\frac{R_{t+1}^i K_t}{w_{t+1}^i} = \frac{\alpha}{1-\alpha} \tag{61}$$

Considering the system (58) and (59), an explicit solution of the long run bubbly equilibrium can be found under the form:

if state 1 occurs in period t,

$$aw^1(K_{t-1}) - K_t = p_t^1 = \xi^1 w^1(K_{t-1}) \tag{62}$$

if state 2 occurs in period t,

$$aw^2(K_{t-1}) - K_t = p_t^2 = \xi^2 w^2(K_{t-1}) \tag{63}$$

Using this assumption in the system (58) and (59), it is obtained:
if state 1 occurs in period t,

$$\frac{1}{a} = \frac{\pi_1 w^1(K_{t-1})}{\xi^1 w^1(K_{t-1}) - \frac{\xi^2 w^2(K_t)}{R^2(K_t)}} + \frac{(1-\pi_1)\, w^1(K_{t-1})}{\xi^1 w^1(K_{t-1}) - \frac{\xi^1 w^1(K_t)}{R^1(K_t)}} \tag{64}$$

if state 2 occurs in period t,

$$\frac{1}{a} = \frac{\pi_1 w^1(K_{t-1})}{\xi^1 w^1(K_{t-1}) - \frac{\xi^2 w^2(K_t)}{R^2(K_t)}} + \frac{(1-\pi_1)\, w^1(K_{t-1})}{\xi^1 w^1(K_{t-1}) - \frac{\xi^1 w^1(K_t)}{R^1(K_t)}} \tag{65}$$

From (61), we obtain:

$$\frac{w^i(K_t)}{R^i(K_t)} = \frac{1-\alpha}{\alpha} K_t$$

From (62) and (63), we also have:
if state 1 occurs in period t,

$$K_t = (a - \xi^1) w^1(K_{t-1})$$

if state 2 occurs in period t,

$$K_t = (a - \xi^1) w^1(K_{t-1})$$

Replacing in (64) and (65) and simplifying, the following system is obtained:

$$\frac{1}{a} = \frac{\pi_1}{\xi^1 - \frac{1-\alpha}{\alpha}\xi^2(a-\xi^1)} + \frac{(1-\pi_1)}{\xi^1 - \frac{1-\alpha}{\alpha}\xi^1(a-\xi^1)} \tag{66}$$

$$\frac{1}{a} = \frac{\pi_1}{\xi^2 - \frac{1-\alpha}{\alpha}\xi^2(a-\xi^2)} + \frac{(1-\pi_1)}{\xi^2 - \frac{1-\alpha}{\alpha}\xi^1(a-\xi^2)} \tag{67}$$

Finally we have proved that the two equations (66) and (67) allows to determine the long run bubbly equilibrium both in case 1 and in case 2. Using the notation $\nu = \frac{1-\alpha}{\alpha}$, the system can be written:

$$\xi^2 = \frac{\xi^1\left(\xi^1 - a\pi_1 + 1/\nu\right)}{(1-\pi_1)a - \xi^1\left[1 - \nu\left(a - \xi^1\right)\right]} \tag{68}$$

$$\xi^1 = \frac{\xi^2\left(\xi^2 - a(1-\pi_1) + 1/\nu\right)}{\pi_1 a - \xi^2\left[1 - \nu\left(a - \xi^2\right)\right]} \tag{69}$$

The symmetry in these two equations allows to study only one equation, (68). The denominator has two roots with opposite signs, denoted by $\check{\xi}^1$ and $\hat{\xi}^1$ with $\check{\xi}^1 < 0 < \hat{\xi}^1$. The numerator has also two roots 0 and $a\pi_1 - 1/\nu$ and $\check{\xi}^1 < a\pi_1 - 1/\nu < \hat{\xi}^1$. One possible solution of this system is $\xi^1 = \xi^2 = 0$. Looking for a bubbly equilibrium, as ξ^1 and ξ^2 must be positive, the only part of the function that matters in when $\xi^1 \in \left(\max(0, a\pi_1 - 1/\nu), \hat{\xi}^1\right)$. The function has a vertical asymptote for $\xi^1 = \hat{\xi}^1$.

The same study can be done for (69) that has an horizontal asymptote for $\xi^2 = \hat{\xi}^2$, $\hat{\xi}^2$ the positive root of the denominator.

There exists a long run bubbly equilibrium if the two curves have an intersection for $\xi^1 \in \left(\max(0, a\pi_1 - 1/\nu), \hat{\xi}^1\right)$ and $\xi^2 \in \left(\max(0, a(1-\pi_1) - 1/\nu), \hat{\xi}^2\right)$. Two cases can occur:

- if $a\pi_1 - 1/\nu > 0$ or $a(1-\pi_1) - 1/\nu > 0$, there exists such an equilibrium. A necessary condition for that is that $a > 1/\nu$.

- if $a\pi_1 - 1/\nu < 0$ or $a(1-\pi_1) - 1/\nu < 0$, the existence of an intersection of the two curves is obtained if

$$\left.\frac{d\xi^2}{d\xi^1}\right|_{\xi^1=0}^{\text{eq (68)}} < \left.\frac{d\xi^2}{d\xi^1}\right|_{\xi^1=0}^{\text{eq (69)}}$$

  This condition leads to

$$\frac{1/\nu - a\pi_1}{(1-\pi_1)a} < \frac{\pi_1 a}{1/\nu - a(1-\pi_1)}$$

  that gives the result: $a > 1/\nu$ or $a(1-\alpha) > \alpha$.

To summarize the two cases, there is only one condition that implies the existence of a bubbly long run equilibrium $a(1-\alpha) > \alpha$. This condition is exactly the condition to obtained overaccumulation in the standard Diamond model without risk. It does not depend on π_1. Therefore, in the special case of a Cobb-Douglas production function, uncertainty does not affect the existence of bubbly equilibria.

6.1.3 The case of a CES production function

In this part, it is assumed that:

$$F(K_t, L_t, A_i) = \left[(A_i K_t)^{\frac{\sigma-1}{\sigma}} + L_t^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} \tag{70}$$

with $A_1 > 1$ and $A_2 < 1$. The wage and the capital gross rate of return are given by:

$$w_t^i = w^i(K_{t-1}) = \left[(A_i K_t)^{\frac{\sigma-1}{\sigma}} + 1 \right]^{\frac{1}{\sigma-1}} \tag{71}$$

$$R_t^i = R^i(K_{t-1}) = A_i \left(A_i K_t \right)^{\frac{-1}{\sigma}} \left[(A_i K_t)^{\frac{\sigma-1}{\sigma}} + 1 \right]^{\frac{1}{\sigma-1}} \tag{72}$$

If $\sigma \to +\infty$, this production function converges to the one used in the basic model (with $R_1 = A_1$ and $R_2 = A_2$).

It is possible to prove the following results. If A_1, A_2 and π_1 are such that there exists a bubbly steady state equilibrium for $\sigma = +\infty$, (conditions (22) and (23) are fulfilled), then, there exists bubbly equilibria in the model with the production function (70), under the assumption that σ is high enough. Moreover, pessimism is favorable to the existence of such equilibria.

The dynamics of the economy with a bubble equilibrium is studied through a phase diagram. Equations (58) and (59) can be written in introducing a new variable $x_{t-1} = K_t$. The dynamics becomes: if state 1 occurs in t,

$$\begin{aligned} x_t &= \phi^1(x_{t-1}, K_{t-1}) \\ K_t &= x_{t-1} \end{aligned}$$

if state 2 occurs in t,

$$\begin{aligned} x_t &= \phi^2(x_{t-1}, K_{t-1}) \\ K_t &= x_{t-1} \end{aligned}$$

with ϕ^1 and ϕ^2 defined implicitly by:

$$1 = \frac{\pi_1 a w^1(K_{t-1})}{\left[a w^1(K_{t-1}) - x_{t-1}\right] - \frac{(a w^2(x_{t-1}) - x_t)}{R^2(x_{t-1})}} + \frac{(1-\pi_1)\, a w^1(K_{t-1})}{\left[a w^1(K_{t-1}) - x_{t-1}\right] - \frac{(a w^1(x_{t-1}) - x_t)}{R^1(x_{t-1})}} \tag{73}$$

and

$$1 = \frac{\pi_1 a w^2(K_{t-1})}{\left[a w^2(K_{t-1}) - x_{t-1}\right] - \frac{(a w^2(x_{t-1}) - x_t)}{R^2(x_{t-1})}} + \frac{(1-\pi_1)\, a w^2(K_{t-1})}{\left[a w^2(K_{t-1}) - x_{t-1}\right] - \frac{(a w^1(x_{t-1}) - x_t)}{R^1(x_{t-1})}} \tag{74}$$

The right-hand side of (73) is denoted by $\chi^1(x_t, x_{t-1}, K_{t-1})$ and the right-hand side of (74) by $\chi^2(x_t, x_{t-1}, K_{t-1})$. As $\chi^1(x_t, x_{t-1}, K_{t-1})$ and $\chi^2(x_t, x_{t-1}, K_{t-1})$ are monotonic functions of x_t, the functions $\phi^i(x_{t-1}, K_{t-1})$ are well defined.

The dynamics is studied for K and x belonging to an interval $\left[\varepsilon, \bar{K}^1\right]$, with $\bar{K}^1$ the (unique) solution of the equation $\bar{K}^1 = a\ w^1(\bar{K}^1)$. $\bar{K}^1$ would be the long run steady state of the economy without bubbles and for a realization of state 1 (the good state of the nature) at all periods. ε is a positive number that is small enough. To fix a lower bound ε to the interval allows that all functions will be uniformly convergent to the functions obtained in the case $\sigma = +\infty$.

For the phase diagram, it is necessary to study for each state $i = 1, 2$ of the nature the sets: $\{(x_{t-1}, K_{t-1})$ such that $1 = \chi^i(x_{t-1}, x_{t-1}, K_{t-1})\}$ and $\{(x_{t-1}, K_{t-1})$ such that $K_{t-1} = x_{t-1}\}$.

As we are interested in the impact of π_1 on the two sets

$\{(x_{t-1}, K_{t-1})$ such that $1 = \chi^i(x_{t-1}, x_{t-1}, K_{t-1})\}$, we will further denote the relation as $1 = \bar{\chi}^i(x_{t-1}, K_{t-1}, \pi_1)$.

Lemma 2 *For σ high enough, the equation $1 = \bar{\chi}^i(x_{t-1}, K_{t-1}, \pi_1)$ implicitly defines a function $x_{t-1} = \psi^i(K_{t-1}, \pi_1)$ with the following properties: ψ^i is an increasing function of K_{t-1} and an increasing function of π_1. $\lim_{K_{t-1}\to 0} \psi^1(K_{t-1}, \pi_1) = \psi^2(K_{t-1}, \pi_1)$ and $\forall K_{t-1} \in \left[\varepsilon, \bar{K}^1\right]$, $\psi^1(K_{t-1}, \pi_1) > \psi^2(K_{t-1}, \pi_1)$.*

Proof. Firstly, it is proved that $x_{t-1} = \psi^i(K_{t-1}, \pi_1)$ is a well defined function. The proof is done for ψ^1 and is similar for ψ^2. The uniqueness of x_{t-1} is ensured by the property that $\bar{\chi}^1(x_{t-1}, K_{t-1}, \pi_1)$ is an increasing function of x_{t-1} for σ high enough, as (22) holds. Existence is obtained in considering each term of $\bar{\chi}^1(x_{t-1}, K_{t-1}, \pi_1)$. The expression

$$aw^1(K_{t-1}) - x_{t-1} - \frac{(aw^2(x_{t-1}) - x_t)}{R^2(x_{t-1})} \tag{75}$$

must be negative for relevant values of x_{t-1} (this property comes from the program of the consumer). For σ high enough, it is an increasing function of x_{t-1}. In ε, for ε small enough, it takes a positive value. For $x_{t-1} = aw^1(K_{t-1})$, it is negative. Therefore, it exists a value $\tilde{x}_t^1$ such that the expression (75) cancels out. The expression

$$aw^1(K_{t-1}) - x_{t-1} - \frac{(aw^1(x_{t-1}) - x_t)}{R^1(x_{t-1})} \tag{76}$$

must be positive for relevant values of x_{t-1} (this property comes from the program of the consumer). For σ high enough, it is an decreasing function of x_{t-1}. In ε, for ε small enough, it takes a positive value. For $x_{t-1} = \bar{K}^1$, it is negative. Therefore, it exists a value $\tilde{x}_t^2$ such that the expression (76) cancels out. Moreover, for σ high enough, $\tilde{x}_t^1 < \tilde{x}_t^2$. Finally, we have shown that

$$\lim_{\substack{x_{t-1} \to \tilde{x}_t^1 \\ x_{t-1} > \tilde{x}_t^1}} \bar{\chi}^1(x_{t-1}, K_{t-1}, \pi_1) \to -\infty \text{ and } \lim_{\substack{x_{t-1} \to \tilde{x}_t^2 \\ x_{t-1} < \tilde{x}_t^2}} \bar{\chi}^1(x_{t-1}, K_{t-1}, \pi_1) \to +\infty.$$

As a consequence, there exists a value $x_{t-1} \in (\tilde{x}_t^1, \tilde{x}_t^2)$ such that $1 = \bar{\chi}^1(x_{t-1}, K_{t-1}, \pi_1)$.

It is straightforward to see that $\bar{\chi}^1(x_{t-1}, K_{t-1}, \pi_1)$ is a decreasing function of K_{t-1} for σ high enough. Therefore, $\psi^1(K_{t-1}, \pi_1)$ is an increasing function of K_{t-1}.

$\bar{\chi}^1(x_{t-1}, K_{t-1}, \pi_1)$ is also a decreasing function of π_1 for σ high enough. Therefore, $\psi^1(K_{t-1}, \pi_1)$ is an increasing function of π_1.

All this study can also be done with $\psi^2(K_{t-1}, \pi_1)$ with analogous results.

The two last results come from the property that $\bar{\chi}^1(x_{t-1}, K_{t-1}, \pi_1)$ and $\bar{\chi}^2(x_{t-1}, K_{t-1}, \pi_1)$ only differ by the term $w^1(K_{t-1})$ in the first function that is replaced by $w^2(K_{t-1})$ in the second one. As $w^1(0) = w^2(0)$, it is obtained

that $\lim_{K_{t-1}\to 0} \psi^1(K_{t-1}, \pi_1) = \psi^2(K_{t-1}, \pi_1)$. Moreover, as $w^1(K_{t-1}) > w^2(K_{t-1})$ for any value of K_{t-1} and as $\bar{\chi}^i(x_{t-1}, K_{t-1}, \pi_1)$ are decreasing functions of $w^i(K_{t-1})$, it is obtained that $\psi^1(K_{t-1}, \pi_1) > \psi^2(K_{t-1}, \pi_1)$, $\forall K_{t-1} \in \left[\varepsilon, \bar{K}^1\right]$.
■

Figure 6 presents the phase diagram that can be drawn taking into account the results obtained in the preceding lemma. To make the figure, the frontiers ψ^1 and ψ^2 have been drawn numerically with the following values of the parameters: $\beta = 2 \Rightarrow a = 2/3$; $\pi_1 = 0.5$; $A_1 = 2$; $A_2 = 0.5$; $\sigma = 30$. Condition (22) is fulfilled as

$$\frac{\pi_1}{R_1} + \frac{(1-\pi_1)}{R_2} = 1.25 > 1$$

When $\sigma \to +\infty$, ψ^1 and ψ^2 converges to the horizontal line $\sigma = +\infty$ that corresponds to the simple case of the linear technology.

Red arrows correspond to the dynamics if state 1 occurs, green arrows for state 2. E_1 would be the bubbly steady state if state 1 occurred at all periods and A_1 the bubbleless steady state. E_2 would be the bubbly steady state if state 2 occurred at all periods and A_2 the bubbleless steady state. E_1 and E_2 are saddle points and A_1 and A_2 are stable.

Please see Figure 6.

K_{t-1} is a backward looking variable and x_{t-1} a forward looking variable.

Considering the dynamical system in which state 1 occurs at all periods, it is possible to define the stable manifold M_1 corresponding to the steady state E_1. For an initial point (x_{-1}, K_{-1}) above M_1, the dynamics converge to A_1, the bubbleless steady state. There exists a bubbly equilibrium but the bubble tends to vanish in the long run. For an initial point (x_{-1}, K_{-1}) under M_1, the dynamics reaches the zone of negative values for K_{t-1}, which is impossible: it cannot be an equilibrium.

The same analysis can be done for the dynamical system in which state 2 occurs at all periods and the stable manifold M_2 can be defined for the steady state E_2.

Now, some given history of the shocks $h = (\sigma_t)_{t\geq 0}$ is considered. An initial point (x_{-1}, K_{-1}) above M_1 will imply the convergence to a bubbleless long run equilibrium. An initial point (x_{-1}, K_{-1}) under M_2 will reach the zone of negative values for capital, which is excluded. A set X is defined as the set of all points (x, K) that are between the two curves M_1 and M_2. For the history h, a bubbly long run equilibrium can be defined as a sequence $(x_{t-1}, K_{t-1})_{t\geq 0}$ that remains in X for all t.

A long run bubbly equilibrium can be built with the following method. For a given initial value K_{-1}, it is possible to define $M(K_{-1})$ as the set of all values x such that $(x, K_{-1}) \in X$. Two sub-sets of $M(K_{-1})$ can be defined: $\bar{M}(K_{-1})$ as the values of x such that the dynamics starting from (x, K_{-1}) goes below X in a finite time; $\underline{M}(K_{-1})$ as the values of x such that the dynamics starting from (x, K_{-1}) goes under X in a finite time. If $x \in \bar{M}(K_{-1})$, any value $x' \in M(K_{-1})$ greater than x belongs to $\bar{M}(K_{-1})$. If $x \in \underline{M}(K_{-1})$, any value $x' \in M(K_{-1})$ smaller than x belongs to $\underline{M}(K_{-1})$. A value of x such that the dynamics starting from (x, K_{-1}) remains in X can be found in the (non-empty) set $\left[\sup \underline{M}(K_{-1}), \inf \bar{M}(K_{-1})\right]$. This method allows to find a long run bubbly equilibrium for the particular history h.

Finally, it is possible to say that pessimism is favorable to bubbly equilibrium. Indeed, pessimism is favorable in the case $\sigma = +\infty$ as condition (22) imposes an upper bound on π_1. On Figure 6, this properties corresponds to the fact that the line $\sigma = +\infty$ goes up when π_1 increases. For a too high value of π_1, this line would be above the bubbleless line aw : it would imply a negative value of the bubble which is impossible. For $\sigma < +\infty$ but high enough, the same property is observed: pessimism is favorable to bubbly equilibria as ψ^i is an increasing function of π_1. Pessimism implies a low value of π_1 that increases the zone of (x_{t-1}, K_{t-1}) corresponding to bubbly equilibria.

A simple numerical illustration is given of the role of π_1. All values are the same as before. The curves corresponding to $\pi_1 = 0.5$ are drawn in blue, and the curves corresponding to $\pi_1 = 0.45$ are drawn in green.

Please see Figure 7.

Finally, the numerical simulations show that, when σ increases in a neighborhood of $\sigma = +\infty$, this has a negative effect on the functions ψ^1 and ψ^2 that go down (see Figure 6). As a consequence, a higher value of σ increases the zone in which dynamic bubbly equilibria are possible, as a fall in π_1.

6.2 Case 2: exploding bubbles

In this part, it is shown that the results of the paper remains true for a CES production function with an elasticity of substitution greater than 1 : pessimism favors small stochastic bubbles (such that $d^1 > d^2$) whereas optimism favors big stochastic bubbles (such that $d^1 < d^2$).

The production remains given by (70) and factor prices by (71) and (72). The existence of the bubble is conditional to the occurrence of state 2. The budget constraints of a generation t agent are:

$$c_t + s_t + p_t x_t = w_t^2 \tag{77}$$

$$d_{t+1}^1 = R_{t+1}^1 s_t \tag{78}$$

$$d_{t+1}^2 = R_{t+1}^2 s_t + p_{t+1} x_t \tag{79}$$

From these constraints is obtained the intertemporal budget constraint:

$$c_t + \frac{d_{t+1}^1}{R_{t+1}^1}\left(1 - R_{t+1}^2 \frac{p_t}{p_{t+1}}\right) + d_{t+1}^2 \frac{p_t}{p_{t+1}} = w_t^2 \tag{80}$$

6.2.1 The case of small stochastic bubbles

It is assumed that the equilibrium is such that $d_{t+1}^1 = R_{t+1}^1 s_t > d_{t+1}^2 = R_{t+1}^2 s_t + p_{t+1} x_t$.The maximization of (5) with respect to (80) gives:

$$c_t = \frac{w_t^2}{1+\beta}$$

$$s_t = \frac{\beta w_t^2}{1+\beta} \frac{\pi_1}{1 - R_{t+1}^2 \frac{p_t}{p_{t+1}}}$$

The equilibrium conditions on the bubble market ($x_t = 1$) and on the capital market ($K_t = s_t$), using (77) leads to

$$K_t = a w_t^2 \frac{\pi_1}{1 - R_{t+1}^2 \frac{p_t}{p_{t+1}}} \tag{81}$$

$$K_t + p_t = a w_t^2 \tag{82}$$

These conditions are studied at the steady state: $K_t = K$ and $p_t = p$. It is obtained:

$$K(1-R^2) = \pi_1 a w^2 \quad (83)$$
$$K + p = a w^2 \quad (84)$$

A bubbly steady state equilibrium is a solution (K, p) to this system of equations such that $p > 0$ and that satisfies the assumption $d^1 > d^2$, or:

$$R^1 K > R^2 K + p$$

(83) can be written

$$1 = \frac{\pi_1 a w^2}{K} + R^2 \quad (85)$$

The right-hand side member of this equation is a decreasing function of K, with an infinite limit in 0 and a limit equal to 0 when $K \to \infty$. Therefore, there exists a unique solution of this equation and this solution is an increasing function of π_1. When π_1 varies from 0 to 1, K increases from a value K_{inf} to a value K_{sup}.

Equation (84) defines a positive value for p if $aw^2 > K$ or

$$\frac{aw^2}{K} > 1$$

The left-hand side member of this equation is a decreasing function of K with an infinite limit in 0 and a limit equal to 0 when $K \to \infty$. Therefore this inequality allows to define a value K_l such that $p > 0 \Leftrightarrow K < K_l$.

It is easy to prove that $K_l < K_{\text{sup}}$. Indeed, if $K_l > K_{\text{sup}}$, for $K = K_{\text{sup}}$, $aw^2(K_{\text{sup}}) > K_{\text{sup}}$. But, by definition of K_{sup}, $aw^2(K_{\text{sup}}) + R^2(K_{\text{sup}})K_{\text{sup}} = K_{\text{sup}}$ which contradicts the preceding inequality. Then, $K_l < K_{\text{sup}}$.

The existence of a bubbly long run equilibrium is now studied with respect to the parameter π_1. There exists a bubbly equilibrium such that $p > 0$ iff $K_l > K_{\text{inf}}$. As K_{inf} is defined as the solution of $R^2(K_{\text{inf}})K_{\text{inf}} = K_{\text{inf}}$, it is also possible to write the condition $aw^2(K_{\text{inf}}) > K_{\text{inf}}$ under the form

$$aw^2(K_{\text{inf}}) > R^2(K_{\text{inf}})K_{\text{inf}} \quad (86)$$

The amount of savings in $K_{\inf}$ must be higher than the amount of capital income. This property is standard for the existence of bubbles in other frameworks.

Assuming that $K_l > K_{\inf}$ (condition 86), there exists an interval $\left(0, \pi_1^l\right)$ such that the long run bubbly equilibrium is well defined. The value π_1^l is defined as the solution of the equation (85) for $K = K_l$:

$$1 = \frac{\pi_1^l a w^2(K_l)}{K_l} + R^2(K_l)$$

Finally, for any value $\pi_1 \in \left(0, \pi_1^l\right)$, there exists a solution (K, p) of the system (83), (84) such that $p > 0$. There remains one condition that must be fulfilled, $R^1 K > R^2 K + p$ or

$$R^1 K > R^2 K + a w^2 - K \tag{87}$$

This condition can always be satisfied if a high enough value is chosen for A_1.

All this study shows that a long run bubbly equilibrium may exist under condition (86) for a low enough value of π_1, $\pi_1 \in \left(0, \pi_1^l\right)$. For a given value of the probability π, as pessimism transforms this value in a lower one π_1, pessimism is favorable to the existence of bubbly equilibria.

6.2.2 The case of big stochastic bubbles

This case is close to the preceding one. It is now assumed that the equilibrium is such that $d_{t+1}^1 = R_{t+1}^1 s_t < d_{t+1}^2 = R_{t+1}^2 s_t + p_{t+1} x_t$. The maximization of (6) with respect to (80) gives:

$$\begin{aligned} c_t &= \frac{w_t^2}{1+\beta} \\ s_t &= \frac{\beta w_t^2}{1+\beta} \frac{\pi_2}{1 - R_{t+1}^2 \frac{p_t}{p_{t+1}}} \end{aligned}$$

The equilibrium conditions leads to

$$K_t = a w_t^2 \frac{\pi_2}{1 - R_{t+1}^2 \frac{p_t}{p_{t+1}}} \tag{88}$$

$$K_t + p_t = a w_t^2 \tag{89}$$

These conditions are studied at the steady state: $K_t = K$ and $p_t = p$. It is obtained:

$$K(1 - R^2) = \pi_2 a w^2 \tag{90}$$
$$K + p = a w^2 \tag{91}$$

A bubbly steady state equilibrium is a solution (K, p) to this system of equations such that $p > 0$ and that satisfies the assumption $d^1 < d^2$, or:

$$R^1 K < R^2 K + p$$

(90) can be written

$$1 = \frac{\pi_2 a w^2}{K} + R^2 \tag{92}$$

There exists a unique solution of this equation and this solution is an increasing function of π_2. When π_2 varies from 0 to 1, K increases from K_{inf} to K_{sup}.

Equation (91) defines a positive value for p if $aw^2 > K$ or

$$\frac{aw^2}{K} > 1$$

This inequality defines the same threshold value K_l^2 such that $p > 0 \Leftrightarrow K < K^l$. As before, it is obtained that $K^l < K_{\text{sup}}$.

The existence of a long run bubbly equilibrium is now studied with respect to the parameter π_2. There exists a bubbly equilibrium such that $p > 0$ iff $K^l > K_{\text{inf}}$. This condition leads to the preceding condition (86)

$$aw^2(K_{\text{inf}}) > R^2(K_{\text{inf}})K_{\text{inf}}$$

The amount of savings in K_{inf} must be higher than the amount of capital income.

Assuming that $K^l > K_{\text{inf}}$, there exists an interval $\left(0, \pi_2^l\right)$ such that the long run bubbly equilibrium is well defined. The value π_2^l is in fact equal to π_1^l as it is defined by the same equation.

Finally, for any value $\pi_2 \in \left(0, \pi_2^l\right)$, there exists a solution (K, p) of the system (90), (91) such that $p > 0$. There remains one condition that must be fulfilled, $R^1 K < R^2 K + p$ or

$$R^1 K < R^2 K + aw^2 - K \tag{93}$$

This condition can be satisfied if the value of A_1 is not too high.

All this study shows that a long run bubbly equilibrium may exist for a low enough value of π_2, $\pi_2 \in \left(0, \pi_2^l\right)$. For a given value of the probability π, as optimism transforms this value in a lower one π_2, optimism is favorable to the existence of bubbly equilibria.

6.2.3 The role of the elasticity of substitution σ

To sum up the results, if (86) holds, it is possible to build equilibria with stochastic bubbles. Such an equilibrium is obtained by choosing any value $K \in \left(K_{\inf}, K^l\right)$. If (87) holds (which is true if A_1 is high enough), it is associated with a bubble such that $d^1 > d^2$. For this value, it is possible to find a transformed probability π_1 and a value p of the bubble consistent with the equilibrium. If (93) holds (which is true if A_1 is low enough), it is associated with a bubble such that $d^1 < d^2$. For this value, it is possible to find a transformed probability π_2 and a value p of the bubble consistent with the equilibrium.

For a CES production function given by (70) and factor prices by (71) and (72), condition (86) takes the following expression:

$$\frac{1}{A_2^{-(\sigma-1)} - 1} < a \tag{94}$$

By assumption, $A_2 < 1$. This inequality gives a lower bound for σ :

$$\sigma > 1 + \frac{-1}{\ln A_2} \ln \left(\frac{1+a}{a}\right)$$

For the Cobb-Douglas production function (60), condition (86) takes the following expression:

$$\frac{\alpha}{1-\alpha} < a \tag{95}$$

which is the usual condition for dynamic inefficiency in a deterministic economy. Note that condition (86) is not obtained as the limit of (94) as σ tends to 1. Indeed, (70) is not defined in such a way that it tends to (60) for σ tending to 1.

For the Cobb-Douglas case ($\sigma = 1$), the condition $aw^2(K_{\text{inf}}) > R^2(K_{\text{inf}})K_{\text{inf}}$ does not depend on the productivity parameter of the technology. A technological shock affects in the same way capital and labor productivity. Therefore the occurrence of shocks and the transformation of probabilities by the agents play no rule in the existence of bubbly equilibria.

For a CES function, in the limit case $\sigma = +\infty$ retained through the paper, the capital augmenting technology shock affects capital productivity but has no impact on labor productivity. This is the most favorable case for the existence of bubbly equilibria.

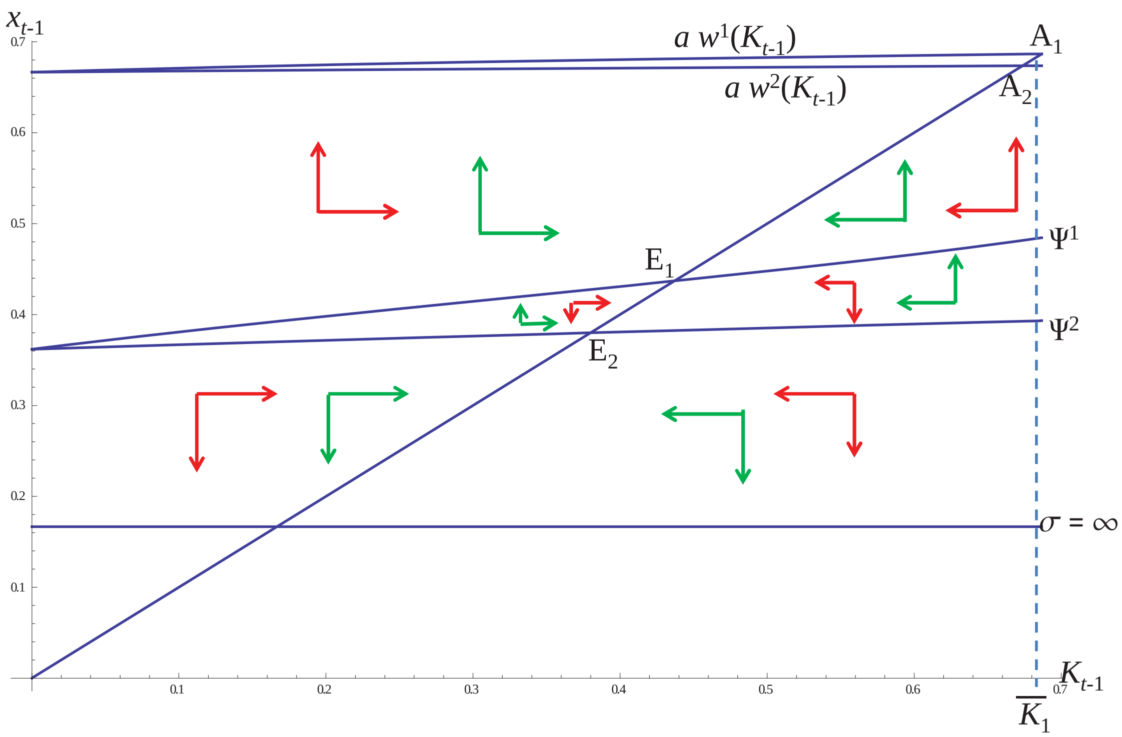

Fig 6: phase diagram with a CES production function

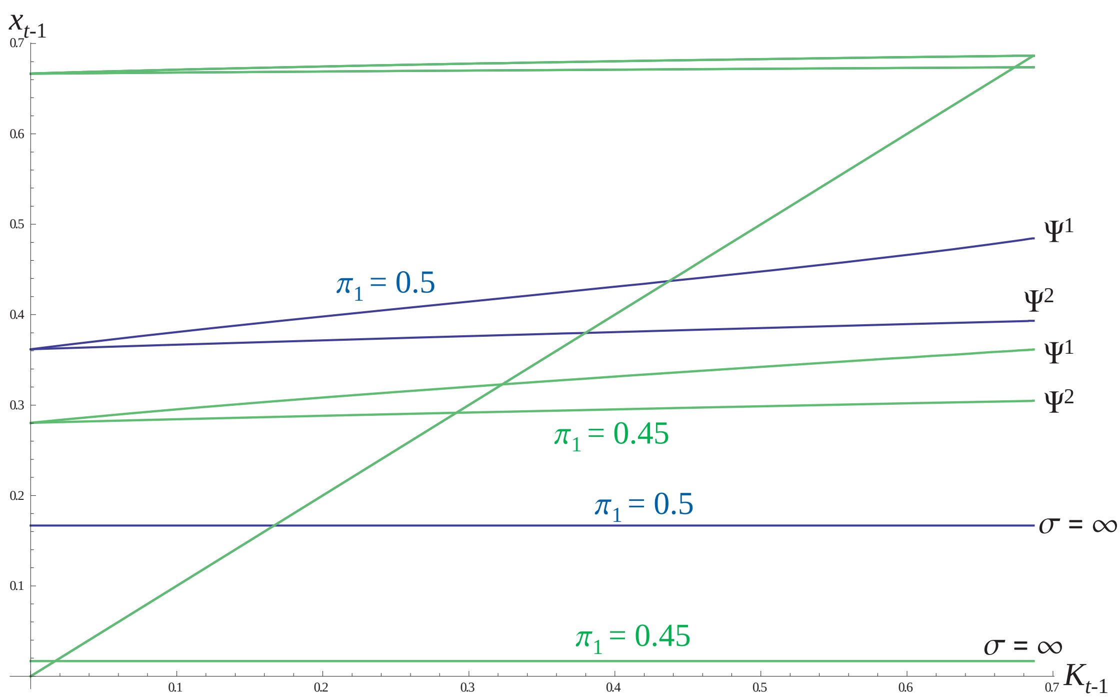

Fig 7: effect of a decrease of π_1